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Steady State Analysis of Impulse Customers and Cancellation Policy in Queueing-Inventory System

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Abstract: This article discusses the queueing-inventory model with a cancellation policy and two classes of customers. The two classes of customers are named ordinary and impulse customers. A customer who does not plan to buy the product when entering the system is called an impulse customer. Suppose the customer enters into the system to buy the product with a plan is called ordinary customer. The system consists of a pool of finite waiting areas of size N and maximum S items in the inventory. The ordinary customer can move to the pooled place if they find that the inventory is empty under the Bernoulli schedule. In such a situation, impulse customers are not allowed to enter into the pooled place. Additionally, the pooled customers buy the product whenever they find positive inventory. If the inventory level falls to s, the replenishment of Q items is to be replaced immediately under the (s, Q) ordering principle. Both arrival streams occur according to the independent Markovian arrival process (MAP), and lead time follows an exponential distribution. In addition, the system allows the cancellation of the purchased item only when there exist fewer than S items in the inventory. Here, the time between two successive cancellations of the purchased item is assumed to be exponentially distributed. The Gaver algorithm is used to obtain the stationary probability vector of the system in the steady-state. Further, the necessary numerical interpretations are investigated to enhance the proposed model.

Keywords: IMPULSE customer; cancellation policy; Markovian arrival process; queueing-inventory model

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1. Introduction

In a queueing-inventory system, customers arrive at the service system on an individual basis if the item is needed. Inventory must be present if customers are to pick up the service. Once the service is completed, the items are removed from inventory for those who have earned the services. This system has emerged as an immense aspect in the mathematical modelling of predicament commotions issuing in a tremendous application, transportation system, grocery store, computer network system, and many other systems. Many researchers have worked with queueing-inventory systems in the last few decades. When ordered products are plentiful, backlogged demands are fulfilled quickly; but, when the item is replenished, backlogged demands may have to wait for fulfillment. This type of backlogged demand is called postponed demand [1].

Customers that buy on the spur of the moment are referred to as impulse customers. Customers who buy on the spur of the moment have little regard for planning, budgeting, or the necessity for a certain item. It is, in reality, a sudden event in which our emotions take over our brain and drive us to go ahead and purchase that object, even though we may not need it right now or even immediately. For instance, perfume and body spray, their attractive smell often encourages customers to buy them even if they do not have any need for it, dress, shoes, and cosmetics for other examples.

Cancellation is a very frequent occurrence that happens in life. For instance, after purchasing any garment or dress, later, if it is found to be unfit or not liked, we go for cancellation. Similarly, we cancel an advance travel reservation, including Bus, Train, and Flight, if we happen to come across some unexpected and sudden circumstances that don't allow us to make our journey as planned.

Aspects of this work are as follows: The model representation of the queueing-inventory with the cancellation policy and impulse customers governed by the Markovian arrival process portrayed in Section 2. An investigation has been made in Section 3 to show the joint probability distribution of the number of pool customers and stock level in a steady-state case. Section 4 is a delineated discussion of a few numerical instances.

Literature Review

Sigman and Levi [2] and Melikov and Molchanov [3] commenced to discourse on the queueing-inventory problem. The MAP is one of the emerging trends in the queueing-inventory system, and Neuts [4] has put forward the Markovian arrival process. Berman et al. [5] have commenced the idea of postponed demand in the inventory model.

Correlated along with the postponed demand article, Sivakumar and Arivarignan [6] scrutinized the perishable inventory model along the infinite size waiting for a place, customers appear under MAP, and when the customer appears in the stock-out, either they move to a pool or fall out. In addition, Paul Manuel et al. [7] deal with a perishable inventory model having infinite waiting space, two kinds of customers arrive according to MAP, and service time follows PH distribution.

Krishnamoorthy et al. [8] commenced inter-cancellation into the queueing-inventory system. Dhanaya Shajin and Krishnamoorthy [9] scrutinized a queueing-inventory model with the MAP, service time under PH distribution, all the items have a common lifetime, and items are overbooked. The waiting customers are fall out when the system is full, forthwith S items are formed. Dhanya Shajin et al. [10] explored advanced reservation and overbooking with MAP in the queueing-inventory model. There are only a few papers related to inter-cancellation. Some notable work done by Sung-Seok Ko [11], Srinivas R. Chakravarthy [12], and Krishnamoorthy et al. [13] on the Markovian arrival process.

Nair and Jose [14] considered the retrial customers, the customers appear according to MAP, the service time is exponentially distributed, and the process of production complies with PH distribution. Production begins and service at a reduced rate up to the zero level of inventory when the inventory goes to a pre-assigned level s. Incoming customers are aimed at a buffer of finite capacity equal to the existing inventory level. Suppose that buffer is complete, customers move to an orbit of infinite size or fall out. Punalal and Babu [15] contemplated a model in which the customers appear, governed with MAP, and all customers are treated as ordinary at the moment of arrival. During busy periods, incoming customers fill the orbit's infinite capacity. Every orbit customer, regardless of others, makes a priority with the time that occurs. Once, the customer got precedence rabidly taken service during the free period; otherwise, the customer went right into a waiting space that is reserved only for precedence generated customers.

Ayyappan and Gowthami [16] considered a queueing model along with the arrival types, such as incoming and outgoing calls. Customers who appear on the system under the MAP make incoming calls, and the server makes outgoing calls during idle time. Service times of incoming or outgoing calls under phase-type distribution. Seokjun Lee et al. [17] contemplated the queueing system with a single server and heterogeneous the arrival flow is under the marked Markov arrival process. There is distinct impatience for customers of several kinds. It is assumed that the difficulty of setting non-preemptive priorities for various types of consumers is solved within the assumption that customers can improve their precedence while being in the buffer. The distribution of service time is of the phase type. Valentina Klimenok et al. [18] analysed the queueing system with a single server. They categorized two kinds of customers who appear under a batch-marked Markov arrival process. Customers with low priority are entitled to obtain higher priority after a random period. Non-priority customers are also allowed in the buffer, but fix the timer. If the timer is expired, the customer falls away from the system with some probability. The high priority acquires the complementary probability. Alexander Dudin and Sergei Dudin [19] analysed the queueing model along with a single server and arrival types, such as customers of type 1 can be queued into the buffer with infinite capacity and customers of type 2 have a finite capacity buffer. Customers of both types can be impatient, and the arrival harmonizes with the marked Markovian arrival process. The author introduces a new form of distribution called phase-type with failures (when a failure can occur while a customer is being served in generalizes of phase-type distribution). The distribution of service time is PHF. In a related bibliography [20–23], the stochastic inventory system with MAP arrivals is shown.

The findings of the previous poll sparked our work, and to our knowledge, there has been little research into impulsive customers with MAP.

2. Model Description

This paper investigates the two classes of customers in a queueing-inventory system with a cancellation policy. The two classes of customers are defined as ordinary and impulse customers. In real-life phenomena, we observe two types of customers who may approach the inventory system to buy a product. Many customers visit an inventory system without planing to buy the product. Suppose at the end of the visiting process, a customer decides whether to buy a product or not. This type of customer is called an impulse customer. On the other hand, once the customers enter into the system, they buy the product compulsorily. These kinds of customers are called ordinary customers. In such a way, the proposed model allows these two classes of customers to purchase inventory items. The arrival pattern of both ordinary and impulse customers are assumed to be independent MAP.

The arrival process of ordinary customer representation is (E_0, E_1), where $E_i (i = 0, 1)$ are square matrices of dimension k_1, such that E_0 governs transitions corresponding to no arrival and E_1 governs transitions corresponding to an arrival. The underlying Markov chain $U_3(t)$ of the MAP has the generator E is a square matrix of dimension k_1, where $E = E_0 + E_1$.

The stationary rate λ_1 of an ordinary customer is defined by $\lambda_1 = \eta_1 E_1 \mathbf{e}$, where stationary row vector η_1 of dimension $1 \times k_1$ is to be obtained by using $\eta_1 E = \mathbf{0}$ and $\eta_1 \mathbf{e} = 1$.

Similarly, (F_0, F_1) represents the arrival pattern of an impulse customer, where $F_i (i = 0, 1)$ are the square matrices of size k_2, such that F_0 governs transitions corresponding to no arrival and F_1 governs transitions corresponding to an arrival. The underlying Markov chain $U_4(t)$ of the MAP has the generator F is a square matrix of size k_2, where $F = F_0 + F_1$.

The stationary rate λ_2 of an impulse customer is defined by $\lambda_2 = \eta_2 F_1 \mathbf{e}$, where stationary row vector η_2 of dimension $1 \times k_2$ is to be obtained by using $\eta_2 F = \mathbf{0}$ and $\eta_2 \mathbf{e} = 1$. The parameters k_1 and k_2 represents the phase of the arrival process of ordinary and impulse customers, respectively.

The service process of the system is assumed to be instantaneous when the inventory level is positive. Any arriving customer finds that there exists a positive inventory, he/she starts purchasing their product. After the purchase completion of the customer, the inventory will be decreased by one unit of an item. In this system, at the end of the visiting process of impulse customers, they leave the system either under the transition rate pF_1 if they buy the product or qF_1 if they do not buy the product. Suppose the inventory system is empty, the arriving impulse customers are considered as lost. In contrast, the ordinary customers may join in the finite size, N of pooled place under the Bernoulli schedule. That is, the arriving ordinary customer enters into the pooled place with probability r_1 or leave the system with probability r_2, where $r_2 = 1 - r_1$. The customers in the pool approach the inventory system whenever they find the positive stock in the inventory with the rate θ. The time between two successive approaches of a pooled customers follows an exponential distribution.

Cancellation policy:

The customers return the purchased product due to their dissatisfaction. Suppose the purchased item is damaged and the system has at most S (maximum inventory level) items in the inventory, they can not be allowed to return the product. Whenever there exists a $(S - i)$ item in the inventory, where $i(1 \leq i \leq S)$ represents the purchased item, the transition rate of the return or cancellation of the product is defined by $i\beta$. The time between two successive cancellations of the product is assumed to be exponentially distributed.

Further, if the storage of the system falls to s, there must be $Q(= S - s > s + 1)$ items immediately replenished with the transition rate $\mu(> 0)$. The lead time follows an exponential distribution.

3. Analysis

In this sector, we construct the transition rate matrix on the queueing-inventory system. The Markov process of the form $\{(U_1(t), U_2(t), U_3(t), U_4(t)), t \geq 0\}$ with state space

$$C = \{(u_1, u_2, u_3, u_4) : u_1 = 0, 1, \cdots, N;\ u_2 = 0, 1, \cdots, S;\ u_3 = 1, 2, \cdots, k_1;\ u_4 = 1, 2, \cdots, k_2\}$$

where

$U_1(t)$: The number of customers in pool of finite size waiting place at time t.
$U_2(t)$: The number of items in the inventory at time t.
$U_3(t)$: Phase of the ordinary customers arrival process at time t.
$U_4(t)$: Phase of the impulse customers arrival process at time t.

Transition rates are:

1. Transition due to ordinary customers arrival
   (a) $(u, v) \to (u, v - 1)$: rate $E_1 \otimes I_{k_2}$, $u = 0, 1, \cdots, N;\ v = 1, 2, \cdots, S$.
   (b) $(u, 0) \to (u + 1, 0)$: rate $r_1 E_1 \otimes I_{k_2}$, $u = 0, 1, \cdots, N - 1$.
2. Transition due to impulse customers arrival
   $(u, v) \to (u, v - 1)$: rate $I_{k_1} \otimes pF_1$, $u = 0, 1, \cdots, N;\ v = 1, 2, \cdots, S$.
3. Transition due to cancellation
   $(u, v) \to (u, v + 1)$: rate $(S - v)\beta I_{k_1} \otimes I_{k_2}$, $u = 0, 1, \cdots, N;\ v = 0, 1, \cdots, S - 1$.
4. Transition due to approach from pooled customers
   $(u, v) \to (u - 1, v - 1)$: rate $\theta I_{k_1} \otimes I_{k_2}$, $u = 1, 2, \cdots, N;\ v = 1, 2, \cdots, S$.
5. Transition due to replenishment
   $(u, v) \to (u, v + Q)$: rate $\mu I_{k_1} \otimes I_{k_2}$, $u = 0, 1, \cdots, N;\ v = 0, 1, \cdots, s$,

where $u = u_1,\ v = u_2$.

The process's infinitesimal generator A is generated by

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} A_{00} & A_{01} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ A_{10} & A_{11} & A_{01} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{10} & A_{11} & A_{01} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{10} & A_{11} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & A_{11} & A_{01} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & A_{10} & A_{22} \end{pmatrix} \end{array},$$

where $[A_{10}]_{vv'} = \begin{cases} \theta I_{k_1} \otimes I_{k_2} & v' = v-1, v = 1,2,\cdots,S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$

$$[A_{01}]_{vv'} = \begin{cases} r_1 E_1 \otimes I_{k_2} & v' = v, v = 0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$[A_{00}]_{vv'} = \begin{cases} E_1 \oplus pF_1 & v' = v-1, v = 1,2,\cdots,S, \\ (S-v)\beta I_{k_1} \otimes I_{k_2} & v' = v+1, v = 0,1,\cdots,S-1, \\ \mu I_{k_1} \otimes I_{k_2} & v' = v+Q, v = 0,1,\cdots,s, \\ (r_2 E_1 + E_0) \oplus F & \\ -(\mu + (S-v)\beta) I_{k_1} \otimes I_{k_2} & v' = v, v = 0, \\ E_0 \oplus (F_0 + qF_1) & \\ -(\mu + (S-v)\beta) I_{k_1} \otimes I_{k_2} & v' = v, v = 1,2,\cdots,s, \\ E_0 \oplus (F_0 + qF_1) & \\ -(S-v)\beta I_{k_1} \otimes I_{k_2} & v' = v, v = s+1, s+2, \cdots, S-1, \\ E_0 \oplus (F_0 + qF_1) & v' = v, v = S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$[A_{11}]_{vv'} = \begin{cases} E_1 \oplus pF_1 & v' = v-1, v = 1,2,\cdots,S, \\ (S-v)\beta I_{k_1} \otimes I_{k_2} & v' = v+1, v = 0,1,\cdots,S-1, \\ \mu I_{k_1} \otimes I_{k_2} & v' = v+Q, v = 0,1,\cdots,s, \\ (r_2 E_1 + E_0) \oplus F & \\ -(\mu + (S-v)\beta) I_{k_1} \otimes I_{k_2} & v' = v, v = 0, \\ E_0 \oplus (F_0 + qF_1) & \\ -(\mu + (S-v)\beta + \theta) I_{k_1} \otimes I_{k_2} & v' = v, v = 1,2,\cdots,s, \\ E_0 \oplus (F_0 + qF_1) & \\ -((S-v)\beta + \theta) I_{k_1} \otimes I_{k_2} & v' = v, v = s+1,\cdots,S-1, \\ E_0 \oplus (F_0 + qF_1) & \\ -\theta I_{k_1} \otimes I_{k_2} & v' = v, v = S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$[A_{22}]_{vv'} = \begin{cases} E_1 \oplus pF_1 & v' = v-1, v = 1, 2, \cdots, S, \\ (S-v)\beta I_{k_1} \otimes I_{k_2} & v' = v+1, v = 0, 1, \cdots, S-1, \\ \mu I_{k_1} \otimes I_{k_2} & v' = v+Q, v = 0, 1, \cdots, s, \\ E \oplus F & \\ -(\mu + (S-v)\beta) I_{k_1} \otimes I_{k_2} & v' = v, v = 0, \\ E_0 \oplus (F_0 + qF_1) & \\ -(\mu + (S-v)\beta + \theta) I_{k_1} \otimes I_{k_2} & v' = v, v = 1, 2, \cdots, s, \\ E_0 \oplus (F_0 + qF_1) & \\ -((S-v)\beta + \theta) I_{k_1} \otimes I_{k_2} & v' = v, v = s+1, \cdots, S-1, \\ E_0 \oplus (F_0 + qF_1) & \\ -\theta I_{k_1} \otimes I_{k_2} & v' = v, v = S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It may be noted that the matrices A_{10}, A_{01}, A_{00}, A_{11}, and A_{22} are all square matrices of dimension $(S+1)k_1k_2$.

3.1. Steady State Probability Vector

The Markov process $\{(U_1(t), U_2(t), U_3(t), U_4(t)), t \geq 0\}$ on the state space C and the limiting distribution $\Theta_{(u_1,u_2,u_3,u_4)} =$

$$\lim_{t \to \infty} Pr[U_1(t) = u_1, U_2(t) = u_2, U_3(t) = u_3, U_4(t) = u_4 | U_1(0), U_2(0), U_3(0), U_4(0)]$$

exists and is independent of the initial state.

Take

$$\Theta = (\Theta_0, \Theta_1, \Theta_2, ..., \Theta_{N-1}, \Theta_N),$$

where

- $\Theta_{u_1} = (\Theta_{(u_1,0)}, \Theta_{(u_1,1)}, \Theta_{(u_1,2)}, \cdots, \Theta_{(u_1,S)}), u_1 = 0, 1, \cdots, N$
- $\Theta_{(u_1,u_2)} = (\Theta_{(u_1,u_2,1)}, \Theta_{(u_1,u_2,2)}, \cdots, \Theta_{(u_1,u_2,k_1)}), u_1 = 0, 1, \cdots, N;\ u_2 = 0, 1, \cdots, S$
- $\Theta_{(u_1,u_2,u_3)} = (\Theta_{(u_1,u_2,u_3,1)}, \Theta_{(u_1,u_2,u_3,2)}, \cdots, \Theta_{(u_1,u_2,u_3,k_2)}), u_1 = 0, 1, \cdots, N;$ $u_2 = 0, 1, \cdots, S;\ u_3 = 1, 2, \cdots, k_1.$

Our general matrix A has same structure in Gaver [24], so we make use of the same arguments to determine the limiting probability vectors.

We present the Gaver algorithm here.

Gaver Algorithm:

1. Determine the matrices Z_n recursively by initializing,
 $Z_0 = A_{00}$
 $Z_n = A_{11} + A_{10}(-Z_{n-1}^{-1})A_{01}, 1 \leq n \leq N-1$
 $Z_N = A_{22} + A_{10}(-Z_{N-1}^{-1})A_{01}.$

2. Compute the limiting probability vectors Θ_n using,
 $\Theta_n = \Theta_{(n+1)} A_{10}(-Z_n^{-1})$, for $n = 0, \cdots, N-1$.

3. Determine the system of equations
 $\Theta_N Z_N = \mathbf{0}$;
 $\sum_{n=0}^{N} \Theta_n \mathbf{e} = 1.$

From the above system of equations $\Theta_N Z_N = \mathbf{0}$, vector Θ_N could be determine distinctively, up to a multiplicative constant. The constant is resolved by $\Theta_n = \Theta_{(n+1)} A_{10}(-Z_n^{-1})$, $n = 0, \cdots, N-1$ and $\sum_{n=0}^{N} \Theta_n \mathbf{e} = 1$.

3.2. Few Significant of the System Peculiarities

In this segment, we acquire a few significant peculiarities measures.

1. Mean inventory level
Let η_I is mean inventory level in the steady state. Since $\Theta_{(i_1,i_2)}$ denote the limiting probability vector with the inventory level represents as i_1 and the number of customers in the pool represents as i_2. This is given by
$$\eta_I = \sum_{i_1=0}^{N}\sum_{i_2=1}^{S} i_2\Theta_{(i_1,i_2)}\mathbf{e}.$$
2. Mean reorder rate
Let η_R denote the mean reorder rate in the steady-state. When the inventory level reduces to s from $s+1$ due to any of the following situations, a reorder is triggered:
   (a) The purchase of an ordinary customer.
   (b) Any one of pooled customers approaches.
   (c) The purchase of an impulse customer.

   This is lead to
$$\eta_R = \sum_{i_1=0}^{N}\Theta_{(i_1,s+1)}(E_1\otimes I_{k_2})\mathbf{e} + \sum_{i_1=1}^{N}\Theta_{(i_1,s+1)}\theta I\mathbf{e} + \sum_{i_1=0}^{N}\Theta_{(i_1,s+1)}(I_{k_1}\otimes pF_1)\mathbf{e}.$$
3. Mean number of customers in the pool
Let η_{PC} denote the mean number of customers in the pool. Since Θ_{i_1} denote the stationary probability vector with the inventory level i_1. Hence, the mean number of customers in the pool is given by
$\eta_{PC} = \sum_{i_1=1}^{N} i_1\Theta_{i_1}\mathbf{e}$. Mean rate of arrival of impulse customers
Let η_{IC} denote the mean rate of arrival of impulse customers in the steady state. Then, η_{IC} is given by
$$\eta_{IC} = \sum_{i_1=0}^{N}\sum_{i_2=0}^{S}\Theta_{(i_1,i_2)}(I_{k_1}\otimes F_1)\mathbf{e}.$$
4. Mean number of lost customers in the system
Let η_L denote the mean number of lost customers in the system. This is given by
$$\eta_L = \sum_{i_1=0}^{N-1}\Theta_{(i_1,0)}(r_2E_1\otimes I_{k_2})\mathbf{e} + \sum_{i_2=0}^{S}\Theta_{(N,i_2)}(E_1\otimes I_{k_2})\mathbf{e} + \sum_{i_1=0}^{N}\Theta_{(i_1,0)}(I_{k_1}\otimes F_1)\mathbf{e}$$
$$+\sum_{i_1=0}^{N}\sum_{i_2=1}^{S}\Theta_{(i_1,i_2)}(I_{k_1}\otimes qF_1)\mathbf{e}.$$
5. Mean cancellation rate of return product
Let η_C denote the mean cancellation rate of return product in the steady state. Then, η_C is given by
$$\eta_C = \sum_{i_1=0}^{N}\sum_{i_2=0}^{S-1}\Theta_{(i_1,i_2)}(S-i_2)\beta\mathbf{e}.$$

3.3. Construction of the Cost Feature

The expected total cost function per unit time is constructed by
$$C(S,s) = c_h\eta_I + c_w\eta_{PC} + c_s\eta_R + c_{cl}\eta_L + c_i\eta_{IC}\ ,$$

where

c_h : The inventory carrying cost per unit time.
c_w : Waiting cost of a customer in the pool per unit time.
c_s : Setup cost per order.
c_{cl} : Cost of a customer lost due to the zero stock per unit time.
c_i : The cost due to the arrival of impulse customer per unit time.

4. Numerical Illustration

We give a few descriptive numerical examples that expose the convexity of the expected cost rate and the MAP for ordinary and impulse customers' appearance. We consider, $E_0 = F_0$ and $E_1 = F_1$.

1. **Hyper-exponential (HEX):**
$$E_0 = F_0 = \begin{bmatrix} -15 & 0 \\ 0 & -5 \end{bmatrix}; E_1 = F_1 = \begin{bmatrix} 13.5 & 1.5 \\ 4.5 & 0.5 \end{bmatrix}$$
2. **Erlang (ER):**
$$E_0 = F_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}; E_1 = F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$
3. **Negative Correlation (NC):**
$$E_0 = F_0 = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}; E_1 = F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0235 & 0 & 2.3265 \\ 3.465 & 0 & 0.035 \end{bmatrix}$$
4. **Positive Correlation (PC):**
$$E_0 = F_0 = \begin{bmatrix} -2.35 & 2.35 & 0 \\ 0 & -2.35 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}; E_1 = F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2.3265 & 0 & 0.0235 \\ 0.035 & 0 & 3.465 \end{bmatrix}$$

The ordinary customer process has negative(positive) correlated arrival with coefficient of variance $c_{var} = 2\lambda_1\eta_1(-E_0)^{-1}\mathbf{e} - 1 =$ **0.9342(0.9342)** and coefficient of correlation $c_{cor} = (\lambda_1\eta_1(-E_0)^{-1}E_1(-E_0)^{-1})\mathbf{e} - 1)/c_{var} =$ **−0.2595(0.2595)** with arrival rate $\lambda_1 = 1.7594$. By our consideration, the values of c_{var} and c_{cor} for the impulse customer are the same as the values of c_{var} and c_{cor} for ordinary customers.

Table 1 gives the behaviour of the cost function of two variables $C(S,s)$ for the case of hyper-exponential distribution. The values are divulged **bold** in each column indicate the minimum cost rate whereas, the least cost rate is specified in each row by <u>underlining</u> the values. Thus a value (bold and underlined) spectacles the local minimum of the function $C(S,s)$. The optimal cost value $C^*(S,s)$ = **34.3798** is achieve at S^* = **23**, s^* = **7** with the values $r_1 = 0.6$, $r_2 = 1 - r_1$, $p = 0.6$, $q = 1 - p$, $\theta = 6.5$, $\lambda_1 = 12.5$, $\lambda_2 = 12.5$, $\mu = 0.5$, $\beta = 1$, $N = 7$, $c_h = 0.9$, $c_w = 3$, $c_s = 10$, $c_{cl} = 0.1$, $c_i = 0.3$. Table 1 and Figure 1 shows that the function $C(S,s)$ is convex.

Table 1. The function of total cost rate with two variables S and s.

S/*s*	5	6	7	8
21	73.9092	50.1316	<u>35.9508</u>	51.6445
22	70.1498	48.7175	<u>35.1863</u>	50.1225
23	66.6530	47.3281	<u>**34.3798**</u>	48.6287
24	**64.2888**	**46.8634**	<u>34.4369</u>	**48.0564**
25	64.3960	46.9908	<u>35.0336</u>	48.0747

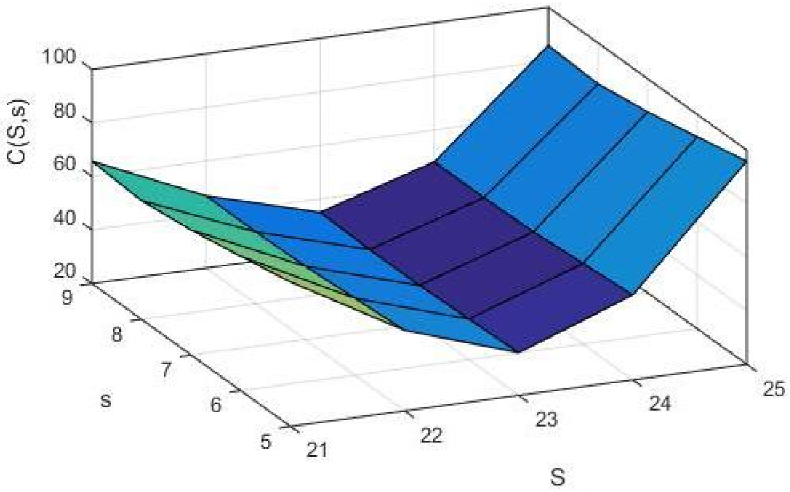

Figure 1. Convexity of the total cost with two variables S and s.

Table 2 scrutinized the ramifications of lead time rate μ and a customer's approach rate from the pool θ, the total cost rate $C(S,s)^*$ and analogous optimal value (S^*, s^*) with values $r_1 = 0.6$, $r_2 = 1 - r_1$, $p = 0.6$, $q = 1 - p$, $\lambda_1 = 12.5$, $\lambda_2 = 12.5$, $\beta = 1$, $N = 7$, $c_h = 0.9$, $c_w = 3$, $c_s = 10$, $c_{cl} = 0.1$, $c_i = 0.3$. We observe, the total expected cost rate decreases whenever θ and μ increases. Tables 3 and 4 are bestow the total expected cost rate is increase when c_w, c_h and c_s increase but total expected cost rate is decrease when c_{cl} increases.

Tables 5–7 give the mean inventory level. The arrival rate of impulse customers, reorder rate, the number of lost customer, and the mean cancellation rate decreases, and the mean number of pooled customers increase whenever θ, μ, and β increase under hyper-exponential, erlang, negative correlation and positive correlation.

Table 8 indicates the ordinary and impulse customer appears under erlang, a negative and positive correlation on the optimal total cost rate values at optimal value S^* and s^*. Table 9 bestow the ordinary customer appears under erlang, and impulse customer appears with various distributions (erlang, negative and positive correlation) on the various measures and also negative and positive correlation. Table 10 bestow the effect of λ_1 for various MAP(ER, NC and PC) with $\lambda_2 = 0.1$. The value of λ_1 increase as $C(S,s)$ increase with different MAP(ER, NC, and PC) and the values $r_1 = 0.4$, $r_2 = 1 - r_1$, $p = 0.6$, $q = 1 - p$, $\theta = 10.5$, $\mu = 2.5$, $\beta = 2$, $N = 7$, $c_h = 9$, $c_w = 3$, $c_s = 10$, $c_{cl} = 0.1$ $c_i = 0.3$. Table 11 bestow the effect of λ_2 for various MAP(ER, NC and PC) with $\lambda_1 = 1$. The value of λ_2 increase as $C(S,s)$ increase with different MAP(ER,NC and PC) and the values $r_1 = 0.4$, $r_2 = 1 - r_1$, $p = 0.6$, $q = 1 - p$, $\theta = 10.5$, $\mu = 0.5$, $\beta = 02$, $N = 7$, $c_h = 9$, $c_w = 3$, $c_s = 10$, $c_{cl} = 0.1$, $c_i = 0.3$. Figure 2 shows the correlation values of ordinary customers are plotted against the $C(S,s)$ values. It is possible to see that the $C(S,s)$ is increase as correlation coefficient values increases and impulse customer pursuant to MAP(ER, NC and PC) with the values $r_1 = 0.5$, $r_2 = 1 - r_1$, $p = 0.5$, $q = 1 - p$, $\lambda_1 = 12.5$, $\lambda_2 = 12.5$, $\beta = 1.5$, $N = 7$, $c_h = 0.9$, $c_w = 3$, $c_s = 9$, $c_{cl} = 0.1$, $c_i = 0.3$. Figure 3 shows the correlation values of impulse customer are plotted against the $C(S,s)$. It is possible to see that the $C(S,s)$ values non-decrease as the correlation coefficient values increases and ordinary customer under MAP(ER, NC, and PC).

Table 2. Ramification of the rate of lead time (μ) and approach from the pool customer (θ) on the optimal values.

μ/θ	**0.5**		**1.0**		**1.5**		**1.5**		**2.0**	
6	23	7	23	7	23	7	23	7	23	7
	35.2053		34.9025		34.5938		34.2810		33.9654	
8	24	7	24	7	24	7	23	7	23	7
	32.0167		31.7418		31.4634		31.1825		30.8967	
10	24	7	24	7	24	7	24	7	24	7
	29.1054		28.8532		28.5981		28.3413		28.0834	
12	24	7	24	7	24	7	24	7	24	7
	26.5210		26.2909		26.0586		25.8250		25.5906	
14	24	7	24	7	24	7	24	7	24	7
	24.2303		24.0214		23.8108		23.5992		23.3871	

Table 3. Ramification of the cost rate of a lost customer during the stock out period (c_{cl}) and cost rate of waiting customer in the pool (c_w) on the optimal values.

c_{cl}/c_w	**1**		**2**		**3**		**4**		**5**	
0.1	24	7	24	7	24	7	24	7	24	7
	25.3237		26.3775		27.4312		28.4850		29.5388	
0.2	23	7	23	7	23	7	23	7	23	7
	20.6208		21.7063		22.7918		23.8773		24.9628	
0.3	23	7	23	7	23	7	23	7	23	7
	15.8716		16.9572		18.0427		19.1282		20.2137	
0.4	23	7	23	7	23	7	23	7	23	7
	11.1225		12.2080		13.2935		14.3790		15.4645	
0.5	22	7	23	7	23	7	23	7	23	7
	6.3611		7.4589		8.5444		9.6299		10.7154	

Table 4. Ramification of the cost of setup cost (c_s) and holding cost (c_h) on the optimal values.

c_s/c_h	**0.7**		**0.8**		**0.9**		**1.0**		**1.1**	
8	23	7	23	7	23	7	23	7	23	7
	24.8072		25.1731		25.5389		25.9047		26.2705	
9	24	7	24	7	24	7	23	7	23	7
	25.7475		26.1340		26.5205		26.9057		27.2716	
10	24	7	24	7	24	7	24	7	24	7
	26.6582		27.0447		27.4312		27.8178		28.2043	
11	24	7	24	7	24	7	24	7	24	7
	27.5689		27.9555		28.3420		28.7285		29.1150	
12	24	7	24	7	24	7	24	7	24	7
	28.4797		28.8662		29.2527		29.6392		30.0258	

Table 5. Ramification of MAP with ER, NC and PC for (S, s, μ, β) = (21, 7, 2.5, 0.1).

Customer Arrivals	θ	η_I	η_R	η_{PC}	η_{IC}	η_L	η_C
MAP with HEX	6	7.1489	5.5542	0.9492	0.1955	0.9655	4.9701
	8	6.1845	5.2418	1.8282	0.1841	0.7165	4.8983
	10	5.5445	5.1297	2.1221	0.1755	0.598	4.8031
	12	5.2971	5.1185	2.3209	0.1587	0.2655	4.7455
	14	4.1842	5.0395	2.7986	0.1484	0.0495	4.2965
MAP with ER	6	10.4716	1.5605	4.7701	0.1907	0.2718	3.9203
	8	10.3639	1.1016	4.8031	0.1618	0.1711	3.7462
	10	10.2159	0.7828	4.8355	0.1421	0.1071	3.3194
	12	9.9523	0.5579	4.8672	0.1271	0.0669	3.2674
	14	9.6010	0.3977	4.8983	0.1148	0.0419	3.0355
MAP with NC	6	11.9361	0.8136	3.6126	0.5072	0.1869	6.5235
	8	11.8047	0.6569	3.6502	0.4694	0.1415	6.1452
	10	11.7142	0.5300	3.6872	0.4375	0.1076	5.4512
	12	10.1372	0.4278	3.7235	0.4099	0.0821	5.2201
	14	9.2649	0.3455	3.7592	0.3857	0.0629	5.1258
MAP with PC	6	11.8179	0.0055	3.0061	0.4916	0.1555	8.2835
	8	11.9699	0.0063	3.0113	0.4738	0.1655	8.1521
	10	11.9704	0.0069	3.0160	0.4590	0.1731	7.9203
	12	11.4244	0.0076	3.0203	0.4464	0.1790	7.5603
	14	10.1432	0.0082	3.0244	0.4352	0.1835	7.3595

Table 6. Ramification of MAP with ER, NC and PC for (S, s, θ, β) = (21, 7, 10.5, 0.1).

Customer Arrivals	μ	η_I	η_R	η_{PC}	η_{IC}	η_L	η_C
MAP with HEX	0.5	9.2511	1.2543	2.3533	0.7530	0.9532	6.2472
	1.0	5.4524	1.0216	2.3751	0.4652	0.7592	6.1552
	1.5	5.2455	1.0178	2.3751	0.3728	0.5313	5.2428
	2.0	3.9425	1.0149	2.3751	0.2086	0.3097	5.1588
	2.5	3.7058	1.0126	2.3751	0.0532	0.0927	5.0214
MAP with ER	0.5	9.5325	1.2029	3.8355	0.6417	0.8013	7.2472
	1.0	9.9282	1.1225	3.8355	0.5124	0.7032	6.8435
	1.5	8.8140	1.0212	3.8356	0.4532	0.6233	6.5472
	2.0	8.2287	1.0129	3.8356	0.3681	0.5237	6.3758
	2.5	8.0266	1.0025	3.8357	0.2522	0.4251	6.2457
MAP with NC	0.5	6.2459	2.0054	3.6872	0.7490	0.1595	7.8643
	1.0	6.1205	2.0056	3.6873	0.6378	0.1645	7.5184
	1.5	13.0476	2.0055	3.6873	0.5431	0.1635	7.4259
	2.0	6.0102	2.0053	3.6874	0.4629	0.1585	7.3445
	2.5	5.3139	2.0051	3.6874	0.3952	0.1510	6.2454
MAP with PC	0.5	5.8184	2.0051	4.5212	1.7060	0.9654	6.9637
	1.0	3.9954	2.0051	4.6122	1.5971	0.7652	6.7235
	1.5	2.3729	2.0049	4.6222	1.5089	0.6449	6.6282
	2.0	2.9522	2.0046	4.7522	1.4371	0.5372	6.4645
	2.5	1.7177	2.0043	4.7622	1.3782	0.3286	6.1278

Table 7. Ramification of MAP with ER, NC and PC for (S, s, θ, μ) = (21, 7, 10.5, 2.5).

Customer Arrivals	β	η_I	η_R	η_{PC}	η_{IC}	η_L	η_C
MAP with HEX	0.5	4.3915	4.7884	1.0554	38.8344	9.9514	4.2575
	1.0	4.3805	4.4750	1.0320	30.7307	8.9913	4.1702
	1.5	4.2681	4.3167	1.1123	30.3223	8.9913	4.1434
	2.0	4.1868	4.0668	1.9921	30.2715	2.0151	1.5565
	2.5	3.6345	3.2879	2.0475	30.1281	1.9440	1.3230
MAP with ER	0.5	6.5733	2.5566	1.0214	1.1602	5.1051	0.7440
	1.0	2.0654	1.4214	1.2613	1.0315	4.0583	0.6516
	1.5	0.8832	1.3771	1.3315	0.7274	1.4230	0.5629
	2.0	0.4074	0.3074	1.3701	0.3122	0.5364	0.2604
	2.5	0.1988	0.0065	1.4128	0.1682	0.2350	0.1353
MAP with NC	0.5	7.7680	4.1544	0.9291	0.0302	0.1130	0.0098
	1.0	2.0747	3.5491	0.9314	0.0139	0.0139	0.0024
	1.5	1.5672	1.5672	0.9437	0.0002	0.0006	0.0001
	2.0	1.7586	0.5786	0.9560	0.0001	0.0005	0.0001
	2.5	0.2107	0.3754	0.9684	0.0001	0.0004	0.0001
MAP with PC	0.5	0.3234	0.1616	1.1871	2.2619	3.4010	1.0797
	1.0	0.3160	0.0788	1.1248	1.0117	1.7219	0.5441
	1.5	0.0590	0.0206	1.2137	0.1521	0.1198	0.1073
	2.0	0.0087	0.0019	1.3521	0.0194	0.0161	0.0158
	2.5	0.0020	0.0003	1.4287	0.0040	0.0034	0.0035

Table 8. Ramification of ordinary and impulse arrival on optimal value.

Ordinary/Impulse Arrivals	MAP with ER		MAP with NC		MAP with PC	
MAP with ER	23	7	23	8	24	7
	23.1112		21.6972		12.0351	
MAP with NC	23	7	23	8	24	7
	22.8082		21.4556		21.1303	
MAP with PC	23	7	23	8	24	7
	21.5698		23.8045		22.7606	

Table 9. Ramification of MAP of ordinary and impulse customer arrivals on various measures.

Impulse Customer Arrivals	η_I	η_R	η_{PC}	η_{IC}	η_L	η_C
MAP of ordinary arrival with ER						
MAP with ER	4.0749	0.3074	1.0370	0.3122	0.5364	0.2604
MAP with NC	2.8386	0.2935	0.9444	0.2783	0.0878	0.0417
MAP with PC	1.8369	0.0038	1.0198	0.0189	0.0562	0.0154
MAP of ordinary arrival with NC						
MAP with ER	1.7586	0.0055	0.5786	0.5431	0.1635	0.1720
MAP with NC	0.6741	0.0161	0.0016	0.9280	0.4787	0.0128
MAP with PC	0.0164	0.0086	0.0029	0.8591	0.2563	0.0031
MAP of ordinary arrival with PC						
MAP with ER	8.0759	0.0019	0.0194	0.0161	0.0158	2.1521
MAP with NC	2.2589	0.0171	0.0015	0.8594	0.5084	1.2315
MAP with PC	0.0909	0.8807	0.2235	0.1300	0.1486	1.1763

Table 10. Effects of λ_1 with various MAP.

<table>
<tr><th>λ_1</th><th>Ordinary/Impulse Arrivals</th><th colspan="2">MAP with ER</th><th colspan="2">MAP with NC</th><th colspan="2">MAP with PC</th></tr>
<tr><td rowspan="6">1.0</td><td rowspan="2">MAP with ER</td><td>30</td><td>9</td><td>30</td><td>8</td><td>31</td><td>9</td></tr>
<tr><td colspan="2">3.2312</td><td colspan="2">2.8333</td><td colspan="2">1.9895</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>33</td><td>7</td><td>33</td><td>7</td><td>31</td><td>9</td></tr>
<tr><td colspan="2">2.7514</td><td colspan="2">2.8149</td><td colspan="2">2.8366</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">3.0706</td><td colspan="2">2.8275</td><td colspan="2">2.8560</td></tr>
<tr><td rowspan="6">1.5</td><td rowspan="2">MAP with ER</td><td>33</td><td>7</td><td>35</td><td>7</td><td>35</td><td>8</td></tr>
<tr><td colspan="2">2.9048</td><td colspan="2">2.9277</td><td colspan="2">2.8302</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>35</td><td>9</td><td>33</td><td>8</td><td>33</td><td>8</td></tr>
<tr><td colspan="2">3.7805</td><td colspan="2">2.8542</td><td colspan="2">2.8954</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">24.7267</td><td colspan="2">2.3890</td><td colspan="2">20.5588</td></tr>
<tr><td rowspan="6">2.0</td><td rowspan="2">MAP with ER</td><td>33</td><td>7</td><td>35</td><td>7</td><td>35</td><td>8</td></tr>
<tr><td colspan="2">2.8922</td><td colspan="2">2.8921</td><td colspan="2">2.8967</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>35</td><td>9</td><td>33</td><td>8</td><td>33</td><td>8</td></tr>
<tr><td colspan="2">2.9265</td><td colspan="2">2.8177</td><td colspan="2">2.8953</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">2.8855</td><td colspan="2">2.8851</td><td colspan="2">2.8920</td></tr>
</table>

Table 11. Effects of λ_2 with various MAP.

<table>
<tr><th>λ_2</th><th>Impulse/Ordinary Arrivals</th><th colspan="2">MAP with ER</th><th colspan="2">MAP with NC</th><th colspan="2">MAP with PC</th></tr>
<tr><td rowspan="6">0.2</td><td rowspan="2">MAP with ER</td><td>30</td><td>9</td><td>30</td><td>8</td><td>31</td><td>9</td></tr>
<tr><td colspan="2">4.2172</td><td colspan="2">1.3851</td><td colspan="2">0.1314</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>33</td><td>7</td><td>33</td><td>7</td><td>31</td><td>9</td></tr>
<tr><td colspan="2">1.3100</td><td colspan="2">1.0523</td><td colspan="2">1.0050</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">1.1253</td><td colspan="2">0.5222</td><td colspan="2">7.0213</td></tr>
<tr><td rowspan="6">0.4</td><td rowspan="2">MAP with ER</td><td>38</td><td>3</td><td>38</td><td>3</td><td>38</td><td>3</td></tr>
<tr><td colspan="2">3.0233</td><td colspan="2">2.2223</td><td colspan="2">2.3503</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>35</td><td>9</td><td>33</td><td>8</td><td>33</td><td>8</td></tr>
<tr><td colspan="2">1.0123</td><td colspan="2">2.2211</td><td colspan="2">2.4125</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">3.4561</td><td colspan="2">2.1847</td><td colspan="2">2.4875</td></tr>
<tr><td rowspan="6">0.6</td><td rowspan="2">MAP with ER</td><td>35</td><td>9</td><td>33</td><td>8</td><td>33</td><td>8</td></tr>
<tr><td colspan="2">2.5654</td><td colspan="2">3.4842</td><td colspan="2">1.1551</td></tr>
<tr><td rowspan="2">MAP with NC</td><td>33</td><td>9</td><td>31</td><td>11</td><td>35</td><td>8</td></tr>
<tr><td colspan="2">3.2484</td><td colspan="2">1.5422</td><td colspan="2">0.4412</td></tr>
<tr><td rowspan="2">MAP with PC</td><td>31</td><td>8</td><td>34</td><td>8</td><td>33</td><td>7</td></tr>
<tr><td colspan="2">3.1654</td><td colspan="2">2.1685</td><td colspan="2">3.1517</td></tr>
</table>

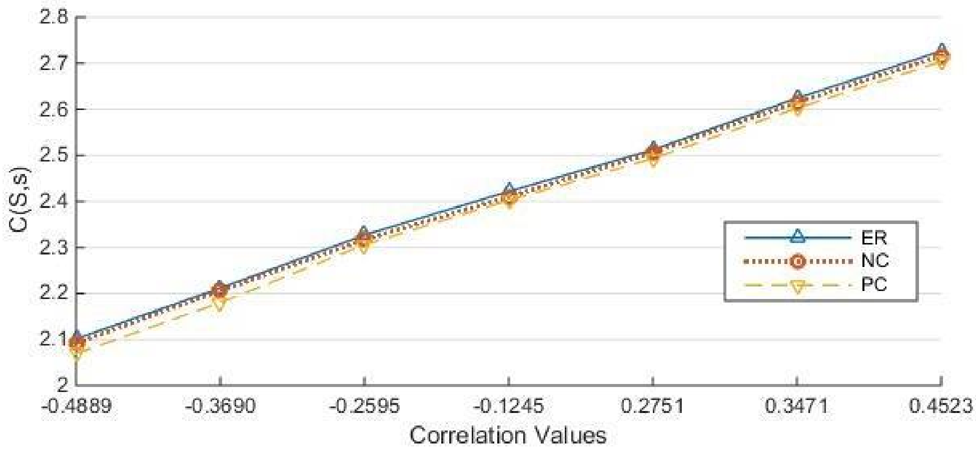

Figure 2. Ordinary customer arrival correlation vs. total cost rate.

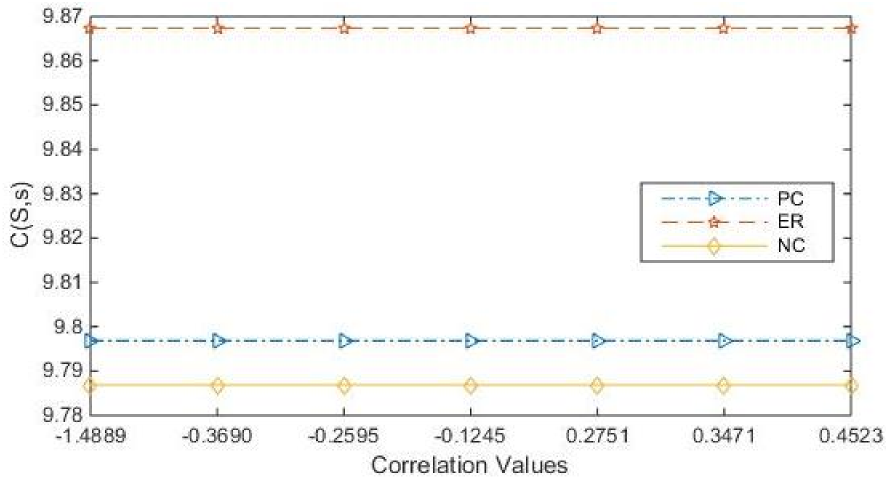

Figure 3. Impulse customer arrival correlation vs. total cost rate.

5. Conclusions

In this article, we introduced the impulse customer with cancellation policy in the queueing-inventory system. This work helps improve the QoS of an inventory management system. We presented the behaviour of the total cost function with variables s and S under hyper exponential distribution. We analysed the total cost function with various distributions like erlang, negative correlation, and positive correlation for ordinary and impulsed customers' arrival streams. We analysed the effects of the pooled customer approach, lead time, and cancellation rates with various arrival streams like hyper-exponential, erlang, negative correlation, and positive correlation. The effects of an average number of impulse customer arrival rates and the average loss rate are indicated. Finally, we showed the effects of ordinary and impulsed customers' arrival correlation with total cost rate.

In the future, we will be interested in extending this model with multi-server, and service time follows PH distribution.

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Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following Notations and Abbreviations are used in this manuscript:

$[A]_{ij}$	The element submatrix at (i,j) the position of A
0	Zero matrix
e	A column vector of 1's appropriate dimension
S	Maximum inventory level
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Kronecker sum of matrices A and B
MAP	Markovian Arrival Process
PH	Phase-type
PHF	Phase-type with Failure.

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