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Coherent-Price Systems and Uncertainty-Neutral Valuation

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Abstract: This paper considers fundamental questions of arbitrage pricing that arises when the uncertainty model incorporates ambiguity about risk. This additional ambiguity motivates a new principle of risk- *and* ambiguity-neutral valuation as an extension of the paper by Ross (1976) (Ross, Stephen A. 1976. The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13: 341–60). In the spirit of Harrison and Kreps (1979) (Harrison, J. Michael, and David M. Kreps. 1979. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20: 381–408), the paper establishes a micro-economic foundation of viability in which ambiguity-neutrality imposes a fair-pricing principle via symmetric multiple prior martingales. The resulting equivalent symmetric martingale measure set exists if the uncertain volatility in asset prices is driven by an ambiguous Brownian motion.

Keywords: ambiguous volatility; nonlinear expectations and prices; arbitrage; asset pricing; preference-free valuation; martingales

1. Introduction

One cornerstone of modern neoclassical finance is the fundamental theorem of asset pricing (FTAP). In its simplest form and if uncertainty is modeled by a probability measure $\mathbb{P}$, the theorem states equivalence between: part (a) the absence of $\mathbb{P}$-arbitrage; and part (b) the existence of a $\mathbb{P}$-equivalent martingale measure (EMM) $\mathbb{Q}$. However, the microeconomic foundation of the FTAP relies on two further pillars: part (c) the price system within a viable equilibrium concept; and part (d) the preference-free valuation perspective.

As masterly elaborated by Harrison and Kreps (1979), part (c) can be directly connected to the infinite-dimensional equilibrium model à la Arrow–Debreu. On the other hand, as explained by Ross (1976), preference-free pricing (part (d)) boils down to risk-neutral pricing, when uncertainty boils down to risk. For an account of all four concepts under single-prior uncertainty, see Dybvig and Ross (2003).

This paper removes the underlying assumption of a given objective, physical, or reference probability measure $\mathbb{P}$. Instead, a set of measures $\mathcal{P}$ describes the (Knightian) uncertainty, which is parameterized by different stochastic volatility models σ. Such an approach deviates from models in which the term structures of volatilities, including stochastic volatility models such as that of Heston (1993), are described by another stochastic process. Doubts about the modeler's ability to be aware of all relevant information lead to the view that knowing the true volatility regime is often impossible. As argued by Epstein and Ji (2013), the hypothetical confidence of a universal dependency between past and future is in question and modeling the volatility in terms of a stochastic process, in which the law of motion is exactly known, is avoided. In a similar vein, Carr and Lee (2009) argued that the choice of a particular model describing the short term volatility is problematic *because the quantity being modeled is not directly observable. Although an estimate for the initially unobserved state variable can be inferred from market prices of derivative securities, noise in the data generates noise in the* *estimate, raising doubts that a modeler can correctly select any parametric stochastic process from the menu of consistent alternatives."*

This ambiguity about the true volatility model translates to a set priors $\mathcal{P}$ that is no longer mutually equivalent but happens to be mutually singular. Singular priors $\mathbb{P}, \mathbb{P}'$ live on disjoint supports, $\mathbb{P}(A) = 1$ and $\mathbb{P}'(A) = 0$, for an event A. This aspect, in particular, requires a reformulation of each of parts (a)–(d) and a careful reconnection of their relations. As such, this paper establishes a FTAP under volatility uncertainty and accounts for a sound microeconomic foundation.

Before introducing the main results of the paper, the new and adjusted parts (a)–(d) deserve some motivation and a detailed account.

(**a**) Typical costless arbitrage is a non-negative contingent claim, that is with positive probability strictly positive. Such a definition clearly depends on the chosen objective prior that forms the law of the underlying asset price. The situation changes when uncertainty is described by a set of possibly mutually singular priors $\mathcal{P}$. As in Vorbrink (2014), a robust form of arbitrage refers to a positive claim that is strictly positive under some prior in $\mathcal{P}$. Geometrically, the set of arbitrage opportunities can be identified by the positive cone of the $\mathcal{P}$-dependent space of contingent claims with a deleted zero.

(**b**) Under sole risk and linear expectations, the martingale concept quantifies a fair game. This standard martingale notion can be extended to the multiple-prior uncertainty setting. Following Peng (2006), this paper considers a multiple-prior martingale notion that is based on a time-consistent conditional sublinear expectation $(\mathcal{E}_t)$. Such an $\mathcal{E}$-martingale is a supermartingale (unfair game) for all single prior conditional expectations and a martingale (fair game) for some maximizing expected value with respect to the related priors. Only if all the priors in $\mathcal{P}$ are maximal is an $\mathcal{E}$-martingale called symmetric. This corresponds to a uniform notion of a fair game. Instead of the existence of an EMM $\mathbb{Q}$, under which the asset prices are martingales, the present objective multiple-prior uncertainty $\mathcal{P}$ now requires the existence of a set of measures $\mathcal{Q}$. This new set has to define a new multiple prior sublinear expectation $\mathcal{E}^{\mathcal{Q}}$ under which the asset price becomes a symmetric martingale. In that case, $\mathcal{Q}$ is called an *equivalent symmetric martingale measure set* (EsMM-set). The prior-by-prior equivalence to the objective world $\mathcal{P}$, now relies on a family of state prices that creates a risk- and ambiguity-adjusted expectation $\mathcal{E}^{\mathcal{Q}}$. The discussion about the role of $\mathcal{E}^{\mathcal{Q}}$ continues in part (d). The relation to viability as a model of equilibrium, stated in part (c), requires the notion of a particular nonlinear equilibrium price system that corresponds to $\mathcal{E}^{\mathcal{Q}}$.

(**c**) By accepting the modified notion of arbitrage in part (a) as a weak dominance principle, one crucial issue points to inconsistencies between the linear price system and the present concept of arbitrage. As in the single-prior setup, linear prices on the space of contingent claims are represented by a prior-dependent state-price density. Again, a risk-neutral measure $\mathbb{Q}$ refers to the normalized version of a linear price system in the sense of Arrow–Debreu. However, under volatility uncertainty, priors in $\mathcal{P}$ can be mutually singular. On the one hand, linear and positive prices only capture strictly positive payoffs in events that are in the support of the representing prior. On the other hand, such price systems are blind to an "arbitrage event" outside their support. For this reason, the $\mathbb{P}$-dependent linear prices are aggregated into a robust pricing scheme. The corresponding coherent price system is sublinear, and the new notion of viability (as a model of an economic equilibrium) no longer shares this problematic feature of linear price systems.[1]

(**d**) The EMM $\mathbb{Q}$ is often referred to as a description of the risk-neutral world. In fact, under this risk-neutral measure $\mathbb{Q}$, the returns of a risk-free and a risky asset coincide. From this perspective, the valuation of any contingent claim does not depend directly on the agent's preferences for risk. The obtained valuation is then preference-free. Departing from this basic insight under sole risk, the next natural step is to ask how the preference-free approach can be applied to an objective

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[1] Ambiguous volatility creates incomplete markets. Ambiguity about the true prior (or pricing measure) and its support result in uncertainties about possible the states of world. Claims on some event A can be an empty promise, whenever A and the support of the true prior have an empty intersection.

multiple prior uncertainty model $\mathcal{P}$. To see this connection, consider an EsMM-set $\mathcal{Q}$ from part (b). Preferences for uncertainty consist of risk and ambiguity preferences. The new component, preferences for ambiguity, becomes neutral when we move to the uncertainty-neutral world $\mathcal{Q}$. The symmetry of $\mathcal{E}^{\mathcal{Q}}$-martingales, that is, the asset price is a $\mathbb{Q}$-martingale under *each* $\mathbb{Q} \in \mathcal{Q}$ with no ambiguity about the expected value, exactly corresponds to ambiguity-neutrality (in the mean). From this perspective, an EsMM-set $\mathcal{Q}$ is then a model of a risk- and ambiguity-neutral world. This reasoning qualifies the valuation principle to be called *uncertainty neutral*.

Based on parts (a)–(d), the FTAP of the present paper is established in a continuous-time setup, in which the risky and ambiguous asset price is driven by a Brownian motion $\mathsf{B} = (B^{\sigma})$ with uncertain volatility σ. This is a zero-mean and stationary process with novel $\mathsf{N}(0, [\underline{\sigma}, \overline{\sigma}])$-normally distributed independent increments. Such random variables are characterized by a nonlinear heat PDE. On the other hand, $\mathsf{N}(0, [\underline{\sigma}, \overline{\sigma}])$-normally distributed random variables are the outcome of a robust central limit theorem for a given confidence interval $[\underline{\sigma}, \overline{\sigma}]$, see Chapter II of Peng (2010). The resulting process is a canonical generalization of the standard Brownian motion, such that the volatility $\sigma = (\sigma_t)$ moves almost arbitrarily within $[\underline{\sigma}, \overline{\sigma}]$. A Samuelson (1965)-type model incorporates this kind of volatility uncertainty in a risky *and* ambiguous asset price process that follows the stochastic differential equation

$$\mathrm{d}S_t = \mu_t \,\mathrm{d}t + V_t \,\mathrm{d}\mathsf{B}_t \,. \tag{1}$$

As in the classic single-prior setting, the increment $\mathrm{d}S_t$ in Equation (1) is divided into a locally certain part and a locally uncertain part $V_t \mathrm{d}\mathsf{B}_t$. The interpretation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{var}_r^{\mathbb{P}}(S_t)\big|_{r=t} \in \left[V_t \cdot \underline{\sigma}\,, V_t \cdot \overline{\sigma}\right], \quad \mathbb{P}\text{-a.s.} \tag{2}$$

where $\mathrm{var}_r^{\mathbb{P}}(S_t)$ refers to the conditional variance under $\mathbb{P}$. In abuse of notation, the issue in Equation (2) is displayed by $\mathrm{var}_t(\mathrm{d}S_t) = V_t^2 \mathrm{d}\langle \mathsf{B}\rangle_t$. Under a standard Brownian motion, the description in Equation (2) of Equation (1) boils down to the textbook description $\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{var}_r(S_t)\big|_{r=t} = V_t$, see Chapter 5 in Duffie (2010).

Section 3 departs from the equivalence in Section 2 of part (a) absence of $\mathcal{P}$-arbitrage; and part (b) existence of an EsMM-set $\mathcal{Q}$. To provide a microeconomic foundation, we discuss the connection between EsMM-sets, coherent price systems and viability as a model of an economic equilibrium from part (c). As mentioned in part (d), the resulting price system is a risk- and ambiguity-neutral valuation.

Related Literature

Harrison and Kreps (1979) described how the arbitrage pricing principle provided an economic foundation. Kreps (1981) continued to lay the foundation. In a later work, Delbaen and Schachermayer (1994) presented a general FTAP in continuous time.

Here, uncertainty and risk are seen as equals. Moving to a dynamic multiple-prior model, the concept of time consistency becomes crucial (see Epstein and Schneider 2003). A common assumption requires mutual equivalence of priors; the resulting drift uncertainty leaves the valuation principle almost unaffected. Cont (2006) noted that such an assumption "*means that all models agree on the universe of possible scenarios [...], if $\mathbb{P}_0$ defines a complete market model, this hypothesis entails that there is no uncertainty on option prices!*"

Jouini and Kallal (1995) studied nonlinear pricing caused by bid–ask spreads and transaction costs. In Araujo et al. (2012), pricing rules with finitely many states are considered. An equilibrium with superlinear prices was discussed by Aliprantis et al. (2001). Regarding the consideration of financial markets under volatility uncertainty, see the works of Avellaneda et al. (1995), Denis and Martini (2006) and Peng (2006) on ambiguous Brownian motions. Epstein and Ji (2013) provided a consumption-based asset pricing model. More recently and also under volatility uncertainty, Bouchard and Nutz (2015) discussed the relation between arbitrage and pricing measures in a discrete-time framework.

2. Risk- and Ambiguity-Neutral Asset Pricing

For the whole section, $\mathcal{P}$ is induced by volatility uncertainty about an underlying one-dimensional state process (B_t). Moreover, assume that $\Omega = C_0([0,T])$, the path space of continuous functions $\omega : [0,T] \to \mathbb{R}$ with $\omega_0 = 0$. For the construction of the ambiguous volatility model, consider the Wiener measure $\mathbb{P}_0$ that makes the coordinate process $B_t(\omega) = \omega_t$ into a Brownian motion. $\mathcal{F}$ denotes the Borel σ-algebra on Ω. Fix the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by $\mathcal{F}_t = \sigma(B_s : s \in [0,t]) \vee \mathcal{N}_{\mathbb{P}_0}$, where $\mathcal{N}_{\mathbb{P}_0}$ denotes the collection of $\mathbb{P}_0$-null sets. Fix an adapted, $\mathbb{P}_0 \otimes dt$-square integrable and positive volatility process (σ_t).

Volatility uncertainty is then based upon martingale laws on $(\Omega, \mathcal{F})$:

$$\mathbb{P}_\sigma := \mathbb{P}_0 \circ (B^\sigma)^{-1}, \quad \text{where} \;\; B_t^\sigma = \int_0^t \sigma_s \mathrm{d}B_s \tag{3}$$

Via the construction in Equation (3), a set $\mathcal{D}$ of volatility regimes builds the uncertainty model $\mathcal{P} = (\mathbb{P}_\sigma)_{\sigma\in\mathcal{D}}$. The quadratic variation $\langle B^\sigma \rangle_t = \int_0^t \sigma_s^2 \mathrm{d}s$ then describes the volatility of B^σ under $\mathbb{P}_0$. The leading Example 1 satisfies the following standing assumption for $\mathcal{P}$. This allows to define in Section 2.2 a sublinear conditional expectation that satisfies the law of iterated expectations. As in Nutz and Soner (2012), for each stopping time τ, define $\mathcal{P}_{\tau,\mathbb{P}} = \{\mathbb{P}' \in \mathcal{P} : \mathbb{P} = \mathbb{P}' \text{ on } \mathcal{F}_\tau\}$, which consists of all the priors in $\mathcal{P}$ that agree with a $\mathbb{P} \in \mathcal{P}$ in the events up to time t.[2]

Assumption 1. *The set of priors $\mathcal{P}$ is* stable under pasting, *if for every $P \in \mathcal{P}$, every $\mathbb{F}$-stopping time τ, $B \in \mathcal{F}_\tau$ and $\mathbb{P}, \mathbb{P}' \in \mathcal{P}_{\tau,\mathbb{P}}$, we have $\mathbb{P}_\tau \in \mathcal{P}$, where $\mathbb{P}_\tau(A) = E^{\mathbb{P}}\left[\mathbb{P}(A|\mathcal{F}_\tau)1_B + \mathbb{P}'(A|\mathcal{F}_\tau)1_{B^c}\right]$, for all $A \in \mathcal{F}_\tau$.*

Example 1. *Suppose that two volatility models σ^1 and σ^2 are consistent with a given dataset. Their respective implications for a trading decision may differ considerably. To address the possibility of different volatility regimes, define the universal extreme cases $\underline{\sigma}_t = \inf(\sigma_t^1, \sigma_t^2)$ and $\overline{\sigma}_t = \sup(\sigma_t^1, \sigma_t^2)$. When thinking about reasonable uncertainty management, no scenario between $\underline{\sigma}$ and $\overline{\sigma}$ should be ignored. These boundaries result in:*

$$\mathcal{P} = \left\{ \mathbb{P}_\sigma : \sigma_t \in [\underline{\sigma}_t, \overline{\sigma}_t] \quad \text{for all} \;\; t \in [0,T] \right\}. \tag{4}$$

For $X \in \mathcal{C}_b(\Omega)$. define the upper expectation $\mathcal{E}^{\mathcal{P}}(X) = \max_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[X]$ and the norm $\|\cdot\|_{\mathcal{P}}$ on $\mathcal{C}_b(\Omega)$, the space of all bounded continuous real-valued functions on Ω, given by $\|X\|_{\mathcal{P}} = \mathcal{E}^{\mathcal{P}}\left(X^2\right)^{\frac{1}{2}}$. The completion of $\mathcal{C}_b(\Omega)$ under $\|\cdot\|_{\mathcal{P}}$ is denoted by $\mathcal{L}_{\mathcal{P}}$, and let $L_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}/\mathcal{N}$ be the quotient space of $\mathcal{L}_{\mathcal{P}}$ with respect to the $\|\cdot\|_{\mathcal{P}}$-null elements $\mathcal{N}$. The elements in $L_{\mathcal{P}}$ have finite variance under every $\mathbb{P} \in \mathcal{P}$ (see Appendix A for a representation of $L_{\mathcal{P}}$). Clearly, $L_{\mathbb{P}} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ is the standard Lebesgue space.

A property holds $\mathcal{P}$-q.s. (quasi surely) if it holds $\mathbb{P}$-almost surely for every $\mathbb{P} \in \mathcal{P}$. A payoff X is positive if $X \geq 0$ $\mathcal{P}$-q.s. This induces an order relation $\geq_{\mathcal{P}}$, denoted by $\geq$, on $L_{\mathcal{P}}$ so that the classical arguments prove that $(L_{\mathcal{P}}, \|\cdot\|_{\mathcal{P}}, \geq)$ is a Banach lattice (see Appendix A). In addition, set $L_{\mathcal{P},t} = \{X \in L_{\mathcal{P}} : X \; \mathcal{F}_t\text{-}measurable\}$. For questions of asset pricing, ψ-deflated claims ψX play a central role. To guarantee $\mathcal{E}^{\mathcal{P}}(\psi X) < \infty$, it suffices to assume $\psi, X \in L_{\mathcal{P}}$.

2.1. Arbitrage and Primitives of the Financial Market

For the sake of simplicity, the price of the riskless asset satisfies $S_t^0 = 1$, that is, the interest rate is zero. Fix a pair $S = (1, S^1)$ on the filtered uncertainty space $(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{F})$, where the price process of the uncertain asset (S_t^1) satisfies $S_t^1 \in L_{\mathcal{P}}$ for each t and is $\mathbb{F}$-adapted. As in Harrison and Kreps (1979),

2 Stability under pasting is the continuous time version of rectangularity from Epstein and Schneider (2003); see the work of Riedel (2009) for a detailed discussion.

trading strategies are simple and omit to exclude doubling strategies under some completion. A simple strategy is an $\mathcal{F}_t$-adapted stochastic process $\eta = (\eta^0, \eta^1)$ such that there is a finite sequence of dates $0 < t_0 \leq \cdots \leq t_N = T$ and η^i, $i = 0, 1$, and can be written as $\eta^i_t = \sum_{k=0}^{N-1} 1_{[t_i, t_{i+1})}(t)\eta^{i,k}$, with $\eta^{i,k} \in L_{\mathcal{P},t_k}$. The fraction invested in the riskless asset is denoted by η^0_t. A trading strategy $\eta = (\eta^0, \eta^1)$ is *self-financing* if $\eta_{t_{n-1}}(1, S^1)_{t_n} = \eta_{t_n}(1, S^1)_{t_n}$ for all dates. The value of the portfolio X^η_t satisfies $\mathcal{E}(X^\eta_t) < \infty$ for every t. The set of self-financing trading strategies is denoted by $\mathcal{A}$. This financial market $(1, S^1)$, taken together with $\mathcal{A}$, is denoted by $\mathtt{FM}(1, S^1)$.

Under model uncertainty, a robust notion of no-arbitrage notion is a basic step. The following arbitrage concept is a slight generalization of Vorbrink (2014).

Definition 1. *Let $\mathcal{R} \subset \mathcal{P}$. There is an $\mathcal{R}$-arbitrage opportunity in $\mathtt{FM}(1, S^1)$ if an admissible pair $\eta \in \mathcal{A}$ exists such that*

$$\eta_0 S_0 \leq 0, \quad \eta_T S_T \geq 0 \ \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{R} \quad \text{and} \quad \mathbb{P}^*(\eta_T S_T > 0) > 0 \text{ for some } \mathbb{P}^* \in \mathcal{R}.$$

The definition rests on the following thought experiment. An arbitrage strategy is riskless for each $\mathbb{P} \in \mathcal{R}$, and if the prior $\mathbb{P}^*$ describes the dynamics of asset prices, one would gain a profit with a strictly positive probability. The $\mathcal{P}$-arbitrage notion can be seen as a rather weak arbitrage opportunity and one could argue that no $\mathcal{R}$-arbitrage is consistent with a weak dominance principle based on $\mathcal{R}$. If $\mathcal{R} = \{\mathbb{P}\}$, the usual arbitrage concept under $\mathbb{P}$ appears.

Remark 1. *As originally discussed by Kreps (1981), the order structure of the underlying space of claims determines the natural candidate for a no-arbitrage condition. The order relation $\geq_{\mathcal{P}}$ on $L_{\mathcal{P}}$ defines the arbitrage cone $L_{\mathcal{P}+} \setminus \{0\}$ by deleting the zero of $L_{\mathcal{P}+} := \{X \in L_{\mathcal{P}} : X \geq_{\mathcal{P}} 0\}$. Every $\mathcal{R} \subset \mathcal{P}$ defines a weaker order relation $\geq_{\mathcal{R}}$ and a larger arbitrage cone. The arbitrage concept in Definition 1 allows any subset $\mathcal{R} \subset \mathcal{P}$ to determine what an arbitrage is.*

2.2. EsMM Sets and Ambiguity Neutrality

Let us move to the dynamics of a continuous-time, multiple-prior uncertainty model. The unique existence of sublinear conditional expectations $\mathcal{E}^{\mathcal{P}}_t : L_{\mathcal{P}} \to L_{\mathcal{P},t}$ is provided through the following construction, which is based on the dynamic programming principle:

$$\mathcal{E}^{\mathcal{P}}_t(X) = \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}} E^{\mathbb{P}'}_t[X] \quad \mathbb{P}\text{-a.s.} \quad \text{for all } \mathbb{P} \in \mathcal{P}. \tag{5}$$

See Section 2 of Epstein and Ji (2014) for details and a list of properties. The sequence of conditioning satisfies the law of iterated expectations, that is, for every $X \in L_{\mathcal{P}}$ the updating rule $\mathcal{E}^{\mathcal{P}}_t(X) = \mathcal{E}^{\mathcal{P}}_t(\mathcal{E}^{\mathcal{P}}_{t+s}(X))$ holds.

A process (X_t) with $X_t \in L_{\mathcal{P},t}$ for each $t \in [0, T]$, is an $\mathcal{E}^{\mathcal{P}}$*-martingale* if

$$\mathcal{E}^{\mathcal{P}}_t(X_{t+s}) = X_t \quad \text{for all } s, t. \tag{6}$$

For an $\mathcal{E}^{\mathcal{P}}$-martingale (X_t), its negative $(-X_t)$ is in general not an $\mathcal{E}^{\mathcal{P}}$-martingale again. If this is the case, the process is a *symmetric $\mathcal{E}^{\mathcal{P}}$-martingale*, which is equivalent to the $E^{\mathbb{P}}$-martingale property of (X_t) for every $\mathbb{P} \in \mathcal{P}$. Symmetry for a process (X_t) implies no ambiguity about the expected value of X_t under $\mathcal{P}$.

Definition 2. *A set of probability measures $\mathcal{Q}$ on Ω is called an equivalent symmetric martingale measure set (EsMM-set) if the following conditions hold:*

1. *For every $\mathbb{Q} \in \mathcal{Q}$, there is a $\mathbb{P} \in \mathcal{P}$ that is equivalent via $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L_{\mathbb{P}}$.*
2. *(S_t) is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale where $\mathcal{E}^{\mathcal{Q}} = \{\mathcal{E}^{\mathcal{Q}}_t : L_{\mathcal{P}} \to L_{\mathcal{P},t}\}_{t \in [0,T]}$ is the conditional sublinear expectation under $\mathcal{Q}$ which is stable under pasting.*

The collection $\texttt{EsMM}(\mathcal{P})$ denotes the collection of all EsMM-sets $\mathcal{Q}$, as in Equation (14). The case $\mathcal{P} = \{\mathbb{P}\}$ boils down to the well-known EMM. The first condition states a direct relation between an element $\mathbb{Q} \in \mathcal{Q}$ and a primitive prior in $\mathcal{P}$. Integrability is a technical condition to guarantee the connection to viability and the commodity-price duality. The second part of Definition 2 points to the adjusted martingale condition. The idea of a fair gamble under $\mathcal{Q}$ reflects the neutrality of preferences for both risk and ambiguity. The symmetric martingale property of S implies, as discussed in the Introduction regarding part (c) and part (d), that the (expected) value of the claim is constant with respect to different measure in the EsMM-set. In other words, the valuation is mean-ambiguity free, i.e., preferences for ambiguity under $\mathcal{Q}$ are neutral.

One can think of the ambiguity-neutral part in terms of risk-neutral maxmin preferences of Gilboa and Schmeidler (1989), i.e., the worst-case expected utility under $\mathcal{Q}$. For a claim X on (S_t), $\mathbb{Q} \mapsto E^{\mathbb{Q}}[X]$ is then constant on $\mathcal{Q}$. Similarly to pricing under risk, in which risk preferences are irrelevant, analogous reasoning remains valid for preferences for ambiguity. As such, uncertainty neutrality immediately leads to the uncertainty-neutral expectation $\mathcal{E}^{\mathcal{Q}}$. The same reasoning remains valid for smooth ambiguity preferences of Klibanoff et al. (2005).

2.3. Existence of EsMM-sets

In this section, asset prices are driven by a G-Brownian motion, in which the volatility uncertainty is contained in the quadratic variation (see Appendix A.2 for an overview). This specific framework allows confirming the existence of EsMM-sets.

Again, the riskless asset S^0 has an interest rate r_t of zero. Volatility uncertainty in the asset prices $S := S^1$ means that every adapted stochastic process (σ_t) taking values in $[\underline{\sigma}, \overline{\sigma}]$ is a possible model for the volatility of the underlying state process. More precisely, S is driven by a G-Brownian motion $\mathsf{B} = (B^{\sigma_t})_{\sigma \in \mathcal{D}}$ (see Appendix A.2 for a detailed exposition). As a special case of Example 1, $\mathcal{P}$ is now induced by $[\underline{\sigma}, \overline{\sigma}]$. The asset price is determined by the following stochastic differential equation

$$\mathrm{d}S_t = \mu_t(S_t)\mathrm{d}\langle \mathsf{B}\rangle_t + V_t(S_t)\mathrm{d}\mathsf{B}_t, \tag{7}$$

where $\langle \mathsf{B}\rangle_t = \langle B^{\sigma}\rangle_t = \int_0^t \sigma_s^2 \mathrm{d}s\ \mathbb{P}_\sigma$-almost surely. The process $t \mapsto V_t(x)$ is strictly positive and bounded. Let $\mu : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $V : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}_+$ be processes such that a unique solution of Equation (7) exists (see Part 5 of Peng (2010) for some classical Lipschitz continuity conditions on μ and V in the state variable).

Remark 2. *As in the classic probabilistic framework, a Girsanov transformation of* $\mathsf{B}_t + \int_0^t \theta_r(S_r)\mathrm{d}\langle \mathsf{B}\rangle_r$, *with* $\theta_r(S_r) = \frac{\mu_r(S_r)}{V_r(S_r)}$, *guarantees the existence of a nontrivial EsMM-set* $\mathcal{Q}$. *For this approach, define a new sublinear expectation on* $L_{\mathcal{P}}$, *via* $\mathcal{Q} = \psi\mathcal{P} = \{\mathbb{Q} \in \Delta(\Omega) : A \mapsto \mathbb{Q}(A) = \int_A \psi \mathrm{d}\mathbb{P}$ *and* $\mathbb{P} \in \mathcal{P}\}$,

$$\mathcal{E}^{\mathcal{Q}}(X) = \max_{\mathbb{Q} \in \mathcal{Q}} E^{\mathbb{Q}}[X] = \max_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[\psi X] = \mathcal{E}^{\mathcal{P}}(\psi X). \tag{8}$$

The state price ψ *in Equation (8) is now an aggregated object under the uncertainty model, i.e.,* $\psi = \psi_{\mathbb{P}}$ $\mathbb{P}$*-a.s. for all* $\mathbb{P} \in \mathcal{P}$. *This universal state-price density* $\psi = \psi_T^\theta$ *is the terminal value of a symmetric exponential martingale* $\mathrm{d}\psi_t^\theta = \psi_t^\theta \theta_t(S_t)\mathrm{d}\mathsf{B}_t$, *with* $\psi_0^\theta = 1$. *The process* (θ_t) *is now the market price of risk and ambiguity. Similarities to the single-prior case are apparent. The results in Appendix A.2 allow us to write* ψ_T^θ *explicitly as*

$$\psi_T^\theta = \exp\left(-\frac{1}{2}\int_0^T \theta_t(S_t)^2 \mathrm{d}\langle \mathsf{B}\rangle_t - \int_0^T \theta_t(S_t)\mathrm{d}\mathsf{B}_t \right). \tag{9}$$

Let the pricing kernel in Equation (9) solve $V_t(S_t)\theta_t(S_t) = \mu_t(S_t)$. Define the symmetric $\mathcal{E}^{\mathcal{P}}$-martingale $S_t^* = S_0^* + \int_0^t V_r(S_r^*)\mathrm{d}\mathsf{B}_r$ (see the second result in Appendix A.2), and assume $\mathcal{E}^{\mathcal{P}}\big(\exp\big(\delta \cdot \int_0^T \theta_t(S_t)^2 \mathrm{d}\langle \mathsf{B}\rangle_t\big)\big) < \infty$ for some $\delta > \frac{1}{2}$. Under these conditions, we have the following

FTAP under ambiguous volatility. The result corresponds to Theorem 3 of Harrison and Kreps (1979) where uncertainty is captured by a single objective probability measure.

Theorem 1. *An EsMM-set $\mathcal{Q} = \psi\mathcal{P}$ exists if and only if $\mathcal{P}$-arbitrage in $\mathtt{FM}(1, S)$ is absent.*

Remark 3. *Theorem 3.3 of Epstein and Ji (2013) obtains analogous state prices by using a consumption-based utility-gradient approach and assuming $\mu_t(S_t) = \mu_t S_t$ and $V_t(S_t) = V_t S_t$. The local form of (7) would then be apparently governed by $\frac{\mathrm{d}S_t}{S_t} = b_t \mathrm{d}\langle \mathsf{B}\rangle_t + V_t \mathrm{d}\mathsf{B}_t$. The relation between the asset price processes, with pricing kernel $(\frac{\mu_t}{V_t}) = (\theta_t)$, is as follows $\int_0^t \mu_s \mathrm{d}\langle \mathsf{B}\rangle_s = \int_0^t \mu_s \hat{a}_s \mathrm{d}s = \int_0^t b_s \mathrm{d}s$, where $\hat{a}_s = \frac{\mathrm{d}}{\mathrm{d}s}\langle \mathsf{B}\rangle_s$ denotes the derivative of the ambiguous quadratic variation.*

We end this section with an example on option pricing under volatility uncertainty.

Example 2. *Let $\theta = \frac{\mu_t(\cdot)}{V_t(\cdot)}$ be constant and $X = \varphi(S_T)$ be an option on the uncertain asset price from (7). If φ is convex, such as a European call $\varphi(x) = (x - \mathtt{s})^+$ for some strike price $\mathtt{s}$, the price from (8) then simplifies to*

$$\Psi(X) = \max_{\mathbb{Q}\in\mathcal{Q}} E^{\mathbb{Q}}[X] = E^{\mathbb{Q}^*}[X] = E^{P_{\overline{\sigma}}}[\psi X],$$

where, as a special of Equation (9) under $P_{\overline{\sigma}}$,

$$\psi = \exp\left(-\frac{1}{2}\int_0^T (\theta\overline{\sigma})^2 \mathrm{d}t - \int_0^T \theta\overline{\sigma}\mathrm{d}B_t \right).$$

Similarly, for concave φ, such as a European put, we get $\Psi(X) = E^{P_{\underline{\sigma}}}[\psi X]$ (see Avellaneda et al. (1995) for a detailed discussion of both cases). In particular, the pricing then coincides with the pricing in a Black–Scholes model.

Nonconvex payoff structures, such as Butterfly options[3], yield a maximizing $\mathbb{Q} \in \mathcal{Q}$ with $\mathrm{d}\mathbb{Q}^* = \psi \mathrm{d}\mathbb{P}_{\sigma^*}$ such that the corresponding σ^* is no longer a constant process. Thus, there is no σ for the Black–Scholes model that matches with the pricing under volatility uncertainty. In Fouque and Ren (2014), with $[\underline{\sigma}, \overline{\sigma}] = [0.15, 0.2]$, the price of a butterfly option under Ψ is computed through approximation techniques of the corresponding nonlinear heat PDE (see also Appendix A.2 for the relation between PDE's and G-Brownian motion).

3. A Micro-Foundation of the FTAP under $\mathcal{P}$

Before introducing the economy, see part (c) in the Introduction, the structure of linear price systems on $L_{\mathcal{P}}$ is discussed and the notion of coherent price systems is introduced.

The states of the world, Ω, build a complete separable metric space. $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω. The uncertainty is described by a weakly compact (with respect to $\mathcal{C}_b(\Omega)$) subset of Borel probability measures $\mathcal{P} \subset \Delta(\Omega)$ on $(\Omega, \mathcal{F})$. Assumption 1 in Section 2 is not imposed.

3.1. Linear Price Systems

To realize the problematic aspect of linear equilibrium price systems under ambiguous volatility, consider the dual space of continuous and linear functionals $\Pi : L_{\mathcal{P}} \to \mathbb{R}$ (see Theorem 2 of Beissner and Denis (2018) for a Riesz-type representation theorem). Non-trivial positive payoffs are costless on events outside the domain of the representing equilibrium pricing measure. The reason for

3 For instance, Fouque and Ren (2014) consider the case $\varphi(x) = (x - 90)^+ - 2(100 - x)^+ + (x - 110)^+$.

this critical aspect relies on the mutual singularity of measures in $\mathcal{P}$. The topological dual space of $L_{\mathcal{P}}$ is denoted by $L^*_{\mathcal{P}}$ (see Appendix A.1), and is given by

$$L^*_{\mathcal{P}} = \left\{ \Pi : L_{\mathcal{P}} \to \mathbb{R} : \Pi(\cdot) = E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] \middle| \mathbb{P} \in \mathcal{P},\ \psi_{\mathbb{P}} \in L_{\mathbb{P}} \right\}. \tag{10}$$

Similar to the case with $L_{\mathbb{P}}$ as the commodity space, an analogous interpretation of the dual space in Equation (10) holds true. The state price density $\psi_{\mathbb{P}}$ is supported by a prior $\mathbb{P} \in \mathcal{P}$, but the stronger norm $\|x\|_{\mathcal{P}}$, i.e., $\|x\|_{\{\mathbb{P}\}} \leq \|x\|_{\mathcal{P}}$ for any $x \in L_{\mathcal{P}}$, causes a richer dual space that is parameterized by $\mathcal{P}$. In abuse of changing domains with respect to Π, this means $L^*_{\mathcal{P}} = \cup_{\mathbb{P} \in \mathcal{P}} L^*_{\mathbb{P}}$. Each $\Pi \in L^*_{\mathcal{P}}$ assigns no value to strictly positive payoffs outside the support of the representing prior $\mathbb{P}_{\Pi}$.

Any linear expectation $E^{\mathbb{Q}}$ under an EMM $\mathbb{Q}$ can be considered as a normalized, linear, strictly positive and continuous price system. When seeking an FTAP under uncertain volatility that incorporates a notion of market viability, the following question serves to clarify the issue: *Is the existence of a single risk neutral measure, $\mathbb{Q}$, equivalent to a certain prior in $\mathcal{P}$ sufficient for the absence of arbitrage?*[4] As such, the *robust* arbitrage notion from Definition 1 is inconsistent with a linear theory of valuation. In other words, one pricing measure $\mathbb{Q}$ is incapable of containing all information about what is possible under $\mathcal{P}$. In the same vein, the idea of "no empty promises" of Willard and Dybvig (1999) points to the unpleasant but possible ignorance of payoffs on zero probability events for only some prior. Section 4 continues with a discussion.

3.2. Coherent Price Systems

Under volatility uncertainty the above-mentioned problematic aspect of normalized linear price systems $\Pi(\cdot) = E^{\mathbb{Q}}[\cdot]$ and single EMM's comes into play, which can be avoided by allowing sublinear prices on $L_{\mathcal{P}}$. For the extension of the price space, this author takes a cue from Aliprantis and Tourky (2002). For each $\mathbb{P} \in \mathcal{P}$, define the set $L^{+,1}_{\mathbb{P}} = \{\psi \in L_{\mathbb{P}} : \psi > 0\ \mathbb{P}-a.s.$ and $E^{\mathbb{P}}[\psi] = 1\}$. Similar to Equation (10), the *space of coherent price systems* is generated by strictly positive and linear functionals and given by

$$L^{\circledast}_{+} = \left\{ \Psi : L_{\mathcal{P}} \to \mathbb{R} :\ \Psi(\cdot) = \sup_{\mathbb{P} \in \Gamma_{\Psi}} E^{\mathbb{P}}[\psi_{\mathbb{P}} \cdot] \middle|\ \psi_{\mathbb{P}} \in L^{+,1}_{\mathbb{P}}\ \forall \mathbb{P} \in \Gamma_{\Psi} \subset \mathcal{P} \right\}, \tag{11}$$

where $\{\mathbb{P}\}_{\mathbb{P} \in \Gamma_{\Psi}}$ determines a Ψ. The set $\Gamma_{\Psi} \subset \mathcal{P}$ in Equation (11) then refers to exactly those priors appearing in the representation of Ψ and is called Γ-*relevant*. Elements in $L^{\circledast}_{+}$ rely on a set of normalized, strictly positive and linear functionals $\{\Pi_{\mathbb{P}}\}_{\mathbb{P} \in \Gamma_{\Psi}}$ with $\Pi_{\mathbb{P}} \in L^*_{\mathbb{P}}$, which are consolidated by combining the pointwise maximum and convex combinations of linear price systems. This leads to a consolidation operation

$$\Gamma : \prod_{\mathbb{P} \in \mathcal{P}} L^{+,1}_{\mathbb{P}} \to L^{\circledast}_{+}, \qquad \{\Pi_{\mathbb{P}}\} \mapsto \Gamma(\{\Pi_{\mathbb{P}}\}) = \Psi \tag{12}$$

to aggregate the linear and prior-dependent price systems $\{\Pi_{\mathbb{P}}\}$. For instance, a linear price system $\Pi_{\mathbb{P}}$ corresponds to a consolidation via the (second-order) point measure $\delta_{\{\mathbb{P}\}} \in \Delta(\mathcal{P})$, such that only $\Gamma_{\mathcal{P}} = \{\mathbb{P}\}$ is relevant (see Equation (A1) in Appendix A for the general case and an example of how a sublinear functional in $L^{\circledast}_{+}$ can be constructed).

Proposition 1. *Each $\Psi \in L^{\circledast}_{+}$ satisfies, $\forall\ X, Y \in L_{\mathcal{P}}$, $c \in \mathbb{R}$ and $\lambda \geq 0$,*

---

[4] A short argument yields a negative answer: Let $X \in M$ be a marketed claim with price $0 = \pi(X)$. Since $E^{\mathbb{Q}} = \pi$ on M, we have $E^{\mathbb{Q}}[X] = 0$. Suppose that $X \in M$ is an arbitrage with $\mathbb{P}'(X > 0) > 0$. In the present setting, $\mathbb{P}' \in \mathcal{P}$ can be singular to $\mathbb{P}$ and, thus, by equivalence, to $\mathbb{Q}$.

1. *Sublinearity* $\Psi(\lambda X + Y) \leq \lambda\Psi(X) + \Psi(Y)$.
2. *Constant preserving* $\Psi(c) = c$.
3. *Strict positivity* $X \geq 0$ *and* $X \neq 0$ *on* Γ_Ψ *implies* $\Psi(X) > 0$.
4. *Monotonicity* *If* Γ_Ψ *is closed,* $X \geq Y$ *implies* $\Psi(X) \geq \Psi(Y)$.
5. *Continuity* $X_n \to X$ *in* $\|\cdot\|_{\mathcal{P}}$ *implies* $\lim_n \Psi(X_n) = \Psi(X)$.

The phrase "$X \neq Y$ on Γ_Ψ" refers to the presence of a $\mathbb{P} \in \Gamma_\Psi$ such that $\mathbb{P}(X \neq Y) > 0$. In view of the sublinearity of Ψ, Corollary 1 in Section 3.3.1 identifies the subspace of $L_{\mathcal{P}}$ where coherent price systems are linear.

3.3. Viable Price Systems

As in Harrison and Kreps (1979), a viable price system is based on a model of an economic equilibrium. To emphasize the key aspects under volatility uncertainty, recall the following.

3.3.1. Viability under Risk

The commodity space is again $L_{\mathbb{P}}$. The price functionals are linear and continuous, and the dual $L^*_{\mathbb{P}}$ can be identified with densities in $L_{\mathbb{P}}$.

Fix a linear price system $\pi : M \to \mathbb{R}$, where the marketed space $M \subset L_{\mathbb{P}}$ contains all frictionless achievable claims. $\mathbb{A}_{\mathbb{P}}$ is the set of convex, strictly monotone and continuous preference relations on $\mathbb{R} \times L_{\mathbb{P}}$. Viability of π means that there is an $\precsim \in \mathbb{A}_{\mathbb{P}}$ and a feasible bundle $(\hat{x}, \hat{X}) \in B_\pi = \{(y, Y) \in \mathbb{R} \times M : y + \pi(Y) \leq 0\}$ with $(\hat{x}, \hat{X}) \succsim (x, X)$ for all $(x, X) \in B_\pi$. Theorem 1 of Harrison and Kreps (1979) shows that *π is viable if and only if there is an extension $\Pi \in L^*_{\mathbb{P}+}$ of π to $L_{\mathbb{P}}$.*

3.3.2. Viability under Ambiguous Volatility

An FTAP with a sound microeconomic foundation contains a third statement (viability) about the existence of an agent (preferring more than less) and being in an optimal state. However, with regard to Remark 1, the notion of strict monotone complete preferences is subtle.[5] Beginning with the introduction of marketed spaces $M_{\mathbb{P}} \subset L_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$, any claim in $M_{\mathbb{P}}$ can be achieved frictionless, if $\mathbb{P}$ is the true prior. The deviation from Kreps (1981) in which the market model is described by the quintuple $(L, \|\cdot\|, L_+\backslash\{0\}, M, \pi)$, relies on the uncertainty model $\mathcal{P}$ and the resulting effect of ambiguity about the true distribution of payoffs.

The following remark makes this conceptually crucial point precise.

Remark 4. *A marketed space $M \subset L_{\mathcal{P}}$ collects all perfectly replicable claims. Such claims are on a stock with simplified price dynamics $\mathrm{d}S_t = V_t \mathrm{dB}_t$ and spans M, via terminal outcomes $X = \eta_T S_T = \int_0^T \eta_t \mathrm{d}S_t = \int_0^T \eta_t V_t \mathrm{dB}_t$, where η is a feasible and replicating trading strategy. The price of $\eta_T S_T$ is then the initial cost $\eta_0 S_0$. With ambiguity in prices, e.g. $\mathtt{B}_t$ has ambiguous volatility $\sigma \in \mathcal{D}$ as in Equation (1), the uncertainty in S reappears in any claim on S:*

$$X = \int_0^T \eta_t \mathrm{d}S_t^\sigma = \int_0^T \eta_t V_t \mathrm{dB}_t^\sigma = \int_0^T \eta_t V_t \,\mathrm{d}\left(\int_0^t \sigma_s \mathrm{d}B_s\right) = \int_0^T \eta_t V_t \sigma_t \mathrm{d}B_t.$$

Here, all the equalities hold only $\mathbb{P}_\sigma$-a.s. The uncertain volatility σ results in an ambiguous payoff and leads to an ambiguous marketed space and an ambiguous price $\eta_0 S_0^\sigma$. To incorporate this ambiguity about replicating portfolio processes, the marketed space $M_{\mathbb{P}_\sigma}$ and price $\pi_{\mathbb{P}_\sigma}$ now depend on the prior.

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5 $\geq$-strict monotonicity means that $X \geq Y$ and $X \neq Y$ implies $X \succ Y$. The most common preferences under ambiguity, such as maxmin, variational or smooth ambiguity representations, are excluded (see Section 2.3 of Beissner (2017) for a detailed discussion).

In view of Remark 4, $\pi_{\mathbb{P}}$ and $\pi_{\mathbb{P}'}$ may have a rich common domain but also different evaluations, namely $\pi_{\mathbb{P}}(X) \neq \pi_{\mathbb{P}'}(X)$ with $X \in M_{\mathbb{P}} \cap M_{\mathbb{P}'}$. Coherence is based on sublinear price systems, as illustrated in the following example and discussed by Heath and Ku (2006).

Example 3. *Let $\mathcal{P} = \{\mathbb{P}, \mathbb{P}'\}$. If $\mathbb{P}$ is the true law, each claim in $M_{\mathbb{P}}$ is priced by a linear functional $\pi_{\mathbb{P}}$. An agent is unable to choose a portfolio in $M_{\mathbb{P}'} + M_{\mathbb{P}}$ due to unawareness of the true prior. Equality of prices on the intersection is then less intuitive, since the different priors create different replication costs and consequently different price structures (see Remark 4.) A robust price for $X \in M_{\mathbb{P}'} \cap M_{\mathbb{P}}$ is $\max\{\pi_{\mathbb{P}'}(X), \pi_{\mathbb{P}}(X)\}$.*

The set $(\pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ of linear scenario-based price functionals inherits all the information of the underlying financial market. In the single-prior setting under $\mathbb{P}$, the market is incomplete if and only if $M_{\mathbb{P}} \neq L_{\mathbb{P}}$. Let $M_{\mathbb{P}} \prod M_{\mathbb{P}'}$ denote the Cartesian product of $M_{\mathbb{P}}$ and $M_{\mathbb{P}'}$. As in Remark 4, the asset span depends on each resulting law $\mathbb{P}_{\sigma^1}$ and $\mathbb{P}_{\sigma^2}$ as constructed in Equation (1).

Definition 3. *Fix a set of linear prices $\{\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}\}_{\mathbb{P}\in\mathcal{P}}$, where $M_{\mathbb{P}}$ is a closed subspace of $L_{\mathbb{P}}$. A price system $[(\pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}; \Gamma]$ consists of a consolidation operator Γ, see Equation (12), and the product of prices $\pi_{\prod\mathcal{P}} := \prod_{\mathbb{P}\in\Gamma_{\mathcal{P}}} \pi_{\mathbb{P}}$ indexed by Γ-relevant priors.*

In the economy, there is a single consumption good, a numeraire. Bundles (x, X) are elements in $\mathbb{R} \times L_{\mathbb{P}}$, the units at time 0 and T. $\mathbb{A}_{\mathbb{P}}$ denotes the set of rational, convex, strictly monotone, and $L_{\mathbb{P}}$-continuous preference relations $\succsim_{\mathbb{P}}$ on $\mathbb{R} \times L_{\mathbb{P}}$. For a price $\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R}$, define the *budget set* by $B_{\pi_{\mathbb{P}}} = \{(y, Y) \in \mathbb{R} \times M_{\mathbb{P}} : y + \pi_{\mathbb{P}}(Y) \leq 0\}$. An appropriate notion of viability defines a minimal consistency criterion, and can be regarded as an inverse no-trade equilibrium condition. Fix $\mathcal{R} \subset \mathcal{P}$.

Definition 4. *A price system is $\mathcal{R}$-viable if for each $\mathbb{P} \in \mathcal{R}$ there is a preference relation $\succsim_{\mathbb{P}} \in \mathbb{A}_{\mathbb{P}}$ and a bundle $(x_{\mathbb{P}}, X_{\mathbb{P}}) \in B_{\pi_{\mathbb{P}}}$ such that $(x_{\mathbb{P}}, X_{\mathbb{P}})$ is $\succsim_{\mathbb{P}}$-maximal on $B_{\pi_{\mathbb{P}}}$.*

The conditions are necessary and sufficient for a classical economic equilibrium in each relevant scenario in $\mathcal{R}$. Note that Definition 4 has to some extent the preference type of Bewley (2002). When $M_{\mathbb{P}} = M$ for every $\mathbb{P}$, scenario-based viability exactly captures the existence of an agent with Bewley preferences and a maximal consumption bundle (x, X). The concept of scenario-based viability, as a model of an economic equilibrium can be related to Theorem 1.

Example 4. *Let $X \in L_{\mathcal{P}}$ be a contingent claim such that it is priced by $\mathcal{P}$-arbitrage. Then, in view of Section 2, the value at any time t is given by $\Psi_t(X) = \frac{1}{\psi_t}\mathcal{E}_t^{\mathcal{P}}(\psi_T X)$, if Γ is assumed to be the maximum operation. In view of Equation (12), this assumption yields $\Gamma(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \Psi(\cdot) = \max_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\psi\cdot]$, where $\psi \in L_{\mathcal{P}}$ is considered as an aggregated collection $(\psi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$. In particular, we can take the deflator $\psi = \psi_T^{\theta}$ from Equation (9).*

The next result connects viability with price systems in $L_+^{\circledast}$. For the proof, we redefine the shifted preference relations $\succsim_{\mathbb{P}}$ such that every feasible net trade is worse off than $(0, 0) \in B_{\pi_{\mathbb{P}}}$. Obviously, an agent given by $\succsim_{\mathbb{P}}$ does not trade.

Theorem 2. *A price system is $\mathcal{R}$-viable if and only if there is a $\Psi \in L_+^{\circledast}$ such that $\pi_{\mathbb{P}} \leq \Psi$ on $L_{\mathcal{P}} \cap M_{\mathbb{P}}$ for each $\mathbb{P} \in \Gamma_{\Psi}$ with $\Gamma_{\Psi} = \mathcal{R}$.*

Apart from the given scenario-based marketed spaces $(M_{\mathbb{P}})$, there is an intrinsic subspace of mean-ambiguity free claims in which the price system Ψ acts in a linear manner. In Section 2, the related symmetry property corresponds to a refined martingale notion, which is induced by EsMM-sets.

Corollary 1. *Every $\Psi \in L^{\circledast}_{+}$ is linear on the* Ψ-marketed space

$$\mathbb{M}_{\Psi} = \left\{ X \in L_{\mathcal{P}} : E^{\mathbb{P}}[\psi_{\mathbb{P}} X] \text{ is constant in } \mathbb{P} \in \Gamma_{\Psi} \right\}. \tag{13}$$

For claims in Equation (13), the law of one price holds. In the finite state case, Beissner and Riedel (2019) considered a stronger notion of viability and introduced a general equilibrium concept with coherent price systems Ψ. In that case, the state price $\psi = \psi_{\mathbb{P}}$ is then the equilibrium outcome and determines the endogenous structure of the marketed space $\mathbb{M}_{\Psi}$. Under ambiguous volatility and $\Gamma_{\Psi} = \mathcal{P}$, the space $\mathbb{M}_{\Psi}$ also determines the implementability of Arrow–Debreu equilibria (with linear prices) via dynamic trading strategies. For a characterization through the net trades of an Arrow–Debreu equilibrium, see the work of Beissner and Riedel (2018).

3.4. Viability, Symmetric Martingales, and the FTAP

Finally, this subsection returns to the setting and results of Section 2. The following theorem justifies the discussion of the connection between viability and symmetric martingales. To guarantee a sublinear conditional expectation $\mathcal{E}_t$ under $\mathcal{P}$ that satisfies the dynamic programming principle of Equation (5), Assumption 1 is again imposed. The G-expectation framework of Theorem 1 satisfies Assumption 1 automatically.

The next result connects Theorems 1 and 2.

Theorem 3. *Suppose that the financial market model* $\mathtt{FM}(1, S)$ *is $\mathcal{P}$-arbitrage free. There is a bijection between viable prices $\Psi \in L^{\circledast}_{+}$ and EsMM-sets, such that Γ_{Ψ} is stable under pasting. $\Psi = \mathcal{E}^{\mathcal{Q}}$ holds and the EsMM-set is*

$$\mathcal{Q} = \{ \mathbb{Q} \in \Delta(\Omega) : \mathrm{d}\mathbb{Q} = \psi_{\mathbb{P}} \mathrm{d}\mathbb{P}, \quad \mathbb{P} \in \Gamma_{\mathcal{P}} \ \textit{and} \ \ \psi_{\mathbb{P}} \in L_{\mathbb{P}+} \}. \tag{14}$$

Recall the corresponding consolidation Γ, defined in Equation (12), satisfies $\Gamma_{\Psi} = \mathcal{R} \subset \mathcal{P}$. Theorem 3 then establishes a one-to-one mapping between $L^{\circledast}_{+}$ and $\mathtt{EsMM}(\mathcal{P})$. Analogous to the single-prior setting, this view yields some further insights by combining Theorems 2 and 3. We say that a financial market $\mathtt{FM}(1, S^1)$ is *viable* if it is $\Gamma_{\mathcal{P}}$-arbitrage free, and the associated price system $\pi_{\prod \mathcal{P}}$ is viable for a given consolidation operator, as stated in Equation (12).

Corollary 2. *Fix an $\mathcal{R} = \Gamma_{\Psi} \subset \mathcal{P}$ that is stable under pasting.*

1. $\mathtt{FM}(1, S^1)$ *is $\mathcal{R}$-viable if and only if there is an EsMM-set in* $\mathtt{EsMM}(\mathcal{R})$.
2. *If* $\mathtt{EsMM}(\mathcal{R})$ *is non-empty, then no $\mathcal{R}$-arbitrage exists.*

Under the uncertainty neutral expectation $\mathcal{E}^{\mathcal{Q}}$, expected returns of the risky and ambiguous asset S^1 equals that of the riskless asset S^0 and contains no ambiguity.

4. Conclusions

This paper presents a theory of derivative security pricing into which volatility uncertainty is incorporated. The classical notion of equivalent martingale measures changes, and the valuation by means of expectations becomes nonlinear. The results of this paper establish a version of the FTAP under volatility uncertainty.

The present uncertainty model is closely related to that of Epstein and Ji (2013), while the present valuation principle follows the preference-free approach by Ross (1976). The price of a claim is the expected value in an uncertainty-neutral world. Expectations for the security price no longer merely depend on one "risk-neutral" prior; the principle of a risk-neutral valuation is insufficient, as different mutually singular priors deliver completely different linear risk-neutral expectations. The shortcoming of linear prices is rearranged. A single prior, as the output of a linear-price equilibrium, can create an invisible threat of convention and may lead to an illusion of security when faced with a risky and ambiguous future. Within any neoclassical infinite states model, this stems from the dual role of an objective and primitive probability measure $\mathbb{P}$. On the one hand, it quantifies uncertainty. On the other hand, this $\mathbb{P}$ reappears in the representation of any linear equilibrium price system and any EMM.

Under the present type of volatility uncertainty, the focus on a single prior creates a hazard. Payoff-relevant events with a positive probability may be costless under a supporting prior of a linear price. For instance, this can result from the first-order condition in a consumption-based asset pricing model.

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Appendix A. Details and Required Results

A property holds *quasi-surely* (q.s.) if it holds outside a polar set, and the $\mathcal{P}$-polar sets evaluated under every prior are zero or one. As shown in Theorem 25 of Denis et al. (2011), the space $L_{\mathcal{P}}$ is characterized by

$$L_{\mathcal{P}} = \left\{ X \in L^0(\Omega) : X \text{ has a q.c. version, } \lim_{n\to\infty} \mathcal{E}^{\mathcal{P}}(|X|^2 1_{\{|X|>n\}}) = 0 \right\}.$$

A mapping $X : \Omega \to \mathbb{R}$ is said to be quasi-continuous (q.c.) if for all $\varepsilon > 0$ there exists an open set O with $\sup_{P\in\mathcal{P}} P(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. The random variable $X : \Omega \to \mathbb{R}$ has a *quasi-continuous version* if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s. Since, for all $X, Y \in L_{\mathcal{P}}$ with $|X| \leq |Y|$ imply $\|X\|_{\mathcal{P}} \leq \|Y\|_{\mathcal{P}}$, we have that $L_{\mathcal{P}}$ is a Banach lattice.

We recall the mentioned criterion for the weak compactness of $\mathcal{P}$. Let $\sigma^1, \sigma^2 : [0, T] \to \mathbb{R}_+$ determine two measures with a Hölder continuous distribution function $t \mapsto \sigma^i([0, t]) = \sigma^i(t)$. As introduced in Equation (1), a measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a martingale probability measure if the coordinate process is a martingale.

Weak compactness of $\mathcal{P}$, Denis and Martini (2006): The set of probability measures $\mathcal{P}(\sigma^1, \sigma^2)$ induced by Equation (1) and $\mathrm{d}\sigma^1_t \leq \mathrm{d}\langle B\rangle^{\mathbb{P}_\sigma}_t \leq \mathrm{d}\sigma^2_t$ is weakly compact.

Appendix A.1. The Sub Order Dual

We discuss the mathematical preliminaries for the price space $L^{\circledast}_{\mathcal{P}}$.

The Topological Dual Space: By Theorem 2 of Beissner and Denis (2018), we have $L^*_{\mathcal{P}} = \{\mu = \int \psi_{\mathbb{P}} \mathrm{d}\mathbb{P} : \mathbb{P} \in \mathcal{P} \text{ and } \psi_{\mathbb{P}} \in L^2(\mathbb{P})\}$.

For the space of coherent price systems $L^{\circledast}_{+}$, every consolidation operator Γ has a domain in $\prod_{P\in\mathcal{P}} L^*_{\mathcal{P}}$ and maps to $L^{\circledast}_{+}$. In the following, we present different operations for consolidation. Let $\Pi_{\mathbb{P}} = E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot] \in L^*_{\mathcal{P}}$, with $\mathbb{P} \in \mathcal{P}$ and $\mu \in \Delta(\mathcal{P})$ has full support on $\mathcal{P}$. In this context, we can consider the additive case in $L^{\circledast}_{+}$, in which a new prior is generated:

$$\Gamma^{\mu}(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \int_{\mathcal{P}} \psi_{\mathbb{P}} \cdot \mu(d\mathbb{P}) = E^{\mathbb{P}_\mu}[\psi_{\mathbb{P}_\mu}\cdot], \tag{A1}$$

where ψ_{P_μ} is constructed as in Example 3. The Dirac measure $\delta_{\mathbb{P}}$ is a particular example of μ in which only one prior $\mathbb{P} \in \mathcal{P}$ drives the pricing scheme. The operation in question is given by $(\Pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}} \mapsto E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$. The second operation in $L^{\circledast}_{+}$ is a point-wise maximum:

$$\Gamma^{\max}(\{\Pi_{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}}) = \max_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot].$$

Combinations of maximum and addition operation are illustrated in Example 4.

Example A1. *Let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a partition of $\mathcal{P}$ and $\mu_n \in \Delta(\mathcal{P})$. For each n, the resulting prior $\mathbb{P}_n(A) = E^{\mu_n}[\mathbb{P}(A)]$ is given by a second-order weighting operation Γ^{μ_n}. Apply Γ^{μ_n} to the densities $\psi_{\mathbb{P}}$ so that $\psi_n = E^{\mu_n}[\psi_{\mathbb{P}}]$. Each group of priors $\mathcal{P}_n$ is consolidated to one pair $(\psi_n, \mathbb{P}_n)$. These resulting pairs $(\psi_n, \mathbb{P}_n)$ can then be consolidated by $\Gamma(\mathcal{P}) = \sup_{n\in\mathbb{N}} E^{\mathbb{P}_n}[\psi_n \cdot]$.*

Representation of sublinearity, Biagini and Frittelli (2010): Let Ψ be a sublinear functional on L, then $\Gamma_\Psi = \{x^* \in L^* : x^*(X) \leq \Psi(X)\ \forall X \in L\}$ is non-empty and

$$\Psi(X) = \max_{x^*\in\Gamma_\Psi} x^*(X).$$

Appendix A.2. Stochastics under Sublinear Expectations

This section recalls some notions and results for the G-Brownian motion. Let $\Omega_T = C_0([0,T])$. A sublinear G-expectation $\mathcal{E}$ on $L_{\mathcal{P}} = L$ is a functional $\mathcal{E} : L \to \mathbb{R}$ satisfying monotonicity, constant preserving, sub-additivity and positive homogeneity. The triple $(\Omega, L, \mathcal{E})$ is called a *sublinear expectation space*. For the construction of the G-expectation and a general overview, see Parts 1 and 2 of Peng (2010). This Appendix focuses only on the very basic concepts.

In the classic probabilistic setting, in which a probability measure $\mathbb{P}$ (or a resulting linear expectation $E^{\mathbb{P}}$) captures the uncertainty, the random vector X is determined by $F_X(A) = \mathbb{P}(X \in A) = E^{\mathbb{P}}[1_{\{X\in A\}}] = E^{\mathbb{P}}[f_A(X)]$.

The "test functions" f_A consists of all the indicator functions with respect to all the elements of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$. In comparison with the (linear) probability theory, the nonlinearity of $\mathcal{E}$ changes the basic notions of stochastics.

Definition A1. *The* distribution *of a random vector $X = (X_1, \ldots, X_n) \in L^n$ under $\mathbb{E}$ is a functional $F_X : C_b(\mathbb{R}^n) \to \mathbb{R}$ given by $\phi \mapsto F_X[\phi] := \mathcal{E}[\phi(X)]$.*

Note that, for each X, the triple $(\mathbb{R}^n, C_b(\mathbb{R}^n), F_X)$ is again a sublinear expectation space. Due to the nonlinearity of $\mathcal{E}$, the (cumulative) distribution function is no longer able to capture all the "information" about the uncertainty of X under $\mathcal{E}$. From this perspective the present type of distribution is determined by all the test functions in $C_b(\mathbb{R}^n)$. In accepting this definition of a distribution, the notion of an identical distribution has an intuitive appeal.

Definition A2. *Two random vectors $X, Y \in L^n$ are* identically distributed *if $\mathcal{E}[\phi(X)] = \mathcal{E}[\phi(Y)]$ for every $\phi \in C_b(\mathbb{R}^n)$ and denoted by $X \sim Y$.*

- *Y is $\mathcal{E}$-*independent *of X if $\forall\phi \in C_b(\mathbb{R}^2)$ we have $\mathcal{E}[\phi(X,Y)] = \mathcal{E}[\mathcal{E}[\phi(x,Y)]|_{x=X}]$.*
- *X' is an independent copy of X. If $aX + bX' \sim \sqrt{a+b} \cdot X'$ for every $a, b \geq 0$, then X is called $N(0, [\overline{\sigma}, \underline{\sigma}])$-*normal distributed.

The connection to G-expectation, i.e., $E_G = \mathcal{E}$ comes from the sublinear and monotone function $G : \mathbb{R} \to \mathbb{R}$ with $G(a) = \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \sigma a = \mathcal{E}[aX^2]$.

As a canonical generalization of the standard normal distribution $N(0,\sigma) = \mathsf{N}(0, [\sigma, \sigma])$, a $\mathsf{N}(0, [\underline{\sigma}, \overline{\sigma}])$-distributed X is characterized by a nonlinear heat partial differential equation $\partial_t u = G(u_{xx})$, with initial condition $\phi = u_{|t=0}$. The unique solution then reads as follows $u(t,x) = \mathcal{E}[\phi(x + \sqrt{t}X)]$.

Next, we state the results on stochastic analysis with G-Brownian Motion used in the proof of Theorem 1. A process $(\mathsf{B}_t)_{t\geq 0}$ on $(\Omega, L, \mathcal{E})$ is called a *G-Brownian motion* with $\mathsf{B}_0 = 0$ if $\mathsf{B}_{t+s} - \mathsf{B}_t$ is $\mathbb{E}$-independent of $(\mathsf{B}_{t_1}, \ldots, \mathsf{B}_{t_n})$ and the increment $\mathsf{B}_{t+s} - \mathsf{B}_t$ is $N(0, [\overline{\sigma} \cdot s, \underline{\sigma} \cdot s])$-distributed, where $t_1 < \ldots < t_n < t$.

The Itô integral $t \mapsto \int_0^t \eta_s \mathrm{dB}_s$ for B can be defined for the integrands in M^2: let H^0 be the space of all simple trading strategies η from Section 2.2. For $\eta \in H^0$, let $\|\eta\|_{M^2} \equiv \left(\mathcal{E}\left[\int_0^T |\eta_s|^2 \mathrm{d}s\right]\right)^{1/2}$ and denote by M^2 the completion of H^0 under $\|\cdot\|_{M^2}$.

Itô-formula, Li and Peng (2011): Let $\Phi \in C^2(\mathbb{R})$ and $\mathrm{d}X_t = \mu_t \mathrm{d}\langle \mathrm{B}\rangle_t + V_t \mathrm{dB}_t$, $t \in [0,T]$, $\mu, V \in M^2$ be bounded processes. Then:

$$\Phi(X_t) - \Phi(X_s) = \int_s^t \Phi_x(X_u) V_u \mathrm{dB}_u + \frac{1}{2}\int_s^t \Big(\Phi_x(X_u)\mu_u + \Phi_{xx}(X_u)V_u^2\Big)\mathrm{d}\langle \mathrm{B}\rangle_u.$$

Martingale representation, Soner et al. (2011): Let $\xi \in L_{\mathcal{P}}$. The $\mathcal{E}$-martingale $X_t \equiv \mathcal{E}[\xi|\mathcal{F}_t]$ has the following unique representation

$$X_t = \mathcal{E}[\xi] + \int_0^t z_s \mathrm{dB}_s - K_t.$$

K is increasing with $K_0 = 0$, $K_T \in L_{\mathcal{P}}$, $z \in M^2$, and $-K$ is an $\mathcal{E}$-martingale. $K \equiv 0$ if and only if (X_t) is a symmetric martingale.

Girsanov for G-expectation, Xu et al. (2011): Assume the following Novikov type condition: there is an $\varepsilon > \frac{1}{2}$ such that $\mathcal{E}[\exp(\varepsilon \cdot \int_0^T \theta_s^2 \mathrm{d}\langle \mathrm{B}\rangle_s)] < \infty$. Then, $\mathrm{B}_t^\theta = \mathrm{B}_t - \int_0^t \theta_s \mathrm{d}\langle \mathrm{B}\rangle_s$ is a G-Brownian motion under the sublinear expectation $\mathcal{E}^\theta(\cdot)$ given by $\mathcal{E}^\theta(X) = \mathcal{E}[\psi_T^\theta \cdot X]$, $\mathcal{P}^\theta = \psi_T^\theta \cdot \mathcal{P}$ with $X \in L_{\mathcal{P}}^\theta$.

Appendix B. Proofs

Proof. $\Rightarrow$ With $\mathcal{R} = \mathcal{P}$, this follows from Corollary 2 (2).

$\Leftarrow$ Let $\mathcal{Q} = \{\mathbb{Q} \in \Delta(\Omega) : \mathrm{d}\mathbb{Q} = \rho \mathrm{d}\mathbb{P}, \mathbb{P} \in \mathcal{P}\}$ be a possible EsMM-set, in which the density ρ satisfies $\rho \in L_{\mathcal{P}}$, $\rho > 0$ $\mathcal{P}$-q.s. and $1 = \mathcal{E}^{\mathcal{P}}(\rho) = -\mathcal{E}^{\mathcal{P}}(-\rho)$. Next, define the stochastic process $(\rho_t)_{t\in[0,T]}$ by $\rho_t = \mathcal{E}_t^{\mathcal{P}}(\rho)$ resulting in a symmetric $\mathcal{E}^{\mathcal{P}}$-martingale to which we apply the martingale representation theorem for the $\mathcal{E}^{\mathcal{P}}$-expectation, stated in Appendix A.2. Hence, there is an adapted process $\gamma \in M^2$ (see Appendix A.2 for a definition of M^2) such that we can write $\rho_t = 1 + \int_0^t \gamma_s \mathrm{dB}_s + 0$. On the other hand, by the Itô formula for G-Brownian motions stated in Appendix A.2, we have, where ϕ is function of γ,

$$\ln(\rho_t) = \int_0^t \phi_s \mathrm{dB}_s + \frac{1}{2}\int_0^t \phi_s^2 \mathrm{d}\langle \mathrm{B}\rangle_s,$$

for every $t \in [0,T]$ and hence $\rho = \psi_T^\phi = \exp\Big(-\frac{1}{2}\int_0^T \theta_s^2 \mathrm{d}\langle \mathrm{B}\rangle_s - \int_0^T \theta_s \mathrm{dB}_s\Big)$.

With this representation of the density process and by the assumed Novikov-type integrability condition $\mathcal{E}^{\mathcal{P}}\big(\exp\big(\delta \cdot \int_0^T \theta_t(S_t)^2 \mathrm{d}\langle \mathrm{B}\rangle_t\big)\big) < \infty$ for some $\delta > \frac{1}{2}$, we can apply the Girsanov theorem, stated in Appendix A.2. Set $\phi_t = \frac{\rho_t}{\gamma_t}$ and consider the process $\mathrm{B}_t^\phi = \mathrm{B}_t - \int_0^t \phi_s \mathrm{d}s$, $t \in [0,T]$. We deduce that B^ϕ is a G-Brownian motion under $\mathcal{E}^\phi(\cdot) = \mathcal{E}^{\mathcal{P}}(\rho\cdot)$ and S satisfies

$$S_t = S_0 + \int_0^t V_s \mathrm{dB}_s^\phi + \int_0^t \Big(\mu_s(S_s) + V_s(S_s)\phi_s\Big)\mathrm{d}\langle \mathrm{B}^\phi\rangle_s, \quad t \in [0,T]$$

on $(\Omega, L_{\mathcal{P}}, \mathcal{E}^\phi)$. Since V is a bounded process, the stochastic integral is a symmetric martingale under $\mathcal{E}^\phi$. S is a symmetric $\mathcal{E}^\phi$-martingale if and only if $\mu_t + V_t\phi_t = 0$ $\mathcal{P}$-q.s. We have shown that $\rho = (\rho_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ is indeed a simultaneous Radon–Nikodym type density of a set $\mathcal{Q}$ that is an EsMM-set. Hence, there is a non-trivial EsMM-set in $\mathtt{EsMM}(\mathcal{P})$, since $\phi_t = \theta_t$ $\mathcal{P}$-q.s for every $t \in [0,T]$. $\square$

Proof. Claims (1) and (2) follow from the construction of the functionals in $L_{\mathcal{P}+}^{\circledast}$. Claim (4) directly follows from the fact that $\mathbb{P}(X \geq Y) = 1$ for every $\mathbb{P} \in \Gamma_\Psi$ and the fact that each $\psi_{\mathbb{P}}$ is positive, i.e., $\Psi(Y) = E^P[\psi_P Y] \leq E^P[\psi_P X] \leq \Psi(X)$ for a $P \in \Gamma_\Psi$. The first equality applies the compactness of $\Gamma_{\mathcal{P}}$. Claim (3) follows a similar argument by using the strict positivity of each $\psi_{\mathbb{P}}$.

By Proposition 1 (2), we have $\Psi(0) = 0$. Since $L_{\mathcal{P}}$ is a Banach lattice, Claim (5) follows from Theorem 1 in Biagini and Frittelli (2010). $\square$

Proof. Fix the price system $\pi_{\prod \mathcal{P}} = \prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} \pi_{\mathbb{P}}$. For the easy direction fix a $\Psi \in L^{\circledast}_{\mathcal{P}+}$ on $L_{\mathcal{P}}$ such that $\pi_{\mathbb{P}} \leq \Psi$ on $M^{\mathcal{P}}_{\mathbb{P}} := M_{\mathbb{P}} \cap L_{\mathcal{P}}$ for each $\mathbb{P} \in \Gamma_{\mathcal{P}}$. We have $\Psi = \max_{\mathbb{P} \in \Gamma_{\mathcal{P}}} \Pi_{\mathbb{P}}$, with $\Pi_{\mathbb{P}} \in L^*_{\mathbb{P}}$. The linear preference relation $\succsim^0_{\mathbb{P}}$ on $\mathbb{R} \times L_{\mathbb{P}}$ defined by

$$(x, X) \succcurlyeq^0_{\mathbb{P}} (x', X') \quad \textit{if} \;\; x + -\Pi_{\mathbb{P}}(-X) \geq x' + -\Pi_{\mathbb{P}}(-X')$$

lies in $\mathbb{A}_{\mathbb{P}}$. For each $\mathbb{P} \in \Gamma_{\mathcal{P}}$, the bundle $(x_{\mathbb{P}}, X_{\mathbb{P}}) = (0, 0)$ satisfies the viability condition of Definition 4, hence $\pi_{\prod \mathcal{P}}$ is scenario-based viable.

In the other direction, let $\pi_{\prod \mathcal{P}}$ be scenario-based viable. The preference relation $\succcurlyeq_{\mathbb{P}} \in \mathbb{A}_{\mathbb{P}}$ satisfies for each $(x_{\mathbb{P}}, X_{\mathbb{P}})$, $\mathbb{P} \in \Gamma_{\mathcal{P}}$, the viability condition. We may assume for each $\mathbb{P}$, $(x_{\mathbb{P}}, X_{\mathbb{P}}) = (0, 0)$, since it is only a geometric deferment in each $L_{\mathbb{P}}$. Let the strict preference $\succ_{\mathbb{P}}$ on $\mathbb{R} \times L_{\mathbb{P}}$ be defined by $(z \succ_{\mathbb{P}} z') :\Leftrightarrow (z \succcurlyeq_{\mathbb{P}} z') \wedge \neg(z' \succcurlyeq_{\mathbb{P}} z)$. Consider the following sets

$$B_{\prod \mathcal{P}} = \prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} B_{\pi_{\mathbb{P}}} \quad \textit{and} \qquad \succ_{\mathcal{P}} = \prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} \Big\{ (x, X) \in \mathbb{R} \times L_{\mathbb{P}} : (x, X) \succ_{\mathbb{P}} (0, 0) \Big\}.$$

$B_{\prod \mathcal{P}}$ and $\succ_{\mathcal{P}}$ are both convex sets. The Riesz space product $\prod L_{\mathbb{P}} = \prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} L_{\mathbb{P}}$ (see paragraph 352 K of Fremlin 2000) is under the norm $\| \cdot \|_{\mathcal{P}}$, again a Banach lattice (see paragraph 354 X part (b) of Fremlin 2000). By the $L_{\mathbb{P}}$-continuity of each $\succsim_{\mathbb{P}}$, the set $\succ_{\mathcal{P}}$ is $\| \cdot \|_{\mathcal{P}}$-open in $\prod L_{\mathbb{P}}$.

An application of the separation theorem for each $\mathbb{P} \in \Gamma_{\mathcal{P}}$ yields a non-zero linear and $\| \cdot \|_{\mathcal{P}}$-continuous functional $\phi_{\mathbb{P}}$ on $\prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} \mathbb{R} \times L_{\mathbb{P}}$ with

1. $\phi_{\mathbb{P}}(x, X) \geq 0$ for all $(x, X) \in \succ_{\mathcal{P}}$
2. $\phi(x, X) \leq 0$ for all $(x, X) \in B_{\prod \mathcal{P}}$
3. $\{(y_{\mathbb{P}}, Y_{\mathbb{P}})\}_{\mathbb{P} \in \Gamma_{\mathcal{P}}} = (y, Y)$ with $\mathrm{pr}_{\mathbb{R} \times L_{\mathbb{P}}}(\phi_{\mathbb{P}})(y, Y) =: \phi_{\restriction \mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) < 0$,

since $\phi_{\mathbb{P}}$ is nontrivial. Note that Condition (3) depends on the chosen $\mathbb{P}$.

Strict monotonicity of $\succsim_{\mathbb{P}}$ implies $(1, 0) \succ_{\mathbb{P}} (0, 0)$. The $L_{\mathbb{P}}$-continuity of each $\succcurlyeq_{\mathbb{P}}$ gives us $(1 + \varepsilon y, \varepsilon Y) \succ_{\mathbb{P}} (0, 0)$, for some $\varepsilon > 0$, hence

$$\phi_{\restriction \mathbb{P}}(1 + \varepsilon y_{\mathbb{P}}, \varepsilon Y_{\mathbb{P}}) = -\phi_{\restriction \mathbb{P}}(1, 0) + \varepsilon \phi_{\restriction \mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) \leq 0$$

and $\phi_{\restriction \mathbb{P}}(1, 0) \geq -\varepsilon \phi_{\restriction \mathbb{P}}(y_{\mathbb{P}}, Y_{\mathbb{P}}) > 0$. After renormalization, let $\phi_{\restriction \mathbb{P}}(1, 0) = 1$. Moreover, we can write $\phi_{\restriction \mathbb{P}}(x_{\mathbb{P}}, X_{\mathbb{P}}) = x_{\mathbb{P}} + \Pi_{\mathbb{P}}(X_{\mathbb{P}})$, where $\Pi_{\mathbb{P}} : L_{\mathbb{P}} \to \mathbb{R}$ can be identified as an element in the topological dual $L^*_{\mathbb{P}}$.

We show strict positivity of $\Pi_{\mathbb{P}}$ on $L_{\mathbb{P}}$. Letting $X \in L_{\mathbb{P}+} \setminus \{0\}$, we have $(0, X) \succ_{\mathbb{P}} (0, 0)$, hence $(-\varepsilon, X) \succ_{\mathbb{P}} (0, 0)$, and therefore $\Pi_{\mathbb{P}}(X) - \varepsilon \geq 0$, for a small $\varepsilon > 0$.

Moreover, $\Pi_{\mathbb{P} \restriction L_{\mathcal{P}}}$ is positive on $L_{\mathcal{P}}$, i.e., $X \geq 0$ $\mathcal{P}$-q.s. implies $\Pi_{\mathbb{P} \restriction L_{\mathcal{P}}} \geq 0$. Since $L_{\mathcal{P}}$ is a Banach lattice, positivity of $\Pi_{\mathbb{P}}$ implies continuity and $\Pi_{\mathbb{P}} \in L^*_{\mathcal{P}}$ follows. Let $X \in M^{\mathcal{P}}_{\mathbb{P}}$, since $(-\pi_{\mathbb{P}}(X), X), (\pi_{\mathbb{P}}(X), -X) \in B_{\pi}$, we have $0 = \phi(\pi_{\mathbb{P}}(X), X) = \pi_{\mathbb{P}}(X) - \Pi_{\mathbb{P}}(X)$ and $\Pi_{\mathbb{P}} = \pi_{\mathbb{P}}$ on $M^{\mathcal{P}}_{\mathbb{P}}$ follows.

$\Gamma((\Pi_{\mathbb{P}})_{\mathbb{P} \in \Gamma_{\mathcal{P}}}) = \Psi$ is by construction in $L^{\circledast}_{\mathcal{P}+}$, as the strict positivity of Ψ follows from the strict positivity of each $\Pi_{\mathbb{P}}$. $\Psi_{\restriction M^{\mathcal{P}}_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$ follows from an inequality in the last part of Proposition 1 and $\Pi_{\mathbb{P}} = \pi_{\mathbb{P}}$ on $M_{\mathbb{P}}$. $\square$

We illustrate the construction of the previous proof in the following diagram:

$$\begin{tikzcd}
{[(\pi_{\mathbb{P}} : M_{\mathbb{P}} \to \mathbb{R})_{\mathbb{P} \in \mathcal{P}}; \Gamma]} \arrow[r, mapsto] & {[\prod_{\mathbb{P} \in \Gamma_{\mathcal{P}}} \pi_{\mathbb{P}}; \Gamma]} \arrow[d, mapsto, "\mathit{Hahn}"', "\mathit{Banach}"] & \\
& \big\{\Pi_{\mathbb{P}} : L_{\mathbb{P}} \to \mathbb{R}\big\}_{\mathbb{P} \in \Gamma_{\mathcal{P}}}, \Gamma \arrow[r, mapsto, "\Gamma"] & \Psi : L_{\mathcal{P}} \to \mathbb{R}
\end{tikzcd}$$

Proof. By construction, every functional Ψ can be represented as the supremum of priors in $\mathcal{P}$. Since $X \in \mathbb{M}_\Psi$, the supremum and the infimum of $\mathbb{P} \mapsto E^{\mathbb{P}}[\psi_{\mathbb{P}} X]$ coincide on $\Gamma_{\mathcal{P}}$. The assertion follows. □

Proof. We fix an EsMM-set $\mathcal{Q}$. The related consolidation Γ gives us the set of relevant priors $\Gamma_{\mathcal{P}} \subset \mathcal{P}$. From Definition 2, there is a $\psi_{\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}} \in L_{\mathbb{P}}$ for each $\mathbb{Q} \in \mathcal{Q}$ and a related $\mathbb{P} \in \mathcal{P}$. Let the associated strictly positive $\Psi = \mathcal{E}^{\mathcal{Q}}$ with $\Psi \in L^{\circledast}_{\mathcal{P}+}$ be given.

Take a marketed claim $X^m \in M^{\mathcal{P}}_{\mathbb{P}} = M_{\mathbb{P}} \cap L_{\mathcal{P}}$ with $\mathbb{P} \in \Gamma_{\mathcal{P}}$ and let $\eta \in \mathcal{A}$ be a self-financing trading strategy that hedges X^m. Since $\eta \in \mathcal{A}$, by the decomposition rule for conditional $\mathcal{E}^{\mathcal{Q}}$-expectation, see for instance Theorem 2.6 (iv) in Epstein and Ji (2014), and since S is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale, the following equalities

$$\mathcal{E}^{\mathcal{Q}}_t(\eta_u S_u) = \eta^+_t \mathcal{E}^{\mathcal{Q}}_t(S_u) + \eta^-_t \mathcal{E}^{\mathcal{Q}}_t(-S_u) = \eta^+_t S_t - \eta^-_t S_t = \eta_t S_t$$

hold, where $\eta = \eta^+ - \eta^-$ with $\eta^+_t, \eta^-_t \geq 0$ and $0 \leq t \leq u \leq T$. Therefore, we achieve, since $\eta_t S_t$ is also an $\mathcal{E}^{\mathcal{Q}}$-martingale,

$$\Psi(X^m) = \mathcal{E}^{\mathcal{Q}}_0(\eta_T S_T) = \eta_0 S_0 \geq \pi_{\mathbb{P}}(X^m), \quad \textit{for every } \mathbb{P} \in \Gamma_{\mathcal{P}}.$$

This show the viability of $\mathcal{E}^{\mathcal{Q}}$.

For the other direction, fix a strictly positive price system $\Psi \in L^{\circledast}_{\mathcal{P}+}$ with $\Psi_{\restriction M_{\mathbb{P}}} \geq \pi_{\mathbb{P}}$, related to a set of linear functionals $(\pi_{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ and $(\Pi_{\mathbb{P}})_{P\in\mathcal{P}}$, such that $\Pi_{\mathbb{P}\restriction M_{\mathbb{P}}} = \pi_{\mathbb{P}}$. This can be inferred from Ψ and the construction in the proof of the second part of Theorem 2. Now, we define $\mathcal{Q}$ in terms of Γ.

To show the first part of Definition 2, we illustrate the possible cases that can appear. For simplicity, we assume $\mathcal{P} = \{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3\}$. Let $\mathbb{P}^{k,j} = \frac{1}{2}\mathbb{P}_k + \frac{1}{2}\mathbb{P}_j$ and $\psi^{k,j} = \frac{1}{2}\psi^k + \frac{1}{2}\psi^j$, recall that we can represent each functional $\Pi_{\mathbb{P}}(\cdot)$ by $E^{\mathbb{P}}[\psi_{\mathbb{P}}\cdot]$. We have, with $\Psi = \Gamma(\{\Pi_i\}) \in L^{\circledast}_{\mathcal{P}+}$,

$$\Gamma(\{\Pi_i\}) = \frac{1}{2}(\Pi_1 + \Pi_2) \wedge \Pi_3 \text{ becomes } \mathcal{Q} = \left\{\psi^{1,2} \cdot \mathbb{P}^{1,2}, \psi_3 \cdot \mathbb{P}_3\right\}.$$

Consequently, $\mathcal{Q} = \{\mathbb{Q} : d\mathbb{Q} = \psi_{\mathbb{P}} d\mathbb{P}, \mathbb{P} \in \Gamma_\Psi, \psi_{\mathbb{P}} \in L_{\mathbb{P}}\}$, where $\psi_{\mathbb{P}}$ with $\mathbb{P} \in \Gamma_\Psi$ is constructed following the procedure of Example 4 in Appendix A.1. The first condition of Definition 2 holds, since the square integrability of each $\psi_{\mathbb{P}}$ follows from the $\|\cdot\|_{\mathcal{P}}$-continuity of each linear functional Π_i.

It remains to prove the symmetric $\mathcal{Q}$-martingale property of the asset price process. Let $B \in \mathcal{F}_t$ and $\eta \in \mathcal{A}$ be a self-financing trading strategy satisfying

$$\eta^1_s = \begin{cases} 1 & s \in [t,u) \text{ and } \omega \in B \\ 0 & \text{else}\,, \end{cases} \qquad \eta^0_s = \begin{cases} S_t, & s \in [t,u) \text{ and } \omega \in B \\ S_u - S_t, & s \in [u,T) \text{ and } \omega \in B \\ 0 & \text{else.} \end{cases}$$

This strategy yields a terminal portfolio value $\eta_T S_T = (S_u - S_t) \cdot 1_B$, and the claim $\eta_T S_T$ is marketed at price zero. Under the uncertainty neutral conditional sublinear expectation $(\mathcal{E}^{\mathcal{Q}}_t)_{t\in[0,T]}$, we have with $t \leq u$

$$\mathcal{E}^{\mathcal{Q}}_t((S_t - S_u)1_B) = 0.$$

By Theorem 4.7 of Xu and Zhang (2010), it follows that $S_t = \mathcal{E}^{\mathcal{Q}}_t(S_u)$.[6] However, this means that $(S_t)_{t\in[0,T]}$ is an $\mathcal{E}^{\mathcal{Q}}$-martingale.

6 The result is proven for the G-framework. The assertion holds also true under Assumption 1 by the martingale representation in Proposition 4.10 of Nutz and Soner (2012).

The same arguments hold for $-S$; hence, (S_t) is a symmetric $\mathcal{E}^{\mathcal{Q}}$-martingale. □

Proof. 1. By Theorem 2, viability is equivalent to the existence of a Ψ in $L_+^{\circledast}$, with $\mathcal{R} = \Gamma_\Psi$. By Theorem 3, the claim follows.

2. Suppose that there is a $\mathcal{Q} \in \mathtt{EsMM}(\mathcal{P})$ and let $\eta \in \mathcal{A}$ such that $\eta_T S_T \geq 0$ in $L_{\mathcal{P}}$ and $\mathbb{P}'(\eta_T S_T > 0) > 0$ for some $\mathbb{P}' \in \mathcal{P}$. Since for all $\mathbb{Q} \in \mathcal{Q}$ there is a $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous, there is a $\mathbb{Q}' \in \mathcal{Q}$ with $\mathbb{Q}'(\eta_T S_T > 0) > 0$. Hence, $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) > 0$ and, from Theorem 2, we observe $\mathcal{E}^{\mathcal{Q}}(\eta_T S_T) = \eta_0 S_0$. This implies that no $\mathcal{P}$-arbitrage exists. □

References

Aliprantis, Charalambos, and Rabee Tourky. 2002. The super order dual of an ordered vector space and the Riesz-Kantorovich formula. *Transactions of the American Mathematical Society* 354: 2055–78. [CrossRef]

Aliprantis, Charalambos, Rabee Tourky, and Nicholas C.Yannelis. 2001. A theory of value with non–linear prices: Equilibrium analysis beyond vector lattices. *Journal of Economic Theory* 100: 22–72. [CrossRef]

Araujo Aloisio, Alain Chateauneuf, and José Heleno Faro. 2012. Pricing rules and Arrow–Debreu ambiguous valuation. *Economic Theory* 49: 1–35. [CrossRef]

Avellaneda, Marco, Arnon Levy, and Antonio Paras. 1995. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance* 2: 73–88. [CrossRef]

Beissner, Patrick. 2017. Equilibrium prices and trade under ambiguous volatility. *Economic Theory* 64: 213–38. [CrossRef]

Beissner, Patrick, and Laurent Denis. 2018. Duality and general equilibrium theory under Knightian uncertainty. *SIAM Journal on Financial Mathematics* 9: 381–400. [CrossRef]

Beissner, Patrick, and Frank Riedel. 2018. Non-Implementability of Arrow-Debreu equilibria by continuous trading under Knightian uncertainty. *Finance and Stochastics* 22: 603–20. [CrossRef]

Beissner, Patrick, and Frank Riedel. 2019. Equilibria under Knightian price uncertainty. *Econometrica* 87: 37–64. [CrossRef]

Bewley, F. Truman 2002. Knightian decision theory. Part I. *Decisions in Economics and Finance* 25: 79–110.

Biagini, Sara, and Marco Frittelli. 2010. On the extension of the Namioka-Klee theorem and on the Fatou property for risk measures. *Optimality and Risk-Modern Trends in Mathematical Finance*. Berlin/Heidelberg: Springer, pp. 1–28.

Bouchard, Bruno, and Marcel Nutz. 2015. Arbitrage and duality in nondominated discrete-time models. *The Annals of Applied Probability* 25: 823–59. [CrossRef]

Carr, Peter, and Roger Lee. 2009. Volatility derivatives. *Annual Review of Financial Economics* 1: 319–39. [CrossRef]

Cont, Rama. 2006. Model uncertainty and its impact on the pricing of derivative instruments. *Mathematical Finance* 16: 519–47. [CrossRef]

Delbaen, Freddy, and Walter Schachermayer. 1994. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* 300: 463–520. [CrossRef]

Denis, Laurent, and Claude Martini. 2006. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *The Annals of Applied Probability* 16: 827–52. [CrossRef]

Denis, Laurent, Mingshang Hu, and Shige Peng. 2011. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Analysis* 34: 139–61. [CrossRef]

Duffie, Darrell. 2010. *Dynamic Asset Pricing Theory*. Princeton: Princeton University Press.

Dybvig, Philip, and Stephen A. Ross. 2003. Arbitrage, state prices and portfolio theory. *Handbook of the Economics of Finance* 1: 605–37.

Epstein, G. Larry, and Shaolin Ji. 2013. Ambiguous volatility and asset pricing in continuous time. *Review of Financial Studies* 26: 1740–86. [CrossRef]

Epstein, G. Larry, and Shaolin Ji. 2014. Ambiguous volatility, possibility and utility in continuous time. *Journal of Mathematical Economics* 50: 269–82. [CrossRef]

Epstein, G. Larry, and Martin Schneider. 2003. Recursive multiple-priors. *Journal of Economic Theory* 113: 1–31. [CrossRef]

Fouque, Jean Pierre, and Bin Ren. 2014. Approximation for option prices under uncertain volatility. *SIAM Journal on Financial Mathematics* 5: 360–83. [CrossRef]

Fremlin, D. H. 2000. *Measure Theory*. Essex: Torres Fremlin, vol. 3.

Gilboa, Itzhak, and David Schmeidler. 1989. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics* 18: 141–53. [CrossRef]

Harrison, J. Michael, and David Kreps. 1979. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20: 381–408. [CrossRef]

Heath, David, and Hyejin Ku. 2006. Consistency among trading desks. *Finance and Stochastics* 10: 331–40. [CrossRef]

Heston L. Steven 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6: 327–43.

Jouini, Elyès, and Hédi Kallal. 1995. Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory* 66: 178–97. [CrossRef]

Klibanoff, Peter, Massimo Marinacci, and Sujoy Mukerji. 2005. A smooth model of decision making under ambiguity. *Econometrica*, 73: 1849–92. [CrossRef]

Kreps, M. David. 1981. Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics* 8: 15–35. [CrossRef]

Li, Xinping, and Shige Peng. 2011. Stopping times and related Itô calculus with G-Brownian motion. *Stochastic Processes and Their Applications* 121: 1492–508. [CrossRef]

Nutz, Marcel, and H. Mete Soner. 2012. Superhedging and dynamic risk measures under Volatility Uncertainty. *SIAM Journal on Control and Optimization* 50: 2065–89 [CrossRef]

Peng, Shige. 2006. G-expectation, G-Brownian motion and related stochastic calculus of Itô type. *Stochastic Analysis and Applications*: 541–67.

Peng, Shige. 2010. Nonlinear expectations and stochastic calculus under uncertainty. *arXiv*. arXiv:1002.4546.

Riedel, Frank. 2009. Optimal stopping with multiple priors. *Econometrica* 77: 857–908.

Ross, A. Stephen. 1976. The arbitrage theory of capital asset pricing. *Journal of Economic Theory* 13: 341–60. [CrossRef]

Samuelson, A. Paul. 1965. Proof that properly anticipated prices fluctuate randomly. *Management Review* 6: 41–49.

Soner, H. Mete, Nizar Touzi, and Jianfeng Zhang. 2011. Martingale representation theorem for the G-expectation. *Stochastic Processes and Their Applications* 121: 265–87. [CrossRef]

Vorbrink, Joerg. 2014. Financial markets with volatility uncertainty. *Journal of Mathematical Economics* 53: 64–78. [CrossRef]

Willard, A. Gregory, and Philip H. Dybvig. 1999. Empty promises and arbitrage. *Review of Financial Studies* 12: 807–34. [CrossRef]

Xu, Jing, and Bo Zhang. 2010. Martingale property and capacity under G-framework. *Electronic Journal of Probability* 15: 2041–68. [CrossRef]

Xu, Jing, Hao Shang, and Bo Zhang. 2011. A Girsanov type theorem under G-f ramework. *Stochastic Analysis and Applications* 29: 386–406. [CrossRef]