

## Article

# Fixed-Time Tracking Control for Nonlinear Cascade Systems with Unknown High Powers

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**Abstract:** This paper investigates the global fixed-time tracking control problem of nonlinear cascade systems with unknown high powers. In the process of control design, an upper bound and a lower bound of high powers are introduced to compensate the unknown system powers, and a state feedback controller is designed under any initial system conditions. Based on the Lyapunov stability analysis method and the fixed-time stability theory, it is verified that the proposed method can regulate the output tracking error to a disc region of the origin within a fixed-time and all the closed-loop signals are bounded. At last, the effectiveness of the proposed scheme is verified by some simulation results.

**Keywords:** nonlinear cascade systems; fixed-time stability; tracking control; unknown powers



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## 1. Introduction

As well known, the problem of output regulation is one of the most important issues in the control field. Based on Lyapunov stability theory and backstepping approach, generous results about output regulations have been achieved for various nonlinear systems. For a class of nonlinear cascade systems with high powers, using adding a power integrator technique [1–3], many scholars have given great efforts to design feedback controllers and achieved massive results on output regulations, see [4–10], and references therein. Comparing with the asymptotic control performance, finite-time control has faster convergence rate and higher accuracy. Therefore, many effective approaches have been developed to finite-time stabilization and/or output tracking control of nonlinear cascade systems with high powers in the past few years [11–13]. For example, Sun et al. established fast finite-time control schemes for nonlinear cascade systems with different circumstances [14–16]; Wang et al. investigated finite-time tracking control problem based on event-triggered mechanism [17]; Liu et al. investigated finite-time stabilization via switching adaptive feedback controller [18].

It should be noted that the settling times of finite-time stability systems depend on the initial states [19], which may lead to some limitations in the control process. As an extension of finite-time stability, the performance of fixed-time stability is superior to finite-time one in that the settling time has an upper bound being regardless of initial conditions. The concept of fixed-time stability is proposed in [20], and is further studied in [21,22]. Due to its advantages in the settling time, many researchers pay great attentions to the fixed-time control and much interesting results have been obtained [23–27]. Up to now, there are some fixed-time control results concerned with high-order nonlinear cascade systems. For example, Chen et al. achieved the fixed-time stabilisation [28]; Yu et al. designed a new fixed-time controller based on a serial of exponential functions and fractional power integration [29]. Ma et al. achieved the tracking control performance for high-order nonlinear

cascade systems within a fixed-time [30]. Despite the above efforts, in the previous results, all the high powers of the nonlinear cascade systems are precisely known and limited to odd integers and/or ratios of odd integers.

With the development of control technology, many scholars have investigated the control problem of nonlinear cascade systems with time-varying high powers. The output feedback stabilization is developed for nonlinear systems with time-varying high powers in [31]. The global stabilization and practical tracking are achieved for unknown time-varying powers in [32]. Wang et al. achieved adaptive stabilization by switching adaptive controller for nonlinear systems with unknown powers [33]. Xie et al. achieved tracking control of nonlinear systems with full-state constraints and unknown powers [34]. The least of perfection is that these results are all about asymptotic stabilization or/and asymptotic tracking control. To the authors' knowledge, no results about fixed-time tracking control have been reported for nonlinear cascade systems with unknown high powers.

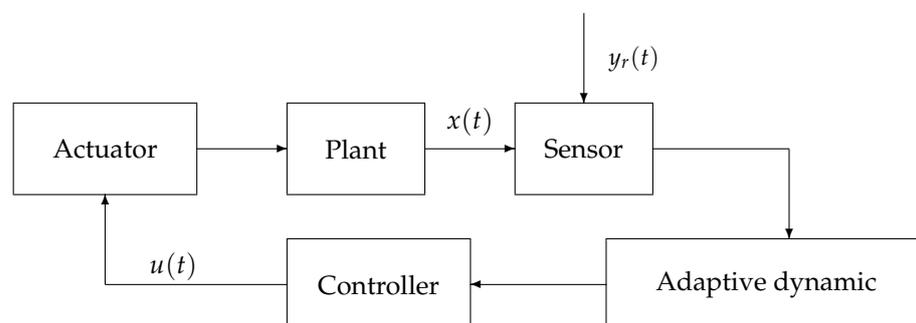
Motivated by the above observations, this paper studies practical fixed-time tracking control for nonlinear cascade systems with unknown high powers. The main contributions of this article are summarized as follows:

- A state feedback fixed-time tracking controller is designed for a class of nonlinear cascade systems with time-varying high powers. Using Lyapunov stability analysis method and fixed-time stability theory, it is verified that the output tracking error can be regulated to a disc region of the origin within a fixed-time by the proposed method, and the settling time is regardless of initial conditions.
- In comparison with most existing results for high-order nonlinear systems, system high powers in this work are relaxed to positive time-varying functions, which makes the considered systems more generally. A upper bound and a lower bound of high powers are introduced in the proposed controller, which are powerful to compensate the time-varying powers.

The organization of this paper is as follows. Section 1 describes the problem formulation and preliminaries of this work. Section 2 provides the control design and performances analysis. Two simulation examples are given in Section 3. Some concluded remarks are given in Section 4.

**Notation 1.**  $\mathbb{R}^j$ : the set of all real  $j$ -dimensional vectors;  $\mathbb{R}_+$ : the set of all nonnegative real numbers;  $[a]^q = \text{sign}(a)|a|^q$ ,  $\text{sign}(a)$  denotes its sign function;  $\varphi_j = [x_1, \dots, x_j]^T \in \mathbb{R}^j$ .

For convenience, the functions are sometimes simplified, for instance, a function  $\Phi(x(t))$  is simplified by  $\Phi(x)$ ,  $\Phi(\cdot)$  or  $\Phi$ . A basic block architecture of tracking control is given in Figure 1.



**Figure 1.** The fixed-time tracking control architecture.

## 2. Problem Formulation and Preliminaries

Consider the following nonlinear cascade systems

$$\begin{cases} \dot{x}_j = \Psi_j(\varphi_j) [x_{j+1}]^{q_j(t)} + F_j(\varphi_j) + \varrho_j(t), & j = 1, \dots, n-1 \\ \dot{x}_n = \Psi_n(\varphi_n) [u]^{q_n(t)} + F_n(\varphi_n) + \varrho_n(t) \\ y = x_1 \end{cases} \quad (1)$$

where  $\varphi_j \in \mathbb{R}^j$  is the system state;  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  are system input and system output, respectively; The nonlinear functions  $F_j : \mathbb{R}^j \rightarrow \mathbb{R}$  and  $\Psi_j : \mathbb{R}^j \rightarrow \mathbb{R}$  are unknown but locally Lipschitz in their arguments;  $\varrho_j : \mathbb{R}_+ \rightarrow \mathbb{R}$  denotes time-varying disturbance; The system high powers  $q_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are time-varying functions satisfying  $q_j(t) \geq 1$ , and there exists at least one  $q_j(t) > 1$  in system (1).

**Control Objective:** Design a state-feedback controller for system (1) such that the output tracking error can be regulated to a disc region of the origin within a fixed-time regardless of initial conditions, and all the closed-loop signals are bounded.

The following assumptions are given to achieve the control objective.

**Assumption 1.** The reference signal  $r_d(t)$  and its first derivative  $\dot{r}_d(t)$  are bounded, i.e., there holds

$$\sup_t (|r_d(t)| + |\dot{r}_d(t)|) \leq \kappa,$$

where  $\kappa$  is an unknown positive constant.

**Assumption 2.** The disturbances  $\varrho_j(t)$ ,  $j = 1, \dots, n$  are bounded, i.e., there holds

$$\sup_t \varrho_j(t) \leq \kappa_1,$$

where  $\kappa_1$  is an unknown positive constant.

**Assumption 3.** For each  $F_j(\varphi_j)$ ,  $j = 1, \dots, n$ , there holds

$$|F_j(\varphi_j)| \leq \theta \bar{F}_j(\varphi_j),$$

where  $\theta$  is an unknown positive constant, and  $\bar{F}_j(\varphi_j)$  is known positive smooth function.

**Assumption 4.** For each  $q_j(t)$ ,  $j = 1, \dots, n$ , there holds  $q \geq q_j(t)$  with  $q$  being a known positive constant.

**Assumption 5.** The functions  $\Psi_j(\varphi_j)$ ,  $j = 1, \dots, n$  are either strictly positive or strictly negative, and there holds

$$\underline{\Psi}_j(\varphi_j) \leq |\Psi_j(\varphi_j)| \leq \bar{\Psi}_j(\varphi_j),$$

where  $\underline{\Psi}_j(\varphi_j)$  and  $\bar{\Psi}_j(\varphi_j)$  are known positive smooth functions. In this work, we assume  $\Psi_j(\varphi_j) > 0$ .

**Remark 1.** Assumptions 1, 2 and 5 can be widely found in the existing tracking control results. Assumption 3 means that the growth conditions of nonlinearities are not needed, so the nonlinearities  $F_j(\varphi_j)$  in system (1) are more general than those in [3,5]. Assumption 4 indicates that time-varying powers  $q_j(t)$  have a known upper bound.

The following definitions and lemmas are given to facilitate the design of tracking controller.

**Definition 1 ([21]).** Consider the nonlinear system:

$$\dot{\omega} = \mathfrak{h}(\omega, u), \quad \omega(0) = \omega_0, \tag{2}$$

where  $\omega \in \mathbb{R}^n$  is the system state,  $\mathfrak{h} : \mathbb{R}_+ \times Y \rightarrow \mathbb{R}^n$  is a continuous function with  $Y$  being an open neighborhood of the origin  $\omega = 0$ . System (2) is globally fixed-time stable if it is stable, and for any  $\omega(0) = \omega_0$ , there exists a function  $T_s(\omega_0) : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that the solution  $\omega(t, \omega_0)$  of (2) satisfying  $\lim_{t \rightarrow T_s(\omega_0)} \omega(t, \omega_0) = 0$  and  $\omega(t, \omega_0) = 0, \forall t \geq T_s(\omega_0)$ . Moreover, there holds  $T_s(\omega_0) \leq T_{\max}$ , where  $T_{\max}$  is a positive constant.

**Lemma 1 ([21,35]).** Consider system (2). If there exists a positive definite function  $V(\omega)$  and some constants  $d_1 > 0, d_2 > 0, b_1 > 1, 0 < b_2 < 1$  and  $0 < \iota < \infty$  such that the time derivative of  $V(\omega)$  satisfies

$$\dot{V}(\omega) \leq -d_1 V^{b_1}(\omega) - d_2 V^{b_2}(\omega) + \iota,$$

then system (2) is practical fixed-time stable. Moreover, the residual set of the trajectory can be described by

$$\left\{ \lim_{t \rightarrow T_s} \omega \mid V(\omega) \leq \min \left\{ \left( \frac{\iota}{(1-k)d_1} \right)^{\frac{1}{b_1}}, \left( \frac{\iota}{(1-k)d_2} \right)^{\frac{1}{b_2}} \right\} \right\}$$

where  $0 < k < 1$ , and the settling time is bounded by  $T_s \leq T_{\max} = 1/kd_1(b_1 - 1) + 1/kd_2(1 - b_2)$ .

**Lemma 2 ([31]).** For any  $\Xi_1, \Xi_2 \in \mathbb{R}$  and any continuous functions  $a_1(t) > 0, a_2(t) > 0$  and  $c > 0$ , there holds

$$|\Xi_1|^{a_1(t)} |\Xi_2|^{a_2(t)} \leq c \cdot \frac{a_1(t)}{a_1(t) + a_2(t)} |\Xi_1|^{a_1(t)+a_2(t)} + c^{-\frac{a_1(t)}{a_2(t)}} \cdot \frac{a_2(t)}{a_1(t) + a_2(t)} |\Xi_2|^{a_1(t)+a_2(t)}.$$

**Lemma 3 ([31]).** For any  $\Xi_j \in \mathbb{R}$  and any continuous function  $a(t) > 0$ , there holds

$$\sum_{j=1}^n |\Xi_j|^{a(t)} \leq \left( \sum_{j=1}^n |\Xi_j| \right)^{a(t)} \leq \max\{n^{a(t)-1}, 1\} \sum_{j=1}^n |\Xi_j|^{a(t)}.$$

**Lemma 4 ([32]).** For any  $\Xi_1, \Xi_2 \in \mathbb{R}$  and any continuous function  $a(t) \geq 1$ , there holds

$$|\lceil \Xi_1 \rceil^{a(t)} - \lceil \Xi_2 \rceil^{a(t)}| \leq |\Xi_1 - \Xi_2|^{a(t)} + a(t) |\Xi_1 - \Xi_2| \cdot (|\Xi_1|^{a(t)-1} + |\Xi_2|^{a(t)-1}).$$

**Lemma 5 ([32]).** For any  $v \in \mathbb{R}$ , If a continuous function  $a(t)$  satisfies  $0 < \underline{a} < a(t) < \bar{a}$ , there holds

$$|v|^{a(t)} \leq |v|^{\underline{a}} + |v|^{\bar{a}}.$$

### 3. Main Results

#### 3.1. Control Design

In this section, we will design a state feedback tracking controller for system (1).

Motivated by the control law designed in the related literature [4,32], the actual state feedback controller is designed as

$$u = -(\underline{\psi}_n^{-1} + \underline{\psi}_n^{-\frac{1}{q}})(\zeta_n + \lceil \zeta_n \rceil^{\frac{1}{q^2}}) - \phi_n(\varrho_n, r_d)(\zeta_n + \lceil \zeta_n \rceil^q) \tag{3}$$

with  $\zeta_n$  being recursively introduced by

$$\begin{cases} \zeta_1 = y - r_d, \\ \zeta_j = x_j - v_{i-1}(\varrho_{j-1}, r_d), \quad j = 2, \dots, n, \end{cases} \tag{4}$$

where  $v_j(\varphi_j, r_d)$  is virtual control signal designed by

$$v_j(\varphi_j, r_d) = -(\underline{\psi}_j^{-1} + \underline{\psi}_j^{-\frac{1}{q}})(\xi_j + \lceil \xi_j \rceil^{\frac{1}{q^2}}) - \phi_j(\varphi_j, r_d)(\xi_j + \lceil \xi_j \rceil^q), \quad j = 1, \dots, n - 1, \quad (5)$$

where  $\phi_j(\varphi_j, r_d), j = 1, \dots, n - 1$  are known positive functions to be designed later.

With suitable choice of  $\phi_j(\cdot)$ , we can obtain the following property of the resulting closed-loop system.

**Proposition 1.** Let  $\xi = [\xi_1, \dots, \xi_n]^T$  and define the Lyapunov function  $V(\xi) = \frac{1}{2} \sum_{j=1}^n \xi_j^2$ . For system (1) under Assumptions 1–5, the actual controller (3) along the trajectories of system (1) can lead to

$$\dot{V} \leq -\sum_{j=1}^n |\xi_j|^{1+\frac{1}{q}} - \sum_{j=1}^n |\xi_j|^{1+q} + C, \quad (6)$$

where  $C$  is an unknown positive constant.

**Proof.** Based on (1) and (4), the time derivative of  $V$  is

$$\begin{aligned} \dot{V} &= -\dot{r}_d \xi_1 + \sum_{j=1}^n (F_j + \varrho_j) \xi_j - \sum_{j=2}^n \frac{\partial v_{j-1}}{\partial r_d} \dot{r}_d \xi_j \\ &\quad - \sum_{j=2}^n \sum_{m=1}^{j-1} \frac{\partial v_{j-1}}{\partial x_m} (\Psi_m[x_{m+1}]^{q_m} + F_m + \varrho_m) \xi_j + \sum_{j=1}^n \Psi_j[x_{j+1}]^{q_j} \xi_j \end{aligned} \quad (7)$$

with  $x_{n+1} = u$ .

Next, we give the appropriate estimations for the right side of (7) (remarked by  $A_1 - A_5$ , respectively).

By Assumptions 1–3 and Lemma 2, one has

$$A_1 = -\dot{r}_d \xi_1 \leq \kappa |\xi_1| \leq M_0 |\xi_1|^{1+q} + \frac{\Theta_1}{L}, \quad (8)$$

$$A_2 = \sum_{j=1}^n (F_j + \varrho_j) \xi_j \leq \sum_{j=1}^n (\theta \bar{F}_j + \kappa_1) |\xi_j| \leq \sum_{j=1}^n M_{1j}(\varphi_j, r_d) |\xi_j|^{1+q} + \frac{\Theta_2}{L}, \quad (9)$$

$$A_3 = \sum_{j=2}^n \frac{\partial v_{j-1}}{\partial r_d} \dot{r}_d \xi_j \leq \sum_{j=2}^n \kappa \left| \frac{\partial v_{j-1}}{\partial r_d} \right| |\xi_j| \leq \sum_{j=2}^n M_{2j}(\varphi_j, r_d) |\xi_j|^{1+q} + \frac{\Theta_3}{L}, \quad (10)$$

where  $\Theta_1, \Theta_2$  and  $\Theta_3$  are unknown positive constants,  $L$  is the constant parameter to be designed,  $M_0, M_{1j}(\cdot)$  and  $M_{2j}(\cdot)$  are all known positive smooth functions. Particularly,  $M_0$  and  $M_{1j}(\cdot)$  are independent of  $\phi_1, \dots, \phi_n$ ,  $M_{2j}(\cdot)$  depends on  $\phi_1, \dots, \phi_{j-1}$  but not on  $\phi_j, \dots, \phi_n$ .

By Assumptions 1–4, and Lemmas 2 and 5, one has

$$\begin{aligned} A_4 &= -\sum_{j=2}^n \sum_{m=1}^{j-1} \frac{\partial v_{j-1}}{\partial x_m} (\Psi_m[x_{m+1}]^{q_m} + F_m + \varrho_m) \xi_j \\ &\leq \sum_{j=2}^n \sum_{m=1}^{j-1} \left| \frac{\partial v_{j-1}}{\partial x_m} \right| (\bar{\psi}_m |x_{m+1}|^{q_m} + \theta \bar{F}_m + \kappa_1) |\xi_j| \\ &\leq \sum_{j=2}^n \sum_{m=1}^{j-1} \left| \frac{\partial v_{j-1}}{\partial x_m} \right| (\bar{\psi}_m (|x_{m+1}|^q + |x_{m+1}|) + \theta \bar{F}_m + \kappa) |\xi_j| \\ &\leq \sum_{j=2}^n M_{3j}(\varphi_j, r_d) |\xi_j|^{1+q} + \frac{\Theta_4}{L}, \end{aligned} \quad (11)$$

where  $\Theta_4$  is an unknown positive constant, and  $M_{3j}(\cdot)$  is a known positive smooth function, which depends on  $\phi_1, \dots, \phi_{j-1}$  but not on  $\phi_j, \dots, \phi_n$ .

From Assumptions 3 and 4, Lemmas 2, 4 and 5, we have

$$\begin{aligned} & \sum_{j=1}^{n-1} \Psi_j([\underline{x}_{j+1}]^{q_j} - [\underline{v}_j]^{q_j})\xi_j \\ &= \sum_{j=2}^n \Psi_{j-1}([\underline{x}_j]^{q_{j-1}} - [\underline{v}_{j-1}]^{q_{j-1}})\xi_{j-1} \\ &\leq \sum_{j=2}^n \bar{\psi}_{j-1} \left( |\xi_j|^{q_{j-1}} + q_{j-1} |\xi_j| (|\underline{x}_j|^{q_{j-1}-1} + |\underline{v}_{j-1}|^{q_{j-1}-1}) \right) |\xi_{j-1}| \\ &\leq \sum_{j=2}^n \bar{\psi}_{j-1} \left( |\xi_j| + |\xi_j|^q + q_{j-1} |\xi_j| \left( (1 + |\underline{x}_j|^2)^{q-1} + (1 + |\underline{v}_{j-1}|^2)^{q-1} \right) \right) |\xi_{j-1}| \\ &\leq \sum_{j=2}^n M_{4j}(\varrho_j, r_d) |\xi_j|^{1+q} + \frac{n-1}{L}, \end{aligned} \tag{12}$$

where  $M_{4j}(\cdot)$  is a known positive smooth function, which depends on  $\phi_1, \dots, \phi_{j-1}$  but not on  $\phi_j, \dots, \phi_n$ .

It follows from (3) and (5) that

$$\begin{cases} \sum_{j=1}^{n-1} \Psi_j [\underline{v}_j]^{q_j} \xi_j = - \sum_{j=1}^{n-1} \Psi_j \phi_j^{q_j} \left[ \frac{\psi_j^{-1} + \underline{\psi}_j^{-\frac{1}{q}}}{\phi_j} (\xi_j + [\xi_j]^{1/q^2}) + \xi_j + [\xi_j]^{q_j} \xi_j \right], \\ \Psi_n [\underline{u}]^{q_n} \xi_n = - \Psi_n \phi_n^{q_n} \left[ \frac{\psi_n^{-1} + \underline{\psi}_n^{-\frac{1}{q}}}{\phi_n} (\xi_n + [\xi_n]^{1/q^2}) + \xi_n + [\xi_n]^{q_n} \xi_n \right]. \end{cases} \tag{13}$$

By Assumption 5, Lemmas 3 and 5, there holds

$$\begin{aligned} & \left[ \frac{\psi_j^{-1} + \underline{\psi}_j^{-\frac{1}{q}}}{\phi_j} (\xi_j + [\xi_j]^{1/q^2}) + \xi_j + [\xi_j]^{q_j} \xi_j \right] \\ &= \left( \frac{\psi_j^{-1} + \underline{\psi}_j^{-\frac{1}{q}}}{\phi_j} (|\xi_j| + |\xi_j|^{1/q^2}) + |\xi_j| + |\xi_j|^q \right)^{q_j} |\xi_j| \\ &\geq \frac{\psi_j^{-q_j} + \underline{\psi}_j^{-\frac{q_j}{q}}}{\phi_j^{q_j}} (|\xi_j|^{1+q_j} + |\xi_j|^{1+\frac{q_j}{q^2}}) + |\xi_j|^{1+q_j} + |\xi_j|^{1+q_j q} \\ &\geq \frac{\psi_j^{-q_j} + \underline{\psi}_j^{-\frac{q_j}{q}}}{\phi_j^{q_j}} |\xi_j|^{1+\frac{1}{q}} + |\xi_j|^{1+q}, \end{aligned} \tag{14}$$

$$\Psi_j (\psi_j^{-q_j} + \underline{\psi}_j^{-\frac{q_j}{q}}) \geq \frac{1}{\underline{\psi}_j^{q_j-1}} + \underline{\psi}_j^{1-\frac{q_j}{q}} \geq 1, \tag{15}$$

which, together with (13), yields

$$\sum_{j=1}^{n-1} \Psi_j [\underline{v}_j]^{q_j} \xi_j + \Psi_n [\underline{u}]^{q_n} \xi_n \leq - \sum_{j=1}^n |\xi_j|^{1+\frac{1}{q}} - \sum_{j=1}^n \Psi_j \phi_j^{q_j} |\xi_j|^{1+q}. \tag{16}$$

From (12) and (16), it follows that

$$\begin{aligned}
 A_5 &= \sum_{j=1}^{n-1} \Psi_j(\lceil x_{j+1} \rceil^{q_j} - \lceil v_j \rceil^{q_j}) \xi_j + \sum_{j=1}^{n-1} \Psi_j \lceil v_j \rceil^{q_j} \xi_j + \Psi_n \lceil u \rceil^{q_n} \xi_n \\
 &\leq - \sum_{j=1}^n |\xi_j|^{1+\frac{1}{q}} - \sum_{j=1}^n \Psi_j \phi_j^{q_j} |\xi_j|^{1+q} + \sum_{j=2}^n M_{4j}(\varrho_j, r_d) |\xi_j|^{1+q} + \frac{n-1}{L}. \tag{17}
 \end{aligned}$$

Finally, substituting the estimations of  $A_1 - A_5$  into (7), we have

$$\begin{aligned}
 \dot{V} &\leq - \sum_{j=1}^n |\xi_j|^{1+\frac{1}{q}} - \sum_{j=1}^n \Psi_j \phi_j^{q_j} |\xi_j|^{1+q} + M_0 |\xi_1|^{1+q} \\
 &\quad + \sum_{j=1}^n M_{1j}(\varrho_j, r_d) |\xi_j|^{1+q} + \sum_{j=2}^n \sum_{i=2}^4 M_{ij}(\varrho_j, r_d) |\xi_j|^{1+q} + C \tag{18}
 \end{aligned}$$

with  $C = (n - 1 + \sum_{i=1}^4 \Theta_j) / L$ .

Next, we design the smooth functions  $\phi_j(\cdot), j = 1, \dots, n$  step-by-step. Firstly, we design

$$\phi_1(x_1, r_d) = (\underline{\psi}_1^{-1} + \underline{\psi}_1^{-\frac{1}{q}})(1 + M_0 + M_{11}(x_1, r_d)). \tag{19}$$

In turn,  $\phi_j(\cdot), j = 2, \dots, n$  are designed as

$$\phi_j(\varrho_j, r_d) = (\underline{\psi}_j^{-1} + \underline{\psi}_j^{-\frac{1}{q}})(1 + \sum_{i=1}^4 M_{ij}(\varrho_k, r_d)). \tag{20}$$

By (15) and Lemma 3, there holds

$$\begin{aligned}
 \Psi_1 \phi_1^{q_1} &\geq \Psi_1 (\underline{\psi}_1^{-q_1} + \underline{\psi}_1^{-\frac{q_1}{q}})(1 + M_0 + M_{11}(x_1, r_d))^{q_1} \\
 &\geq 1 + M_0 + M_{11}(x_1, r_d), \tag{21}
 \end{aligned}$$

$$\begin{aligned}
 \Psi_j \phi_j^{q_j} &\geq \Psi_j (\underline{\psi}_j^{-q_j} + \underline{\psi}_j^{-\frac{q_j}{q}})(1 + \sum_{i=1}^4 M_{ij}(\varrho_j, r_d))^{q_j} \\
 &\geq 1 + \sum_{i=1}^4 M_{ij}(\varrho_j, r_d), \quad j = 2, \dots, n. \tag{22}
 \end{aligned}$$

Then, substituting (19) and (20) into (18), we directly obtain (6).

The proof of Proposition 1 is completed.  $\square$

### 3.2. Stability Analysis

**Theorem 1.** Consider the nonlinear system (1). Under the Assumptions 1–5, the proposed state-feedback controller (3) can guarantee that the output tracking error can be regulated to a disc region of the origin within a fixed-time and all the closed-loop signals are bounded.

**Proof.** From (6) and Lemma 3, one has

$$\begin{aligned}
 \dot{V} &\leq - \sum_{j=1}^n |\xi_j|^{1+q} - \sum_{j=1}^n |\xi_j|^{1+\frac{1}{q}} + C \\
 &\leq -d_1 \left(\sum_{j=1}^n \frac{1}{2} \xi_j^2\right)^{b_1} - d_2 \left(\sum_{j=1}^n \frac{1}{2} \xi_j^2\right)^{b_2} + C \tag{23} \\
 &= -d_1 V^{b_1} - d_2 V^{b_2} + C
 \end{aligned}$$

with

$$d_1 = n\left(\frac{2}{n}\right)^{\frac{1+q}{2}}, \quad d_2 = 2^{\frac{1+\frac{1}{q}}{2}}, \quad b_1 = \frac{1+q}{2}, \quad b_2 = \frac{1+\frac{1}{q}}{2}.$$

Define the set

$$\Omega_1 = \left\{ \xi \mid V^{b_2}(\xi) < \frac{C}{d_2} \right\}. \tag{24}$$

We can deduce from (23) that

$$\dot{V}(\xi) < 0, \quad \forall \xi \in \mathbb{R}^n - \Omega_1. \tag{25}$$

Thus, the signals  $\xi_1, \dots, \xi_n$  enter  $\Omega_1$  in a finite time and stay in  $\Omega_1$  thereafter. Therefore,  $\xi_j, j = 1, \dots, n$  are bounded. Invoking (3) and (4), we can recursively conclude that the system state  $x_j$  and the actual input  $u$  are bounded by that of  $\xi_j$  and  $r_d$ , and the smoothness of  $\phi_j(\cdot)$ . Therefore, all the closed-loop signals are bounded.

According to Lemma 1 and the definition of  $V$ , it follows from (23) that

$$|\xi_1(t)| \leq \sqrt{2\left(\frac{C}{(1-k)d_2}\right)^{\frac{1}{b_2}}}, \quad \forall t \geq T_{\max} \tag{26}$$

with

$$T_{\max} = \frac{1}{kd_1(b_1 - 1)} + \frac{1}{kd_2(1 - b_2)}, \tag{27}$$

which means that the output tracking error  $\xi_1$  converges into a region  $\Omega_2 = \{ \xi_1 \mid |\xi_1| \leq \sqrt{2(C/(1-k)d_2)^{\frac{1}{b_2}}} \}$  within a fixed-time.

This completes the proof.  $\square$

**Remark 2.** This work concentrates on the tracking problem of nonlinear systems with parameter unknowns and external disturbances, so the tracking error is regulated to an adjustable compact set rather than zero within a fixed time. The radius of the region  $\Omega_2$  is related to the constant  $C$ , where  $C = (n - 1 + \sum_{j=1}^4 \Theta_j) / L$ , so we can choose large  $L$  to obtain smaller radius of  $\Omega_2$ . It means that the tracking error can be rendered arbitrarily small by adjusting  $L$  large enough. Although the size of the region can be reduced, the larger parameter  $L$  also result to larger control effort, thus the tradeoff between tracking precision and control effort should be made according to the actual situation.

#### 4. Simulation Results

**Example 1.** Consider the following high-order nonlinear cascade system

$$\begin{cases} \dot{x}_1 = [x_2]^{q_1(t)} + \theta_1 x_1^2 + \varrho_1(t) \\ \dot{x}_2 = [u]^{q_2(t)} - \theta_2 \cos(x_1 x_2) + \varrho_2(t) \\ y = x_1 \end{cases} \tag{28}$$

where  $\theta_1$  and  $\theta_2$  are unknown constants,  $1 \leq q_j(t) \leq 2$ ,  $\varrho_1(t) = \cos t$  and  $\varrho_2(t) = 0.6 \sin t$  are disturbances. The desired trajectory  $r_d = 0.5 \sin t + \cos t$ . Obviously, system (28) satisfies Assumptions 1–5 with  $q = 2$ ,  $\kappa = 1.5$  and  $\kappa_1 = 1$ .

In accordance with the control design in Section 3, we design  $v_1(x_1, r_d)$  and  $u(\varphi_2, r_d)$  as follows:

$$\begin{cases} v_1(x_1, r_d) = -2(\xi_1 + [\xi_1]^{q^2})^{\frac{1}{q^2}} + (1 + M_0 + M_{11})(\xi_1 + [\xi_1]^q), \\ u(\varphi_2, r_d) = -2(\xi_2 + [\xi_2]^{q^2})^{\frac{1}{q^2}} + (1 + M_{12} + \sum_{i=2}^4 M_{i2})(\xi_2 + [\xi_2]^q) \end{cases} \quad (29)$$

with

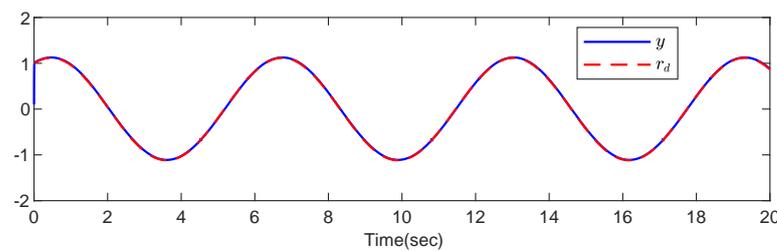
$$\begin{cases} M_0 = L^q, \\ M_{11}(x_1, r_d) = L^q \bar{F}_1^{1+q}, \\ M_{12}(x_1, x_2, r_d) = L^q \bar{F}_2^{1+q}, \\ M_{22}(x_1, x_2, r_d) = L^q \left| \frac{\partial v_1}{\partial r_d} \right|^{1+q}, \\ M_{32}(x_1, x_2, r_d) = L^q \left| \frac{\partial v_1}{\partial x_1} \right|^{1+q} (1 + (|x_2|^q + |x_2|)^{1+q} + \bar{F}_1^{1+q}), \\ M_{42}(x_1, x_2, r_d) = L^q (q|\xi_1|(1 + (1 + v_1^2)^{q-1} + (1 + x_2^2)^{q-1}))^{1+q} + L^{\frac{1}{q}} |\xi_1|^{1+\frac{1}{q}} \end{cases} \quad (30)$$

In the simulation, we select  $L = 5$ ,  $\theta_1 = 2$ ,  $\theta_2 = 6$ ,  $q_1 = 1.5$ ,  $q_2 = 1.8$ , and the initial conditions as follows:

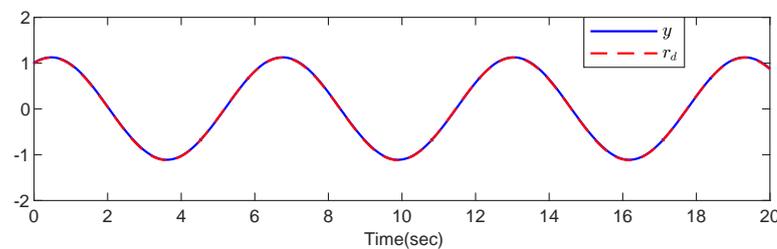
Case 1:  $x_1(0) = 0.1, x_2(0) = 0.1$ .

Case 2:  $x_1(0) = 1, x_2(0) = 1$ .

Figures 2–5 present the simulation results. Figure 2 displays the trajectories of output  $y$  and reference signal  $r_d$  under the two cases. Figure 3 displays the curve of the output tracking error  $\xi_1$ . The trajectories of the state  $x_2$  is displayed in Figure 4. Figure 5 presents the control input  $u(t)$ . From the simulation results, it can be observed that all the closed-loop signals of the considered system are globally bounded, and the output tracking error is regulated to a disc region within a fixed-time. This clearly clarify the effectiveness of the proposed control approach.

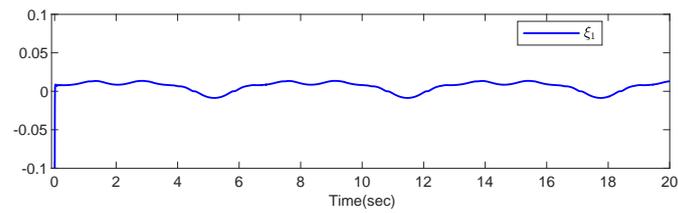


(a)

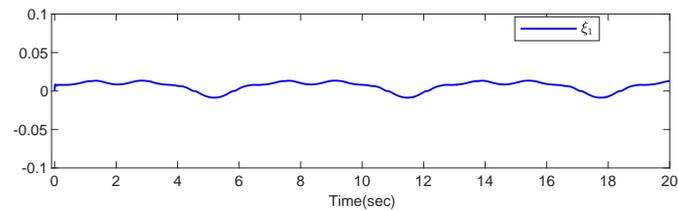


(b)

**Figure 2.** (a) The trajectories of  $y(t)$  and  $r_d(t)$  under case 1. (b) The trajectories of  $y(t)$  and  $r_d(t)$  under case 2.

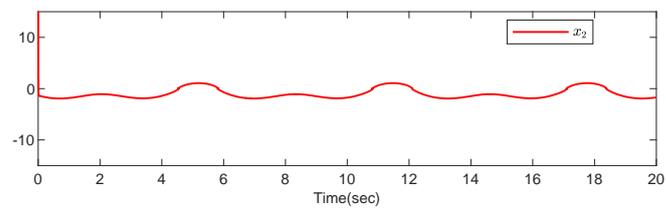


(a)

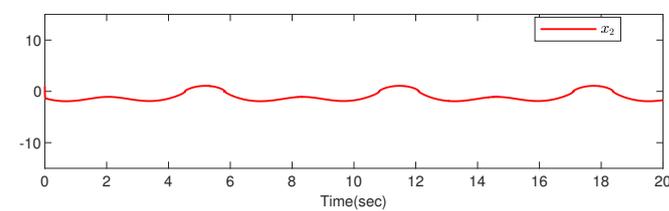


(b)

**Figure 3.** (a) Tracking error  $\xi_1$  under *case 1*. (b) Tracking error  $\xi_1$  under *case 2*.

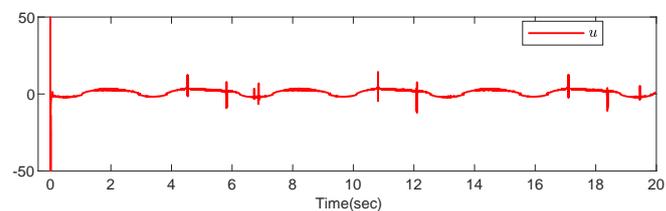


(a)

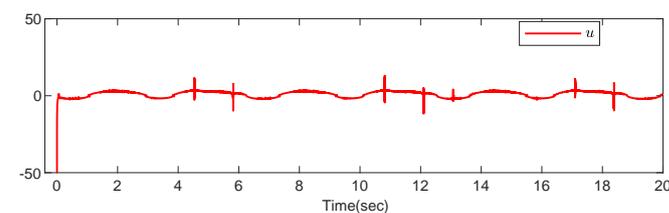


(b)

**Figure 4.** (a) System state  $x_2$  under *case 1*. (b) System state  $x_2$  under *case 2*.



(a)



(b)

**Figure 5.** (a) Control input  $u$  under *case 1*. (b) Control input  $u$  under *case 2*.

**Example 2.** Consider the following high-order nonlinear cascade system

$$\begin{cases} \dot{x}_1 = \Psi_1 [x_2]^{q_1} + F_1 + \varrho_1(t) \\ \dot{x}_2 = \Psi_2 [x_3]^{q_2} + F_2 + \varrho_2(t) \\ \dot{x}_3 = [u]^{q_3} + F_3 + \varrho_3(t) \\ y = x_1 \end{cases} \tag{31}$$

where  $\Psi_1 = 1 + \cos^2 x_1$ ,  $\Psi_2 = 1 + \sin^2 x_2$ ,  $F_1 = 2x_1^2$ ,  $F_2 = \cos x_2$ ,  $F_3 = \cos x_2 - 3x_1^2$ ,  $1 \leq q_j \leq 2$ ,  $\varrho_1(t) = 2 \cos t$ ,  $\varrho_2(t) = 5 \sin t$ ,  $\varrho_3(t) = -6 \sin t$ . The desired trajectory  $r_d = \sin 2t$ . System (31) satisfies Assumptions 1–5 with  $q = 2$ ,  $\kappa = 2$ ,  $\kappa_1 = 6$ ,  $\underline{\psi}_1 = 1$ ,  $\bar{\psi}_1 = 2$ ,  $\underline{\psi}_2 = 1$  and  $\bar{\psi}_2 = 2$ .

In accordance with the process of design in Section 3, we design the smooth functions  $v_1(x_1, r_d)$ ,  $v_2(\varphi_2, r_d)$  and  $u(\varphi_3, r_d)$  as follows:

$$\begin{cases} v_1(x_1, r_d) = -2(\xi_1 + [\xi_1]^{1/q}) - \phi_1(\xi_1 + [\xi_1]^q) \\ v_2(\varphi_2, r_d) = -2(\xi_2 + [\xi_2]^{1/q}) - \phi_2(\xi_2 + [\xi_2]^q) \\ u(\varphi_3, r_d) = -2(\xi_3 + [\xi_3]^{1/q}) - \phi_3(\xi_3 + [\xi_3]^q), \end{cases} \tag{32}$$

where

$$\begin{aligned} \phi_1 &= (\underline{\psi}_1^{-1} + \underline{\psi}_1^{-1/q})(1 + L^q + L^q \bar{F}_1^{1+q}) \\ \phi_2 &= (\underline{\psi}_2^{-1} + \underline{\psi}_2^{-1/q})(1 + L^q \bar{F}_2^{1+q} + L^q \left| \frac{\partial v_1}{\partial r_d} \right|^{1+q} + M_{32}(\cdot) + M_{42}(\cdot)) \\ \phi_3 &= (\underline{\psi}_3^{-1} + \underline{\psi}_3^{-1/q})(1 + L^q \left| \frac{\partial v_2}{\partial r_d} \right|^{1+q} + M_{33}(\cdot) + M_{43}(\cdot)) \end{aligned} \tag{33}$$

with

$$\begin{aligned} M_{32}(\cdot) &= L^q \left| \frac{\partial v_1}{\partial x_1} \right|^{1+q} (1 + (\bar{\psi}_1 (|x_2|^q + |x_2|))^{1+q} + \bar{F}_1^{1+q}) \\ M_{42}(\cdot) &= L^q (q |\bar{\psi}_1 \xi_1| (1 + (1 + v_1^2)^{q-1} + (1 + x_2^2)^{q-1}))^{1+q} + L^{1/q} |\bar{\psi}_1 \xi_1|^{1+1/q} \\ M_{33}(\cdot) &= L^q \sum_{m=1}^2 \left| \frac{\partial v_2}{\partial x_m} \right|^{1+q} (1 + (\bar{\psi}_m (|x_{m+1}|^q + |x_{m+1}|))^{1+q} + \bar{F}_m^{1+q}) \\ M_{43}(\cdot) &= L^q (q |\bar{\psi}_2 \xi_2| (1 + (1 + v_2^2)^{q-1} + (1 + x_3^2)^{q-1}))^{1+q} + L^{1/q} |\bar{\psi}_2 \xi_2|^{1+1/q}. \end{aligned}$$

In the simulation, we design parameter  $L = 5$ , select  $q_1 = 1.5$ ,  $q_2 = 1.8$ ,  $q_3 = 1.95$ , and  $x_1(0) = 0.1$ ,  $x_2(0) = 0.1$  and  $x_3(0) = 0.15$ .

Figures 6–10 present the simulation results. It can be observed that all the closed-loop signals are globally bounded, and the output tracking error is able to be regulated to a disc region within a fixed-time.

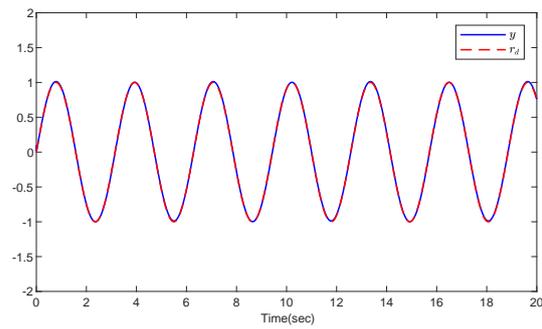


Figure 6. The trajectories of  $y$  and  $r_d$ .

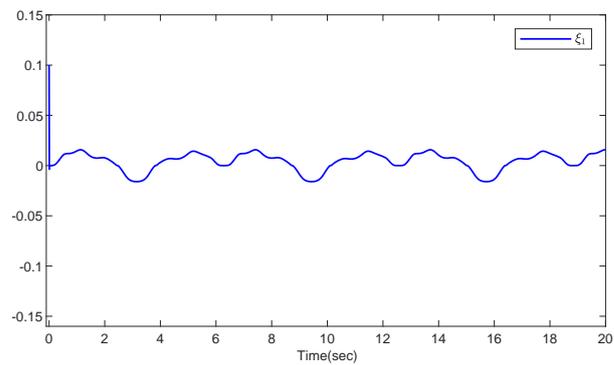


Figure 7. Tracking error  $\xi_1$ .

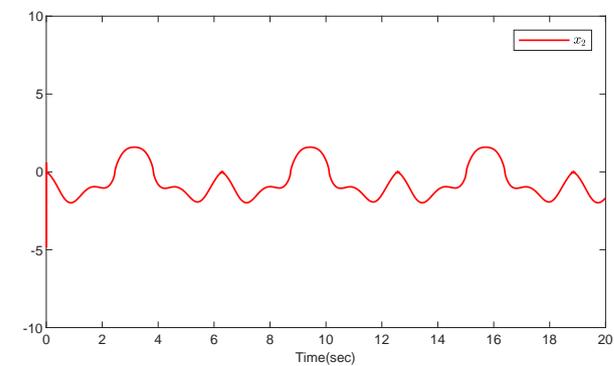


Figure 8. System state  $x_2$ .

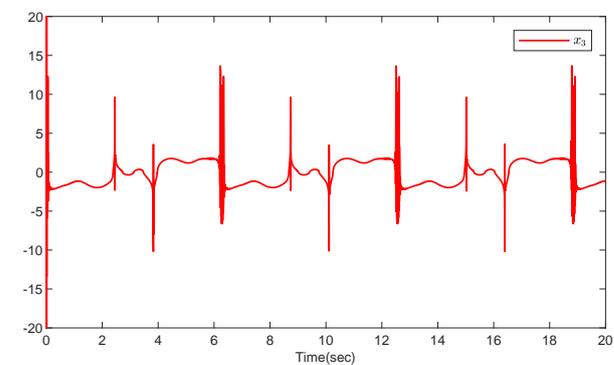
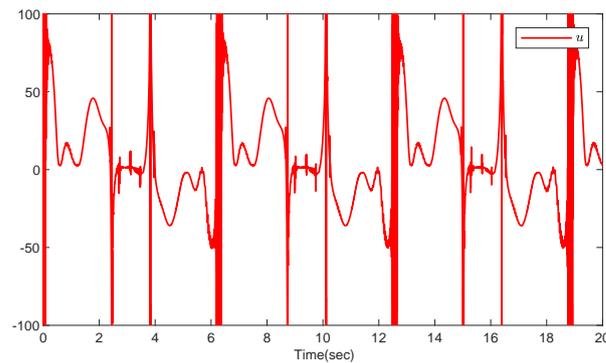


Figure 9. System state  $x_3$ .



**Figure 10.** Control input  $u$ .

## 5. Conclusions

In this work, the fixed-time practical tracking control problem has been developed for uncertain nonlinear cascade systems with unknown high powers. In the control design, an upper bound and a lower bound of high powers are introduced to compensate the unknown system powers, and a state feedback controller is designed for any initial system conditions. According to the fixed-time control theory, it is verified that the output tracking error can be regulated to a disc region of the origin within a fixed-time and all the closed-loop signals are bounded. The above control scheme has been developed under the requirement that the control directions are known a priori. Further studies will extend the fixed-time tracking control to the systems with unknown control directions. In future, the developed control scheme also can be extended to stochastic nonlinear systems [36,37], high-order time-delay nonlinear systems and high-order switched nonlinear systems.

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