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ON THE INVENTORY MODEL WITH TWO DELAYING BARRIERS

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Abstract: In this paper the process of semi-Markovian random walk with negative drift under angle α $(0^{\circ} < \alpha < 90^{\circ})$, and positive jumps with probability ρ $(0 < \rho < 1)$ having two delaying screens at level zero and $a\,(a > 0)$ is constructed. The exact expressions for Laplace transforms of the distributions of the first moments in order to reach to these screens by the process and, in particularly, the expectations and the variances of indicated distributions are obtained.
Keywords: Semi-Markovian random walk, inventory level, delaying screen, positive jumps, Laplace transforms.

1. INTRODUCTION

It is known that a numerous interesting problems in the fields of reliability, queuing, inventory theories, biomedicine etc., are given in terms of the stochastic processes with discrete chance interference. Particularly, these problems can often be modeled by using random walk with one or two barriers. There is large literature on theory and application about random walk with one or two barriers. For example, Spitzer (1964), Feller (1966), Skorohod & Slobedenyuk (1970), Borovkov (1975), Korolyuk & Turbin (1976), Nasirova (1984, 1999), Lotov (1991), Alsmeyer (1992), El-Shehaway (1992), Zhang (1992), Khaniev & Ozdemir (1995,1998), Khaniev & Kucuk (2004), Unver (1997), etc. On the other hand the process with negative drift under the angle a $(0^{\circ} < a < 90^{\circ})$ having positive jumps with probability ρ $(0 < \rho < 1)$ with two delaying screens have not be investigated properly yet. In this paper, the process of semi-Markovian random walk with negative drift under the angle $a(0^{\circ} < a < 90^{\circ})$, and positive jumps with probability ρ $(0 < \rho < 1)$ having two delaying screens at level zero and a $(a > 0)$ is constructed. The exact expressions for Laplace transforms of the distributions of the first moments in order to reach to these screens by the process, in particularly, the expectations and the variances of indicated distributions are obtained.

2. DESCRIPTION OF THE PROBABILISTIC MODEL

Suppose that, initially, there is z $(0 < z < a)$ stock in a warehouse, with size a. Let ξ_1^0 denotes the amount which customer demands. The customers are served with a fixed speed. After serving to all customers, the expected stock level at the end of the inventory cycle equals to $z - \xi_1^0$. An inventory cycle is defined as the time period between two successive arrivals of orders. If the warehouse meets the demands of all customers and still keeps a stock of amount $z - \xi_1^0 > 0$ then the stock ceases serving until new customers arrive. When a new demand arrives, there are two possibilities to continue:

a) A new demand is immediately served with the probability $1-\rho$ $(0 \le \rho \le 1)$

b) A stock with the amount ζ_1 and probability ρ is stored at the warehouse and then the new demand is served.

Since storage capacity in the warehouse is limited, amount of inventory in the warehouse can't be greater than *a*. In this case, in the beginning of the inventory cycle (right after quantity order ζ_1 of size is received), the expected stock level is $\min(a, z-\xi_1^0+\zeta_1)$. After serving all the customers, inventory in the warehouse is consumed $(z-\xi_1^0 \le 0)$ new arriving customer service is started after an order of size $\min(a, \zeta_1^0)$ is received. The level of the stock is a stochastic process and we denote it by X(t). For this inventory model, it is important to determine the distributions of the first moments of the exhaustion and to fill the warehouse.

3. STRUCTURE OF THE PROCESS AND MATHEMATICAL STATEMENT OF THE PROBLEM

Suppose that, $\{\xi_k^0, \eta_k, \zeta_k\}_{k=\overline{1,\infty}}$ is a sequence of independent and identically distributed random variables defined on probability space $(\Omega, \Im, P)$ such that $\xi_k^0 > 0, \eta_k > 0, \zeta_k \ge 0$ are independent random variables. Construct the following process:

$$X_1(t) = \begin{cases} z + \sum_{i=1}^{k-1} \zeta_i - [t - \sum_{i=1}^{k-1} \eta_i] \mathrm{ctg}\alpha, & \sum_{i=1}^{k-1} (\xi_i + \eta_i) \le t < \sum_{i=1}^{k-1} (\xi_i + \eta_i) + \xi_k \\ z + \sum_{i=1}^{k-1} \zeta_i - \sum_{i=1}^{k} \xi_i^0, & \sum_{i=1}^{k-1} (\xi_i + \eta_i) + \xi_k \le t < \sum_{i=1}^{k} (\xi_i + \eta_i) \end{cases}$$

where $\xi_k = \xi_k^0 \mathrm{tg}\alpha, \quad k = \overline{1,\infty}, z > 0$. Delaying the process $X_1(t)$ with screen "0", we have

$X_2(t) = X_1(t) - \inf\limits_{0 \le s \le t}(0, X_1(t)))$ and delaying the process $X_2(t)$ with screen "a", we have

$$X(t) = a + X_2(t) - \sup_{0 \le s \le t}(a, X_2(s)).$$

The process $X(t)$ is called the process of semi-Markovian random walk with negative drift under the angle α $(0^\circ < \alpha < 90^\circ)$, positive jumps with probability ρ $(0 \le \rho \le 1)$ and two delaying screens at the level of zero and a (a>0). The first moments for reaching screens at the level of zero and a (a>0) by this process can be obtained by writing

$$\tau_1^0 = \sum_{i=1}^{\nu_1^0 - 1} (\xi_i + \eta_i) + \xi'_{\nu_1^0} \quad \text{and} \quad \tau_1^a = \sum_{i=1}^{\nu_1^a} (\xi_i + \eta_i), \tag{3.1}$$

where ν_1^0 and ν_1^a are the numbers of the steps (inventory cycles) for the first moments of reaching screens at level zero and a (a>0) by process $X(t)$, respectively; $\xi'_{\nu_1^0}$ is the part of the random variable $\xi_{\nu_1^0}$.

4. DETERMINATION OF LAPLACE TRANSFORM $L_{\tau_1^0}(\theta)$

Denote the Laplace transforms of the random variables τ_1^0, ξ_1, η_1 by

$$L_{\tau_1^0}(\theta) = Ee^{-\theta\tau_1^0},\ L_{\xi_1}(\theta) = Ee^{-\theta\xi_1} \text{ and } L_{\tau_1^0}(\theta) = Ee^{-\theta\tau_1^0},\ \theta > 0. \tag{4.1}$$

Using the Wald equality from (3.1), (4.1.), (4,2) and (4.3), we obtain

$$L_{\tau_1^0}(\theta) = L_{\xi'_{\nu_1^0}}(\theta)\frac{{}_0\Psi(L_{\xi_1}(\theta)L_{\eta_1}(\theta))}{L_{\xi_1}(\theta)L_{\eta_1}(\theta)} \tag{4.2}$$

We denote the generating function of the random variable ν_1^0 by ${}_0\Psi(u) = Eu^{\nu_1^0}$, $0 < u \le 1$. After that, by using the total probability formula for expectations, we have

$${}_0\Psi(u) = \int_0^a {}_0\Psi(u,z)P\{X(0) \in dz\},\ z > 0, \tag{4.3}$$

where ${}_0\Psi(u,z) = \sum_{k=1}^{\infty} u^k P\{\nu_1^0 = k / X(0) = z\},\ 0 < u \le 1$.

Using total probability formula for $k \ge 2$, we have

$$P\{\nu_1^0 = k / X(0) = z\} = \int_{y=0+}^{a} P\{z - \xi_1^0 > 0; \zeta_1 = 0; z - \xi_1^0 \in dy\}P\{\nu_1^0 = k - 1 / X(0) = y\} +$$

$$+ \int_{y=0+}^{a} P\{z - \xi_1^0 > 0; \zeta_1 > 0; \min(a, z - \xi_1^0 + \zeta_1) \in dy\}P\{\nu_1^0 = k - 1 / X(0) = y\}.$$

Multiplying both sides of this equality by u^k $(0 < u \le 1)$, summing up for $k = \overline{2,\infty}$ and after a simple transformation we obtain

$$\begin{aligned}{}_0\Psi(u,z) &= uP\{\xi_1^0 > z\} + {}_0\Psi(u,a)uP\{z - \xi_1^0 > 0; \zeta_1 > 0; z - \xi_1^0 + \zeta_1 > a\} +\\ &+ uP\{\zeta_1 = 0\}\int_{y=0+}^{a} {}_0\Psi(u,y)d_yP\{z - \xi_1^0 < y\} +\\ &+ u\int_{y=0+}^{a} {}_0\Psi(u,y)d_yP\{z - \xi_1^0 > 0; \zeta_1 > 0; z - \xi_1^0 + \zeta_1 < y\}.\end{aligned} \tag{4.4}$$

Suppose that the distribution of the random variable ξ_1^0 has the density function $P_{\xi_1^0}(x),\ x > 0$ and the distribution of the random variable ζ_1 has the density function $P_{\zeta_1}(x),\ x > 0$ and has a jump with size $1-\rho$ at $x = 0$. Then (4.4) can be written as follows:

$$\begin{aligned}{}_0\Psi(u,z) &= uP\{\xi_1^0 > z\} + {}_0\Psi(u,a)u\int_{x=0}^{a} P\{\zeta_1 > a + x - z\}P_{\xi_1^0}dx +\\ &+ uP\{\zeta_1 = 0\}\int_{y=+}^{a} P_{\xi_1^0}(z-y)\,{}_0\Psi(u,y)dy + u\int_{y=0+}^{z}\Big[\int_{x=z-y}^{z} P_{\xi_1^0}(x)P_{\zeta_1^0}(x+y-z)dx\Big]{}_0\Psi(u,y)dy +\\ &+ u\int_{y=0+}^{a}\Big[\int_{x=0}^{z} P_{\xi_1^0}(x)P_{\zeta_1^0}(x+y-z)dx\Big]{}_0\Psi(u,y)dy.\end{aligned} \tag{4.5}$$

We can solve this equation by using the Erlang class of distributions.

$$P\{\xi_1^0 < t\} = 1 - e^{\mu t}, \quad P\{\zeta_1 < t\} = 1 - \rho e^{\lambda t}, \quad t > 0, \mu > 0, \lambda > 0, \quad 0 < \rho \le 1 \tag{4.6}$$

Using this distribution we can write expression (4.5) as

$$\Psi_0(u,z) = e^{-\mu z} + u\frac{\mu\rho}{\lambda+\mu} e^{-\lambda a}(e^{\lambda z} - e^{-\mu z})_0\Psi(u,a) +$$

$$+\mu[\frac{\lambda\rho}{\lambda+\mu} + (1-\rho)]e^{-\mu z}u\int_{y=0+}^{z} e^{\mu y}\,{}_0\Psi(u,y)dy +$$

$$+\frac{\lambda\mu\rho}{\lambda+\mu}e^{\lambda z}u\int_{y=z}^{a} e^{-\lambda y}\,{}_0\Psi(u,y)dy - \frac{\lambda\mu\rho}{\lambda+\mu}e^{-\mu z}u\int_{y=0+}^{a} e^{-\lambda y}\,{}_0\Psi(u,y)dy\,.$$

From this integral equation we have the following differential equation

$${}_0\Psi_z''(u,z) - [\lambda - \mu + \mu(1-\rho)u]_0\Psi_z'(u,z) - \lambda\mu(1-u)_0\Psi(u,z) \tag{4.7}$$

which has the solution:

$${}_0\Psi(u,z) = c_1(u)e^{zk_1(u)} + c_2(u)e^{zk_2(u)}, \tag{4.8}$$

where $k_1(u)$ and $k_2(u)$ are the roots of the characteristic equation

$$k^2(u) - [\lambda - \mu + \mu(1-\rho)u]k(u) - \lambda\mu(1-u) = 0\,.$$

Hence, we have

$$k_{1,2}(u) = \left\{[\lambda - \mu + \mu(1-\rho)u] \mp \sqrt{[\lambda - \mu + \mu(1-\rho)u]^2 + 4\lambda\mu(1-u)}\right\}/2 \tag{4.9}$$

The functions $c_1(u)$ and $c_2(u)$ can be obtained from the following boundary conditions

$${}_0\Psi(u,0) = u; \quad {}_0\Psi_z'(u,0) = -\mu u + \mu\rho e^{-\lambda a}\,{}_0\Psi(u,a)u + \mu(1-\rho)u^2 + \lambda\mu\rho u\int_{y=0+}^{a} e^{-\lambda y}\,{}_0\Psi(u,y)dy$$

from which we obtain

$$c_1(u) + c_2(u) = u; \quad [k_1(u) + \mu(1-u)]e^{ak_1(u)}c_1(u) + [k_2(u) + \mu(1-u)]e^{ak_2(u)}c_2(u) = 0 \tag{4.10}$$

Hence, we have

$$c_1(u) = -\frac{u[k_2(u) + \mu(1-u)]e^{ak_2(u)}}{[k_1(u) + \mu(1-u)]e^{ak_1(u)} - [k_2(u) + \mu(1-u)]e^{ak_2(u)}u}, \tag{4.11}$$

$$c_2(u) = u - c_1(u)\,. \tag{4.12}$$

Using the fact that the distribution of the random variable $X(0)$ coincides with the distribution of the random variable $\min(a,\zeta_1)$ and in addition to this using the distribution of the random variable ζ_1, which has a jump with of size $(1-\rho)$, we can write (4.3) as

$${}_0\Psi(u) = (1-\rho)_0\Psi(u,0) + \rho e^{-\lambda a}\,{}_0\Psi(u,a) + \lambda\rho\int_0^a e^{-\lambda z}\,{}_0\Psi(u,z)dz\,. \tag{4.13}$$

where ${}_0\Psi(u,z)$ is given by (4.8) together with (4.11) and (4.12). Substituting (4.8) into (4.13) and using the expression ${}_0\Psi(u,0) = u$ that was obtained earlier, we finally have

$${}_0\Psi(u) = (1-\rho)u + \rho\sum_{i=1}^{2}\frac{1}{\lambda - k_i(u)}[\lambda - k_i(u)e^{-[\lambda - k_i(u)]a}]c_i(u), i = 1,2\,. \tag{4.14}$$

Now we can find the Laplace transform of the random variable τ_1^0. In fact, the random variable ξ_1 has the exponential distribution, and we have $L_{\xi_{\nu_1^0}}(\theta) = L_{\xi_1}(\theta)$. Then we can write (4.2) as

$$L_{\tau_1^0}(\theta) = L_{\eta_1}^{-1}(\theta)_0\Psi(u), \tag{4.15}$$

where $u = L_{\xi_1}(\theta)L_{\eta_1}(\theta) = \dfrac{\mu}{\mu+\theta}L_{\eta_1}(\theta)$. Finally substituting (4.14) into (4.15), we get

$$L_{\tau_1^0}(\theta) = L_{\eta_1}^{-1}(\theta)[(1-\rho)u + \rho\sum_{i=1}^{2}\frac{1}{\lambda - k_i(u)}[\lambda - k_i(u)e^{-[\lambda - k_i(u)]a}]c_i(u)]. \tag{4.16}$$

5. CALCULATION OF $E\tau_1^0$ AND $Var\tau_1^0$

From (3.1), we have

$$E\tau_1^0 = (E\xi_1 + E\eta_1)(E\nu_1^0 - 1) + E\xi'_{\nu_1^0}, \tag{5.1}$$

$$Var\tau_1^0 = (Var\xi_1 + Var\eta_1)(E\nu_1^0 - 1) + [E\xi_1 + E\eta_1]Var\nu_1^0 + Var\xi'_{\nu_1^0}. \tag{5.2}$$

Note that, these formulas can also be obtained from (4.16). As the random variable ξ_1 is exponential the distributed, we have $E\xi'_{\tau_1^0} = E\xi_1, Var\xi'_{\tau_1^0} = Var\xi_1$. Then

$$E\tau_1^0 = (E\xi_1 + E\eta_1)E\nu_1^0 - E\eta_1, \tag{5.3}$$

$$Var\tau_1^0 = (Var\xi_1 + Var\eta_1)E\nu_1^0 + [E\xi_1 + E\eta_1]Var\nu_1^0 - Var\eta_1. \tag{5.4}$$

By using the property of generating function of a random variable, we can write

$$E\nu_1^0 = {}_0\Psi'_u(u) \text{ and } Var\nu_1^0 = {}_0\Psi''_u(1) + {}_0\Psi'_u(1)[1 - {}_0\Psi'_u(1)] \tag{5.5}$$

The expressions for ${}_0\Psi'(1)$ and ${}_0\Psi''(1)$ can be found by using (4.14). But we will use (4.13) to determine these derivations easily. Firstly, we have to determine expressions for $k_i(1), k'_i(1), k''_i(1), c_i(1), c'_i(1), c''_i(1),\ i = 1,2$. These expressions can be obtained from (4.9), (4.11) and (4.12) at u=l (Table 5.1).

By differentiating (4.13) with u=1 and substituting obtained expressions for ${}_0\Psi'(1)$, ${}_0\Psi''(1)$ into (5.5) together with ${}_0\Psi(u,0) = 0, k_1(1) = 0, c_1(1) = 0, c_2(1) = 0$ we get

$$E\nu_1^0 = [k'_1(1) + k_2(1)c'_2(1)]/\mu, \tag{5.6}$$

$$Var\nu_1^0 = \frac{1}{\mu}\{k'''_1(1) + 2[k'_1(1)c'_1(1) + k'_2(1)c'_2(1)] + k_2(1)c''_2(1)\} - E\nu_1^0[1 + E\nu_1^0]. \tag{5.7}$$

Table 5.1. Expressions for $k_i(1), k'_i(1), k''_i(1), c_i(1), c'_i(1), c''_i(1),\ i = 1,2$

<table>
<tr><td>$k_1(1)$</td><td>0</td><td>$k'_1(1)$</td><td>$\dfrac{\lambda\mu}{\lambda-\mu\rho}$</td><td>$c_1(1)$</td><td>1</td><td>$c'_1(1)$</td><td>$1+(\dfrac{\mu}{\lambda-\mu\rho})^2\rho e^{-(\lambda-\mu\rho)a}$</td></tr>
<tr><td>$k''_1(1)$</td><td colspan="3">$\dfrac{\mu^2[\lambda+\mu\rho+(\lambda-\mu\rho)\rho]^2}{2(\lambda-\mu\rho)^3} - \dfrac{[\mu(1-\rho)]^2}{2(\lambda-\mu\rho)}$</td><td>$c''_1(1)$</td><td colspan="3">$\dfrac{1}{k_2(1)}\{[k''_1(1)+2ak'_1(1)[k'_1(1)-\mu]]e^{-k_2(1)a} + 2[k'_1(1)-\mu]e^{-k_2(1)a}c + 2[k'_2(1)+ak_1(1)k'_1(1)-\mu]c'_1(1)$</td></tr>
<tr><td>$k_2(1)$</td><td colspan="3">$\lambda-\mu\rho$</td><td>$c_2(1)$</td><td colspan="3">0</td></tr>
<tr><td>$k'_2(1)$</td><td colspan="3">$\mu(1-\rho)-[\lambda\mu/(\lambda-\mu\rho)]$</td><td>$c'_2(1)$</td><td colspan="3">$1-c'_1(1)$</td></tr>
<tr><td>$k''_2(1)$</td><td colspan="3">$-k''_1(1)$</td><td>$c''_2(1)$</td><td colspan="3">$-c''_1(1)$</td></tr>
</table>

Finally, from (5.3) and (5.4) together with (5.6) and (5.7), we have

$$E\tau_1^0 = (\frac{1}{\mu} + E\eta_1)E\nu_1^0 - E\eta_1, \qquad (5.8)$$

$$Var\tau_1^0 = (\frac{1}{\mu^2} + Var\eta_1)E\nu_1^0 + [\frac{1}{\mu} + E\eta_1]^2 Var\nu_1^0 - Var\eta_1, \qquad (5.9)$$

where $\eta_1 > 0$ has a general distribution.

6. DETERMINATION OF THE LAPLACE TRANSFORM $L_{\tau_1^a}(\theta)$

Denote the Laplace transform of the random variable τ_1^a by

$$L_{\tau_1^a}(\theta) = Ee^{-\theta\tau_1^a}, \theta > 0 \qquad (6.1)$$

Using the Wald equality from (6.1), (3.1), (4,1), we obtain

$$L_{\tau_1^a}(\theta) = {}_a\Psi(L_{\xi_1}(\theta)L_{\eta_1}(\theta)) \qquad (6.2)$$

Denote the generating function of random variable ν_1^a by ${}_a\Psi(u) = Eu^{\nu_1^a}, 0 < u \le 1$. Then, by using total probability formula for expectations, we have

$${}_a\Psi(u) = \int_0^a {}_a\Psi(u,z)P\{X(0) \in dz\}, z > 0 \qquad (6.3)$$

where ${}_a\Psi(u,z) = \sum_{k=1}^{\infty} u^k P\{\nu_1^0 = k / X(0) = z\}, 0 < u \le 1$. Using the total probability formula for $k \ge 2$ and multiplying both sides of this equality by $u^k (0 < u \le 1)$, summing up for $k = \overline{2,\infty}$, and transforming, we obtain

$${}_a\Psi(u,z) = u\int_{x=0}^{z} P\{\zeta_1 > a - z + x\}P\{\xi_1^0 \in dx\} + uP\{\zeta_1 > a\}P\{\xi_1^0 > z\} +$$

$$+ uP\{\zeta_1 = 0\}P\{\xi_1^0 > z\}{}_a\Psi(u,0) + u\int_{y=z}^{a} {}_a\Psi(u,y)\int_{x=0}^{z} P\{\zeta_1 > 0; \zeta_1 < x + y - z\}P\{\xi_1^0 \in dx\} +$$

$$+ u\int_{y=0}^{z} {}_a\Psi(u,y)dy \int_{x=z-y}^{a} P\{\zeta_1 > 0; \zeta_1 < x + y - z\}P\{\xi_1^0 \in dx\} +$$

$$+ uP\{\zeta_1^0 > 0\}\int_{y=0}^{a} {}_a\Psi(u,y)d_y P\{\zeta_1 > 0; \zeta_1 < y\} +$$

$$+ uP\{\zeta_1 = 0\}\int_{y=0}^{a} {}_a\Psi(u,y)d_y P\{z - \xi_1^0 > 0; \xi_1^0 \in dy\}. \qquad (6.4)$$

Analogous to the Sec. 5 by using the distributions of the random variables ξ_1^0 and ζ_1, which are given by (4.8) we can write

$${}_a\Psi(u,z) = u\frac{\mu\rho}{\lambda + \mu}e^{-\lambda a}(e^{\lambda z} - e^{\mu z}) + u\rho e^{-\lambda a - \mu z} +$$

$$+ \frac{\lambda\mu\rho u}{\lambda + \mu}e^{\lambda z}\int_{y=z}^{a} e^{-\lambda y}{}_a\Psi(u,y)dy - \frac{\lambda\mu\rho u}{\lambda + \mu}e^{-\mu z}\int_{y=0}^{a} e^{-\lambda y}{}_a\Psi(u,y)dy +$$

$$+\frac{\lambda\mu\rho u}{\lambda+\mu}e^{-\mu z}\int_{y=0}^{z}e^{\mu y}{}_a\Psi(u,y)dy-\lambda\rho u e^{-\mu z}\int_{y=0}^{a}e^{-\lambda y}{}_a\Psi(u,y)dy+$$

$$+u\mu P\{\zeta_1=0\}e^{-\mu z}\int_{y=0}^{z}e^{\mu y}{}_a\Psi(u,y)dy+uP\{\zeta_1=0\}e^{-\mu z}{}_a\Psi(u,0). \qquad (6.5)$$

From this integral equation we have the following differential equation

$${}_a\Psi_z''(u,z)-[\lambda-\mu+\mu(1-\rho)u]_a\Psi_z'(u,z)-\lambda\mu(1-u)_a\Psi_z(u,z)=0, \qquad (6.6)$$

which has the solution

$${}_a\Psi(u,z)=d_1(u)e^{zk_1(u)}+d_2(u)e^{zk_2(u)}, \qquad (6.7)$$

where $k_1(u)$ and $k_2(u)$ are given by (4.9). The expressions for $d_1(u)$ and $d_2(u)$ are obtained from the following boundary conditions

$${}_a\Psi_z'(u,0)=0; \qquad {}_a\Psi(u,0)=u\rho e^{-\lambda a}+u(1-\rho)_a\Psi(u,a)+\lambda\rho u\int_{y=0+}^{a}e^{-\lambda y}{}_a\Psi(u,y)dy$$

from which we obtain

$$d_1(u)=u\mu\rho k_2(u)e^{-\lambda a}\cdot\{\mu[1-u(1-\rho)][k_2(u)-k_1(u)]-k_2(u)[\lambda-k_2(u)][1-e^{-[\lambda-k_1(u)]a}]+k_1(u)[\lambda-k_1(u)][1-e^{-[\lambda-k_2(u)]a}]\}^{-1}$$

$$d_2(u)=-[k_1(u)/k_2(u)]d_1(u) \qquad (6.8)$$

Analogous to the Sec. 4 [Eq. (4.13)] generating function for ${}_a\Psi(u)$ can be written as

$${}_a\Psi(u)=(1-\rho)_a\Psi(u,0)+\rho e^{-\lambda a}{}_a\Psi(u,a)+\lambda\rho\int_0^a e^{-\lambda z}{}_a\Psi(u,z)dz. \qquad (6.9)$$

Inserting (6.7) together with (6.8) into (6.9), we have

$${}_a\Psi(u)=\sum_{i=1}^{2}\{(1-\rho)+\rho e^{-[\lambda-k_i(u)]a}+\frac{\lambda\rho}{\lambda-k_i(u)}[1-e^{-[\lambda-k_i(u)]a}]\}d_i(u), \qquad (6.10)$$

Finally substituting (6.10) into (6.2), we have

$$L_{\tau_1^a}(\theta)=\sum_{i=1}^{2}\{(1-\rho)+\rho e^{-[\lambda-k_i(u)]a}+\frac{\lambda\rho}{\lambda-k_i(u)}[1-e^{-[\lambda-k_i(u)]a}]\}d_i(u), \qquad (6.11)$$

where $u=(\mu/\mu+\theta)L_{\eta_1}(\theta)$.

7. CALCULATION OF THE $E\tau_1^a$ AND THE $Var\tau_1^a$

From (3.2), we have

$$E\tau_1^a=(E\xi_1+E\eta_1)E\nu_1^a, \quad Var\tau_1^a=(Var\xi_1+Var\eta_1)E\nu_1^a+[E\xi_1+E\eta_1]^2 Var\nu_1^a \quad (7.1)$$

Note that these formulas also can be obtained from (6.3). By analogy with Sec.5 we can write

$$E\nu_1^a={}_a\Psi_u'(1) \text{ and } Var\nu_1^a={}_a\Psi_u''(1)+{}_a\Psi_u'(1)[1-{}_a\Psi_u'(1)]. \qquad (7.2)$$

The expressions for ${}_a\Psi_u'(1)$ and ${}_a\Psi_u''(1)$ can be found by using (6.11). But we will use (6.9) to determine these derivations easily. Firstly, we determine expressions for $d_i(1), d_i'(1), d_i''(1), i=1,2$.

These expressions can be obtained from (6.8) and (6.9) with $u=1$ (Table 7.1).

Table 7.1. Expression for $d_i(1), d_i'(1), d_i''(1), i=1,2$

$d_1(1)$	1	$d_1'(1)$	$\frac{e^{\lambda a}}{\rho}-[a+\frac{1}{\lambda}(1-e^{\lambda a})k_1'(1)+\frac{1}{\mu\rho}[k_2(1)-\lambda e^{-\mu\rho a}]e^{\lambda a}d_2'(1)$
$d_1''(1)$	$\frac{\lambda k_1''(1)+2k_1'(1)[\lambda+k_1'(1)]}{\lambda^2}(1-e^{\lambda a})+2[1+\frac{k_1'(1)}{\lambda}]ak_1(1)+$ $+[ak_1''(1)+(ak_1'(1))^2][1+d_2''(1)]-$ $-2[(1-\frac{1}{\rho})e^{\lambda a}+(1+\frac{k_1'(1)}{\lambda})(1-\frac{e^{\lambda a}}{\lambda})+\frac{ak_1'(1)}{\rho}]d_1'(1)$ - $-2[(1-\frac{1}{\rho})e^{\lambda a}+(1+\frac{k_2'(1)}{\mu\rho})(e^{ak_1(1)}-\frac{e^{\lambda a}}{\mu\rho})+\frac{ak_2'(1)}{\rho}]d_2'(1)$		
$d_2(1)$	0	$d_2'(1)$	$-k_1'(1)/k_2'(1)$
$d_2''(1)$	$-[1/k_2(1)]\{k_1''(1)+2[k_1'(1)d_1'(1)+k_2'(1)d_2'(1)]\}$		

By differentiation of (6.9) with $u=1$ and substituting the obtained expressions for ${}_a\Psi_u'(1)$, ${}_a\Psi_u''(1)$ into (7.2) together with $k_1(1)=0, d_1(1)=1, d_2(1)=0$ we get

$$E\nu_1^a=-1+a\rho k_1'(1)e^{-\lambda a}+[1+\rho e^{-\lambda a}]d_1'(1)+[1+\rho e^{-\mu\rho a}]d_2'(1), \tag{7.3}$$

$$\begin{aligned}\mathrm{Var}\nu_1^a &= 2-2[d_1'(1)+d_2'(1)]+[d_1''(1)+d_2''(1)]+\\ &+\rho e^{-\lambda a}\{d_1''(1)+e^{k_2(1)a}d_2''(1)+2a[k_1'(1)d_1'(1)+k_2'(1)e^{k_2(1)a}d_2'(1)]+k_1''(1)+a^2[k_1'(1)]^2\}+\\ &+E\nu_1^a[1-E\nu_1^a].\end{aligned} \tag{7.4}$$

Finally, from (7.1) together with (7.3) and (7.4), we have

$$E\tau_1^a=(\frac{1}{\mu}+E\eta_1)E\nu_1^a, \quad \mathrm{Var}\tau_1^a=(\frac{1}{\mu^2}+\mathrm{Var}\eta_1)E\nu_1^a+[\frac{1}{\mu}+E\eta_1]^2\mathrm{Var}\nu_1^a \tag{7.5}$$

where $\eta_1>0$ has a general distribution.

8. NUMERICAL RESULTS

This section presents numerical results which are obtained by using Matlab5. Fig.8.1 shows the plot of the expected values $E\nu_1^0$ and $E\nu_1^a$ obtained from (5.6) and (7.3), as a function of $\lambda/\mu\rho$.

In Fig.8.2 we plot (7.5) as a function of $\lambda/\mu\rho$. The values of $\lambda/\mu\rho$ are changed in the interval (0.3, 2.5). From this figure, we see that for $E\xi_1=E\zeta_1^0$ the expected values $E\nu_1^a$ increases.

The explanations above are valid for $\mathrm{Var}\tau_1^0$ as well. Fig.8.3. shows the plot of the expected values $E\nu_1^a$ as a function of capacity of warehouse.

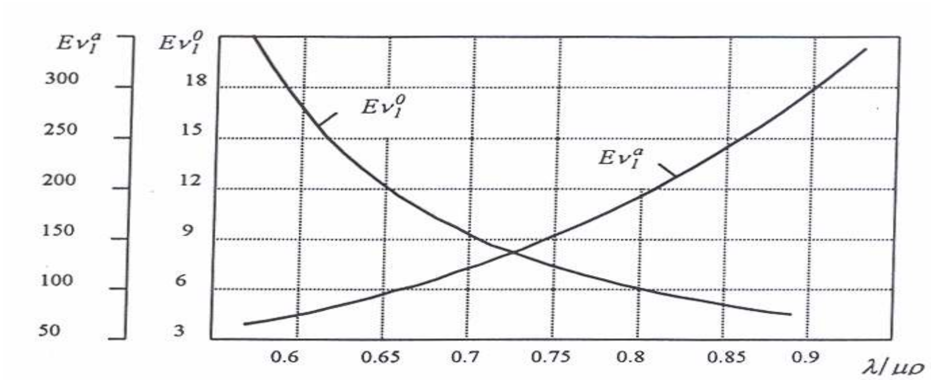

Figure 8.1. The plots of the expectations of the random variables ν_1^0 and ν_1^a as the functions of $\lambda/\mu\rho$.

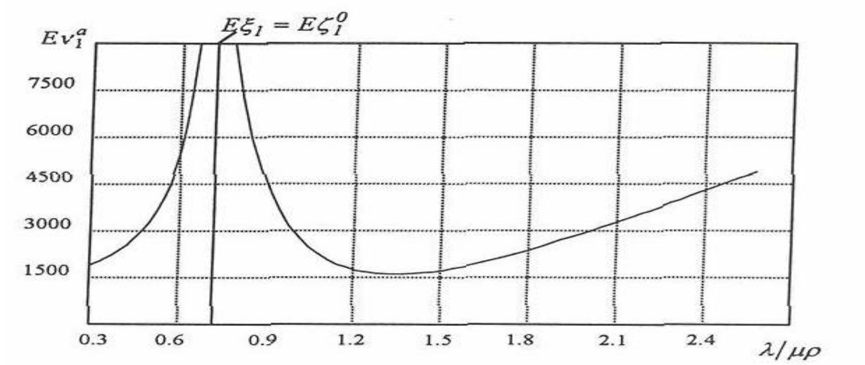

Figure 8.2. The plot of the expectations of the random variable ν_1^a as the function of $\lambda/\mu\rho$.

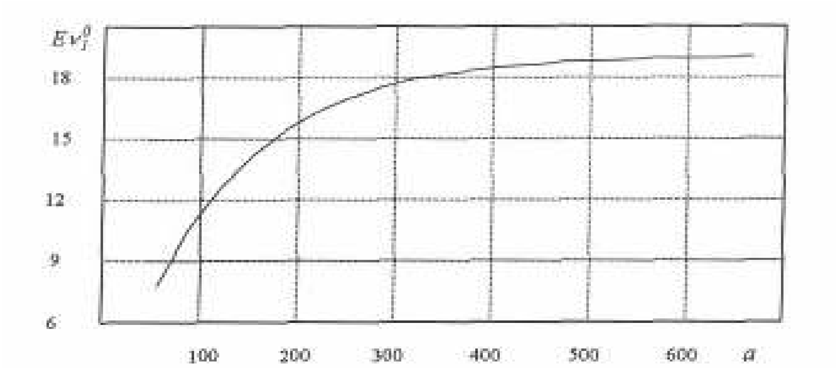

Figure 8.3. The plot of the expectations of the random variable ν_1^0 as the function of a.

In the following the parameters $E\tau_1^0$ and $Var\tau_1^0$ are calculated using formulas (5.8) and (5.9). Results are given in Table 8.1.

Table 8.1 Results dates (for a=300, $E\eta_1 = 5$, $Var\eta_1 = 15$)

	$E\nu_1^0$	$Var\nu_1^0$	$E\tau_1^0$	$Var\tau_1^0$
for $\lambda < \mu\rho$	1.29×10^3	1.09×10^6	2.32×10^3	3.50×10^6
for $\lambda > \mu\rho$	3.28	46.79	72.1	2.73×10^3

According to Table 8.1, the expected value $E\nu_1^0$ is much greater when $\lambda < \mu\rho$ $(E\xi_1 < E\zeta_1)$ than that of $\lambda > \mu\rho$, $(E\xi_1 > E\zeta_1)$. Similarly the expected value $E\tau_1^0$ is much greater in this case than that of $\lambda > \mu\rho$.

9. CONCLUSION

In this paper the process of semi-Markovian random walk with negative drift under the angle α $(0^\circ < \alpha < 90^\circ)$ having positive jumps with probability ρ $(0^\circ < \rho < 90^\circ)$ and with two delaying screens at level zero and a $(a > 0)$ is constructed. The exact expressions for Laplace transforms of the distributions of the first moments of reaching these screens by this process and in particularly the expectations and the variances of indicated distributions are obtained. The results obtained in this study can be practiced in queering and reliability theory.

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