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Moments of Compound Renewal Sums with Dependent Risks Using Mixing Exponential Models

Fouad Marri [1,*], Franck Adékambi [2] and Khouzeima Moutanabbir [3]

1. Department of Statistics and Actuarial Science, Institut National de Statistique et d'Economie Appliquée, INSEA, Rabat 10112, Morocco
2. School of Economics, University of Johannesburg, Johannesburg 2006, South Africa; fadekambi@uj.ac.za
3. Department of Mathematics and Actuarial Science, The American University in Cairo, New Cairo 11835, Egypt; k.moutanabbir@aucegypt.edu

* Correspondence: fmarri@insea.ac.ma

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Abstract: In this paper, we study the discounted renewal aggregate claims with a full dependence structure. Based on a mixing exponential model, the dependence among the inter-claim times, the claim sizes, as well as the dependence between the inter-claim times and the claim sizes are included. The main contribution of this paper is the derivation of the closed-form expressions for the higher moments of the discounted aggregate renewal claims. Then, explicit expressions of these moments are provided for specific copulas families and some numerical illustrations are given to analyze the impact of dependency on the moments of the discounted aggregate amount of claims.

Keywords: renewal process; discounted aggregate claims; copulas; archimedean copulas

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1. Introduction

Over the past few years, extensive studies on the risk aggregation problem for insurance portfolios have appeared in the literature. Among these studies we find Albrecher and Boxma (2004), Albrecher and Teugels (2006) and Boudreault et al. (2006) which analyze ruin-related problems; Léveillé et al. (2010), Léveillé and Adékambi (2011, 2012), investigate the risk aggregation and the distribution of the discounted aggregate amount of claims; Léveillé and Garrido (2001a, 2001b) use the renewal theory to derive a closed expressions for the first two moments of the discounted aggregated claims; and Léveillé and Hamel (2013) study the aggregate discount payment and expenses process for medical malpractice insurance. Most recently, Jang et al. (2018) study the family of renewal shot-noise processes. Based on the piecewise deterministic Markov process theory and the martingale methodology, they obtained the Feynmann-Kac formula and then derived the Laplace transforms of the conditional moments and asymptotic moments of the processes.

For the risk management of non-life insurance portfolios, the mathematical expectation of the discounted aggregate claims plays an important role in determining the pure premium, in addition to giving a measure of the central tendency of its distribution. Moments centered at the 2nd, 3rd and 4th order average are the other moments usually considered, as they generally give a good indication of the pace of the distribution. The 2nd order centered moment gives us a measure of the dispersion around its mean, the 3rd order moment gives us a measure of the asymmetry of the distribution of and the 4th order moment gives us a measure of the flattening of the distribution of the discounted aggregate sums. Moments, whether simple, joint, or conditional, may be useful for constructing predictors, regression curves, or approximations of the distribution of the discounted aggregate claims.

The papers cited above assume that the inter-arrival times and the claim amounts are independent. Such an assumption is not supported by empirical observations which reduces the practicality of these works. For example, in non-life insurance, the same catastrophic event such as a flood or an earthquake could lead to frequent and high losses. This means that in such context a positive dependence between the claim sizes and the inter-claim times should be observed.

During the last decade, few papers in the actuarial literature considered incorporating this type of dependence. For example, Barges et al. (2011) introduce the dependence between the claim sizes and the inter-claim times using a Farlie-Gumbel-Morgenstern (FGM) copula and derive a close-from expression for the moments of the discounted aggregate claims. Guo et al. (2013) incorporate time dependence in a mixed Poisson process to study loss models. Landriault et al. (2014) consider a non-homogeneous birth process for the claim counting process to study time dependent aggregate claims.

For a given portfolio, we consider the renewal risk process suggested by Andersen (1957) and described as follows. Let $\{N(t)\}_{t\geq 0}$ be a renewal process that counts the number of claims. The positive random variable (rv) W_k represents the time between the $(k-1)-$th and $k-$th claims, $k \in \mathbb{N}^\star = \{1, 2, \cdots\}$, and the amount of the k-th claim is given by the positive rv X_k. We also define $\{T_k,\ k \in \mathbb{N}^\star\}$ as a sequence of rvs such that $T_k = \sum_{i=1}^{k} W_i,\ T_0 = 0$. The rv T_k represents the occurrence time of the $k-$th received claim. For any given integer n and $t \geq 0$, we have $\{N(t) \geq n\} = \{T_n \leq t\}$. The main variable of interest in this paper is the discounted aggregate amount of claims up to a certain time $\mathcal{Z}(t)$ defined as follows

$$\mathcal{Z}(t) = \sum_{i=1}^{N(t)} e^{-\delta T_i} X_i, \quad t \geq 0, \tag{1}$$

with $\mathcal{Z}(t) = 0$ if $N(t) = 0$, where δ is the force of net interest (See e.g., Léveillé and Garrido 2001a). In the rest of the paper, it is assumed that

- $\{W_k, k \in \mathbb{N}^\star = \{1, 2, \cdots\}\}$ forms a sequence of continuous positive dependent and identically distributed rvs with a common cumulative distribution function (cdf) $F_W(.)$ and a survival function (sf) $\bar{F}_W(.) = 1 - F_W(.)$,
- The claim amounts $\{X_k, k \in \mathbb{N}^\star\}$ are positive dependent and identically distributed rvs with a common cdf $F_X(.)$ and a common sf $\bar{F}_X(.) = 1 - F_X(.)$, and
- $\{(W_k, X_k), k \in \mathbb{N}^\star\}$ forms a sequence of identically distributed random vectors distributed as the canonical random vector (W, X) in which the components may be dependent.

In this paper, we specify three sources of dependence: among the claims X_k, among the subsequent inter-claims time W_k, and a dependence between the subsequent inter-claims time W_k and the claims X_k. For the dependence between the inter-claim times $\{W_k, k \in \mathbb{N}^\star = \{1, 2, \cdots\}\}$, we assume the existence of a positive continuous rv Θ such that given $\Theta = \theta$ the rvs W_k are iid and exponentially distributed with a mean $\frac{1}{\theta}$. Similarly, we introduce the dependence between the amounts of claims $\{X_k, k \in \mathbb{N}^\star\}$ through a positive continuous rv Λ such that conditional on $\Lambda = \lambda$ the rvs X_k are iid and exponentially distributed with a mean $\frac{1}{\lambda}$. In other words, the conditional distributions of the components of W and X are only influenced by the rv Θ and Λ respectively. The rvs Θ and Λ represent the factors that introduce the dependence between risks (e.g., climate conditions, age, $\cdots$, etc.).

In what follows, let $F_{\Theta,\Lambda}$ be the joint cdf of the positive random vector (Θ, Λ) and the marginal cdfs are F_Θ and F_Λ. We also define the joint Laplace transform $f^\star_{\Theta,\Lambda}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-(\theta s_1 + \lambda s_2)} dF_{\Theta,\Lambda}(\theta, \lambda)$, for $s_1, s_2 \geq 0$, as well as the univariate Laplace transforms $f^\star_\Theta(s) = \int_0^\infty e^{-\theta s} dF_\Theta(\theta)$ and $f^\star_\Lambda(s) = \int_0^\infty e^{-\lambda s} dF_\Lambda(\lambda)$, for $s \geq 0$. Following the model's specifications, the univariate distributions of W_i and X_i are given as a mixture of exponential distributions with survival functions given by

$$\bar{F}_W(x) \quad = \quad \int_0^\infty e^{-\theta x} dF_\Theta(\theta) = f^\star_\Theta(x), \tag{2}$$

and

$$\bar{F}_X(x) \quad = \quad \int_0^\infty e^{-\lambda x} dF_\Lambda(\lambda) = f^\star_\Lambda(x), \tag{3}$$

for $x \geq 0$. This implies that the marginal distributions of W_i and X_i are completely monotone. We refer to Albrecher et al. (2011) for more details on the mixed exponential model and the completely monotone marginal distributions. The general mixed risk model that we consider in this paper is an extension of the risk model described in Albrecher et al. (2011).

This paper is structured as follows: In Section 2, we describe the dependence structure of our risk model. Moments of the aggregate discounted claims are derived in Section 3. Section 4 provides few examples of risk models for which explicit expressions for the moment are given. Numerical examples are provided to illustrate the impact of dependency on the moments of discounted aggregate claims. Section 5 concludes the paper.

2. The Dependence Structure

In this section, a description of the dependence between the different components of our model is provided. For a given n and under our conditional exponential model, the joint conditional survival function of $W_1, W_2, \cdots, W_n, X_1, X_2 \cdots, X_n$ is given by

$$\Pr\left(W_1 \geq t_1, \cdots, W_n \geq t_n, X_1 \geq s_1, \cdots, X_n \geq s_n \mid \Theta = \theta, \Lambda = \lambda\right) = e^{-\theta \sum\limits_{i=1}^{n} t_i} e^{-\lambda \sum\limits_{i=1}^{n} s_i},$$

for $n \in \{2, 3, \cdots\}$, $t_1, \cdots, t_n \geq 0$ and $s_1, \cdots, s_n \geq 0$. it is immediate that the multivariate survival function of $W_1, W_2, \cdots, W_n, X_1, X_2 \cdots, X_n$ could be expressed in terms of the bivariate Laplace transform $f^{\star}_{\Theta,\Lambda}$ such that

$$\begin{aligned} \bar{F}_{W_1,\cdots,W_n,X_1,\cdots,X_n}\left(t_1, \cdots, t_n, s_1, \cdots, s_n\right) &= \int_0^{\infty} \int_0^{\infty} e^{-\theta \sum\limits_{i=1}^{n} t_i} e^{-\lambda \sum\limits_{i=1}^{n} s_i} dF_{\Theta,\Lambda}(\theta, \lambda) \\ &= f^{\star}_{\Theta,\Lambda}\left(\sum_{i=1}^{n} t_i, \sum_{i=1}^{n} s_i\right). \end{aligned} \tag{4}$$

On the other hand, according to Sklar's theorem for survival functions, see e.g., Sklar (1959), the joint distribution of the tail of $W_1, \cdots, W_n, X_1, \cdots, X_n$ can be written as a function of the marginal survival functions $\bar{F}_{W_i}$, $\bar{F}_{X_i}$, $i = 1, \cdots, n$, and the copula C describing the dependence structure as follows

$$\bar{F}_{W_1,\cdots,W_n,X_1,\cdots,X_n}\left(t_1, \cdots, t_n, s_1, \cdots, s_n\right) = C\left(\bar{F}_{W_1}(t_1), \cdots, \bar{F}_{W_n}(t_n), \bar{F}_{X_1}(s_1), \cdots, \bar{F}_{X_n}(s_n)\right),$$

for $n \in \{2, 3, \cdots\}$, $t_1, \cdots, t_n \geq 0$ and $s_1, \cdots, s_n \geq 0$. By combining (2), (3) and (4) with the last expression, one deduces that for $(u_1, \cdots, u_n, v_1, \cdots, v_n) \in [0, 1]^{2n}$

$$C(u_1, \cdots, u_n, v_1, \cdots, v_n) = f^{\star}_{\Theta,\Lambda}\left(\sum_{i=1}^{n} f^{\star -1}_{\Theta}(u_i), \sum_{i=1}^{n} f^{\star -1}_{\Lambda}(v_i)\right). \tag{5}$$

According to (4), the bivariate survival function of (W_i, X_i), for $i = 1, \cdots, n$, is given by

$$\bar{F}_{W_i,X_i}(t, s) = f^{\star}_{\Theta,\Lambda}(t, s), \tag{6}$$

for $t \geq 0$ and $s \geq 0$. Hence, using Sklar's theorem, the dependency relation between W_i and X_i is generated by a copula C_{12} given by

$$C_{12}(u, v) = f^{\star}_{\Theta,\Lambda}\left(f^{\star -1}_{\Theta}(u), f^{\star -1}_{\Lambda}(v)\right), \tag{7}$$

for $(u, v) \in [0, 1]^2$. Otherwise, it is clear from (4) that the multivariate survival function of $(W_1, \cdots, W_n)$ is given by

$$\bar{F}_{W_1,\cdots,W_n}(t_1, \cdots, t_n) = f_\Theta^\star\left(\sum_{i=1}^n t_i\right), \tag{8}$$

for $t_1, \cdots, t_n \geq 0$. Consequently, an application of Sklar's theorem shows that the joint distribution of the tail of $W_1, \cdots, W_n$ can be written as a function of the marginal survival functions $\bar{F}_{W_i}$, $i = 1, \cdots, n$, and a copula C_1 describing the dependence structure as follows

$$\bar{F}_{W_1,\cdots,W_n}(t_1, \cdots, t_n) = C_1\left(\bar{F}_{W_1}(t_1), \cdots, \bar{F}_{W_n}(t_n)\right).$$

An expression for C_1 is identified and for $(u_1, \cdots, u_n) \in [0, 1]^n$, we obtain

$$C_1(u_1, \cdots, u_n) = f_\Theta^\star\left(\sum_{i=1}^n f_\Theta^{\star\,-1}(u_i)\right). \tag{9}$$

Similarly, the joint distribution of the tail of $X_1, \cdots, X_n$ is given by

$$\bar{F}_{X_1,\cdots,X_n}(t_1, \cdots, t_n) = f_\Lambda^\star\left(\sum_{i=1}^n t_i\right), \tag{10}$$

for $t_1, \cdots, t_n \geq 0$, and using Sklar's theorem yields the following survival copula for the Xs

$$C_2(u_1, \cdots, u_n) = f_\Lambda^\star\left(\sum_{i=1}^n f_\Lambda^{\star\,-1}(u_i)\right), \tag{11}$$

for $(u_1, \cdots, u_n) \in [0, 1]^n$. From the expressions for the copulas C_1 and C_2 obtained above, one can identify that these two copulas belong to the large class of Archimedean copulas (e.g., Nelsen 1999) with the corresponding generators $f_\Theta^{\star\,-1}$ and $f_\Lambda^{\star\,-1}$. Note that although the dependence among the claim sizes and among the inter-claim times are described by Archimedean copulas. The dependence between W and X is not restricted to this family of copulas. Moreover, the mixture of exponentials model introduces a positive dependence between the inter-claim times Ws as well as a positive dependence between the amount Xs. First, we recall the following definition

Definition 1. *Let X and Y be random variables. X and Y are positively quadrant dependent (PQD) if for all (x, y) in $\mathbb{R}^2$,*

$$Pr\left[X \leq x, Y \leq y\right] \geq Pr\left[X \leq x\right] Pr\left[Y \leq y\right],$$

or equivalently

$$Pr\left[X > x, Y > y\right] \geq Pr\left[X > x\right] Pr\left[Y > y\right].$$

Proposition 2.1. *Consider the model described by (8) and (10). Then, W_i and W_j (X_i and X_j) are PQD for all $i, j = 1, 2, \cdots$.*

Proof. We refer the reader to Chapter 4 in Joe (1997) for the proof of this proposition. □

Combining (5), (7), (9)and (11), one gets

$$C(u_1, \cdots, u_n, v_1, \cdots, v_n) = C_{12}\Big(C_1(u_1, \cdots, u_n), C_2(v_1, \cdots, v_n)\Big),$$

for $(u_1, \cdots, u_n, v_1, \cdots, v_n) \in [0,1]^{2n}$. Throughout the paper, we suppose that the Laplace transform $f^{\star}_{\Theta,\Lambda}$ exists over a subset $K \times K \subset \mathbb{R}^2$ including a neighborhood of the origin. In the following section, the moments of the rv $\mathcal{Z}(t)$ are derived.

3. Moments of the Discounted Aggregate Claims

In order to find the moments of the discounted aggregate claims, we first derive an expression for the moments generating function (mgf) of the rv $\mathcal{Z}(t)$ under the dependent model introduced in the previous section.

Theorem 3.1. *Consider the discounted aggregate claims under the assumptions of the model in Section 2. Then, for any $t \geq 0$ and $\delta > 0$, the mgf of $\mathcal{Z}(t)$ is given by*

$$M_{\mathcal{Z}(t)}(s) \quad = \quad E\left[\frac{\Lambda - se^{-\delta t}}{\Lambda - s}\right]^{\frac{\Theta}{\delta}}. \tag{12}$$

Proof. Given $\Theta = \theta$ and $\Lambda = \lambda$, the aggregate discounted processes, $\mathcal{Z}(t)$ is a compound Poisson processes with independent subsequent inter-claim times. According to Léveillé et al. (2010), the mgf of $\mathcal{Z}(t)$ given $\Theta = \theta$ and $\Lambda = \lambda$ can be written as

$$\begin{aligned} M_{\mathcal{Z}(t)|\Theta=\theta,\Lambda=\lambda}(s) &= E\left[e^{s\mathcal{Z}(t)} \mid \Theta = \theta, \Lambda = \lambda\right] \\ &= e^{s\theta \int_0^t \left[\frac{e^{-\delta v}}{\lambda - se^{-\delta v}}\right] dv} = \left(\frac{\lambda - se^{-\delta t}}{\lambda - s}\right)^{\frac{\theta}{\delta}}. \end{aligned} \tag{13}$$

Otherwise $M_{\mathcal{Z}(t)}(s) = \int_0^\infty \int_0^\infty M_{\mathcal{Z}(t)|\Theta=\theta,\Lambda=\lambda}(s) dF_{\Theta,\Lambda}(\theta, \lambda)$. Substituting (13) into the last expression yields (12). □

The following theorem provides closed formulas for the higher moments of the discounted aggregate claims $\mathcal{Z}(t)$.

Theorem 3.2. *Consider the discounted aggregate claims under the assumptions of the model in Section 2. Then, for any $t \geq 0$, $n \in \mathbb{N}^\star$ and $\delta > 0$, the $n-$th moment of $\mathcal{Z}(t)$ is given by*

$$E\left[\mathcal{Z}^n(t)\right] \quad = \quad \sum \frac{n!}{k_1!k_2! \cdots k_n!} \bar{a}^k_{\overline{t|}\delta} E\left[\frac{\Theta(\Theta - \delta) \cdots (\Theta - \delta(k-1))}{\Lambda^n}\right], \tag{14}$$

where $\bar{a}_{\overline{t|}\delta} = \frac{1 - e^{-t\delta}}{\delta}$ is the standard actuarial notation and the sum is over all nonnegative integer solutions of the Diophantine equation $k_1 + 2k_2 + \cdots + nk_n = n$, $k := k_1 + k_2 + \cdots + k_n$.

Proof. Conditional on the two rvs Θ and Λ, we have

$$E\left[\mathcal{Z}^n(t)\right] \quad = \quad \int_0^\infty \int_0^\infty E\left[\mathcal{Z}^n(t) \mid \Theta = \theta, \Lambda = \lambda\right] dF_{\Theta,\Lambda}(\theta, \lambda). \tag{15}$$

Taking the $n-$th order derivative of (13) with respect to s and using Faà di Bruno's rule (see Faa di Bruno 1855) yield

$$M^{(n)}_{\mathcal{Z}(t)|\Theta=\theta,\Lambda=\lambda}(s) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} h^{(k)}\left(g(s)\right) \prod_{j=1}^{n} \left(\frac{g^{(j)}(s)}{j!}\right)^{k_j}, \tag{16}$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $k_1 + 2k_2 + \cdots + nk_n = n$, $\quad k := k_1 + k_2 + \cdots + k_n$, $g(s) = \frac{\lambda - se^{-\delta t}}{\lambda - s}$ and $h(s) = s^{\frac{\theta}{\delta}}$. Otherwise, the $k-$th derivatives of g and h are given respectively by

$$g^{(k)}(s) \quad = \quad \lambda(1 - e^{-\delta t}) \frac{k!}{(\lambda - s)^{k+1}}, \tag{17}$$

and

$$h^{(k)}(s) \quad = \quad \frac{\Gamma(\frac{\theta}{\delta} + 1)}{\Gamma(\frac{\theta}{\delta} - k + 1)} s^{\frac{\theta}{\delta} - k}, \tag{18}$$

for $k = 1, \cdots, n$. By substituting (17) and (18) into (16) with $s = 0$, one concludes that

$$\begin{aligned} E\left[\mathcal{Z}^n(t) \mid \Theta = \theta, \Lambda = \lambda\right] &= \frac{1}{\lambda^n} \sum \frac{n!}{k_1! k_2! \cdots k_n!} \left(1 - e^{-\delta t}\right)^k \frac{\Gamma(\frac{\theta}{\delta} + 1)}{\Gamma(\frac{\theta}{\delta} - k + 1)} \\ &= \sum \frac{n!}{k_1! k_2! \cdots k_n!} \left(1 - e^{-\delta t}\right)^k \frac{\frac{\theta}{\delta}\left(\frac{\theta}{\delta} - 1\right) \cdots \left(\frac{\theta}{\delta} - (k-1)\right)}{\lambda^n} \\ &= \sum \frac{n!}{k_1! k_2! \cdots k_n!} \bar{a}^k_{\overline{t}|\delta} \frac{\theta(\theta - \delta) \cdots (\theta - \delta(k-1))}{\lambda^n}. \end{aligned} \tag{19}$$

Finally, substitution of (20) into (15) yields the required result. □

The moments of $\mathcal{Z}(t)$ given in (14) could be simplified and expressed in terms of the expected value of $E\left[\frac{\Theta^l}{\Lambda^n}\right]$. First, we write

$$\frac{\theta}{\delta}\left(\frac{\theta}{\delta} - 1\right) \cdots \left(\frac{\theta}{\delta} - (k-1)\right) = \left(\frac{\theta}{\delta}\right)_k,$$

where $(x)_k$ is the falling factorial. It is known that the falling factorial could be expanded as follows

$$(x)_k = \sum_{l=1}^{k} \begin{bmatrix} k \\ l \end{bmatrix} x^l, \tag{20}$$

where the coefficients $\left[{k \atop l}\right]$ are the Stirling numbers of the first order (see e.g., Ginsburg 1928). Using (20), we find

$$\frac{\theta}{\delta}\left(\frac{\theta}{\delta} - 1\right) \cdots \left(\frac{\theta}{\delta} - (k-1)\right) = \sum_{l=1}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \left(\frac{\theta}{\delta}\right)^l.$$

Thus,

$$E\left[\mathcal{Z}^n(t)\right] \quad = \quad \sum \frac{n!}{k_1! k_2! \cdots k_n!} \bar{a}^k_{\overline{t}|\delta} \sum_{l=1}^{k} \delta^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} E\left[\frac{\Theta^l}{\Lambda^n}\right]. \tag{21}$$

In the rest of the paper, it is assumed that there exist an integer n such that the expected value of $\frac{\Theta^i}{\Lambda^j}$ is finite for positive integers i and j with $i, j \leq n$. Using the previous theorem, we give the explicit expressions of the first two moments of $\mathcal{Z}(t)$.

Corollary 3.1. *For a given time t and a positive constant forces of interest δ, we have*

$$E\left[\mathcal{Z}(t)\right] \quad = \quad \bar{a}_{\overline{t}|\delta} E\left[\frac{\Theta}{\Lambda}\right], \tag{22}$$

and

$$E\left[\mathcal{Z}^2(t)\right] = 2\bar{a}_{\bar{t}|2\delta}E\left[\frac{\Theta}{\Lambda^2}\right] + \bar{a}^2_{\bar{t}|\delta}E\left[\frac{\Theta^2}{\Lambda^2}\right]. \tag{23}$$

Proof. The results follow from Theorem (3.2). When $n = 1$, then $k_1 = k = 1$, which yields (22). When $n = 2$, we find that the nonnegative integer solutions of the equation $k_1 + 2k_2 = 2$ are $(k_1, k_2) = (2, 0)$ or $(0, 1)$ with corresponding values of k being 2 or 1 respectively, we get the required result. □

In the following corollary, we derive expressions for the first two moments of $\mathcal{Z}(t)$ when Θ and Λ are independent.

Corollary 3.2. *If the dependency relation between Θ and Λ is generated by the independence copula then*

$$E\left[\mathcal{Z}(t)\right] = \bar{a}_{\bar{t}|\delta}E\left[\Theta\right]E\left[\frac{1}{\Lambda}\right],$$

and

$$E\left[\mathcal{Z}^2(t)\right] = 2\bar{a}_{\bar{t}|2\delta}E\left[\Theta\right]E\left[\frac{1}{\Lambda^2}\right] + \bar{a}^2_{\bar{t}|\delta}E\left[\Theta^2\right]E\left[\frac{1}{\Lambda^2}\right].$$

Proof. The result follows easily from Corollary (3.1). □

Note that the moments of $\mathcal{Z}(t)$ are given in terms of the expected values of $\frac{\Theta^l}{\Lambda^n}$, for $l, n \in \mathbb{N}^\star \times \mathbb{N}^\star$. According to Cressie et al. (1981), the expression of $E\left[\frac{\Theta^l}{\Lambda^n}\right]$ can be derived from the $M_{\Theta,\Lambda}(t,s)$, the joint mgf of (Θ, Λ). We have

$$E\left[\frac{\Theta^l}{\Lambda^n}\right] = \frac{1}{\Gamma(n)}\int_0^\infty x^{n-1}\lim_{s\to 0}\frac{\partial^l M_{\Theta,\Lambda}(s,-x)}{\partial s^l}dx,$$

where the joint mgf $M_{\Theta,\Lambda}$ is given by

$$M_{\Theta,\Lambda}(s,x) = f^*_{\Theta,\Lambda}(-s,-x) = C_{12}\left(f^*_\Theta(-s), f^*_\Lambda(-x)\right).$$

It follows that

$$E\left[\frac{\Theta^l}{\Lambda^n}\right] = \frac{1}{\Gamma(n)}\int_0^\infty x^{n-1}\lim_{s\to 0}\frac{\partial^l f^*_{\Theta,\Lambda}(-s,x)}{\partial s^l}dx. \tag{24}$$

Application of Faà di Bruno's rule for the $l-$th derivative of $f^*_{\Theta,\Lambda}(-t,s)$ gives

$$\frac{\partial^l M_{\Theta,\Lambda}(s,-x)}{\partial s^l} = \sum \frac{l!}{m_1!m_2!\cdots m_l!}\frac{\partial^m C_{12}\left(f^*_\Theta(-s), f^*_\Lambda(x)\right)}{\partial u^m}\prod_{j=1}^{l}\left(\frac{\partial^j f^*_\Theta(-s)}{\partial s^j}\frac{1}{j!}\right)^{m_j},$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $m_1 + 2m_2 + \cdots + lm_l = l$, $m := m_1 + m_2 + \cdots + m_l$. It follows that

$$E\left[\frac{\Theta^l}{\Lambda^n}\right] = \frac{1}{\Gamma(n)}\sum \frac{l!}{m_1!m_2!\cdots m_l!}\prod_{j=1}^{l}\left(\frac{E\left[\Theta^j\right]}{j!}\right)^{m_j}\int_0^\infty x^{n-1}\frac{\partial^m C_{12}\left(1, f^*_\Lambda(x)\right)}{\partial u^m}dx. \tag{25}$$

4. Examples

In the previous section, a general formula for the moments of $\mathcal{Z}(t)$ is derived. In order to illustrate our findings and to discuss further features of our risk model, we provide some examples when additional assumptions on the marginal distributions and the copulas are added. For each example, first the joint Laplace distribution of the mixing distribution $F_{\Theta,\Lambda}$ is specified then the expressions of the copulas C_1, C_2 and C_{12} are identified. Applying our closed-form, the moments of $\mathcal{Z}(t)$ are given for these specific models. Some numerical illustrations are provided in order to stress the impact of dependence between different components of the risk models on the distribution of the discounted aggregated amount of claims.

4.1. Clayton Copula with Pareto Claims and Inter-Claim Times

Assume that the mixing random vector (Θ, Λ) has a bivariate Gamma distribution with a Laplace transform $f^{\star}_{\Theta,\Lambda}$ defined by

$$f^{\star}_{\Theta,\Lambda}(s,x) = \left[(1+as)^{\tilde{\alpha}_1} + (1+bx)^{\tilde{\alpha}_2} - 1\right]^{-\alpha}, \quad s \geq 0, \quad x \geq 0, \tag{26}$$

with $\alpha, a, b, \alpha_1, \alpha_2 > 0$ and $\tilde{\alpha}_i = \frac{\alpha_i}{\alpha}$, $i = 1, 2$. Then, the random variables Θ and Λ are distributed as gamma distributions, $\Theta \sim \mathcal{G}a(\alpha_1, \frac{1}{a})$ and $\Lambda \sim \mathcal{G}a(\alpha_2, \frac{1}{b})$. Also, from (2) and (3), the claim amounts X_i and the inter-claim times W_i, for $i = 1, 2, \cdots$, follow Pareto distributions $X \sim \mathcal{P}a(\alpha_2, \frac{1}{b})$ and $W \sim \mathcal{P}a(\alpha_1, \frac{1}{a})$. From (9) and (11), we identify the copulas C_1 and C_2 to be Clayton copulas with parameters $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$, respectively. We have

$$C_1(u_1, \cdots, u_n) = \left[u_1^{\frac{-1}{\alpha_1}} + \cdots + u_n^{\frac{-1}{\alpha_1}} - (n-1)\right]^{-\alpha_1},$$

and

$$C_2(u_1, \cdots, u_n) = \left[u_1^{\frac{-1}{\alpha_2}} + \cdots + u_n^{\frac{-1}{\alpha_2}} - (n-1)\right]^{-\alpha_2},$$

for $(u_1, \cdots, u_n) \in [0,1]^n$. The Clayton copula is first introduced by Clayton (1978). The dependence between the Clayton copula parameter and Kendall's tau rank measure, τ_i, is given by (see e.g., Joe 1997 and Nelsen 1999):

$$\tau_i = \frac{1}{1 + 2\alpha_i}, \quad i = 1, 2. \tag{27}$$

This suggests that the Clayton copula does not allow for negative dependence. If $\alpha_i \to \infty$, $i = 1, 2$, then the marginal distributions become independent, when $\alpha_i = 0$, $i = 1, 2$, the Clayton copula approximates the Fréchet–Hoeffding upper bound.

From (7), the joint copula C_{12} is also a Clayton copula with a parameter $\frac{1}{\alpha}$ and we have

$$C_{12}(u, v) = \left[u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1\right]^{-\alpha},$$

for $(u, v) \in [0,1]^2$. Let τ_{12} be the Kendall's tau dependence measure for the copula C_{12}. It follows that

$$\tau_{12} = \frac{1}{1 + 2\alpha}. \tag{28}$$

The following corollary gives the expressions of the first two moments of $\mathcal{Z}(t)$ for this model.

Corollary 4.1. *For a given horizon t and a positive constant forces of real interest δ, we have*

$$E\left[\mathcal{Z}(t)\right] = \frac{a\alpha_1}{b\left(\tilde{\alpha}_2(\alpha+1)-1\right)}\bar{a}_{\overline{t}|\delta},$$

for $\tilde{\alpha}_2 \geq \frac{1}{1+\alpha}$, and

$$\begin{aligned} E\left[\mathcal{Z}^2(t)\right] &= \frac{2a\alpha_1}{b^2\left(\tilde{\alpha}_2(\alpha+1)-1\right)\left(\tilde{\alpha}_2(\alpha+1)-2\right)}\bar{a}_{\overline{t}|2\delta} \\ &+\frac{a^2}{b^2}\left[\frac{\alpha_1(1-\tilde{\alpha}_1)}{\left(\tilde{\alpha}_2(\alpha+1)-1\right)\left(\tilde{\alpha}_2(\alpha+1)-2\right)}+\frac{\alpha_1\tilde{\alpha}_1(1+\alpha)}{\left(\tilde{\alpha}_2(\alpha+2)-1\right)\left(\tilde{\alpha}_2(\alpha+2)-2\right)}\right]\bar{a}^2_{\overline{t}|\delta}, \end{aligned}$$

for $\tilde{\alpha}_1 \geq \frac{1}{1+\alpha}$.

Proof. We have from (4.1)

$$\lim_{s\to 0}\frac{\partial f^*_{\Theta,\Lambda}(-s,x)}{\partial s} = a\alpha_1\left[1+bx\right]^{-\tilde{\alpha}_2(1+\alpha)}, \tag{29}$$

and

$$\lim_{s\to 0}\frac{\partial^2 f^*_{\Theta,\Lambda}(-s,x)}{\partial s^2} = a^2\left[\alpha_1(1-\tilde{\alpha_1})\left(1+bx\right)^{-\tilde{\alpha_2}(1+\alpha)}+\alpha_1\tilde{\alpha_1}(1+\alpha)\left(1+bx\right)^{-\tilde{\alpha_2}(2+\alpha)}\right]. \tag{30}$$

Let $I(n,\alpha,b)$ be defined as

$$I(n,\alpha,b) = \int_0^\infty s^{n-1}(1+bs)^{-\alpha}ds, \qquad n\in\mathbb{N}^\star, \quad \alpha>0.$$

Set $x=(1+bs)^{-1}$, the integral becomes

$$I(n,\alpha,b) = \frac{1}{b^n}\int_0^1 x^{\alpha-n-1}(1-x)^{n-1}dx = \frac{\Gamma(n)\Gamma(\alpha-n)}{b^n\Gamma(\alpha)}, \tag{31}$$

for $\alpha>n$. Combination of (24), (29) and (31) yields

$$E\left[\frac{\Theta}{\Lambda}\right] = \frac{a\alpha_1}{\Gamma(1)}I\left(1,\tilde{\alpha}_2(\alpha+1),b\right) = \frac{a\alpha_1}{b\left(\tilde{\alpha}_2(\alpha+1)-1\right)}.$$

Substitution of (29) into (24) and use of (31) gives

$$E\left[\frac{\Theta}{\Lambda^2}\right] = \frac{a\alpha_1}{\Gamma(2)}I\left(2,\tilde{\alpha}_2(\alpha+1),b\right) = \frac{a\alpha_1}{b^2\left(\tilde{\alpha}_2(\alpha+1)-1\right)\left(\tilde{\alpha}_2(\alpha+1)-2\right)}.$$

Similarly, susbtitution of (30) into (24) and use of (31) gives

$$\begin{aligned} E\left[\frac{\Theta^2}{\Lambda^2}\right] &= \frac{a^2\alpha_1(1-\tilde{\alpha})}{\Gamma(2)}I\left(2,\tilde{\alpha}_2(\alpha+1),b\right)+\frac{a^2\alpha_1\tilde{\alpha}_1(1+\alpha)}{\Gamma(2)}I\left(2,\tilde{\alpha}_2(\alpha+2),b\right), \\ &= \frac{a^2}{b^2}\left[\frac{\alpha_1(1-\tilde{\alpha}_1)}{\left(\tilde{\alpha}_2(\alpha+1)-1\right)\left(\tilde{\alpha}_2(\alpha+1)-2\right)}+\frac{\alpha_1\tilde{\alpha}_1(1+\alpha)}{\left(\tilde{\alpha}_2(\alpha+2)-1\right)\left(\tilde{\alpha}_2(\alpha+2)-2\right)}\right]. \end{aligned}$$

Finally, we find the expressions for $E\left[\mathcal{Z}\right]$ and $E\left[\mathcal{Z}^2(t)\right]$ by applying the Corollary (3.1). □

Corollary 4.2. *For the special case $\alpha_1 = \alpha_2 = \alpha$, we have*

$$E\left[\mathcal{Z}(t)\right] = \frac{a}{b}\bar{a}_{\overline{t}|\delta}, \tag{32}$$

and

$$E\left[\mathcal{Z}^2(t)\right] = \frac{2a}{b^2(\alpha-1)}\bar{a}_{\overline{t}|2\delta} + \frac{a^2}{b^2}\bar{a}^2_{\overline{t}|\delta}. \tag{33}$$

Proof. The result follows directly from Corollary (4.1). □

4.2. Lomax Copula with Pareto Marginal Distributions

In the previous example and for the special case $\alpha_1 = \alpha_2 = \alpha$, we have

$$f^{\star}_{\Theta,\Lambda}(s,x) = (1 + as + bx)^{-\alpha}, \quad s \geq 0, \quad x \geq 0.$$

This specification of the joint Laplace transform leads to the Clayton copula model with the same parameter for the copulas C_1, C_2 and C_{12}. It is possible to modify this model in order to include more flexibility in the model. In this example, it is assumed that the random vector (Θ, Λ) has a bivariate Gamma distribution with the following Laplace transform

$$f^{\star}_{\Theta,\Lambda}(s,x) = (1 + as + bx + csx)^{-\alpha}, \quad s \geq 0, \quad x \geq 0, \tag{34}$$

with $c \geq 0$. The extra parameter c introduces more flexible dependence between the mixing distributions and between the Xs and Ws. For example, it is possible to obtain the independence between Θ and Λ which implies that W and X are independent when $c = ab$. The univariate Laplace transforms are given by

$$f^{\star}_{\Theta}(s) = (1 + as)^{-\alpha},$$

and

$$f^{\star}_{\Lambda}(x) = (1 + bx)^{-\alpha}.$$

It follows that the copulas C_1 and C_2 are Clayton copulas with dependence parameter $\frac{1}{\alpha}$. The joint survival copula of (W, X) is given by

$$\begin{aligned} C_{12}(u,v) &= f^{\star}_{\Theta,\Lambda}\left(a^{-1}(u^{\frac{-1}{\alpha}} - 1), b^{-1}(v^{\frac{-1}{\alpha}} - 1)\right) \\ &= \left(u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1 + \tfrac{c}{ab}\left(u^{\frac{-1}{\alpha}} - 1\right)\left(v^{\frac{-1}{\alpha}} - 1\right)\right)^{-\alpha} \\ &= uv\left(u^{\frac{1}{\alpha}} + v^{\frac{1}{\alpha}} - u^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}} + \tfrac{c}{ab}u^{\frac{1}{\alpha}}v^{\frac{1}{\alpha}}\left(u^{\frac{-1}{\alpha}} - 1\right)\left(v^{\frac{-1}{\alpha}} - 1\right)\right)^{-\alpha} \\ &= uv\left(1 - \gamma(1 - u^{\frac{1}{\alpha}})(1 - v^{\frac{1}{\alpha}})\right)^{-\alpha}, \end{aligned} \tag{35}$$

which is the Lomax copula defined in Fang et al. (2000) with Kendall's tau, τ_{12}, given by (see e.g., Fang et al. 2000):

$$\tau_{12} = \frac{2\alpha\gamma}{(2\alpha+1)^2}\sum_{k=0}^{\infty}\frac{k!\gamma^k}{(2\alpha+2)_k}, \tag{36}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$, and $(a)_0 = 1$ where a is a real number (See e.g., Erdélyi et al. 1953). Some properties of the family of copulas in (35) are the following:

- when $c = ab$, $(\gamma = 0)$, $C_{12}(uv) = uv$ corresponds to the case of independence.
- as $\alpha = 1$, C_{12} in (35) becomes $C_{12}(u,v) = \frac{uv}{1-\gamma(1-u)(1-v)}$, which is the Ali-Mikhail-Haq (AMH) copula.

- when $c = 0$, $(\gamma = 1)$, $C_{12}(u, v) = \left(u^{-\frac{1}{\alpha}} + v^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}$ is the Clayton's copula.

Note that from (8) and (10), the joint survival function of $(W_1, W_2, \cdots, W_n)$ and $(X_1, X_2, \cdots, X_n)$ can then be written, for $x_i \geq 0$, $i = 1, \cdots, n$, as

$$\bar{F}_{W_1,\cdots,W_n}(s_1, \cdots, s_n) = \left(1 + a\sum_{i=1}^{n} s_i\right)^{-\alpha}, \tag{37}$$

and

$$\bar{F}_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \left(1 + b\sum_{i=1}^{n} x_i\right)^{-\alpha}, \tag{38}$$

which are the joint survival function of a Pareto II distribution proposed by Arnold (1983, 2015).

The following corollary gives the expressions of the first two moments of $\mathcal{Z}(t)$ for this model.

Corollary 4.3. *For a given time $t \geq 0$ and a positive constant forces of real interest δ, we have*

$$E\left[\mathcal{Z}(t)\right] = \left(\frac{a}{b} + \frac{c}{b^2(\alpha - 1)}\right)\bar{a}_{\overline{t}|\delta},$$

for $\alpha > 1$, and

$$E\left[\mathcal{Z}^2(t)\right] = 2\left(\frac{ab\alpha + 2(c - ab)}{b^3(\alpha - 1)(\alpha - 2)}\right)\bar{a}_{\overline{t}|2\delta} + \left(\frac{a^2}{b^2} + \frac{4ac}{b^3(\alpha - 1)} + \frac{6c^2}{b^4(\alpha - 1)(\alpha - 2)}\right)\bar{a}^2_{\overline{t}|\delta},$$

for $\alpha > 2$.

Proof. Use of (24) and (34), show that

$$\begin{aligned} E\left[\frac{\Theta^l}{\Lambda^n}\right] &= \tfrac{\Gamma(\alpha+l)}{\Gamma(n)\Gamma(\alpha)} \textstyle\int_0^\infty x^{n-1}(a + cx)^l(1 + bx)^{-(\alpha+l)}dx \\ &= \frac{\Gamma(\alpha + l)}{\Gamma(n)\Gamma(\alpha)} \textstyle\sum_{j=0}^{l} \binom{l}{j} a^{l-j}c^j I(n + j, \alpha + l, b), \end{aligned} \tag{39}$$

where $I(n, \alpha, b) = \int_0^\infty x^{n-1}(1 + bx)^{-\alpha}dx$. With the help of (31) and (39), one gets

$$E\left[\frac{\Theta}{\Lambda}\right] = \alpha\left[aI(1, \alpha + 1, b) + cI(2, \alpha + 1, b)\right] = \frac{a}{b} + \frac{c}{b^2(\alpha - 1)},$$

$$E\left[\frac{\Theta}{\Lambda^2}\right] = \alpha\left[aI(2, \alpha + 1, b) + cI(3, \alpha + 1, b)\right] = \frac{ab\alpha + 2(c - ab)}{b^3(\alpha - 1)(\alpha - 2)},$$

and

$$\begin{aligned} E\left[\frac{\Theta^2}{\Lambda^2}\right] &= \alpha(\alpha + 1)\left[a^2I(2, \alpha + 2, b) + 2acI(3, \alpha + 2, b) + c^2I(4, \alpha + 2, b)\right] \\ &= \frac{a^2}{b^2} + \frac{4ac}{b^3(\alpha - 1)} + \frac{6c^2}{b^4(\alpha - 1)(\alpha - 2)}. \end{aligned}$$

Applying corollary (3.1), we obtain expressions for the first two moments $E\left[\mathcal{Z}(t)\right]$ and $E\left[\mathcal{Z}^2(t)\right]$. □

4.3. Lomax Copulas and Mixed Exponential-Negative Binomial Marginal Distributions

The next model that we consider in our examples is the mixed exponential-Negative Binomial marginal distributions with Lomax copulas. For this purpose it is assumed that (Θ, Λ) has a bivariate shifted Negative Binomial distribution (see e.g., Marshall and Olkin 1988), the Laplace transform of (Θ, Λ) is defined by

$$f^{\star}_{\Theta,\Lambda}(s,x) = \left(\frac{p}{e^{s+x}-q}\right)^{\alpha}, \quad s\,, x \geq 0, \tag{40}$$

where $\alpha > 0$, $0 < p < 1$ and $q = 1 - p$. Then, the random variables Θ and Λ are distributed as shifted Negative Binomial distributions $\Theta \sim \mathcal{NB}(p, \alpha)$ and $\Lambda \sim \mathcal{NB}(p, \alpha)$. With the help of (8), the multivariate survival function of $(W_1, W_2, \cdots, W_n)$ can be written, for $s_i \geq 0$, $i = 1, \cdots, n$, as

$$\bar{F}_{W_1,\cdots,W_n}(s_1,\cdots,s_n) = \left(\frac{p}{e^{\sum\limits_{i=1}^{n} s_i} - q}\right)^{\alpha}. \tag{41}$$

Then, the marginal survival functions of W_i is given, for $s \geq 0$, by

$$\bar{F}_{W_i}(s) = \left(\frac{p}{e^{s}-q}\right)^{\alpha}, \quad i = 1, \cdots, n. \tag{42}$$

The corresponding copula takes the form

$$C_1(u_1,\cdots,u_n) = \left(\frac{p}{\prod\limits_{i=1}^{n}\left(p u_i^{\frac{-1}{\alpha}} + q\right) - q}\right)^{\alpha}, \tag{43}$$

for $(u_1, \cdots, u_n) \in [0,1]^n$. Similarly, the joint survival function of $(X_1, X_2, \cdots, X_n)$ can be written, for $x_i \geq 0$, $i = 1, \cdots, n$, as

$$\bar{F}_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \left(\frac{p}{e^{\sum\limits_{i=1}^{n} x_i} - q}\right)^{\alpha}. \tag{44}$$

The marginal survival functions of X_i is given by

$$\bar{F}_{X_i}(x) = \left(\frac{p}{e^{x}-q}\right)^{\alpha}, \quad i = 1, \cdots, n, \tag{45}$$

for $x \geq 0$ and $i = 1, \cdots, n$. The corresponding dependence structure takes the form

$$C_2(u_1,\cdots,u_n) = \left(\frac{p}{\prod\limits_{i=1}^{n}\left(p u_i^{\frac{-1}{\alpha}} + q\right) - q}\right)^{\alpha}. \tag{46}$$

Note that the marginal survival functions of W_i and X_i, $i = 1, \cdots, n$, in (42) and (45) correspond to the survival function of the univariate mixed exponential-geometric distribution introduced in Adamidis and Loukas (1998). It is useful to note that the mixed exponential-geometric distribution is completely monotone (see Marshall and Olkin 1988). The copulas C_1 and C_2 in (43) and (46) are multivariate shifted negative binomial copulas presented in Joe (2014).

The joint survival function of the bivariate random vector (W_i, X_i) is given by

$$\bar{F}_{W_i,X_i}(s,x) = \left(\frac{p}{e^{s+x}-q}\right)^{\alpha}, \quad s,\, x \geq 0,$$

for $i = 1, \cdots, n$. Then, the corresponding dependence structure is the copula C_{12} given by

$$\begin{aligned} C_{12}(u_1,u_2) &= \left(\frac{p}{(q+pu_1^{-\frac{1}{\alpha}})(q+pu_2^{-\frac{1}{\alpha}})-q}\right)^{\alpha} \\ &= \left(\frac{pu_1^{\frac{1}{\alpha}}u_2^{\frac{1}{\alpha}}}{(qu_1^{\frac{1}{\alpha}}+p)(qu_2^{\frac{1}{\alpha}}+p)-qu_1^{\frac{1}{\alpha}}u_2^{\frac{1}{\alpha}}}\right)^{\alpha} \\ &= \frac{u_1u_2}{\left(1-q(1-u_1^{\frac{1}{\alpha}})(1-u_2^{\frac{1}{\alpha}})\right)^{\alpha}}, \end{aligned} \tag{47}$$

which corresponds to the Lomax copula.

We now state a Corollary for calculating the first an second moments of the discounted aggregate renewal claims.

Corollary 4.4. *For a positive constant forces of real interest δ:*

$$E\left[\mathcal{Z}(t)\right] = \bar{a}_{\bar{t}|\delta}, \tag{48}$$

and

$$E\left[\mathcal{Z}^2(t)\right] = \bar{a}^2_{\bar{t}|\delta} + 2\left(\frac{p}{q}\right)^{\alpha} B(q;\alpha,1-\alpha)\bar{a}_{\bar{t}|2\delta}, \tag{49}$$

where $B(z;\alpha,\beta) = \int_0^z u^{\alpha-1}(1-u)^{\beta-1}du$ is the incomplete Beta function.

Proof. From elementary calculus, one gets from (40)

$$\lim_{s\to 0}\frac{\partial f^{\star}_{\Theta,\Lambda}(-s,x)}{\partial s} = \alpha p^{\alpha}\frac{e^x}{(e^x-q)^{\alpha+1}}. \tag{50}$$

Substituting the last expression into (24) with ($n = l = 1$) yields $E\left[\frac{\Theta}{\Lambda}\right] = 1$. Combining this with Corollary (3.1), one gets (48). Otherwise, we get from (24) with ($n = 2$ and $l = 1$)

$$\begin{aligned} E\left[\frac{\Theta}{\Lambda^2}\right] &= \alpha p^{\alpha}\int_0^{\infty} x\frac{e^x}{(e^x-q)^{\alpha+1}}dx = p^{\alpha}\int_0^{\infty}\frac{1}{(e^x-q)^{\alpha}}dx \\ &= \left(\frac{p}{q}\right)^{\alpha}\int_0^q u^{\alpha-1}(1-u)^{-\alpha}du = \left(\frac{p}{q}\right)^{\alpha} B(q;\alpha,1-\alpha), \end{aligned} \tag{51}$$

where $B(z;\alpha,\beta) = \int_0^z u^{\alpha-1}(1-u)^{\beta-1}du$ is the incomplete Beta function. Otherwise, $\lim_{s\to 0}\frac{\partial^2 f^{\star}_{\Theta,\Lambda}(-s,x)}{\partial^2 s} = \alpha p^{\alpha}\frac{qe^x+\alpha e^{2x}}{(e^x-q)^{\alpha+2}}$. Substituting the last expression into (24) with ($n = 2$ and $l = 2$), one gets

$$E\left[\frac{\Theta^2}{\Lambda^2}\right] = \alpha q p^{\alpha}\int_0^{\infty}\frac{xe^x}{(e^x-q)^{\alpha+2}}dx + \alpha^2 p^{\alpha}\int_0^{\infty}\frac{xe^{2x}}{(e^x-q)^{\alpha+2}}dx. \tag{52}$$

Otherwise, integration by parts gives

$$\begin{aligned} \int_0^\infty \frac{xe^x}{(e^x-q)^{\alpha+2}}dx &= \frac{1}{\alpha+1}\int_0^\infty \frac{1}{(e^x-q)^{\alpha+1}}dx \\ &= \frac{1}{\alpha+1}\frac{1}{q^{\alpha+1}}B(q;\alpha+1,-\alpha). \end{aligned} \tag{53}$$

Similarly, integrating by parts

$$\begin{aligned} \int_0^\infty \frac{xe^{2x}}{(e^x-q)^{\alpha+2}}dx &= \frac{1}{\alpha+1}\int_0^\infty \frac{e^x+xe^x}{(e^x-q)^{\alpha+1}}dx \\ &= \frac{1}{\alpha+1}\left(\frac{1}{\alpha p^\alpha}+\frac{1}{\alpha}\frac{1}{q^\alpha}B(q;\alpha,-\alpha+1)\right). \end{aligned} \tag{54}$$

Hence, through (52), (53) and (54), we obtain

$$E\left[\frac{\Theta^2}{\Lambda^2}\right] = \frac{\alpha}{(\alpha+1)}+\frac{\alpha p^\alpha}{(\alpha+1)q^\alpha}\left(B(q;\alpha+1,-\alpha)+B(q;\alpha,1-\alpha)\right)=1.$$

Finally, we combine the last expression with (51) and Corollary (3.1) to obtain (49). □

Note that if $\alpha=1$, the copula C_{12} in (48) reduces to the AMH copula with Kendall's tau, τ_{12}, given by (see e.g., Nelsen 1999)

$$\tau_{12} = \frac{3q-2}{3q}-\frac{2(1-q)^2 ln(1-q)}{3q^2}.$$

For this special case, we obtain $E\left[\mathcal{Z}(t)\right]=\bar{a}_{\bar{t}|\delta}$, and $E\left[\mathcal{Z}^2(t)\right]=\bar{a}^2_{\bar{t}|\delta}-2(\frac{p}{q})log(p)\bar{a}_{\bar{t}|2\delta}$.

4.4. Numerical Illustrations

In this subsection, we present numerical examples to illustrate how the distribution of the discounted renewal aggregate claims behaves when we change the dependency parameters. The computations provided are related to the general case of Clayton copulas. For the discounted aggregate amount of claims, as in Section 4.1, we assume that the force of interest is fixed at the value of $\delta=5\%$ and we set $a=1$ and $b=0.2$. The sensitivity analysis is done by varying Kendall's tau dependence measures τ_i, $i=1,2$ and τ_{12} given by (27) and (28) respectively. In order to investigate the impact of the dependence structure on the distribution of $\mathcal{Z}(t)$, we compute the mean $E[\mathcal{Z}(t)]$, the standard deviation $SD[\mathcal{Z}(t)]$, the skewness $Skew[\mathcal{Z}(t)]$ and the kurtosis $Kurt[\mathcal{Z}(t)]$ using different values for the Kendall tau's of the copulas C_{12}, C_1 and C_2. Both the expressions of $E[\mathcal{Z}(t)]$ and $SD[\mathcal{Z}(t)]$ are given in Section 4.1. The third and the fourth moments are computed numerically. Using the software Matlab, we evaluate the integral in (25) then we use the closed form in (3.1) for $n=3$ and 4. The results are presented using different time horizons where t is set to be $110, 100$ and ∞.

Tables 1–3 display the obtained results. For a fixed t, τ_1 and τ_{12}, increasing the dependence between the claims leads to a higher level of risk, i.e., large values of $E\left[\mathcal{Z}(t)\right]$ and $SD\left[\mathcal{Z}(t)\right]$. On the other hand, increasing the dependence between the inter-claim times reduces the level of risk for the whole portfolio. We also notice that both the expected value and volatility of the aggregate discounted claims decrease as τ_{12} increases. A strong positive dependence between the inter-claim times and the claim sizes means that the portfolio generates either large and less frequent losses or small and very frequent losses. This leads to a small value of $E\left[\mathcal{Z}(t)\right]$ and less volatile $\mathcal{Z}(t)$. Increasing the dependence parameter τ_{12} or τ_1 generates longer and fatter right tails. Decreasing τ_2 has the same impact on the shape of the tails as increasing the Kendall's tau measures of the copulas C_{12} and C_1.

Table 1. Impact of changing τ_{12} on the distribution of $\mathcal{Z}(t)$ with $\tau_1 = 0.8$ and $\tau_2 = 0.3$.

$E[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.45	0.2937	2.3694	5.9813	6.0219
	0.55	0.2020	1.6294	4.1132	4.1411
	0.65	0.1355	1.0930	2.7591	2.7778
	0.75	0.0851	0.6863	1.7324	1.7442
$SD[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.45	0.8740	4.0281	9.2282	9.2864
	0.55	0.7306	3.4214	7.8775	7.9274
	0.65	0.5970	2.7841	6.4017	6.4422
	0.75	0.4650	2.0910	4.7519	4.7816
$Skew[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.45	1.8969	1.2997	1.3984	1.3991
	0.55	2.5299	2.0445	2.2137	2.2148
	0.65	3.3014	3.0643	3.3997	3.4018
	0.75	4.3794	4.9922	5.8956	5.9010
$Kurt[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.45	3.9901	2.1244	1.9211	1.9199
	0.55	6.7845	5.3719	5.7602	5.7624
	0.65	11.5174	13.0582	15.3542	15.3674
	0.75	21.2007	39.8072	52.3969	52.4717

Table 2. Impact of changing τ_1 on the distribution of $\mathcal{Z}(t)$ with $\tau_{12} = 0.4$ and $\tau_2 = 0.2$.

$E[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.7	0.2850	2.2995	5.8048	5.8442
	0.75	0.2217	1.7885	4.5148	4.5455
	0.8	0.1663	1.3414	3.3861	3.4091
	0.85	0.1174	0.9469	2.3902	2.4064
$SD[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.7	0.8683	4.0706	9.3752	9.4346
	0.75	0.7748	3.7135	8.6088	8.6637
	0.8	0.6777	3.3068	7.7043	7.7536
	0.85	0.5744	2.8438	6.6520	6.6947
$Skew[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.7	1.9920	1.4973	1.6216	1.6225
	0.75	2.3856	1.7579	1.8353	1.8358
	0.8	2.8825	2.1144	2.1588	2.1592
	0.85	3.5647	2.6220	2.6404	2.6406
$Kurt[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.7	4.1974	1.0229	0.1587	0.1537
	0.75	5.6748	1.3470	0.3437	0.3381
	0.8	7.9450	1.9977	0.7742	0.7675
	0.85	11.7835	3.2307	1.6279	1.6194

Table 3. Impact of changing τ_2 on the distribution of $\mathcal{Z}(t)$ with $\tau_{12} = 0.55$ and $\tau_1 = 0.85$.

$E[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.05	0.0136	0.1094	0.2763	0.2781
	0.15	0.0491	0.3964	1.0006	1.0073
	0.25	0.1033	0.8332	2.1034	2.1176
	0.35	0.1957	1.5792	3.9865	4.0136
$SD[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.05	0.1974	0.9952	2.3387	2.3537
	0.15	0.3730	1.8589	4.3553	4.3833
	0.25	0.5349	2.6167	6.1008	6.1399
	0.35	0.7224	3.4130	7.8788	7.9288
$Skew[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.05	11.4633	9.3207	9.4760	9.4772
	0.15	5.8516	4.7009	4.8002	4.8009
	0.25	3.8518	3.0297	3.1204	3.1211
	0.35	2.5621	1.9254	2.0290	2.0298
$Kurt[\mathcal{Z}(t)]$	τ_{12}	$t = 1$	$t = 10$	$t = 100$	$t = \infty$
	0.05	116.7367	44.8970	32.4352	32.3701
	0.15	31.1083	12.4559	9.3112	9.2946
	0.25	14.0675	6.5740	5.5232	5.5176
	0.35	6.9009	5.2676	5.6800	5.6824

5. Conclusions

In this paper, we derived explicit expressions for the higher moments of the discounted aggregate renewal claims with dependence. Closed expressions for the moments of the aggregate discounted claims are obtained when the claims and the subsequent inter-claim are distributed as Pareto and Mixed exponential-geometric distributions. Numerical examples are given to illustrate the impact of dependency on the moments of the discounted aggregate renewal mixed process.

Since the assumption of constant force of interest is quite restrictive, studying the discounted renewal aggregate claims with a stochastic force of interest would be interesting. A more challenging problem would be the extension of the mixed exponential risk model to incorporate other forms of dependence structure between the model components.

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