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ON THE NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN PRIMITIVE VARIABLES USING GRID GENERATION TECHNIQUES

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Abstract- In this paper presents a new model procedure for the solution of the incompressible Navier-Stokes equations in primitive variables, using grid generation techniques. The time dependent momentum equations are solved explicitly for the velocity field using the explicit marching procedure, the continuity equation is implied at each grid point in the solution of pressure equation, while the SOR method is used for the Neumann problem for pressure. Results obtained for the model problem of driven flow in a square cavity demonstrate that the method yields accurate solutions. The results of the numerical computations in a driven cavity, which are presented for the history of the residues at several Reynolds numbers Re=100,1000,4000 and 5000 all the computed results are obtained without any artificial dissipation. This feature of the present procedure demonstrates its excellent convergence and stability characteristics. Numerically results obtained for the steady state static pressure in the driven cavity are presented for the first time at Re=4000 and 5000 using non-staggered grid.
Key words- successive over-relaxation iteration, Navier-Stokes equations, driven cavity flows

1. INTRODUCTION

We shall describe a combination of numerical grid generation techniques using stretching function by Marcel Vinkur [9] for solving the incompressible Navier-Stokes equations in primitive variables. , As an illustration, use them to compute driven cavity flows. Two dimensional laminar incompressible flows have been studied extensively by several investigators using the Navier-Stokes equations formulated in terms of vorticity ω, and stream function ψ, as the dependent variables. Roach [10] has given an excellent survey of the techniques of solution of Navier-Stokes equations using the (ω, ψ) formulation. Accurate and efficient numerical methods given by: R.Schreiber [7]. A numerical method using the primitive variables (u, v, p) to solve the Navier-Stokes equations are given by Ghia [1] and S.Abdallah [2]. Burggraf [6] studied the cavity-flow problem with great care and the results he provided for Reynolds number, (Re), ranging from zero to 400 are still among the most accurate results available for this problem. Several investigators [1,2,5,6,7] have used the cavity problem as model problem to test new numerical schemes, because it comprises a relatively simple flow configuration. Numerical grid generation technique introduced by W.D.Barfield [14], Wen-Hwa Chu [15], A.A.Amsden and C.W.Hirt [18], Joe.Thompson, Thames, Mastin [16] and Middlecoff. [8] Discussed the problem of grid generation is that of determining the physical domain to the computational domain, i.e., a transformation from physical domain. To computational domain is introduced this transformation is accomplished by specifying a generalized coordinate system which will map the nonrectangular grid system in the physical domain to rectangular uniform grid spacing in the computational domain. In the present method in all computations a grid was numerically calculated for the physical domain. The grid lines were clustered in all walls the purpose of clustering was to capture the grid strong gradient of the phenomenon, which are concentrated at all walls. A method of controlling the spacing of the coordinate lines given by Middlecoff [8] has been domain in the square cavity in order to treat higher Reynolds number flow, since the coordinate lines must concentrate near the surface to a greater degree as the Reynolds number increases. The solution shows excellent at Reynolds numbers Re=100,400 and 1000 with the results at Refs. [2,4,5,7] And another new results given at Re=2000,3000,4000 and 5000. Therefore, in the present study, the model problem of driven flow in a square cavity is investigated using the primitive variables (u, v, p) using grid generation technique. Results for velocity as well as pressure fields are obtained for Reynolds number Re ranging from 100 to 5000 and are compared with those of the (ω,ψ) system and (u, v, p) system without grid generation technique.

2. MATHEMATICAL FORMULATION

The different equations governing the motion of an incompressible viscous fluid inside a square cavity are the two- dimensional Navier-Stokes equations together with the continuity equation in the convective form, i.e., nonconservative form, these equations are

X-Momentum equation

$$u_t + uu_x + vu_y = -p_x + \left(\frac{1}{\mathrm{Re}}\right)\left(u_{xx} + u_{yy}\right) \tag{1}$$

Y-Momentum equation

$$v_t + uv_x + vv_y = -p_y + \left(\frac{1}{\mathrm{Re}}\right)\left(v_{xx} + v_{yy}\right) \tag{2}$$

Continuity equation

$$u_x + v_y = 0 \tag{3}$$

Here, the Reynolds number Re is defined as $\mathrm{Re} = \dfrac{\rho\, Ul}{\mu}$, with U being the velocity of the moving wall, in the above equations velocities have been made dimensionless with respect to u, pressure with respect to $\rho\, u^2$ and distance with respect to the width of the square cavity l.

The pressure p, which is nested in this system doesn't appear as dominant variable in any of these equations, the pressure p in the (u, v, p) system may be determined from a Poisson equation obtained by appropriately forming the divergence of the vector momentum equation .

2.1 Pressure Poisson Equation

By differentiating equation (1) with respect to x and equation (2) with respect to y and adding the resulting two equations, we get the pressure equation in the form

$$p_{xx} + p_{yy} = S_p - \frac{\partial(u_x + v_y)}{\partial t} \tag{4}$$

Where

$$S_p = \frac{\partial}{\partial x}\left[-\left(uu_x + vu_y\right) + \left(\frac{1}{\mathrm{Re}}\right)\left(u_{xx} + u_{yy}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[-\left(uv_x + vv_y\right) + \left(\frac{1}{\mathrm{Re}}\right)\left(v_{xx} + v_{yy}\right)\right] \qquad (5)$$

The derivatives of the dilation $D \cong u_x + v_y$ appearing in the equation (4) and (5) have been intentionally not set to zero.

2.2 The boundary conditions

The boundary conditions [1] for the cavity-flow problem in the (u, v, p) system are relatively straight forward. For the momentum equations, the normal velocities are zero at the nonporous walls while the tangential velocities satisfy the condition consist of the normal gradient evaluated from the appropriate momentum equation. Thus the boundary conditions on the cavity walls, see Fig (1) is:

At wall BC:

$$y = 0 \ , \ and \ , \ 0 \le x \le 1 \ , u = v = 0$$
$$-\frac{\partial p}{\partial y} = uv_x + vv_y - \left(\frac{1}{\mathrm{Re}}\right)\omega_x \qquad (6)$$

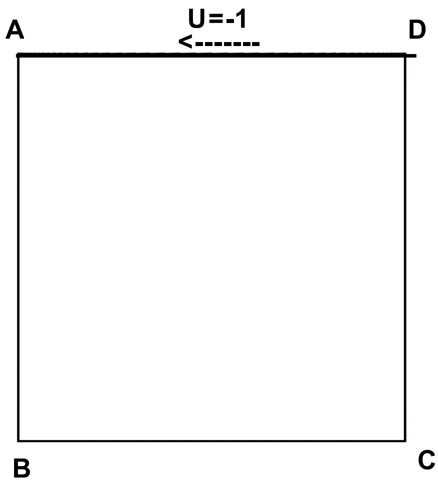
At wall AB:

$$x = 0 \ , \ and \ , \ 0 \le y \le 1 \ , u = v = 0$$
$$-\frac{\partial p}{\partial x} = uu_x + vu_y + \left(\frac{1}{\mathrm{Re}}\right)\omega_y \qquad (7)$$

At wall AD:

$$y = 1 \ , \ and \ , \ 0 \le x \le 1 \ , u = -1, v = 0$$
$$-\frac{\partial p}{\partial y} = uv_x + vv_y - \left(\frac{1}{\mathrm{Re}}\right)\omega_x \qquad (8)$$

At wall CD:



. Figure1: Cavity geometry

$$x = 1 \ , \ and \ , \ 0 \le y \le 1 \ , u = v = 0$$
$$-\frac{\partial p}{\partial x} = uu_x + vu_y + \left(\frac{1}{\mathrm{Re}}\right)\omega_y \qquad (9)$$

Such that $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

Solution for the pressure Poisson equation (4) with Neuman boundary conditions (6---9) requires the satisfaction of the compatibility condition

$$\int_A S_p(x,y)dA = \int_C \frac{\partial p}{\partial n} ds \qquad (10)$$

Where A is the area of the solution region, C is the perimeter of this area and s & n are the local tangent and outward normal to C.

3. MATHEMATICAL TRANSFORMATION

Two-dimensional elliptic boundary value problem is considered. The general transformation from the physical plane (x, y) to the transformed plane (ξ, η) is

$$\xi = \xi(x,y) \ , \qquad \eta = \eta(x,y) \qquad (11)$$

And the inverse transform is

$$x = x(\xi,\eta) \ , \qquad y = y(\xi,\eta) \qquad (12)$$

The Jacobian of the transformation is

$$J = J\left[\frac{x,y}{\xi,\eta}\right] = x_\xi y_\eta - x_\eta y_\xi \ , \ x_\xi = \frac{\partial x}{\partial \xi} \ , y_\eta = \frac{\partial y}{\partial \eta} \ , etc \qquad (13)$$

It is easy to show that

$$\xi = \frac{y_\eta}{J} \ , \ \xi_y = -\frac{x_\eta}{J} \ , \ \eta_x = -\frac{y_\xi}{J} \ , \ \eta_y = \frac{x_\xi}{J} \qquad (14)$$

Then

$$\frac{\partial}{\partial x} = \left(\frac{y_\eta}{J}\frac{\partial}{\partial \xi} - \frac{y_\xi}{J}\frac{\partial}{\partial \eta}\right) \ , \ \frac{\partial}{\partial y} = \left(\frac{x_\xi}{J}\frac{\partial}{\partial \eta} - \frac{x_\eta}{J}\frac{\partial}{\partial \xi}\right) \qquad (15)$$

Higher derivative can be obtained by repeated operations.

3.1 Mapping of the physical domain

The choice of mapping is largely dependent on its simplicity and the effort required for a desired accuracy. The boundary of physical domain is specified at discrete points (x_b, y_b). These points correspond to the boundary points (ξ_b, η_b) in the transform plane (ξ,η). Since we desire to have a prescribed, convenient mesh in the (ξ,η) plane (ξ,η) must be used as variables; their values are governed by any suitable elliptic partial differential equation. As independent a first-boundary value problem. ξ,η must satisfy the Poisson equation in the physical plane:

$$\nabla^2 \xi = P(\xi,\eta) \ , \qquad \nabla^2 \eta = Q(\xi,\eta) \qquad (16)$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad (17)$$

The dependent and independent variables can be interchanged by applying equation (14) and (15) one finds that

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = O1 \qquad (18)$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = O1 \qquad (19)$$

Where

$$O1 = -J^2\left[x_\xi P(\xi,\eta) + x_\eta Q(\xi,\eta)\right] ,$$

$$O1 = -J^2\left[y_\xi P(\xi,\eta) + y_\eta Q(\xi,\eta)\right] , \qquad (20)$$

$$\alpha = x_\eta{}^2 + y_\eta{}^2 \ , \ \beta = x_\xi x_\eta + y_\xi y_\eta \ , \ \gamma = x_\xi{}^2 + y_\xi{}^2$$

Equations (18) and (19) are clearly coupled quasi-linear elliptic equations, and only in special cases $(x_\xi = y_\eta \ and \ x_\eta = -y_\xi)$ can be reduced to Poisson equations for which mapping is conformal.

Equations (18) and (19) such that when O1=O2=0 when P=Q=0 in the general case can be conveniently solved by the finite difference method with successive overrelaxation (SOR) [12] of the dependent variable.

4. THE NUMERICAL SOLUTION USING GRID GENERATION TECHNIQUES

To obtain the numerical solution using grid generation technique, we transform the governing equation (1--- 4) from the physical domain into the computational domain. So using the equations (11-15) then equations (1) and (2) will take the forms

$$\begin{aligned} & u_t - \frac{1}{J}[(y_\eta u_\xi - y_\xi u_\eta)x_t + (x_\xi u_\eta - x_\eta u_\xi)y_t] \\ & + \frac{1}{J}[u(y_\eta u_\xi - y_\xi u_\eta) + v(x_\xi u_\eta - x_\eta u_\xi)] \\ & = -\frac{1}{J}(y_\eta p_\xi - y_\xi p_\eta) + \frac{1}{J^2 \operatorname{Re}}(\alpha u_{\xi\xi} - 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + du_\eta + eu_\xi) \end{aligned} \qquad (31)$$

$$\begin{aligned} & v_t - \frac{1}{J}[(y_\eta v_\xi - y_\xi v_\eta)x_t + (x_\xi v_\eta - x_\eta v_\xi)y_t] \\ & + \frac{1}{J}[u(y_\eta v_\xi - y_\xi v_\eta) + v(x_\xi v_\eta - x_\eta v_\xi)] \\ & = -\frac{1}{J}(x_\xi p_\eta - x_\eta p_\xi) + \frac{1}{J^2 \operatorname{Re}}(\alpha v_{\xi\xi} - 2\beta v_{\xi\eta} + \gamma v_{\eta\eta} + dv_\eta + ev_\xi) \end{aligned} \qquad (32)$$

Where $\alpha \ , \beta \ and \ \gamma$ are defined in (19) and

$$d = \frac{1}{J}(y_\xi D_x - x_\xi D_y) \ , \qquad e = \frac{1}{J}(x_\eta D_y - y_\eta D_x)$$

Such that

$$D_x = \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} \ , \qquad D_y = \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta}$$

And the equation (3) takes the form

$$\frac{1}{J}(y_\eta u_\xi - y_\xi u_\eta) + \frac{1}{J}(x_\xi v_\eta - x_\eta v_\xi) = 0 \tag{33}$$

In order to transform the pressure equation from the physical domain to computational domain we use the transformation equations (14-15), so we get

$$uu_x + vu_y = \frac{1}{J}(\alpha_1 u_\xi + \beta_1 u_\eta) \tag{34}$$

$$uv_x + vv_y = \frac{1}{J}(\alpha_1 v_\xi + \beta_1 v_\eta) \tag{35}$$

Where

$$\alpha_1 = \frac{1}{J}(uy_\eta - vx_\eta) \ , \qquad \beta_1 = \frac{1}{J}(vx_\xi - uy_\xi) = 0$$

By substituting from (34) and (35) into (5) and using once again, we get the transformed pressure equation in the form

$$\begin{aligned} &\frac{1}{J^2}(\alpha p_{\xi\xi} - 2\beta p_{\xi\eta} + \gamma p_{\eta\eta} + dp_\eta + ep_\xi) \\ &= -\frac{1}{J}[y_\eta \frac{\partial}{\partial \xi}(\alpha_1 u_\xi + \beta_1 u_\eta) - y_\xi \frac{\partial}{\partial \eta}(\alpha_1 u_\xi + \beta_1 u_\eta)] \\ &- \frac{1}{J}[x_\xi \frac{\partial}{\partial \eta}(\alpha_1 v_\xi + \beta_1 v_\eta) - x_\eta \frac{\partial}{\partial \xi}(\alpha_1 v_\xi + \beta_1 v_\eta)] - \frac{\partial D}{\partial t} \\ &- \frac{1}{J}[(y_\xi D_\eta - y_\eta D_\xi)x_t + (x_\eta D_\xi - x_\xi D_\eta)y_t] \end{aligned} \tag{36}$$

Where

$$D = \frac{1}{J}(y_\eta u_\xi - y_\xi u_\eta + x_\xi v_\eta - x_\eta v_\xi)$$

The values of the stream function Ψ are obtained by solving Poisson equation

$$\Psi_{xx} + \Psi_{yy} = -\omega \tag{37}$$

And the transformation of this equation will take the form

$$\alpha \Psi_{\xi\xi} - 2\beta \Psi_{\xi\eta} + \gamma \Psi_{\eta\eta} + d\Psi_\eta + e\Psi_\xi = -J^2 \omega \tag{38}$$

In order to solve this system of equations we solve it in sequential procedure, and the solution is considered to have converged when all u, v, p have satisfied suitable convergence criteria. We note that the boundary conditions in the transformed plane will take the form

$$
\begin{aligned}
-p_\xi &= \left(\frac{1}{J \ \mathrm{Re}}\right)(\gamma\omega_\eta - \beta\omega_\xi) \\
&+ \frac{u}{J}(x_\xi y_\eta u_\xi - x_\xi y_\xi u_\eta - y_\xi{}^2 v_\eta + y_\xi y_\eta v_\xi) \\
&+ \frac{v}{J}(x_\xi{}^2 u_\eta - x_\xi x_\eta u_\xi + y_\xi x_\xi v_\eta - x_\eta y_\xi v_\xi)
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
-p_\eta &= \left(\frac{1}{J \ \mathrm{Re}}\right)(\beta\omega_\eta - \alpha\omega_\xi) \\
&+ \frac{u}{J}(x_\eta y_\eta u_\xi - y_\xi x_\eta u_\eta + y_\eta{}^2 v_\eta - y_\xi y_\eta v_\eta) \\
&+ \frac{v}{J}(-x_\eta{}^2 u_\xi + x_\xi x_\eta u_\eta + y_\eta x_\xi v_\eta - x_\eta y_\eta v_\xi)
\end{aligned} \tag{40}
$$

To solve numerically equations (1), (2) and (3) with boundary conditions (on unit square) given by the equations (6), (7), (8) and (9) in computational domain we will solve the transformed equations (31), (32), (33) and (36) we use the second order central finite difference formula. The time derivative terms are approximated using forward difference.

5. RESULTS AND DISCUSSION

The driven cavity problem, which has been widely used for validating solution techniques for the incompressible Navier-Stokes equations, is selected to validate the present method. The results for the computed stream function, Vorticity, velocity and pressure in the driven cavity are presented in Figures. (2) Through (5) for Reynolds numbers 100, 1000, 4000 and 5000 the computed results can compare with the available results of other investigators, Ref. [5], [6] and [7] in order to validate the accuracy of the numerical procedure. The Vorticity contours and the velocity profiles are presented in Figures (3) and (4). Both studies obtained their results using the stream function Vorticity formulation. Burggraf and Mansour [5] used 41×41 grid and Schreiber [7] used 180×180 grid and a fourth-order accurate technique one can observe the agreement between the computed Vorticity contours and Burggraf's [6], Mansour [5] results at Re=100 and Schreiber's [7], Mansour [5] at Re=1000. The velocity profiles for u and v at Re=1000 can compare with the results by Schreiber [7] and Mansour [5] in Figures (4).

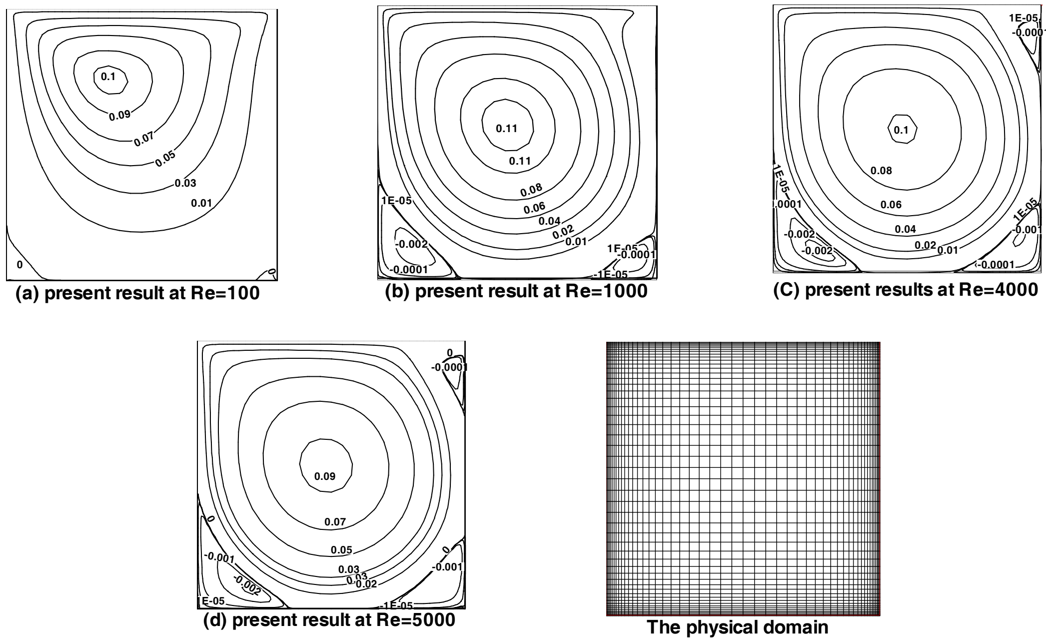


Figure 2: (a-d) Stream function contours and physical domain Clustering grid (51×51)

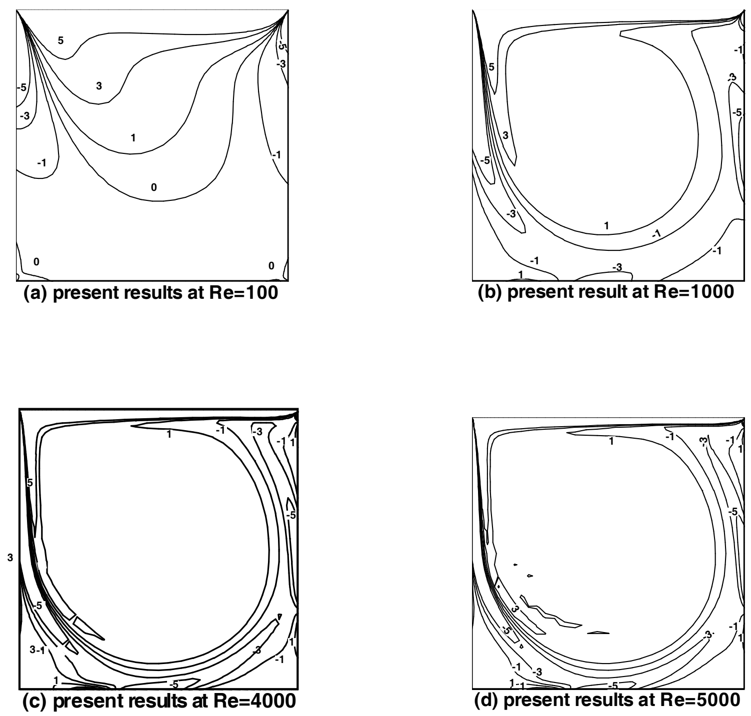
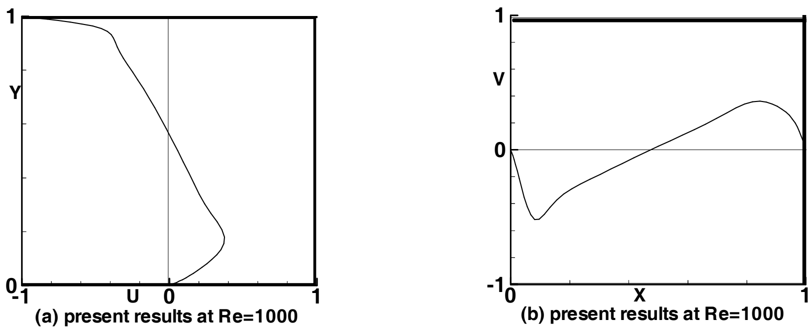


Figure 3: Vorticity contours



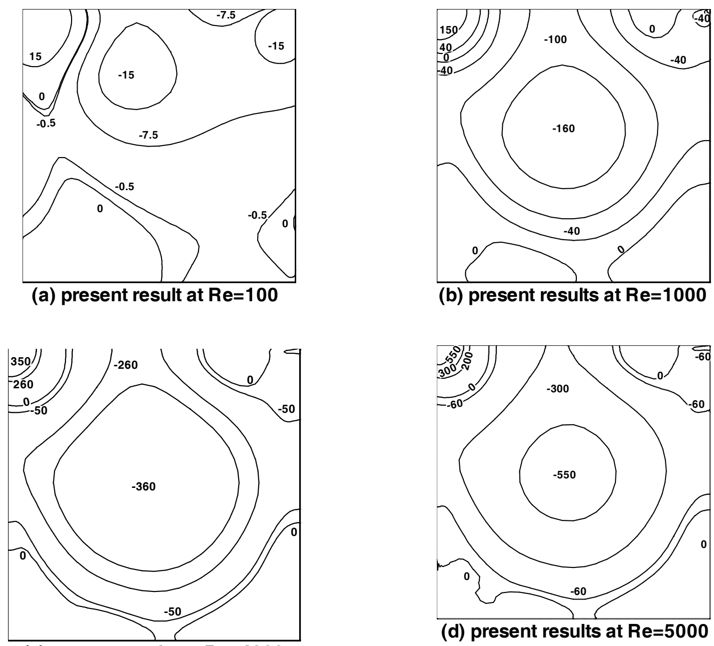
(a) present results at Re=1000 (b) present results at Re=1000

Figure 4: (a) U-Velocity profiles at the centerline of the cavity
(b) V-Velocity profiles at the centerline of the cavity

The comparison shows excellent agreement with clustering grid from all sides, using grid system generated by elliptic PDEs in the present computations. Figures (5) present the computed static pressure coefficient and the results of Ref. [5], at Re=100 and 1000. The numerical solution for the pressure in Ref. [5] was obtained on a (41×41) clustering grid. The static pressure coefficient is defined as

$$C_p = 2\,\mathrm{Re}(p - p_0)/U^2$$

Where p_0 is the reference pressure at the center of the cavity's lower boundary. One can see that the results of the present method (G.G. Method) are in excellent agreement with the results obtained by Burggraf [6] and Mansour [5] at 100 and 400 Reynolds number. The computed static pressure coefficient at Re=100 and 1000 are shown in Figure (5). The new results are given by present method for the computed Vorticity, stream function, pressure at Re=4000 and Re=5000 are shown in Figures (2), (3) and (5).



(a) present result at Re=100 (b) present results at Re=1000

(c) present results at Re=4000 (d) present results at Re=5000

Figure 5: (a-d) Pressure coefficient contours

6. CONCLUSIONS

Numerical grid generation is one of the most effective methods for solving partial differential equations in irregular domains.

In this paper we use the elliptic grid generation for solving incompressible Navier-Stokes equations in primitive variables (u, v, p) using different Reynolds numbers for clustering grid (51×51) using explicit difference schemes. We must note that the momentum equations are solved for the velocity field by marching in time using Gauss-Siedel method while the pressure, system function equations were solved using successive over relaxation method with parameter 1.7, and these methods are considered to be explicit methods.

In comparing the results obtained by solving the partial differential equations using grid generation technique and those obtained by using finite difference methods we get that the grid generation method is an effective method for dealing with partial differential equations with irregular boundary conditions since we can transform the physical domain into the computational domain in such a way that any change in the physical domain with time, causes the same change in the computational domain and hence we can get more accurate solution by using grid generation method.

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