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n-Dimensional Laplace Transforms of Occupation Times for Spectrally Negative Lévy Processes

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Abstract: Using a Poisson approach, we find Laplace transforms of joint occupation times over n disjoint intervals for spectrally negative Lévy processes. They generalize previous results for dimension two.

Keywords: spectrally negative Lévy process; occupation time; scale function

1. Introduction

A spectrally negative Lévy process is a stochastic process with stationary independent increments and with sample paths of no positive jumps. It often serves as a surplus process in risk theory. Occupation time also finds applications in risk theory. In the so-called Omega risk model, the Laplace transform of occupation time is associated with the bankruptcy probability; see Gerber et al. [1] for more details.

Due to the Wiener–Hopf factorization and excursion theory, many fluctuation results of the spectrally negative Lévy process can be expressed semi-explicitly in terms of the corresponding scale functions. Expressions of Laplace transforms of occupation times for spectrally negative Lévy processes have been obtained in recent years with different approaches; see, for example, Cai et al. [2], Landriault et al. [3], Loeffen et al. [4] and Li and Palmowski [5].

Using techniques developed in Albrecher et al. [6], in Li and Zhou [7], a Poisson approach is adopted to find joint Laplace transforms for occupation times over two disjoint intervals for general spectrally negative Lévy processes. This approach uses a property of Poisson random measure and can be effectively implemented. With this method, we have also recovered the main results of Loeffen et al. [4] in Kuang and Zhou [8]. The method can also be easily adapted to study occupation times of other stochastic processes, as long as the exit problems are solvable and expressions of the potential measures are available.

In this paper, for a spectrally negative Lévy process, we adopt the Poisson approach to further find joint Laplace transforms of occupation times (up to the first exit times) over n-disjoint subintervals resulting from a partition of a finite interval. Equivalently, we find Laplace transforms for weighted occupation times with step weight functions. The Laplace transforms are expressed in terms of iterated integrals of the scale functions. In particular, they generalize the results in Li and Zhou [7]. For the proofs, we use induction and improve the previous arguments of Li and Zhou [7]. Although our results can also be obtained by solving the integral equations in Li and Palmowski [5], our generic approach can be easily adapted to handle situations not covered by the integral equations of Li and Palmowski [5].

This paper is arranged as follows. In Section 2, we review the basics of spectrally negative Lévy processes that we need for this paper and introduce generalized versions of the scale functions. Section 3 contains the main results with proofs, and Section 4 provides the conclusions.

2. Spectrally Negative Lévy Processes

Let $X = (X_t)_{t\geq 0}$ be a spectrally negative Lévy process, that is, a stochastic process with stationary independent increments and with no positive jumps, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by X. We also assume that X is not the negative of a subordinator. Denote by $\mathbb{P}_x$ the probability law of X given $X_0 = x$, and the corresponding expectation by $\mathbb{E}_x$. Write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. Because of the Lévy process, X allows no positive jumps, and its Laplace transform always exists and is given by

$$\mathbb{E}e^{\lambda X_t} = e^{\psi(\lambda)t},$$

for $\lambda \geq 0$, where

$$\psi(\lambda) = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1_{\{x>-1\}})\pi(\mathrm{d}x),$$

for $\mu \in \mathbb{R}, \sigma \geq 0$ and the σ-finite Lévy measure π on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)}(1 \wedge x^2)\pi(\mathrm{d}x) < \infty$. Furthermore, there exists a function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ defined by

$$\Phi(q) := \sup\{\lambda \geq 0 : \ \psi(\lambda) = q\}, \qquad \text{for} \quad q \geq 0.$$

We first recall the definition of a q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of process X is defined on $[0,\infty)$ as the continuous function with Laplace transform specified by

$$\int_0^\infty e^{-\lambda y} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi(\lambda) - q} \qquad \text{for} \quad \lambda > \Phi(q), \tag{1}$$

with initial value $W^{(q)}(0) := \lim_{x\downarrow 0} W^{(q)}(x)$. The function $W^{(q)}$ is unique, positive and strictly increasing for $x \geq 0$. For convenience, we extend the domain of $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. Write $W = W^{(0)}$ whenever $q = 0$. It is known that $W^{(q)}(0) = 0$ if and only if process X has sample paths of unbounded variation.

Write

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y.$$

The next identities on scale function are first noticed in Loeffen et al. [4]. For $a > 0$,

$$\begin{aligned}(q-p)\int_0^a W^{(p)}(a-y)W^{(q)}(y)\mathrm{d}y &= W^{(q)}(a) - W^{(p)}(a),\\ (q-p)\int_0^a W^{(p)}(a-y)Z^{(q)}(y)\mathrm{d}y &= Z^{(q)}(a) - Z^{(p)}(a).\end{aligned} \tag{2}$$

Many fluctuation results for spectrally negative Lévy processes can be expressed in terms of scale functions; see, for example, Kyprianou [9] and Kuznetsov et al. [10]. We list some of those that are needed in this paper. Define exit times

$$\tau_a^- := \inf\{t > 0 : X_t < a\}, \qquad \text{and} \qquad \tau_a^+ := \inf\{t > 0 : X_t > a\},$$

with the convention $\inf \emptyset = \infty$. For $0 < x < a$ and $q \geq 0$, it is well known that

$$\mathbb{E}_x[e^{-q\tau_a^+}; \tau_a^+ < \tau_0^-] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3}$$

and

$$\mathbb{E}_x[e^{-q\tau_0^-}; \tau_0^- < \tau_a^+] = Z^{(q)}(x) - W^{(q)}(x)\frac{Z^{(q)}(a)}{W^{(q)}(a)}. \tag{4}$$

The following expression is for potential measure of process X killed upon exiting interval $[0, b]$. For $0 < x < b$ and $p \geq 0$,

$$\int_0^\infty \mathbb{P}_x\{t < \tau_b^+ \wedge \tau_0^-, X_t \in \mathrm{d}y\} e^{-pt}\mathrm{d}t = \left(\frac{W^{(p)}(x)W^{(p)}(b-y)}{W^{(p)}(b)} - W^{(p)}(x-y) \right) \mathrm{d}y. \tag{5}$$

In this paper, we generalize scale functions $W^{(q)}$ and $Z^{(q)}$ as follows. For any $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$, $0 = a_0 \leq a_1 \leq \ldots \leq a_m$ and $x \in \mathbb{R}$, write

$$W_{(a_0)}^{(\lambda_0)}(x) := W^{(\lambda_0)}(x), \quad Z_{(a_0)}^{(\lambda_0)}(x) := Z^{(\lambda_0)}(x),$$

and for $n = 1, \ldots, m$

$$W_{(a_0,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_n)}(x) := W_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(x) + (\lambda_n - \lambda_{n-1}) \int_{a_n}^x W^{(\lambda_n)}(x-y) W_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(y)\mathrm{d}y, \tag{6}$$

$$Z_{(a_0,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_n)}(x) := Z_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(x) + (\lambda_n - \lambda_{n-1}) \int_{a_n}^x W^{(\lambda_n)}(x-y) Z_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(y)\mathrm{d}y, \tag{7}$$

where, for $x < a_n$, the integral is understood as 0. Observe from the above definitions that

$$W_{(a_0,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_n)}(x) = W_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(x) \qquad \text{and} \qquad Z_{(a_0,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_n)}(x) = Z_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(x) \tag{8}$$

for $1 \leq j \leq n-1$ and $x \in (-\infty, a_{j+1}]$. In addition, for $0 \leq i \leq n$ and $\lambda_i = \lambda_{i+1} = \ldots = \lambda_n$,

$$W_{(a_0,\ldots,a_i,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_i,\ldots,\lambda_i)} = W_{(a_0,\ldots,a_i)}^{(\lambda_0,\ldots,\lambda_i)} \quad \text{and} \quad Z_{(a_0,\ldots,a_i,\ldots,a_n)}^{(\lambda_0,\ldots,\lambda_i,\ldots,\lambda_i)} = Z_{(a_0,\ldots,a_i)}^{(\lambda_0,\ldots,\lambda_i)}.$$

Using the Poisson approach, Li and Zhou [7] show that for $\lambda_0, \lambda_1 > 0$, $\;0 = a_0 < a_1 < a_2$ and $0 = a_0 < x < a_2$,

$$\mathbb{E}_x\left[e^{-\lambda_0 \int_0^{\tau_{a_2}^+} 1_{X_s \in (0,a_1)}\mathrm{d}s - \lambda_1 \int_0^{\tau_{a_2}^+} 1_{X_s \in (a_1,a_2)}};\ \tau_{a_2}^+ < \tau_0^- \right] = \frac{W_{(0,a_1)}^{(\lambda_0,\lambda_1)}(x)}{W_{(0,a_1)}^{(\lambda_0,\lambda_1)}(a_2)}$$

and

$$\mathbb{E}_x\left[e^{-\lambda_0 \int_0^{\tau_0^-} 1_{X_s \in (0,a_1)}\mathrm{d}s - \lambda_1 \int_0^{\tau_0^-} 1_{X_s \in (a_1,a_2)}};\ \tau_0^- < \tau_{a_2}^+ \right] = Z_{(a_0,a_1)}^{(\lambda_0,\lambda_1)}(x) - \frac{W_{(0,a_1)}^{(\lambda_0,\lambda_1)}(x)}{W_{(0,a_1)}^{(\lambda_0,\lambda_1)}(a_2)} Z_{(0,a_1)}^{(\lambda_0,\lambda_1)}(a_2).$$

We end this section by presenting explicit expressions of the above-mentioned generalized scale functions for two examples.

If X is a standard one-dimensional Brownian motion with scale function

$$W^{(q)}(x) = \frac{1}{\sqrt{2q}}(e^{\sqrt{2q}x} - e^{-\sqrt{2q}x}), \qquad \text{for} \quad x \geq 0.$$

One can easily verify that

$$\begin{aligned} W^{(\lambda_0,\lambda_1)}_{(0,a_1)}(x) = &\frac{\lambda_0 - \lambda_1}{2\sqrt{\lambda_0\lambda_1}} \left(\frac{e^{(\sqrt{2\lambda_0}-\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} - \sqrt{2\lambda_1}} + \frac{e^{-(\sqrt{2\lambda_0}+\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} + \sqrt{2\lambda_1}} \right) e^{\sqrt{2\lambda_1}x} \\ &- \frac{\lambda_0 - \lambda_1}{2\sqrt{\lambda_0\lambda_1}} \left(\frac{e^{(\sqrt{2\lambda_0}+\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} + \sqrt{2\lambda_1}} + \frac{e^{-(\sqrt{2\lambda_0}-\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} - \sqrt{2\lambda_1}} \right) e^{-\sqrt{2\lambda_1}x} \end{aligned} \tag{9}$$

and

$$\begin{aligned} W^{(\lambda_0,\lambda_1,\lambda_2)}_{(0,a_1,a_2)}(x) = &\frac{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)}{2\sqrt{2\lambda_0\lambda_1\lambda_2}} \left(\frac{e^{(\sqrt{2\lambda_0}+\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} + \sqrt{2\lambda_1}} + \frac{e^{-(\sqrt{2\lambda_0}-\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} - \sqrt{2\lambda_1}} \right) \\ &\times \left(\frac{e^{-(\sqrt{2\lambda_1}+\sqrt{2\lambda_2})a_2}}{\sqrt{2\lambda_1} + \sqrt{2\lambda_2}} e^{\sqrt{2\lambda_2}x} - \frac{e^{-(\sqrt{2\lambda_1}-\sqrt{2\lambda_2})a_2}}{\sqrt{2\lambda_1} - \sqrt{2\lambda_2}} e^{-\sqrt{2\lambda_2}x} \right) \\ &- \frac{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)}{2\sqrt{2\lambda_0\lambda_1\lambda_2}} \left(\frac{e^{(\sqrt{2\lambda_0}-\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} - \sqrt{2\lambda_1}} + \frac{e^{-(\sqrt{2\lambda_0}+\sqrt{2\lambda_1})a_1}}{\sqrt{2\lambda_0} + \sqrt{2\lambda_1}} \right) \\ &\times \left(\frac{e^{(\sqrt{2\lambda_1}+\sqrt{2\lambda_2})a_2}}{\sqrt{2\lambda_1} + \sqrt{2\lambda_2}} e^{-\sqrt{2\lambda_2}x} - \frac{e^{(\sqrt{2\lambda_1}-\sqrt{2\lambda_2})a_2}}{\sqrt{2\lambda_1} - \sqrt{2\lambda_2}} e^{\sqrt{2\lambda_2}x} \right). \end{aligned}$$

The corresponding Laplace transforms for occupation times then follow readily from Theorem 1 and Theorem 2.

If X is a compound Poisson process, i.e.,

$$X_t = \mu t - \sum_{i=1}^{N_t} \xi_i, \qquad t > 0,$$

where $\mu > 0$, ξ_i are i.i.d variables, which are exponentially distributed with parameter $\rho > 0$ and N_t is an independent Poisson process with intensity $a > 0$. Then, the Laplace exponent is given by

$$\psi(t) = \mu t - \frac{at}{\rho + t}, \qquad t > 0.$$

For $q \geq 0$, the equation $\psi(t) = q$ has two real solutions $\{-g(q), \Phi(q)\}$ such that

$$g(q) = \frac{1}{2\mu} \left(\sqrt{(a + q - \mu\rho)^2 + 4\mu q\rho} - (a + q - \mu\rho) \right)$$

and

$$\Phi(q) = \frac{1}{2\mu} \left(\sqrt{(a + q - \mu\rho)^2 + 4\mu q\rho} + (a + q - \mu\rho) \right).$$

The scale function is

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \frac{e^{-g(q)x}}{\psi'(-g(q))}, \qquad x \geq 0.$$

Thus, one can easily verify that

$$\begin{aligned}
W^{(\lambda_0,\lambda_1)}_{(0,a_1)}(x) &= \Phi'(\lambda_0)e^{\Phi(\lambda_0)x} + \frac{e^{-g(\lambda_0)x}}{\psi'(-g(\lambda_0))} \\
&+ \frac{(\lambda_1-\lambda_0)\Phi'(\lambda_1)\Phi'(\lambda_0)e^{\Phi(\lambda_0)x}}{\Phi(\lambda_0)-\Phi(\lambda_1)} - (\lambda_1-\lambda_0)\Phi'(\lambda_1)\Phi'(\lambda_0)e^{\Phi(\lambda_1)x}\frac{e^{(\Phi(\lambda_0)-\Phi(\lambda_1))a_1}}{\Phi(\lambda_0)-\Phi(\lambda_1)} \\
&+ \frac{(\lambda_1-\lambda_0)\Phi'(\lambda_0)e^{\Phi(\lambda_0)x}}{\psi'(-g(\lambda_1))(\Phi(\lambda_0)+g(\lambda_1))} - (\lambda_1-\lambda_0)\frac{\Phi'(\lambda_0)e^{-g(\lambda_1)x}}{\psi'(-g(\lambda_1))} \times \frac{e^{(\Phi(\lambda_0)+g(\lambda_1))a_1}}{\Phi(\lambda_0)+g(\lambda_1)} \\
&- \frac{(\lambda_1-\lambda_0)\Phi'(\lambda_1)e^{-g(\lambda_0)x}}{\psi'(-g(\lambda_0))(g(\lambda_0)+\Phi(\lambda_1))} + (\lambda_1-\lambda_0)\frac{\Phi'(\lambda_1)e^{\Phi(\lambda_1)x}}{\psi'(-g(\lambda_0))} \times \frac{e^{-(g(\lambda_0)+\Phi(\lambda_1))a_1}}{g(\lambda_0)+\Phi(\lambda_1)} \\
&- \frac{(\lambda_1-\lambda_0)e^{-g(\lambda_0)x}}{\psi'(-g(\lambda_1))\psi'(-g(\lambda_0))(g(\lambda_0)-g(\lambda_1))} + \frac{(\lambda_1-\lambda_0)e^{-g(\lambda_1)x}}{\psi'(-g(\lambda_1))\psi'(-g(\lambda_0))} \times \frac{e^{-(g(\lambda_0)-g(\lambda_1))a_1}}{g(\lambda_0)-g(\lambda_1)}.
\end{aligned}$$

It is evident that the expression of $W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}$ for general n would be a rather complicated linear combination of exponential functions.

3. Main Results

We first present several auxiliary lemmas that are of independent interest. The next lemma generalizes identity (2).

Lemma 1. *For $0 = a_0 \le a_1 \le \ldots \le a_n \le x$, $\lambda_0, \lambda_1, \ldots, \lambda_n \ge 0$, $q \ge 0$, and $n \in \mathbb{N}$*

$$(q-\lambda_n)\int_{a_n}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y = W^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_n)}(x) - W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(x) \tag{10}$$

and

$$(q-\lambda_n)\int_{a_n}^x W^{(q)}(x-y)Z^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y = Z^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_n)}(x) - Z^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(x). \tag{11}$$

Proof. The proof of identity (11) is similar to that of identity (10). We only prove identity (10).

Applying definition (6), changing the order of integrals, and then, by identity (2), we have

$$\begin{aligned}
&(q-\lambda_k)\int_{a_k}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y \\
&= (q-\lambda_k)\int_{a_k}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y \\
&\quad + (q-\lambda_k)(\lambda_k-\lambda_{k-1})\int_{a_k}^x W^{(q)}(x-y)\mathrm{d}y\int_{a_k}^y W^{(\lambda_k)}(y-z)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(z)\mathrm{d}z \\
&= (q-\lambda_k)\int_{a_k}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y \\
&\quad + (q-\lambda_k)(\lambda_k-\lambda_{k-1})\int_{a_k}^x W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(z)\mathrm{d}z\int_z^x W^{(q)}(x-y)W^{(\lambda_k)}(y-z)\mathrm{d}y \\
&= (q-\lambda_k)\int_{a_k}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y \\
&\quad + (\lambda_k-\lambda_{k-1})\int_{a_k}^x W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(z)\left(W^{(q)}(x-z)-W^{(\lambda_k)}(x-z)\right)\mathrm{d}z \\
&= (q-\lambda_{k-1})\int_{a_k}^x W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y - (\lambda_k-\lambda_{k-1})\int_{a_k}^x W^{(\lambda_k)}(x-y)W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y \\
&= W^{(\lambda_0,\ldots,\lambda_{k-1},q)}_{(a_0,\ldots,a_k)}(x) - W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(x),
\end{aligned}$$

where we use definition (6) again for the last equation. □

It follows from Lemma 1 that, for $a_{n+1} = a_n$,

$$W^{(\lambda_0,\ldots,\lambda_n,\lambda_{n+1})}_{(a_0,\ldots,a_n,a_{n+1})} = W^{(\lambda_0,\ldots,\lambda_{n-1},\lambda_{n+1})}_{(a_0,\ldots,a_{n-1},a_{n+1})} \quad \text{and} \quad Z^{(\lambda_0,\ldots,\lambda_n,\lambda_{n+1})}_{(a_0,\ldots,a_n,a_{n+1})} = Z^{(\lambda_0,\ldots,\lambda_{n-1},\lambda_{n+1})}_{(a_0,\ldots,a_{n-1},a_{n+1})}.$$

Lemma 2. *For any $0 = a_0 \le a_1 \le \ldots \le a_{n+1}$, $\lambda_0, \lambda_1, \ldots, \lambda_n \ge 0$, $q \ge 0$ and $x \in \mathbb{R}$, we have*

$$(q-\lambda_n)\int_{a_n}^{a_{n+1}} W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y = W^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_{n-1},a_n)}(x) - W^{(\lambda_0,\ldots,\lambda_n,q)}_{(a_0,\ldots,a_n,a_{n+1})}(x) \tag{12}$$

and

$$(q-\lambda_n)\int_{a_n}^{a_{n+1}} W^{(q)}(x-y)Z^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y = Z^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_{n-1},a_n)}(x) - Z^{(\lambda_0,\ldots,\lambda_n,q)}_{(a_0,\ldots,a_n,a_{n+1})}(x). \tag{13}$$

Proof. Identities (12) and (13) for $x < a_n$ follow from Equation (8).

For $x \ge a_n$, by definition (6) and Lemma 1, we have

$$\begin{aligned}
&(q-\lambda_n)\int_{a_n}^{a_{n+1}} W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y \\
&= (q-\lambda_n)\int_{a_n}^{x} W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y - (q-\lambda_n)\int_{a_{n+1}}^{x} W^{(q)}(x-y)W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(y)\mathrm{d}y \\
&= W^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_n)}(x) - W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(x) - \left(W^{(\lambda_0,\ldots,\lambda_n,q)}_{(a_0,\ldots,a_{n+1})}(x) - W^{(\lambda_0,\ldots,\lambda_n)}_{(a_0,\ldots,a_n)}(x)\right) \\
&= W^{(\lambda_0,\ldots,\lambda_{n-1},q)}_{(a_0,\ldots,a_n)}(x) - W^{(\lambda_0,\ldots,\lambda_n,q)}_{(a_0,\ldots,a_{n+1})}(x).
\end{aligned}$$

The proof of identity (13) is similar. □

The following result has been first pointed out in Loeffen et al. [4].

Lemma 3. *For any $\lambda_0, \lambda_1 \ge 0$, and $0 \le a \le x \le b, 0 \le y < a$,*

$$\begin{aligned}
&\mathbb{E}_x[e^{-\lambda_1\tau_a^-}W^{(\lambda_0)}(X_{\tau_a^-}-y);\tau_a^- < \tau_b^+] \\
&= W^{(\lambda_0)}(x-y) + (\lambda_1-\lambda_0)\int_a^x W^{(\lambda_1)}(x-z)W^{(\lambda_0)}(z-y)\mathrm{d}z \\
&- \frac{W^{(\lambda_1)}(x-a)}{W^{(\lambda_1)}(b-a)}\left(W^{(\lambda_0)}(b-y) + (\lambda_1-\lambda_0)\int_a^b W^{(\lambda_1)}(b-z)W^{(\lambda_0)}(z-y)\mathrm{d}z\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_x[e^{-\lambda_1\tau_a^-}Z^{(\lambda_0)}(X_{\tau_a^-}-y);\tau_a^- < \tau_b^+] \\
&= Z^{(\lambda_0)}(x-y) + (\lambda_1-\lambda_0)\int_a^x W^{(\lambda_1)}(x-z)Z^{(\lambda_0)}(z-y)\mathrm{d}z \\
&- \frac{W^{(\lambda_1)}(x-a)}{W^{(\lambda_1)}(b-a)}\left(Z^{(\lambda_0)}(b-y) + (\lambda_1-\lambda_0)\int_a^b W^{(\lambda_1)}(b-z)Z^{(\lambda_0)}(z-y)\mathrm{d}z\right).
\end{aligned}$$

Lemma 4. *For any $\lambda_0, \ldots, \lambda_n \ge 0$, $0 = a_0 < a_1 \ldots < a_n$ and $a_{n-1} \le x \le a_n, n \ge 2$,*

$$\begin{aligned}
&\mathbb{E}_x[e^{-\lambda_{n-1}\tau_{a_{n-1}}^-}W^{(\lambda_0,\ldots,\lambda_{n-2})}_{(a_0,\ldots,a_{n-2})}(X_{\tau_{a_{n-1}}^-});\tau_{a_{n-1}}^- < \tau_{a_n}^+] \\
&= W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(x) - \frac{W^{(\lambda_{n-1})}(x-a_{n-1})}{W^{(\lambda_{n-1})}(a_n-a_{n-1})}W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)
\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\mathbb{E}_x[e^{-\lambda_{n-1}\tau_{a_{n-1}}^-}Z_{(a_0,\ldots,a_{n-2})}^{(\lambda_0,\ldots,\lambda_{n-2})}(X_{\tau_{a_{n-1}}^-});\tau_{a_{n-1}}^-<\tau_{a_n}^+]\\&=Z_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(x)-\frac{W^{(\lambda_{n-1})}(x-a_{n-1})}{W^{(\lambda_{n-1})}(a_n-a_{n-1})}Z_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(a_n).\end{aligned}\tag{15}$$

Proof. We only prove identity (15) by induction. The case of $n=2$ follows from Lemma 3. Suppose that identity (15) holds for $n=k$. Then, by Lemma 2, for $n=k+1$, we have

$$\begin{aligned}&\mathbb{E}_x[e^{-\lambda_k\tau_{a_k}^-}Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(X_{\tau_{a_k}^-});\tau_{a_k}^-<\tau_{a_{k+1}}^+]\\&=\mathbb{E}_x[e^{-\lambda_k\tau_{a_k}^-}Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(X_{\tau_{a_k}^-});\tau_{a_k}^-<\tau_{a_{k+1}}^+]\\&-(\lambda_{k-1}-\lambda_{k-2})\int_{a_{k-2}}^{a_{k-1}}Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\mathbb{E}_x[e^{-\lambda_k\tau_{a_k}^-}W^{(\lambda_{k-1})}(X_{\tau_{a_k}^-}-y);\tau_{a_k}^-<\tau_{a_{k+1}}^+]\mathrm{d}y.\end{aligned}\tag{16}$$

Suppose that identity (15) holds for $n=k$, we use definition (7) to expand the right-hand side of the identity (15). By Lemma 3 and the inductive assumption, the above quantity is equal to

$$\begin{aligned}&Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(x)+(\lambda_k-\lambda_{k-1})\int_{a_k}^{x}W^{(\lambda_k)}(x-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(y)\mathrm{d}y\\&-\frac{W^{(\lambda_k)}(x-a_k)}{W^{(\lambda_k)}(a_{k+1}-a_k)}\left(Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(a_{k+1})+(\lambda_k-\lambda_{k-1})\int_{a_k}^{a_{k+1}}W^{(\lambda_k)}(a_{k+1}-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(y)\mathrm{d}y\right)\\&-(\lambda_{k-1}-\lambda_{k-2})\int_{a_{k-2}}^{a_{k-1}}W^{(\lambda_{k-1})}(x-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\mathrm{d}y\\&-(\lambda_k-\lambda_{k-1})(\lambda_{k-1}-\lambda_{k-2})\int_{a_{k-2}}^{a_{k-1}}Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\int_{a_k}^{x}W^{(\lambda_k)}(x-z)W^{(\lambda_{k-1})}(z-y)\mathrm{d}z\mathrm{d}y\\&+\frac{W^{(\lambda_k)}(x-a_k)}{W^{(\lambda_k)}(a_{k+1}-a_k)}(\lambda_{k-1}-\lambda_{k-2})\int_{a_{k-2}}^{a_{k-1}}W^{(\lambda_{k-1})}(a_{k+1}-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\mathrm{d}y\\&+\frac{W^{(\lambda_k)}(x-a_k)}{W^{(\lambda_k)}(a_{k+1}-a_k)}(\lambda_k-\lambda_{k-1})(\lambda_{k-1}-\lambda_{k-2})\\&\times\int_{a_{k-2}}^{a_{k-1}}Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\int_{a_k}^{a_{k+1}}W^{(\lambda_k)}(a_{k+1}-z)W^{(\lambda_{k-1})}(z-y)\mathrm{d}z\mathrm{d}y.\end{aligned}$$

Notice that

$$\begin{aligned}&(\lambda_{k-1}-\lambda_{k-2})\int_{a_{k-2}}^{a_{k-1}}Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\int_{a_k}^{x}W^{(\lambda_k)}(x-z)W^{(\lambda_{k-1})}(z-y)\mathrm{d}z\mathrm{d}y\\&=(\lambda_{k-1}-\lambda_{k-2})\int_{a_k}^{x}W^{(\lambda_k)}(x-z)\int_{a_{k-2}}^{a_{k-1}}W^{(\lambda_{k-1})}(z-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-2})}(y)\mathrm{d}y\mathrm{d}z\\&=\int_{a_k}^{x}W^{(\lambda_k)}(x-y)Z_{(a_0,\ldots,a_{k-2})}^{(\lambda_0,\ldots,\lambda_{k-3},\lambda_{k-1})}(y)\mathrm{d}y-\int_{a_k}^{x}W^{(\lambda_k)}(x-y)Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(y)\mathrm{d}y.\end{aligned}$$

With cancelations the right-hand side of Equation (16) is equal to

$$\begin{aligned}&Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(x)+(\lambda_k-\lambda_{k-1})\int_{a_k}^{x}W^{(\lambda_k)}(x-y)Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(y)\mathrm{d}y\\&-\frac{W^{(\lambda_k)}(x-a_k)}{W^{(\lambda_k)}(a_{k+1}-a_k)}\left(Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_{k+1})+(\lambda_k-\lambda_{k-1})\int_{a_k}^{a_{k+1}}W^{(\lambda_k)}(a_{k+1}-y)Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(y)\mathrm{d}y\right)\\&=Z_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_k)}(x)-\frac{W^{(\lambda_k)}(x-a_k)}{W^{(\lambda_k)}(a_{k+1}-a_k)}Z_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_k)}(a_{k+1}).\end{aligned}$$

Therefore, identity (15) holds for $n=k+1$. □

We next present the main results on Laplace transform of joint occupation times for spectrally negative Lévy processes.

Theorem 1. *For $0 = a_0 < a_1 < \ldots < a_n$, $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \geq 0$ and $0 \leq x \leq a_n$, we have*

$$\mathbb{E}_x \left[e^{-\sum\limits_{i=0}^{n-1} \lambda_i \int_0^{\tau_{a_n}^+} 1_{(a_i, a_{i+1})}(X_s) \mathrm{d}s}; \tau_{a_n}^+ < \tau_0^- \right] = \frac{W_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(x)}{W_{(a_0,\ldots,a_{n-1})}^{(\lambda_0,\ldots,\lambda_{n-1})}(a_n)}. \tag{17}$$

Proof. Write $\omega_n(x)$ for the left-hand side of Equation (17). By the strong Markov property and lack of positive jumps, for $x < a_{n-1}$, we have

$$\omega_n(x) = \omega_{n-1}(x)\omega_n(a_{n-1}). \tag{18}$$

We proceed to prove this theorem by induction. The case for $n = 1$ is obvious. Let us suppose that identity (17) holds for $n \leq k$. By identities (3) and (18), the assumption for $n = k$ and Lemma 4, for $n = k + 1$, $\varepsilon > 0$ and $a_k + \varepsilon \leq a_{k+1}$, we have

$$\begin{aligned}
&\omega_{k+1}(a_k + \varepsilon) \\
&= \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k \tau_{a_{k+1}}^+}; \tau_{a_{k+1}}^+ < \tau_{a_k}^-] + \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k \tau_{a_k}^-} \omega_{k+1}(X_{\tau_{a_k}^-}); \tau_{a_k}^- < \tau_{a_{k+1}}^+] \\
&= \frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1} - a_k)} + \frac{\omega_{k+1}(a_k)}{W_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)} \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k \tau_{a_k}^-} W_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(X_{\tau_{a_k}^-}); \tau_{a_k}^- < \tau_{a_{k+1}}^+] \\
&= \frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1} - a_k)} + \frac{\omega_{k+1}(a_k)}{W_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)} \left\{ W_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_k)}(a_k + \varepsilon) - \frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1} - a_k)} W_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_k)}(a_{k+1}) \right\}.
\end{aligned} \tag{19}$$

For each $i = 0, \ldots, k$, let N_i be an independent Poisson process with rates λ_i and write $0 < T_1^i < T_2^i < \ldots$ for the sequence of arrival times for N_i. We also assume that the Poisson processes (N_i) are independent of process X. Observe that

$$\omega_{k+1}(x) = \mathbb{P}_x \left\{ \cap_{i=0}^k \left\{ \{T_j^i\} \cap \{s : s < \tau_{a_{k+1}}^+ < \tau_0^-, X_s \in (a_i, a_{i+1})\} = \varnothing \right\} \right\},$$

where we have used a property of the Poisson process. The simplest version of this property is

$$\mathbb{P}\{\{T_j^1\} \cap B = \varnothing\} = e^{-\lambda_1 L(B)}$$

for any Borel set $B \subset [0, \infty)$ and Lebesgue measure L.

For convenience, let

$$\mathcal{D}_j := \{0, \ldots, j-1, j+1, \ldots, k\}, \quad \overline{\lambda} := \sum_{i=0}^k \lambda_i, \quad \overline{\lambda}_j := \overline{\lambda} - \lambda_j, \tag{20}$$

and

$$\begin{aligned}
A :&= \sum_{i=0}^{k-1} \lambda_i \int_{a_k}^{a_k+\varepsilon} \int_0^\infty e^{-\overline{\lambda} t} \mathbb{P}_{a_k}\{t < \tau_{a_k+\varepsilon}^+ \wedge \tau_0^-, X(t) \in \mathrm{d}y\} \omega_{k+1}(y) \mathrm{d}t \\
&= \sum_{i=0}^{k-1} \lambda_i \int_{a_k}^{a_k+\varepsilon} \left(\frac{W^{(\overline{\lambda})}(a_k) W^{(\overline{\lambda})}(a_k + \varepsilon - y)}{W^{(\overline{\lambda})}(a_k + \varepsilon)} - W^{(\overline{\lambda})}(a_k - y) \right) \omega_{k+1}(y) \mathrm{d}y.
\end{aligned}$$

Using identities (3) and (5), for any $\varepsilon > 0$,

$$\begin{aligned}
&\omega_{k+1}(a_k) \\
&\quad = \mathbb{P}_{a_k}\{\tau^+_{a_k+\varepsilon} < \min\{\tau_0^-, T^0, \ldots, T^k\}\}\omega_{k+1}(a_k+\varepsilon) \\
&\quad + \int_{a_k}^{a_k+\varepsilon} \mathbb{P}_{a_k}\left\{\cup_{i=0}^{k-1}\{T^i < \min\{\tau_0^-, \tau^+_{a_k+\varepsilon}, T^0, \ldots, T^{i-1}, T^{i+1}, \ldots, T^k\}, X(T^i) \in \mathrm{d}y\}\right\}\omega_{k+1}(y) \\
&\quad + \sum_{j=0}^{k-1}\int_{a_j}^{a_{j+1}} \mathbb{P}_{a_k}\left\{\cup_{i\in\mathcal{D}_j}\{T^i < \min\{\tau_0^-, \tau^+_{a_k+\varepsilon}, T^0, \ldots, T^{i-1}, T^{i+1}, \ldots, T^k\}, X(T^i) \in \mathrm{d}y\}\right\}\omega_{k+1}(y).
\end{aligned}$$

Then

$$\begin{aligned}
\omega_{k+1}(a_k) &= \mathbb{E}_{a_k}[e^{-\overline{\lambda}\tau^+_{a_k+\varepsilon}}; \tau^+_{a_k+\varepsilon} < \tau_0^-]\omega_{k+1}(a_k+\varepsilon) + A \\
&+ \sum_{j=0}^{k-1}\overline{\lambda}_j \int_{a_j}^{a_{j+1}}\int_0^\infty e^{-\overline{\lambda}t}\mathbb{P}_{a_k}\{t < \tau^+_{a_k+\varepsilon}\wedge\tau_0^-, X(t) \in \mathrm{d}y\}\omega_{k+1}(y)\mathrm{d}t \\
&= \frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\omega_{k+1}(a_k+\varepsilon) + A \\
&+ \sum_{j=0}^{k-1}\overline{\lambda}_j\int_{a_j}^{a_{j+1}}\left(\frac{W^{(\overline{\lambda})}(a_k)W^{(\overline{\lambda})}(a_k+\varepsilon-y)}{W^{(\overline{\lambda})}(a_k+\varepsilon)} - W^{(\overline{\lambda})}(a_k-y)\right)\omega_{k+1}(y)\mathrm{d}y,
\end{aligned}$$

where, by identity (18) and Lemma 2, the second term is equal to

$$\begin{aligned}
&\frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\left\{\frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\sum_{j=0}^{k-1}\overline{\lambda}_j\int_{a_j}^{a_{j+1}}W^{(\overline{\lambda})}(a_k+\varepsilon-y)W^{(\lambda_0,\ldots,\lambda_j)}_{(a_0,\ldots,a_j)}(y)\mathrm{d}y\right. \\
&\quad\left. - \sum_{j=0}^{k-1}\overline{\lambda}_j\int_{a_j}^{a_{j+1}}W^{(\overline{\lambda})}(a_k-y)W^{(\lambda_0,\ldots,\lambda_j)}_{(a_0,\ldots,a_j)}(y)\mathrm{d}y\right\} \\
&= \frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\left\{\frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\left(W^{(\overline{\lambda})}(a_k+\varepsilon) - W^{(\lambda_0,\ldots,\lambda_{k-1},\overline{\lambda})}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)\right)\right. \\
&\quad\left. - \left(W^{(\overline{\lambda})}(a_k) - W^{(\lambda_0,\ldots,\lambda_{k-1},\overline{\lambda})}_{(a_0,\ldots,a_k)}(a_k)\right)\right\} \\
&= \omega_{k+1}(a_k) - \frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}W^{(\lambda_0,\ldots,\lambda_{k-1},\overline{\lambda})}_{(a_0,\ldots,a_k)}(a_k+\varepsilon).
\end{aligned}$$

Therefore,

$$0 = \frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\omega_{k+1}(a_k+\varepsilon) + A - \frac{W^{(\overline{\lambda})}(a_k)}{W^{(\overline{\lambda})}(a_k+\varepsilon)}\frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}W^{(\lambda_0,\ldots,\lambda_{k-1},\overline{\lambda})}_{(a_0,\ldots,a_k)}(a_k+\varepsilon). \tag{21}$$

Combining Equations (21) and (19), by Lemma 1, we have

$$\begin{aligned}
&\frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1}-a_k)}\left\{\frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1}) - 1\right\} \\
&= \frac{W^{(\overline{\lambda})}(a_k+\varepsilon)}{W^{(\overline{\lambda})}(a_k)}A - \frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\left(W^{(\lambda_0,\ldots,\lambda_{k-1},\overline{\lambda})}_{(a_0,\ldots,a_k)}(a_k+\varepsilon) - W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)\right) \\
&= \frac{W^{(\overline{\lambda})}(a_k+\varepsilon)}{W^{(\overline{\lambda})}(a_k)}A - \frac{\omega_{k+1}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\sum_{i=0}^{k-1}\lambda_i\int_{a_k}^{a_k+\varepsilon}W^{(\overline{\lambda})}(a_k+\varepsilon-y)W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y.
\end{aligned} \tag{22}$$

Notice that $\lim_{x\to 0+} W^{(p)}(x)/W(x) = 1$; see equation (56) and Lemma 3.1 of Kuznetsov et al. [10]. Then, we have

$$A = o(W^{(\lambda_k)}(\varepsilon)) \quad \text{and} \quad \int_{a_k}^{a_k+\varepsilon} W^{(\bar{\lambda})}(a_k+\varepsilon-y)W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y = o(W^{(\lambda_k)}(\varepsilon))$$

as $\varepsilon \to 0+$, where $o(W^{(\lambda_k)}(\varepsilon))$ is simply $o(1)$ for X of bounded variation. Dividing both sides of Equation (22) by $W^{(\lambda_k)}(\varepsilon)$, letting $\varepsilon \to 0+$ in Equation (22), we have

$$\omega_{k+1}(a_k) = \frac{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1})}. \tag{23}$$

Plugging Equation (23) into Equation (19), we prove that identity (17) holds for $n = k+1$. $\square$

Theorem 2. *For $0 = a_0 < a_1 < \ldots < a_n$, $\lambda_0, \lambda_1, \ldots, \lambda_n \geq 0$, $0 \leq x \leq a_n$,*

$$\begin{aligned} &\mathbb{E}_x\left[e^{-\sum_{i=0}^{n-1}\lambda_i\int_0^{\tau_0^-} 1_{(a_i,a_{i+1})}(X_s)\mathrm{d}s}; \tau_0^- < \tau_{a_n}^+\right] \\ &= Z^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(x) - \frac{W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(x)}{W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)} Z^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n). \end{aligned} \tag{24}$$

Proof. Write $f_n(x)$ for the left-hand side of Equation (24). Then for $x < a_{n-1}$,

$$f_n(x) = f_{n-1}(x) + \omega_{n-1}(x) f_n(a_{n-1}). \tag{25}$$

We want to prove identity (24) by induction. The case for $n = 1$ is obvious. Suppose identity (24) holds for $n \leq k$. By identity (25), the assumption for $n = k$ and Lemma 4, if $n = k+1, \varepsilon > 0$ and $a_k + \varepsilon \leq a_{k+1}$, we have

$$\begin{aligned} &f_{k+1}(a_k+\varepsilon) \\ &= \int_{-\infty}^{a_k} \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k\tau_{a_k}^-}; \tau_{a_k}^- < \tau_{a_{k+1}}^+, X_{\tau_{a_k}^-} \in \mathrm{d}y] f_{k+1}(y) \\ &= \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k\tau_{a_k}^-} f_{k+1}(X_{\tau_{a_k}^-}); \tau_{a_k}^- < \tau_{a_{k+1}}^+] \\ &= \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k\tau_{a_k}^-} Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(X_{\tau_{a_k}^-}); \tau_{a_k}^- < \tau_{a_{k+1}}^+] \\ &\quad + \frac{f_{k+1}(a_k) - Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)} \mathbb{E}_{a_k+\varepsilon}[e^{-\lambda_k\tau_{a_k}^-} W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(X_{\tau_{a_k}^-}); \tau_{a_k}^- < \tau_{a_{k+1}}^+] \\ &= Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon) - \frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1}-a_k)} Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1}) \\ &\quad + \frac{f_{k+1}(a_k) - Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)} \left\{ W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon) - \frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1}-a_k)} W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1}) \right\}. \end{aligned} \tag{26}$$

For the (T_j^i) defined in the proof of Theorem 1, observe that

$$f_{k+1}(x) = \mathbb{P}_x\left\{\cap_{i=0}^k \left\{\{T_u^i\} \cap \{s : s < \tau_0^- < \tau_{a_{k+1}}^+, X_s \in (a_i, a_{i+1})\} = \emptyset\right\}\right\}.$$

We can obtain the expressions of $f_{k+1}(a_k)$ as follows. Put

$$\begin{aligned} B :=& \int_{a_k}^{a_k+\varepsilon} \mathbb{P}_{a_k}\left\{\cup_{i=0}^{k-1}\{T^i < \min\{\tau_0^-, \tau_{a_k+\varepsilon}^+, T^0, \ldots, T^{i-1}, T^{i+1}, \ldots, T^k\}, X(T^i) \in \mathrm{d}y\}\right\} f_{k+1}(y) \\ =& \sum_{i=0}^{k-1} \lambda_i \int_{a_k}^{a_k+\varepsilon} \int_0^\infty e^{-\bar{\lambda}t} \mathbb{P}_{a_k}\{t < \tau_{a_k+\varepsilon}^+ \wedge \tau_0^-, X(t) \in \mathrm{d}y\} f_{k+1}(y)\mathrm{d}t, \end{aligned}$$

where $\bar{\lambda}$ is defined in definition (20). Similar to the proof of Theorem 1, for any $\varepsilon > 0$,

$$\begin{aligned} & f_{k+1}(a_k) \\ &= \mathbb{P}_{a_k}\{\tau_0^- < \min\{\tau_{a_k+\varepsilon}^+, T^0, \ldots, T^k\}\} + \mathbb{P}_{a_k}\{\tau_{a_k+\varepsilon}^+ < \min\{\tau_0^-, T^0, \ldots, T^k\}\} f_{k+1}(a_k+\varepsilon) \\ &\quad + \int_{a_k}^{a_k+\varepsilon} \mathbb{P}_{a_k}\left\{\cup_{i=0}^{k-1}\{T^i < \min\{\tau_0^-, \tau_{a_k+\varepsilon}^+, T^0, \ldots, T^{i-1}, T^{i+1}, \ldots, T^k\}, X(T^i) \in \mathrm{d}y\}\right\} f_{k+1}(y) \\ &\quad + \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} \mathbb{P}_{a_k}\left\{\cup_{i\in\mathcal{D}_j}\{T^i < \min\{\tau_0^-, \tau_{a_k+\varepsilon}^+, T^0, \ldots, T^{i-1}, T^{i+1}, \ldots, T^k\}, X(T^i) \in \mathrm{d}y\}\right\} f_{k+1}(y) \\ &= \mathbb{E}_{a_k}[e^{-\bar{\lambda}\tau_0^-}; \tau_0^- < \tau_{a_k+\varepsilon}^+] + \mathbb{E}_{a_k}[e^{-\bar{\lambda}\tau_{a_k+\varepsilon}^+}; \tau_{a_k+\varepsilon}^+ < \tau_0^-] f_{k+1}(a_k+\varepsilon) + B \\ &\quad + \sum_{j=0}^{k-1} \overline{\lambda}_j \int_{a_j}^{a_{j+1}} \int_0^\infty e^{-\bar{\lambda}t} \mathbb{P}_{a_k}\{t < \tau_{a_k+\varepsilon}^+ \wedge \tau_0^-, X(t) \in \mathrm{d}y\} f_{k+1}(y)\mathrm{d}t, \end{aligned}$$

which by Equations (3)–(5) and (25) is further equal to

$$\begin{aligned} & Z^{(\bar{\lambda})}(a_k) - \frac{Z^{(\bar{\lambda})}(a_k+\varepsilon)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} W^{(\bar{\lambda})}(a_k) + \frac{W^{(\bar{\lambda})}(a_k)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} f_{k+1}(a_k+\varepsilon) + B \\ &\quad + \sum_{j=0}^{k-1} \overline{\lambda}_j \int_{a_j}^{a_{j+1}} \left(\frac{W^{(\bar{\lambda})}(a_k) W^{(\bar{\lambda})}(a_k+\varepsilon-y)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} - W^{(\bar{\lambda})}(a_k-y) \right) \\ &\qquad \times \left(Z_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(y) + \frac{f_{k+1}(a_k) - Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)}{W_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)} W_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(y) \right) \mathrm{d}y. \end{aligned}$$

By Lemma 2,

$$\sum_{j=0}^{k-1} \overline{\lambda}_j \int_{a_j}^{a_{j+1}} W^{(\bar{\lambda})}(a_k+\varepsilon-y) Z_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(y)\mathrm{d}y = Z^{(\bar{\lambda})}(a_k+\varepsilon) - Z_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_{k-1},\bar{\lambda})}(a_k+\varepsilon)$$

and

$$\sum_{j=0}^{k-1} \overline{\lambda}_j \int_{a_j}^{a_{j+1}} W^{(\bar{\lambda})}(a_k-y) Z_{(a_0,\ldots,a_j)}^{(\lambda_0,\ldots,\lambda_j)}(y)\mathrm{d}y = Z^{(\bar{\lambda})}(a_k) - Z_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_{k-1},\bar{\lambda})}(a_k).$$

Then

$$\begin{aligned} 0 =& \frac{W^{(\bar{\lambda})}(a_k)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} f_{k+1}(a_k+\varepsilon) + B - \frac{W^{(\bar{\lambda})}(a_k)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} Z_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_{k-1},\bar{\lambda})}(a_k+\varepsilon) \\ & - \frac{W^{(\bar{\lambda})}(a_k)}{W^{(\bar{\lambda})}(a_k+\varepsilon)} \frac{f_{k+1}(a_k) - Z_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)}{W_{(a_0,\ldots,a_{k-1})}^{(\lambda_0,\ldots,\lambda_{k-1})}(a_k)} W_{(a_0,\ldots,a_k)}^{(\lambda_0,\ldots,\lambda_{k-1},\bar{\lambda})}(a_k+\varepsilon). \end{aligned} \tag{27}$$

Combining Equations (26) and (27), by Lemma 1, we have

$$
\begin{aligned}
&\frac{W^{(\lambda_k)}(\varepsilon)}{W^{(\lambda_k)}(a_{k+1}-a_k)}\left\{Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1})+\frac{f_{k+1}(a_k)-Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1})\right\}\\
&=Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)-Z^{(\lambda_0,\ldots,\lambda_{k-1},\bar\lambda)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)+\frac{W^{(\bar\lambda)}(a_k+\varepsilon)}{W^{(\bar\lambda)}(a_k)}B\\
&\quad+\frac{f_{k+1}(a_k)-Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\left\{W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)-W^{(\lambda_0,\ldots,\lambda_{k-1},\bar\lambda)}_{(a_0,\ldots,a_k)}(a_k+\varepsilon)\right\}\\
&=\frac{W^{(\bar\lambda)}(a_k+\varepsilon)}{W^{(\bar\lambda)}(a_k)}B-\sum_{i=0}^{k-1}\lambda_i\int_{a_k}^{a_k+\varepsilon}W^{(\bar\lambda)}(a_k+\varepsilon-y)Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y\\
&\quad-\frac{f_{k+1}(a_k)-Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}{W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)}\sum_{i=0}^{k-1}\lambda_i\int_{a_k}^{a_k+\varepsilon}W^{(\bar\lambda)}(a_k+\varepsilon-y)W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y.
\end{aligned}
$$

Notice that

$$\int_{a_k}^{a_k+\varepsilon}W^{(\bar\lambda)}(a_k+\varepsilon-y)Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y=o(W^{(\lambda_k)}(\varepsilon)),$$

$$\int_{a_k}^{a_k+\varepsilon}W^{(\bar\lambda)}(a_k+\varepsilon-y)W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(y)\mathrm{d}y=o(W^{(\lambda_k)}(\varepsilon))\ \text{ and }\ B=o(W^{(\lambda_k)}(\varepsilon)).$$

Letting $\varepsilon\to 0+$, we have

$$f_{k+1}(a_k)=Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k)-\frac{Z^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1})}{W^{(\lambda_0,\ldots,\lambda_k)}_{(a_0,\ldots,a_k)}(a_{k+1})}W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(a_k). \tag{28}$$

Using Equation (28) in Equation (26), we have thus proved identity (24) for $n=k+1$. □

For general spectrally negative Lévy processes, it appears challenging to find explicit expressions for Laplace transforms of weighted occupation times with weight functions more general than step functions. We end this paper with a corollary.

Corollary 1. *Given $n\geq 1$ and $\lambda_{-1},\lambda_0,\ldots,\lambda_{n-1}\geq 0$, for $-\infty=a_{-1}<0=a_0<a_1<\ldots<a_{n-1}<a_n<\infty$, we have*

$$\mathbb{E}_x\left[e^{-\sum\limits_{i=-1}^{n-1}\lambda_i\int_0^{\tau_{a_n}^+}1_{(a_i,a_{i+1})}(X_s)\mathrm{d}s};\tau_{a_n}^+<\infty\right]=\frac{\mathbf{W}_{n-1}(x)}{\mathbf{W}_{n-1}(a_n)}, \tag{29}$$

where $x<a_n$,

$$\mathbf{W}_{-1}(x)=e^{\Phi(\lambda_{-1})x},\ \ \mathbf{W}_k(x)=\mathbf{W}_{k-1}(x)+(\lambda_k-\lambda_{k-1})\int_{a_k}^x W^{(\lambda_k)}(x-y)\mathbf{W}_{k-1}(y)\mathrm{d}y,\ \ k\geq 0; \tag{30}$$

for $0=a_0<a_1<\ldots<a_{n-1}<a_n=\infty$, we have

$$\mathbb{E}_x\left[e^{-\sum\limits_{i=0}^{n-1}\lambda_i\int_0^{\tau_0^-}1_{(a_i,a_{i+1})}(X_s)\mathrm{d}s};\tau_0^-<\infty\right]=Z^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(x)-\frac{z_{n-1}}{w_{n-1}}W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(x), \tag{31}$$

where $x > 0$ and (w_{n-1}, z_{n-1}) is given by

$$w_{n-1} = 1 - \sum_{k=1}^{n-1}(\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} e^{-\Phi(\lambda_{n-1})y} W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y$$

and

$$z_{n-1} = \frac{\lambda_{n-1}}{\Phi(\lambda_{n-1})} - \sum_{k=1}^{n-1}(\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} e^{-\Phi(\lambda_{n-1})y} Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y.$$

Proof. For $k \geq 0$ and $a > 0$,

$$\begin{aligned}
&W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_k)}_{(0,a,a+a_1,\ldots,a+a_k)}(a+x) \\
&= W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{k-1})}_{(0,a,a+a_1,\ldots,a+a_{k-1})}(a+x) + (\lambda_k - \lambda_{k-1}) \int_{a+a_k}^{a+x} W^{(\lambda_k)}(a+x-y) W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{k-1})}_{(0,a,a+a_1,\ldots,a+a_{k-1})}(y)\mathrm{d}y \\
&= W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{k-1})}_{(0,a,a+a_1,\ldots,a+a_{k-1})}(a+x) + (\lambda_k - \lambda_{k-1}) \int_{a_k}^{x} W^{(\lambda_k)}(x-z) W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{k-1})}_{(0,a,a+a_1,\ldots,a+a_{k-1})}(a+z)\mathrm{d}z.
\end{aligned} \tag{32}$$

Since

$$\mathbf{W}_{-1}(x) := \lim_{a\to\infty} \frac{W^{(\lambda_{-1})}(a+x)}{W^{(\lambda_{-1})}(a)} = e^{\Phi(\lambda_{-1})x}$$

for any x, we can show by induction and Equation (32) that, for $x < a_n$ and $k \geq 0$,

$$\mathbf{W}_k(x) := \lim_{a\to\infty} W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_k)}_{(0,a,a+a_1,\ldots,a+a_k)}(a+x)/W^{(\lambda_{-1})}(a)$$

exists and satisfies Equation (30). Since X is spatially homogeneous, by Theorem 1,

$$\mathbb{E}_x\left[e^{-\sum\limits_{i=-1}^{n-1}\lambda_i\int_0^{\tau_{a_n}^+} 1_{(a_i,a_{i+1})}(X_s)\mathrm{d}s}; \tau_{a_n}^+ < \infty\right] = \lim_{a\to\infty} \frac{W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{n-1})}_{(0,a,a+a_1,\ldots,a+a_{n-1})}(a+x)}{W^{(\lambda_{-1},\lambda_0,\lambda_1,\ldots,\lambda_{n-1})}_{(0,a,a+a_1,\ldots,a+a_{n-1})}(a+a_n)}.$$

Then identity (29) follows.

Similarly, let

$$w_0 = 1, \quad z_0 = \lim_{y\to\infty} \frac{Z^{(\lambda_{n-1})}(y)}{W^{(\lambda_{n-1})}(y)} = \frac{\lambda_{n-1}}{\Phi(\lambda_{n-1})},$$

and, for $k = 1, \ldots, n-1$, put

$$w_k := \lim_{y\to\infty} W^{(\lambda_0,\ldots,\lambda_{k-1},\lambda_{n-1})}_{(a_0,\ldots,a_{k-1},a_k)}(y)/W^{(\lambda_{n-1})}(y)$$

and

$$z_k := \lim_{y\to\infty} Z^{(\lambda_0,\ldots,\lambda_{k-1},\lambda_{n-1})}_{(a_0,\ldots,a_{k-1},a_k)}(y)/W^{(\lambda_{n-1})}(y).$$

By Lemma 2, we have

$$\begin{aligned}
W^{(\lambda_0,\ldots,\lambda_{k-1},\lambda_{n-1})}_{(a_0,\ldots,a_{k-1},a_k)}(x) &= W^{(\lambda_0,\ldots,\lambda_{k-2},\lambda_{n-1})}_{(a_0,\ldots,a_{k-2},a_{k-1})}(x) - (\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} W^{(\lambda_{n-1})}(x-y) W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y, \\
Z^{(\lambda_0,\ldots,\lambda_{k-1}\lambda_{n-1})}_{(a_0,\ldots,a_{k-1},a_k)}(x) &= Z^{(\lambda_0,\ldots,\lambda_{k-2},\lambda_{n-1})}_{(a_0,\ldots,a_{k-2},a_{k-1})}(x) - (\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} W^{(\lambda_{n-1})}(x-y) Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y.
\end{aligned}$$

Then, by asymptotic results for scale functions, we can show by induction that (w_k, z_k) also exists and satisfies

$$\begin{aligned}
&w_0 = 1, \;\; w_k = w_{k-1} - (\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} e^{-\Phi(\lambda_{n-1})y} W^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y, \\
&z_0 = \frac{\lambda_{n-1}}{\Phi(\lambda_{n-1})}, \;\; z_k = z_{k-1} - (\lambda_{n-1} - \lambda_{k-1}) \int_{a_{k-1}}^{a_k} e^{-\Phi(\lambda_{n-1})y} Z^{(\lambda_0,\ldots,\lambda_{k-1})}_{(a_0,\ldots,a_{k-1})}(y)\mathrm{d}y,
\end{aligned} \tag{33}$$

for $k = 1, \ldots, n-1$. Since

$$\lim_{a_n \to \infty} \frac{W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)}{Z^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)} = \lim_{a_n \to \infty} \frac{W^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)/W^{(\lambda_{n-1})}(a_n)}{Z^{(\lambda_0,\ldots,\lambda_{n-1})}_{(a_0,\ldots,a_{n-1})}(a_n)/W^{(\lambda_{n-1})}(a_n)} = \frac{w_{n-1}}{z_{n-1}},$$

identity (31) then follows from Theorem 2 by letting $a_n \to \infty$, and the expression of (w_{n-1}, z_{n-1}) follows from Equation (33). □

4. Conclusions

In this paper, for spectrally negative Lévy processes we obtain Laplace transforms of weighted occupation times with step weight functions up to the first exit times. The results are expressed using multiple integrals of the associated scale functions. In the proofs we modify the Poisson approach of Li and Zhou [7], which can be further adapted to study other problems involving Laplace transforms of occupation times. The results have possible applications in risk theory for insurance.

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