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Parameter Estimation in Stable Law

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Abstract: For general stable distribution, cumulant function based parameter estimators are proposed. Extensive simulation experiments are carried out to validate the effectiveness of the estimates over the entire parameter space. An application to non-life insurance losses distribution is made.

Keywords: bootstrap; characteristic function; cumulant function; parameter estimation; simulation; severity distribution

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1. Introduction

The four-parameter stable law arises as the limiting distribution of normalized sum of independent, identically distributed random variables. Stable distributions allow skewness and heavy tails and are proposed as models for various processes in physics, finance and elsewhere (see, e.g., [1]). However, modelling is complicated due to lack of a closed form for density of stable law. A number of parameter estimation techniques are based on the characteristic function. Regression-type characteristic function based methods are proposed in [2–4] and a minimum distance approach in [5]. Based on the logarithm of characteristic function, Press [6] proposes explicit point estimators for four parameters of stable law. His estimates depend on an arbitrary choice of two pairs of arguments of empirical characteristic function, and the method has not been recommended in practice. However, we found only a few papers (e.g, [5,7] for symmetric stable laws) introducing simulations on Press's method while the optimal selection of arguments is still unresolved (for symmetric stable laws, some suggestions are given in [8]). In this paper, we show that the parameters of stable law can be expressed through cumulant function of one pair of arguments and hence the method of Press can be applied for one, not two pairs of arguments. We study the selection of arguments by an empirical search. To assess the effectiveness of estimates, we perform extensive simulations over the parameter space as well as present an application to non-life insurance losses.

The paper is organized as follows. In Section 2, we give some preliminary results about the stable laws. In Section 3, we discuss the main results on cumulant function based estimation. In Section 4, we present an empirical search for the selection of arguments, and in Section 5, we discuss simulations for a selected pair of arguments. Section 6 is devoted to an example in non-life insurance, and Section 7 provides conclusions.

2. Preliminaries

A random variable X is referred to as stable (see, e.g., [1,9,10]) if there exist constants $d_n > 0$ and $c_n \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} X_i \overset{d}{=} c_n + d_n X, \tag{1}$$

where $X_1, X_2, \ldots, X_n$ are independent random variables each having the same distribution as X. It has been shown (e.g., [1,11]) that in Equation (1), we have necessarily $d_n = n^{1/\alpha}$ for some $0 < \alpha \leq 2$ only. The variance of both sides of Equation (1) gives $n\,\mathrm{Var}(X) = n^{2/\alpha}\,\mathrm{Var}(X)$. For non-degenerate

($\mathrm{Var}(X) \neq 0$) distributions with finite variance, the index α must be equal to 2. If $\alpha \neq 2$, the relation can be formally satisfied only with $\mathrm{Var}(X) = \infty$. Indeed, all stable distributions with $\alpha < 2$ have infinite variances, and when $\alpha \leq 1$, they have an infinite mean as well. It is well known that normal ($\alpha = 2$), Cauchy ($\alpha = 1$) and Levy ($\alpha = 1/2$) distributions belong to the class of stable laws. Naturally, α describes the rate of decay of the tails of stable distribution, the smaller the α, the slower is the decay and the heavier the tails. The parameter α is called a characteristic exponent, or index of stability. A skewness or asymmetry parameter $\beta \in [-1, 1]$ characterizes the degree of asymmetry of the distributions being different from normal law (β is irrelevant when $\alpha = 2$). For $\beta = 0$, we have symmetric stable distributions and for $\beta = \pm 1$ totally skewed stable distributions. Like normal law, all stable distributions also remain stable under linear transformations, hence the scale parameter $\gamma > 0$ and the location parameter $\delta \in \mathbb{R}$ are introduced. It is worth mentioning that the scale parameter is not the standard deviation, and the location parameter is not generally the mean [e.g., [1,10]]. However, for $\gamma = 1, \delta = 0$ we say standard stable distributions.

The density function of four-parameter stable distributions cannot be written analytically, and it is not as convenient to use as compared with characteristic (or cumulant) functions that also contain the complete information. The characteristic function $\varphi_X(u)$ of a stable random variable X is $\varphi_X(u) = \mathbb{E}[e^{iuX}]$ and the cumulant function $\Psi_X(u)$ is $\Psi_X(u) = \ln \mathbb{E}[e^{iuX}]$. The explicit representation of the characteristic (and cumulant) function of X depends on the parametrizations. In [1], several parametrizations are introduced while Nolan [10] proposes even more. Hence, when discussing the characteristic or cumulant function of stable law, the parametrization should always be stressed. Hereby, we denote [10] stable laws by $S(\alpha, \beta, \gamma, \delta; 1)$, where 1 specifies the parametrization we use. The representation of characteristic function under 1-parametrization, similar to parametrization (A) in [1], is one of the most presented (as in [9], for example). The Definition 1 for the cumulant function of stable law follows the definition of 1-parametrization in [10].

Definition 1. *A random variable X is distributed according to distribution $S(\alpha, \beta, \gamma, \delta; 1)$ if*

$$X = \begin{cases} \gamma Z + \delta, & \alpha \neq 1, \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma), & \alpha = 1, \end{cases}$$

where Z is a random variable with cumulant function

$$\Psi_Z(u) = \begin{cases} -|u|^\alpha [1 - i\beta \tan(\frac{\pi\alpha}{2}) \operatorname{sign}(u)], & \alpha \neq 1, \\ -|u|[1 + i\beta \frac{2}{\pi} \operatorname{sign}(u) \log(|u|)], & \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\gamma > 0$ and $\delta \in \mathbb{R}$, and $i = \sqrt{-1}$. Then, X has the cumulant function

$$\Psi_X(u) = \begin{cases} -\gamma^\alpha |u|^\alpha [1 - i\beta (\operatorname{sign} u) \tan \frac{\pi\alpha}{2}] + i\delta u, & \alpha \neq 1, \\ -\gamma |u| [1 + i\beta \frac{2}{\pi} (\operatorname{sign} u) \ln |u|] + i\delta u, & \alpha = 1. \end{cases} \tag{2}$$

Our estimation procedure is based on the empirical cumulant function, i.e., on the logarithm of empirical characteristic function. For a sample $X_1, \ldots, X_n$ of independent and identically distributed random variables, the empirical characteristic function $\hat{\varphi}_{X_1,\ldots,X_n}(u) = \hat{\varphi}_n(u)$ is given as $\hat{\varphi}_n(u) = \frac{1}{n} \sum_{j=1}^n e^{iuX_j}$. It is easy to see that if $X_1, \ldots, X_n$ are distributed as X, then $\mathbb{E}[\hat{\varphi}_n(u)] = \frac{1}{n} \sum_{j=1}^n \mathbb{E}[e^{iuX_j}] = \varphi_X(u)$. Hence, by the strong law of large numbers, the theoretical empirical characteristic function almost surely converges to the characteristic function for $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \hat{\varphi}_n(u) \overset{a.s.}{=} \varphi_X(u), \tag{3}$$

and $\hat{\varphi}_n(u)$ is a consistent estimator for $\varphi_X(u)$. For studies on parameter estimation of stable laws based on empirical characteristic function, we refer, for example, to [2–8,12,13].

For a sample $X_1, \ldots, X_n$ i.i.d. of random variables, the empirical cumulant function $\hat{\Psi}_{X_1,\ldots,X_n}(u) = \hat{\Psi}_n(u)$ is

$$\hat{\Psi}_n(u) = \ln \hat{\varphi}_n(u) = \ln \left(\frac{1}{n} \sum_{j=1}^{n} e^{iuX_j} \right). \tag{4}$$

Complex numbers are complete metric space and the natural logarithm function is continuous; hence, as Equation (3) holds, then by continuous mapping theorem, see, e.g., [14]

$$\lim_{n\to\infty} \hat{\Psi}_n(u) \stackrel{a.s.}{=} \Psi_X(u), \tag{5}$$

and $\hat{\Psi}_n(u)$ is a consistent estimator for $\Psi_X(u)$. For more on the theory of empirical cumulant function based estimation, see, for example, [15].

3. Main Results

In this section, we introduce cumulant function estimation procedure. In Theorem 1, we show that the parameters of $S(\alpha, \beta, \gamma, \delta; 1)$ can be expressed through the cumulant function, and then, based on the empirical cumulant function, we propose the cumulant function based estimators.

Theorem 1. *Let $u_1 > 0, u_2 > 0 \in \mathbb{R}, u_1 \neq u_2$. The parameters of $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ can be expressed through the cumulant function (2) at u_1, u_2,*

$$\alpha = \frac{\ln(-\operatorname{Re}(\Psi_X(u_1))) - \ln(-\operatorname{Re}(\Psi_X(u_2)))}{\ln u_1 - \ln u_2}, \tag{6}$$

$$\gamma = \exp\left\{ \frac{\ln u_1 \ln(-\operatorname{Re}(\Psi_X(u_2))) - \ln u_2 \ln(-\operatorname{Re}(\Psi_X(u_1)))}{\ln(-\operatorname{Re}(\Psi_X(u_1))) - \ln(-\operatorname{Re}(\Psi_X(u_2)))} \right\}, \tag{7}$$

and in the case of $\alpha \neq 1$,

$$\beta = \frac{u_2 \operatorname{Im}(\Psi_X(u_1)) - u_1 \operatorname{Im}(\Psi_X(u_2))}{\gamma^\alpha \left(u_2 u_1^\alpha - u_1 u_2^\alpha\right) \tan \frac{\pi\alpha}{2}}, \tag{8}$$

$$\delta = \frac{u_1^\alpha \operatorname{Im}(\Psi_X(u_2)) - u_2^\alpha \operatorname{Im}(\Psi_X(u_1))}{u_2 u_1^\alpha - u_1 u_2^\alpha}, \tag{9}$$

where α is given by Equation (6) and γ by Equation (7), and, in the case of $\alpha = 1$,

$$\beta = \pi \frac{u_2 \operatorname{Im}(\Psi_X(u_1)) - u_1 \operatorname{Im}(\Psi_X(u_2))}{2\gamma u_1 u_2 (\ln u_2 - \ln u_1)}, \tag{10}$$

$$\delta = \frac{u_2 \operatorname{Im}(\Psi_X(u_1)) \ln u_2 - u_1 \operatorname{Im}(\Psi_X(u_2)) \ln u_1}{u_1 u_2 (\ln u_2 - \ln u_1)}, \tag{11}$$

where γ is given by Equation (7), while Re, Im *stand, respectively, for the real and imaginary parts of the cumulant function.*

Proof. Let us choose constants $u_1, u_2 \in \mathbb{R}$ so that $u_1 > 0, u_2 > 0, u_1 \neq u_2$. Assuming that the parameters of a stable random variable $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ are fixed, we can write the following system of equations:

$$\begin{cases} \Psi_X(u_1) = -\gamma^\alpha u_1^\alpha + i \begin{cases} (\beta\gamma^\alpha u_1^\alpha \tan\frac{\pi\alpha}{2} + \delta u_1), & \alpha \neq 1, \\ (-\frac{2}{\pi}\beta\gamma u_1 \log u_1 + \delta u_1), & \alpha = 1, \end{cases} \\ \Psi_X(u_2) = -\gamma^\alpha u_2^\alpha + i \begin{cases} (\beta\gamma^\alpha u_2^\alpha \tan\frac{\pi\alpha}{2} + \delta u_2), & \alpha \neq 1, \\ (-\frac{2}{\pi}\beta\gamma u_2 \log u_2 + \delta u_2), & \alpha = 1. \end{cases} \end{cases} \tag{12}$$

As the system of Equation (12) is a system of complex numbers, it must simultaneously hold for real and imaginary parts of $\Psi_X(u_1)$ and $\Psi_X(u_2)$. The real parts of $\Psi_X(u_1)$ and $\Psi_X(u_2)$ in Equation (12) give

$$\begin{cases} \mathrm{Re}(\Psi_X(u_1)) = -\gamma^\alpha u_1^\alpha, \\ \mathrm{Re}(\Psi_X(u_2)) = -\gamma^\alpha u_2^\alpha. \end{cases} \tag{13}$$

Solving system (13) for α and γ gives Equations (6) and (7). The imaginary parts of $\Psi_X(u_1)$ and $\Psi_X(u_2)$ in (12) give two systems of equations. First, in the case of $\alpha \neq 1$, the imaginary parts in Equation (12) give

$$\begin{cases} \mathrm{Im}(\Psi_X(u_1)) = \beta\gamma^\alpha u_1^\alpha \tan\frac{\pi\alpha}{2} + \delta u_1, \\ \mathrm{Im}(\Psi_X(u_2)) = \beta\gamma^\alpha u_2^\alpha \tan\frac{\pi\alpha}{2} + \delta u_2. \end{cases} \tag{14}$$

Solving system (14) for δ and β gives Equations (8) and (9), where α and γ are solved from system (13) (and given by Equation (6) and Equation (7)). In the case of $\alpha = 1$, the imaginary parts in Equation (12) give

$$\begin{cases} \mathrm{Im}(\Psi_X(u_1) = -\frac{2}{\pi}\beta\gamma u_1 \ln u_1 + \delta u_1, \\ \mathrm{Im}(\Psi_X(u_2) = -\frac{2}{\pi}\beta\gamma u_2 \ln u_2 + \delta u_2. \end{cases} \tag{15}$$

Solving system (15) for δ and β gives the Equations (10) and (11) where γ is solved from Equation (13) and given by Equation (7). □

Next, we propose cumulant function based estimators. Techniques based on the logarithm of characteristic function are usually classified as characteristic function based methods. We, however, propose term cumulant estimators.

Definition 2. *If $X_1, X_2, \ldots, X_n$ form a sample of independent and identically distributed random variables having the same distribution as $X \sim S(\alpha,\ \beta,\ \gamma, \delta; 1)$, and $u_1 > 0, u_2 > 0 \in \mathbb{R}, u_1 \neq u_2$, then cumulant estimators*

$$\hat{\alpha}_n = \hat{\alpha}(u_1, u_2, X_1, \ldots, X_n), \tag{16}$$
$$\hat{\beta}_n = \hat{\beta}(u_1, u_2, X_1, \ldots, X_n), \tag{17}$$
$$\hat{\gamma}_n = \hat{\gamma}(u_1, u_2, X_1, \ldots, X_n), \tag{18}$$
$$\hat{\delta}_n = \hat{\delta}(u_1, u_2, X_1, \ldots, X_n), \tag{19}$$

for the parameters of $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ are defined to satisfy Equations (6)–(11), where the real and imaginary parts of cumulant functions (2) are replaced with the real and imaginary parts of the empirical cumulant function (4).

As Equation (5) holds, then $\lim_{n\to\infty} \mathrm{Re}(\hat{\Psi}_n(u)) \overset{a.s.}{=} \mathrm{Re}(\Psi_X(u))$ and $\lim_{n\to\infty} \mathrm{Im}(\hat{\Psi}_n(u)) \overset{a.s.}{=} \mathrm{Im}(\Psi_X(u))$, and by continuous mapping theorem, see, e.g., [14], the cumulant estimators $\hat{\boldsymbol{\alpha}}_n, \hat{\boldsymbol{\beta}}_n, \hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{\delta}}_n$ are consistent for the parameters of $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ (as also discussed in [6]).

Proposition 1. *The real and imaginary parts of empirical cumulant function (4) satisfy the representation's* $\mathrm{Re}(\hat{\Psi}_n(u))=\ln\sqrt{(\frac{1}{n}\sum_{j=1}^n \cos uX_j)^2 + (\frac{1}{n}\sum_{j=1}^n \sin uX_j)^2}$ *and* $\mathrm{Im}(\hat{\Psi}_n(u))=\mathrm{atan2}(\frac{1}{n}\sum_{j=1}^n \sin uX_j, \frac{1}{n}\sum_{j=1}^n \cos uX_j)$, *where* $X_1, \ldots, X_n$ *form a sample of i.i.d. random variables.*

Proof. From Euler's formula, we have $\hat{\Psi}_n(u) = \ln\left(\sum_{j=1}^n \frac{1}{n}\cos(uX_j) + i\sum_{j=1}^n \frac{1}{n}\sin(uX_j)\right)$. The logarithm of a complex number $z = x + iy$ is $\ln z = \ln|z| + i\arg z$, where $|z| = \sqrt{x^2+y^2}$ and $\arg z$ is calculated as arctangent function with two arguments (see, e.g., Kasana [16]), denoted by $\mathrm{atan2}(x, y)$. Hence, the real and imaginary part of $\hat{\Psi}_n(u)$ are as given in Proposition 1. □

It has been proposed to standardize the data with some estimates for the location δ and scale parameters γ before estimation procedure. Fama and Roll [17], and later [2,3] used the truncated sample mean for δ and sample quantiles for γ, while [4,5] proposed search methods for the initial estimates of δ and γ. However, we propose scaling by sample median, i.e., apply cumulant estimation procedure on the reduced (by sample median) data. For simplicity, we denote cumulant as well as reduced values' cumulant estimators by $\hat{\boldsymbol{\alpha}}_n, \hat{\boldsymbol{\beta}}_n, \hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{\delta}}_n$, while the estimation method will be specified in context.

Definition 3. *Reduced values' cumulant estimators* $\hat{\boldsymbol{\alpha}}_n, \hat{\boldsymbol{\beta}}_n, \hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{\delta}}_n$ *for the parameters of* $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ *are defined through the cumulant estimators (Definition 2) on* $\frac{1}{m}X_1, \ldots, \frac{1}{m}X_n$ *,i.e., for the parameters of* $\frac{1}{m}X$, *where* $m > 0$ *is the absolute value of the median of the sample* $X_1, \ldots, X_n$ *i.i.d. random variables.*

Note that reduced values' cumulant estimators can only be used for samples with non-zero median.

Proposition 2. *If* $X \sim S(\alpha, \beta, \gamma, \delta; 1)$, *then for any* $m > 0$,

$$\frac{1}{m}X \sim \begin{cases} S(\alpha, \beta, \frac{1}{m}\gamma, \frac{1}{m}\delta; 1), & \alpha \neq 1, \\ S(\alpha, \beta, \frac{1}{m}\gamma, \frac{1}{m}\delta - \frac{2}{\pi}\beta\gamma\frac{1}{m}\ln\frac{1}{m}; 1), & \alpha = 1. \end{cases} \tag{20}$$

Proof. The proof is based on the representation of the property of cumulant function, $\Psi_{aX+b}(u) = \ln \mathbb{E}[e^{iu(aX+b)}] = \ln\left(e^{ibu}\mathbb{E}[e^{i(au)X}]\right) = ibu + \Psi_X(au)$ for any $a \neq 0, b \in \mathbb{R}$. □

In the Section, following Section 4, we empirically search for the arguments $u_1 > 0, u_2 > 0 \in \mathbb{R}, u_1 \neq u_2$ for the estimators $\hat{\boldsymbol{\alpha}}_n, \hat{\boldsymbol{\beta}}_n, \hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{\delta}}_n$. In what follows, we discuss **cumulant estimates**, denoted by $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ (not in bold), i.e., the non-random values computed on a particular realization $x_1, \ldots, x_n$ of a sample $X_1, \ldots, X_n$ i.i.d. random variables.

4. Empirical Search for the Optimal Arguments of Cumulant Estimators

Without loss of generality, we fix location parameter $\delta = 0$ and scale parameter $\gamma = 1$, i.e., study standard stable distributions and by reflection property (e.g., [1,10]) perform simulations for $\beta \in [0, 1]$ only. All simulations are carried out with package "stabledist" [18] in the open-source environment for statistical computing and graphics R [19].

Under each fixed $\alpha \in (0, 2]$, $\beta \in [0, 1]$, we simulate 200 realizations (**replicates**) of the sample $X_1, ..., X_n$, $n = 10^5$ i.i.d stable random variables, $X_i \sim S(\alpha, \beta, \gamma = 1, \delta = 0; 1)$. For each replicate, we calculate the squared errors of cumulant and reduced values' cumulant estimates at several selections of $u_1, u_2 \in \mathbb{R}, u_1 > 0, u_2 > 0, u_1 \neq u_2$. We assess the quality of estimates to be the mean of squared errors (of 200 estimates), denoted by MSE ($\hat{\alpha}_n$), MSE($\hat{\beta}_n$), MSE($\hat{\gamma}_n$), and MSE($\hat{\delta}_n$).

In general, the selection of arguments $u_1 > 0, u_2 > 0 \in \mathbb{R}, u_1 \neq u_2$ is arbitrary. In our empirical search, we focus on pairs where the first argument $u_1 > 0$ is an arbitrary number on the order of magnitude -2, -1 or 0, and the second argument $u_2 > 0$ is a multiple of the first argument. We present (and suggest) in Table 1 an arbitrary example of such set of pairs.

Table 1. Selection of pairs of u_1, u_2.

Argument	Values											
u_1	0.03	0.03	0.03	0.03	0.3	0.3	0.3	0.3	3	3	3	3
u_2	0.09	0.9	9	90	0.09	0.9	9	90	0.09	0.9	9	90

There is no specific reason for the selection of pairs in Table 1. We constructed (and performed simulations on) several similar sets, while in the main simulation used the set in Table 1. For the sake of space, we do not present all simulations (distributions) for pairs of arguments in Table 1. We will just choose four disparate standard stable distributions as examples. In Appendix A, the MSEs of reduced values' estimates for $S(0.5, 0.1, 1, 0; 1)$, $S(0.5, 1, 1, 0; 1)$, $S(1.5, 0.1, 1, 0; 1)$ and $S(1.5, 1, 1, 0; 1)$ for the pairs of arguments are presented in Table 1. In addition, we do not present the MSEs of cumulant estimates, as in all cases they turned out higher than the MSEs of reduced values' cumulant estimates. Naturally, for each stable law in Appendix A, there is a unique best (i.e., the values of MSE($\hat{\alpha}_n$), MSE($\hat{\beta}_n$), MSE($\hat{\gamma}_n$), MSE($\hat{\delta}_n$) are lowest) pair of arguments. However, based on our empirical search (Appendix A and additional simulations), we propose the pair of $u_1 = 0.03, u_2 = 0.09$ as optimal (not necessarily the best).

Remark 1. *In our empirical search, we used the set of pairs arguments presented in Table 1 and proposed the pair of $u_1 = 0.03$ and $u_2 = 0.09$ as optimal.*

In the remark, we accentuate $u_1 = 0.03$ and $u_2 = 0.09$, as they are the only good pair of arguments for $S(1.5, 0.1, 1, 0; 1)$ as well as giving MSEs less than 0.004 (at least) to all considered distributions. However, the quality of estimates may not be as good as for a single simulation as well as for real data. In Section 5, we present simulation over $\alpha \in (0, 2]$, $\beta \in [0, 1]$ at $u_1 = 0.03$, $u_2 = 0.09$, while in Section 6, for the application to non-life insurance losses, we use all the pairs of arguments in Table 1.

5. Simulations on the Effectiveness of Cumulant Estimates at $u_1 = 0.03, u_2 = 0.09$

In this section, we present simulations to assess the effectiveness of $\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n, \hat{\delta}_n$ at the selected pair of arguments $u_1 = 0.03, u_2 = 0.09$ (as proposed in Remark, Remark 1). We fix location parameter $\delta = 0$ and scale parameter $\gamma = 1$, and, by reflection property (e.g., [1,10]), perform simulations for $\beta \in [0, 1]$ only. Similarly to the previous section, under each fixed $\alpha \in (0, 2]$, $\beta \in [0, 1]$, we simulate 200 realizations (**replicates**). We simulate samples of sizes of $n = 10^2, 10^3, 10^4, 10^5$, while, to save space, present the mean squared errors, MSE($\hat{\alpha}_n$), MSE($\hat{\beta}_n$), MSE($\hat{\gamma}_n$), MSE($\hat{\delta}_n$), for $n = 10^5$ only. We note that the quality of estimates strongly depended on the sample size, as the smaller the sample, the lower the quality of estimates. All simulations are carried out with package "stabledist" [18] in the open-source environment for statistical computing and graphics R [19].

5.1. Simulations for $\alpha = 0.25, 0.5, 0.75, 1.25, 1.5, 1.75$

We present in the Appendix B the mean squared errors of cumulant (Definition 2) and of reduced values' cumulant estimates (Definition 3) with $n = 10^5$ in the cases of $\alpha = 0.25, 0.5, 0.75, 1.25, 1.5, 1.75$, $\beta = 0.1, 0.25, 0.5, 0.75, 1$.

Remark 2. *The mean squared errors of reduced values' estimates (at $u_1 = 0.03, u_2 = 0.09$ with $n = 10^5$) for the parameters of $S(\alpha, \beta, \gamma = 1, \delta = 0; 1)$ in the cases of $\alpha = 0.25, 0.5, 0.75, 1.25, 1.5, 1.75$, $\beta = 0.1$, $0.25, 0.5, 0.75, 1$ turned out to be on the order of magnitude -6 (the values of $MSE(\hat{\delta}_n)$ were on the order of magnitude from -6 to -2).*

Based on Appendix B and additional simulations (not presented here), we give the following remark.

Remark 3. *In our simulations, reduced values' estimates turned out of better quality than cumulant estimates in the cases of $\alpha < 1.75$ and $\beta > 0.1$, while not in the cases of $\alpha \geqslant 1.75$ and $\beta \leqslant 0.1$.*

Similarly to our Remark 3 it has been discussed [2–5,7,8,17] that standardising, or scaling, improves estimates as it extenuates the dependence of the estimates on δ, and γ.

5.2. Simulations in the Neighbourhood of $\alpha = 1$

We study reduced values' cumulant estimates in situations where the index of stability α is close to 1, i.e., $0.95 < \alpha < 1.05$, while $\beta = 0.1, 0.25, 0.5, 0.75, 1$. We present in Table 2 the mean squarer errors of some estimates where β was calculated by the Formulas (8), and (10) and δ was calculated by the Formulas (9) and (11).

Table 2. MSEs of cumulant estimates for $S(\alpha, \beta, 1, 0; 1)$, $n = 10^{-5}$.

Formula			**(8)**	**(10)**		**(9)**	**(11)**
α	β	**MSE** $(\hat{\alpha}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\beta}_n)$	**MSE** $(\hat{\gamma}_n)$	**MSE** $(\hat{\delta}_n)$	**MSE** $(\hat{\delta}_n)$
0.95	0.1	0.0000	0.0001	0.0003	0.0000	0.0458	1.6567
0.95	1	0.0003	0.0008	0.0688	0.0023	9×10^1	2×10^2
0.96	0.1	0.0000	0.0001	0.0002	0.0001	0.1537	2.6231
0.96	1	0.0005	0.0012	0.0496	0.0044	5×10^4	3×10^2
0.98	0.1	0.0001	0.0002	0.0003	0.0003	7×10^1	1×10^1
0.98	1	0.0009	0.0027	0.0309	0.0135	4×10^5	1×10^3
0.99	0.1	0.0002	0.0005	0.0005	0.0007	1×10^6	4×10^1
0.99	1	0.0017	0.0045	0.0553	0.0333	2×10^4	4×10^3
1.01	0.1	0.0002	0.0004	0.0004	0.0007	2×10^5	4×10^1
1.01	1	0.0016	0.0060	0.0383	0.0278	4×10^6	4×10^3
1.02	0.1	0.0001	0.0002	0.0002	0.0002	1×10^3	1×10^1
1.02	1	0.0009	0.0027	0.0202	0.0108	4×10^4	9×10^2
1.04	0.1	0.0001	0.0001	0.0002	0.0001	0.1101	2.4367
1.04	1	0.0006	0.0015	0.0340	0.0046	2×10^4	2×10^2
1.05	0.1	0.0000	0.0001	0.0001	0.0000	0.0458	1.5814
1.05	1	0.0005	0.0013	0.0392	0.0032	3×10^2	1×10^2

It follows that reduced values' cumulant estimates fail for the location parameter δ in the neighbourhood of $\alpha = 1$. However, the estimates for other parameters are of better quality. Note that, for skewness parameter β, Formula (8) does not give better estimates than Formula (10), even if α is very close to 1. In addition, again, the MSEs of cumulant estimates (not presented here) were lower than reduced values' cumulant estimates. In addition, we performed cumulant and reduced values' cumulant estimates for all pairs of arguments given in Table 1. It follows that some other pairs of arguments (in Table 1) gave better estimates for the location parameter δ but concurrently fail in estimating the remaining parameters.

5.3. Simulations for $\alpha = 1$

In Table 3, the MSEs of cumulant estimates for the case of $\alpha = 1$ and $\beta = 0.1, 0.25, 0.5, 0.75, 1$ are presented. As in the estimation procedure, $\hat{\alpha}_n$ is not exactly 1. Then, for comparison, we present cumulant estimates for β with Formulas (8) and (10) and for δ with Formulas (9) and (11).

Table 3. MSEs of cumulant estimates for $S(\alpha = 1, \beta, 1, 0; 1)$, $n = 10^{-5}$.

Formula		**(8)**	**(10)**		**(9)**	**(11)**
β	**MSE** ($\hat{\alpha}_n$)	**MSE** ($\hat{\beta}_n$)	**MSE** ($\hat{\beta}_n$)	**MSE** ($\hat{\gamma}_n$)	**MSE** ($\hat{\delta}_n$)	**MSE** ($\hat{\delta}_n$)
0.1	0.0002	0.0004	0.0004	0.0013	5×10^1	0.0012
0.25	0.0002	0.0609	0.0609	0.0216	3×10^3	0.0072
0.5	0.0002	0.2421	0.2422	0.0348	2×10^1	0.0430
0.75	0.0002	0.5476	0.5473	0.0443	1×10^3	0.0938
1	0.0002	0.9717	0.9713	0.3854	5×10^3	0.1925

In Table 3, cumulant estimates for the parameters of $S(\alpha = 1, \beta, 1, 0; 1)$ are of good quality for the index of stability α (as well as for the other pairs of u_1, u_2 in Table 1). As expected, the estimates for the location parameter δ are better with Formula (11) than with Formula (9).

5.4. Simulations for $\alpha \downarrow 0$ and $\alpha \uparrow 2$

In the case of $\alpha \downarrow 0$, stable distributions are very condensed and scale factor γ has not much influence on the shape of the distribution (and may be difficult to estimate). We studied reduced values' estimates for several cases of $\alpha < 0.2$, $\beta \in [0, 1]$. To save space, we do not present simulation results here. We will just summarise our findings by the following remarks.

Remark 4. *For $\alpha < 0.2$ (with $n = 10^5$), MSE($\hat{\alpha}_n$) and MSE($\hat{\delta}_n$) of reduced values' estimates turned out to be on the order of magnitude from -6 to -3, while the method fails for the scale parameter γ and (based on Equation (8)) for the skewness parameter β.*

For the case of $\alpha \uparrow 2$, it is discussed (see, e.g., [10,20]) that the value of β loses its effect (and may be difficult to estimate) as stable distributions get close to the normal distributions. We studied reduced values' estimates for several cases of $\alpha > 1.8$, $\beta \in [0, 1]$ and summarise our findings with the following remark.

Remark 5. *For $\alpha > 1.8$ (with $n = 10^5$), the MSE($\hat{\alpha}_n$), MSE($\hat{\gamma}_n$), and MSE($\hat{\delta}_n$) of reduced values' estimates turned out to be on the order of magnitude from -6 to -3, while the method fails for the skewness parameter, β.*

6. Application in Non-Life Insurance

We consider an Estonian data set on fire, natural forces and other property insurance claim sizes of legal persons in a calendar year. The sample contains 2802 losses (EUR), and the summary statistics are in Table 4.

Table 4. Summary of non-life insurance claims (EUR) $n = 2802$.

Min	**1st Quartile**	**Median**	**Mean**	**3rd Quartile**	**Maximum**
15.3	358.0	955.0	6703.0	2781.0	1166000.0

Based on the histogram of losses in Figure 1 and statistics in Table 4, the data is skewed, i.e., we expect the estimate for β to be close to 1. In addition, as the data is not condensed, it is natural to expect that estimates for α are not close to 0, as well as not close to 2 (claim sizes clearly are not normally distributed). We fit losses using cumulant and reduced values' cumulant estimates for all pairs of arguments suggested in Table 1. Before estimating, we apply a simple non-parametric bootstrap with replacement.

6.1. Cumulant Estimates for Claim Sizes

We present in Table 5 the mean values of cumulant estimates for 200 bootstrap replicates. We remind readers that cumulant estimates are the same as Press's [6] estimates, where we used the same pair of arguments for all parameters.

Table 5. Cumulant estimates for the data of claims ($n = 2802$).

u_1	u_2	Mean ($\hat{\alpha}_n$)	Mean ($\hat{\beta}_n$)	Mean ($\hat{\gamma}_n$)	Mean ($\hat{\delta}_n$)
0.03	0.09	0.13	1.62	9×10^{77}	–3.77
0.03	0.9	0.01	6.70	Inf	1.53
0.03	9	0.03	2.98	Inf	–0.10
0.03	90	0.02	–0.76	Inf	–0.02
0.3	0.09	0.04	–0.42	Inf	3.05
0.3	0.9	–0.17	–1.22	Inf	1.93
0.3	9	–0.01	–1.46	Inf	–0.17
0.3	90	–0.01	0.79	Inf	–0.02
3	0.09	0.01	–2.35	Inf	0.12
3	0.9	0.14	2.18	3×10^{125}	–0.87
3	9	–0.04	0.36	5×10^{184}	–0.29
3	90	–0.03	0.59	Inf	–0.02

Inf–infinity.

As previously discussed, the estimates for α should not be close to 0, while estimates for β should be close to 1. In addition, cumulant estimates for scale parameter γ turned out infinite (Inf). Hence, cumulant estimates in Table 5 are not meaningful, as also mentioned in [5] about Press's [6] method.

6.2. Reduced Values' Cumulant Estimates for Claim Sizes

We present in Table 6 the mean and coefficient of variation of reduced values' cumulant estimates for 200 bootstrap replicates from claimsdata.

Table 6. Reduced values' cumulant estimates for data of claims ($n = 2802$).

u_1	u_2	Mean ($\hat{\alpha}_n$)	CV ($\hat{\alpha}_n$)	Mean ($\hat{\beta}_n$)	CV ($\hat{\beta}_n$)	Mean ($\hat{\gamma}_n$)	CV ($\hat{\gamma}_n$)	Mean ($\hat{\delta}_n$)	CV ($\hat{\delta}_n$)
0.03	9	0.71	0.030	1.19	0.059	382.46	0.073	–432.25	0.206
3	0.09	0.72	0.030	1.12	0.042	444.48	0.045	–574.15	0.209
0.3	9	0.67	0.032	1.17	0.046	410.33	0.059	–335.48	0.181
0.03	0.9	0.77	0.039	1.05	0.057	568.07	0.057	–1113.55	0.328
0.03	90	0.56	0.039	1.78	0.079	119.88	0.192	–103.29	0.181
0.3	0.9	0.80	0.048	1.06	0.055	578.91	0.059	–1460.33	0.342
0.3	90	0.48	0.058	1.87	0.085	181.45	0.175	–85.05	0.183
3	0.9	0.60	0.064	1.18	0.061	475.25	0.046	–283.09	0.345
0.3	0.09	0.75	0.069	1.00	0.072	523.66	0.145	–819.13	0.793
0.03	0.09	0.78	0.099	1.09	0.101	581.48	0.304	–1989.91	1.272
3	9	0.60	0.107	0.94	0.133	484.41	0.092	–139.00	0.608
3	90	0.33	0.151	2.00	0.157	690.79	0.194	–60.62	0.204

CV–coefficient of variation.

Table 6 is sorted increasingly by the variation of coefficient of $\hat{\alpha}_n$. Reduced values' estimates for the parameters of stable distribution in Table 6 are similar for several pairs of arguments. Based on the smallest coefficients of variation, we choose estimates at $u_1 = 3$, $u_2 = 0.09$, i.e., the stable distribution $S(\alpha = 0.72, \beta = 1, \gamma = 444, \delta = -574; 1)$. However, similar to how it is discussed by Borak et al. [20] for Press"s [6] procedure, the optimal selection of arguments is still an open question. Clearly, optimal selection of u_1 and u_2 is related to scaling, i.e., reducing by the median. However, the relationship is not clear and needs further study.

6.3. Comparison to Other Estimation Methods

We compare the reduced values' cumulant estimates with other commonly used estimation procedures, i.e., maximum likelihood method by Nolan [21], quantile based method by McCulloch [22] and empirical characteristic function based method by Koutrouvelis [2,3], and Kogon and Williams [4]. The estimates are performed with the STABLE program version, manufacturer,... (version 3.14.02, John P. Nolan, Washington, DC, USA) [23]. We present the estimated stable distributions in Table 7.

Table 7. Modelling non-life insurance losses ($n = 2802$) via stable distributions.

Fitting Method	Estimated Stable Distribution
Reduced values' cumulant estimates	$S(\alpha = 0.72, \beta = 1, \gamma = 444, \delta = -574; 1)$
Characteristic function based [23]	$S(\alpha = 0.78, \beta = 1, \gamma = 581, \delta = -1117; 1)$
Maximum likelihood based [23]	$S(\alpha = 0.60, \beta = 1, \gamma = 606, \delta = -189; 1)$
Quantile based [23]	$S(\alpha = 0.82, \beta = 1, \gamma = 1213, \delta = -3258; 1)$

All estimation methods in Table 7 propose skewed stable distribution ($\beta = 1$) with an index of stability less than 1 ($\alpha < 1$) and negative location parameter ($\delta < 0$). To illustrate the matches, we present in Figure 1 the histogram of losses and the density functions of stable distributions from Table 6. Densities in figures are numerically computed with package "stabledist" [18] in R [19].

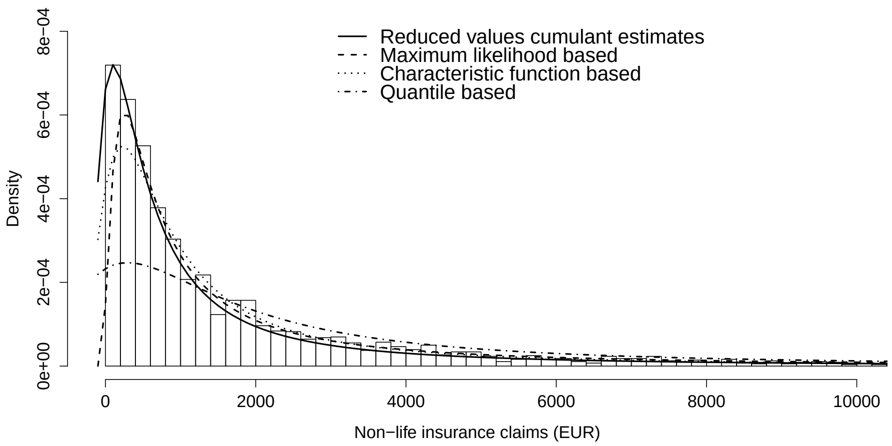

Figure 1. Modelling non-life insurance losses ($n = 2802$) via stable distributions (Table 6).

We also present the tail behaviour in Figure 2. According to Figures 1 and 2, the stable distribution estimated by reduced values' cumulant estimates seem to match best with claims data. Naturally, the best fit should be measured with some goodness-of-fit test. In addition, as the estimated stable laws in Table 6 turned out different then our conclusions, similar to [24], more than one technique should be applied to fit any data with stable distributions.

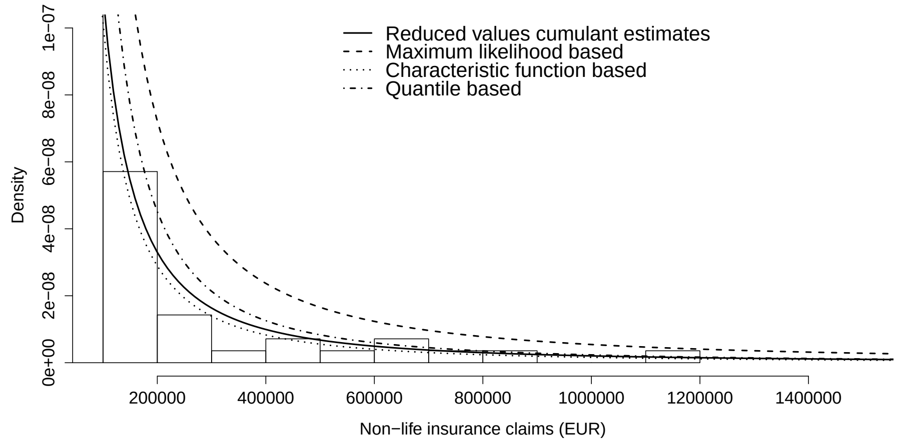

Figure 2. Modelling non-life insurance losses ($n = 2802$) via stable distributions (Table 6).

7. Summary

In this paper we review the procedure of Press [6] for estimating the parameters of stable law. The method is based on cumulant function and leads explicit point estimators for all parameters. Press's [6] estimates depend on an arbitrary choice of two pairs of arguments of empirical characteristic function, and it has not been recommended in practice. In this paper, we

- show that the parameters of stable law can be expressed through cumulant function of one pair of arguments, and hence
- propose the method of Press [6] at one pair of arguments only;
- suggest data scaling by median, i.e., introduce *reduced values' cumulant estimates*;
- perform an empirical search for the selection of two arguments;
- carry out simulation experiments over parameter space at arguments of $u_1 = 0.03$ and $u_2 = 0.09$;
- present an application to non-life insurance losses;

According to our simulations reduced values' estimates are of good quality for large samples ($n > 10^4$) at empirically selected arguments of $u_1 = 3$ $u_2 = 0.09$. In our simulations reduced values' cumulant estimates turned out of better quality than Press's [6] estimates at almost all empirically studied cases. Also, based on our application reduced values' cumulant estimates can be suggested in practice.

An area for further research is to study the cumulant function based method under some other parametrization of stable laws, as well as some different shifting and scaling of data. Also, the cumulant function based method may be generalized to the multivariate case. Withal, the optimal selection of the two arguments still is an open question.

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Appendix A

Table A1. The MSEs of reduced values' cumulant estimates for the selection of u_1, u_2 for 200 replicates (sample size $n = 10^5$) from $S(\alpha, \beta, \gamma = 1, \delta = 0; 1)$.

$S(\alpha = 0.5, \beta = 0, \gamma = 1, \delta = 0; 1)$												
u_1	0.03	0.03	0.03	0.03	0.3	0.3	0.3	0.3	3	3	3	3
u_2	0.09	0.9	9	90	0.09	0.9	9	90	0.09	0.9	9	90
MSE ($\hat{\alpha}_n$)	0.0000	0.0000	0.0059	0.0392	0.0000	0.0002	0.0163	0.0755	0.0007	0.0045	0.1520	0.2151
MSE ($\hat{\beta}_n$)	0.0002	0.0001	0.0012	0.0091	0.0002	0.0007	0.0086	0.0311	0.0005	0.0231	4×10^1	3×10^1
MSE ($\hat{\gamma}_n$)	0.0001	0.0002	0.0425	0.3183	0.0001	0.0011	0.2077	7.7057	0.0002	0.1267	Inf	Inf
MSE ($\hat{\delta}_n$)	0.0003	0.0000	0.0006	0.0000	0.0001	0.0002	0.0006	0.0000	0.0008	0.0030	0.0014	0.0000
$S(\alpha = 0.5, \beta = 1, \gamma = 1, \delta = 0; 1)$												
u_1	0.03	0.03	0.03	0.03	0.3	0.3	0.3	0.3	3	3	3	3
u_2	0.09	0.9	9	90	0.09	0.9	9	90	0.09	0.9	9	90
MSE ($\hat{\alpha}_n$)	0.0001	0.0000	0.0000	0.0003	0.0000	0.0000	0.0000	0.0005	0.0000	0.0000	0.0001	0.0014
MSE ($\hat{\beta}_n$)	0.0003	0.0001	0.0001	0.0043	0.0003	0.0002	0.0001	0.0146	0.0001	0.0002	0.0004	0.0993
MSE ($\hat{\gamma}_n$)	0.0036	0.0002	0.0002	0.0183	0.0012	0.0005	0.0002	0.0081	0.0001	0.0001	0.0003	0.0009
MSE ($\hat{\delta}_n$)	0.0230	0.0006	0.0001	0.0218	0.0047	0.0013	0.0001	0.0235	0.0002	0.0004	0.0003	0.0318
$S(\alpha = 1.5, \beta = 0, \gamma = 1, \delta = 0; 1)$												
u_1	0.03	0.03	0.03	0.03	0.3	0.3	0.3	0.3	3	3	3	3
u_2	0.09	0.9	9	90	0.09	0.9	9	90	0.09	0.9	9	90
MSE ($\hat{\alpha}_n$)	0.0001	0.3339	0.8957	1.2256	0.0933	2.2773	2.2475	2.2518	1.1632	2.2708	2.2755	2.2477
MSE ($\hat{\beta}_n$)	0.0002	0.0414	0.0369	0.0594	0.2565	2×10^2	1×10^2	4×10^2	0.0653	2×10^2	2×10^2	4×10^2
MSE ($\hat{\gamma}_n$)	0.0000	0.1838	0.6175	0.8490	0.0056	Inf	Inf	Inf	1.0659	Inf	Inf	Inf
MSE ($\hat{\delta}_n$)	0.0001	5×10^3	0.0003	0.0000	8×10^1	0.1019	0.0004	0.0000	0.0029	0.0090	0.0010	0.0000
$S(\alpha = 1.5, \beta = 1, \gamma = 1, \delta = 0; 1)$												
u_1	0.03	0.03	0.03	0.03	0.3	0.3	0.3	0.3	3	3	3	3
u_2	0.09	0.9	9	90	0.09	0.9	9	90	0.09	0.9	9	90
MSE ($\hat{\alpha}_n$)	0.0004	0.0001	0.1231	0.4640	0.0001	0.0000	0.3508	0.9181	0.0114	0.0928	2.2727	2.2629
MSE ($\hat{\beta}_n$)	0.0011	0.0003	1.2741	1.9049	0.0002	0.0002	2.4033	4.4211	1.6320	5.1234	2×10^2	4×10^2
MSE ($\hat{\gamma}_n$)	0.0008	0.0000	0.3852	0.8615	0.0000	0.0000	0.1876	0.6170	0.0216	0.0039	Inf	Inf
MSE ($\hat{\delta}_n$)	0.0003	0.0002	0.1416	0.0045	0.0002	0.0002	8.5030	0.0030	0.2452	5×10^1	0.0973	0.0005

Inf–infinity.

Appendix B

Table B1. The MSEs of reduced values' cumulant estimates (RVCE) and cumulant estimates (CE) at $u_1 = 0.03, u_2 = 0.09$ for 200 replicates (sample size $n = 10^5$) from $S(\alpha, \beta, \gamma = 1, \delta = 0; 1)$.

α	β	Method	MSE ($\hat{\alpha}_n$)	MSE ($\hat{\beta}_n$)	MSE ($\hat{\gamma}_n$)	MSE ($\hat{\delta}_n$)
0.25	0.1	RVCE	5.7×10^{-6}	6.9×10^{-5}	1.5×10^{-4}	1.7×10^{-6}
0.25	0.1	CE	3.8×10^{-5}	6.3×10^{-4}	5.5×10^{-3}	5.5×10^{-3}
0.25	0.25	RVCE	4.1×10^{-6}	4.9×10^{-5}	6.6×10^{-5}	1.4×10^{-5}
0.25	0.25	CE	3.5×10^{-5}	4.6×10^{-4}	4.6×10^{-3}	4.6×10^{-3}
0.25	0.5	RVCE	4.3×10^{-6}	5.2×10^{-5}	3.8×10^{-4}	1.7×10^{-4}
0.25	0.5	CE	3.9×10^{-5}	6.1×10^{-4}	5.8×10^{-3}	5.8×10^{-3}
0.25	0.75	RVCE	3.4×10^{-6}	5.9×10^{-5}	5.8×10^{-4}	8.9×10^{-4}
0.25	0.75	CE	3.6×10^{-5}	6.1×10^{-4}	5.4×10^{-3}	5.4×10^{-3}
0.25	1	RVCE	4.2×10^{-6}	1.0×10^{-4}	1.1×10^{-3}	4.3×10^{-3}
0.25	1	CE	4.3×10^{-5}	7.1×10^{-4}	6.3×10^{-3}	6.3×10^{-3}
0.5	0.1	RVCE	4.1×10^{-6}	1.8×10^{-5}	1.5×10^{-5}	3.0×10^{-5}
0.5	0.1	CE	4.6×10^{-5}	3.2×10^{-4}	4.3×10^{-3}	4.3×10^{-3}

Table B1. *Cont.*

α	β	Method	MSE ($\hat{\alpha}_n$)	MSE ($\hat{\beta}_n$)	MSE ($\hat{\gamma}_n$)	MSE ($\hat{\delta}_n$)
0.5	0.25	RVCE	3.7×10^{-6}	2.5×10^{-5}	5.5×10^{-5}	1.2×10^{-4}
0.5	0.25	CE	6.7×10^{-5}	2.4×10^{-4}	3.6×10^{-3}	3.6×10^{-3}
0.5	0.5	RVCE	5.5×10^{-6}	2.4×10^{-5}	1.5×10^{-4}	3.2×10^{-4}
0.5	0.5	CE	5.8×10^{-5}	2.6×10^{-4}	4.4×10^{-3}	4.4×10^{-3}
0.5	0.75	RVCE	7.4×10^{-6}	3.6×10^{-5}	2.9×10^{-4}	1.1×10^{-3}
0.5	0.75	CE	5.5×10^{-5}	2.8×10^{-4}	6.1×10^{-3}	6.1×10^{-3}
0.5	1	RVCE	8.5×10^{-6}	3.9×10^{-5}	4.5×10^{-4}	2.5×10^{-3}
0.5	1	CE	5.5×10^{-5}	3.0×10^{-4}	8.4×10^{-3}	8.4×10^{-3}
0.75	0.1	RVCE	4.5×10^{-6}	1.6×10^{-5}	2.0×10^{-5}	1.9×10^{-4}
0.75	0.1	CE	9.9×10^{-5}	3.1×10^{-4}	7.3×10^{-3}	7.3×10^{-3}
0.75	0.25	RVCE	8.3×10^{-6}	2.5×10^{-5}	8.8×10^{-5}	6.7×10^{-4}
0.75	0.25	CE	1.2×10^{-4}	3.1×10^{-4}	9.6×10^{-3}	9.6×10^{-3}
0.75	0.5	RVCE	1.2×10^{-5}	3.9×10^{-5}	2.0×10^{-4}	2.5×10^{-3}
0.75	0.5	CE	9.9×10^{-5}	2.5×10^{-4}	1.7×10^{-2}	1.7×10^{-2}
0.75	0.75	RVCE	2.0×10^{-5}	4.0×10^{-5}	4.2×10^{-4}	8.3×10^{-3}
0.75	0.75	CE	9.5×10^{-5}	2.3×10^{-4}	2.8×10^{-2}	2.8×10^{-2}
0.75	1	RVCE	2.1×10^{-5}	5.4×10^{-5}	5.3×10^{-4}	1.6×10^{-2}
0.75	1	CE	9.4×10^{-5}	2.3×10^{-4}	4.3×10^{-2}	4.3×10^{-2}
1.25	0.1	RVCE	5.8×10^{-6}	1.5×10^{-5}	5.7×10^{-6}	5.7×10^{-5}
1.25	0.1	CE	3.2×10^{-4}	8.5×10^{-4}	1.4×10^{-3}	1.4×10^{-3}
1.25	0.25	RVCE	1.8×10^{-5}	3.9×10^{-5}	4.3×10^{-5}	1.3×10^{-4}
1.25	0.25	CE	4.1×10^{-4}	1.0×10^{-3}	2.7×10^{-3}	2.7×10^{-3}
1.25	0.5	RVCE	3.8×10^{-5}	7.8×10^{-5}	1.5×10^{-4}	4.0×10^{-4}
1.25	0.5	CE	4.3×10^{-4}	7.7×10^{-4}	4.8×10^{-3}	4.8×10^{-3}
1.25	0.75	RVCE	5.6×10^{-5}	1.4×10^{-4}	3.0×10^{-4}	8.9×10^{-4}
1.25	0.75	CE	4.0×10^{-4}	7.2×10^{-4}	7.9×10^{-3}	7.9×10^{-3}
1.25	1	RVCE	8.8×10^{-5}	1.3×10^{-4}	5.5×10^{-4}	1.9×10^{-3}
1.25	1	CE	3.8×10^{-4}	6.0×10^{-4}	1.2×10^{-2}	1.2×10^{-2}
1.5	0.1	RVCE	3.8×10^{-6}	1.8×10^{-5}	1.2×10^{-6}	1.2×10^{-5}
1.5	0.1	CE	5.9×10^{-4}	1.9×10^{-3}	2.2×10^{-4}	2.2×10^{-4}
1.5	0.25	RVCE	5.0×10^{-6}	2.2×10^{-5}	2.9×10^{-6}	1.2×10^{-5}
1.5	0.25	CE	6.0×10^{-4}	2.0×10^{-3}	2.4×10^{-4}	2.4×10^{-4}
1.5	0.5	RVCE	1.6×10^{-5}	4.7×10^{-5}	1.7×10^{-5}	1.8×10^{-5}
1.5	0.5	CE	6.2×10^{-4}	1.9×10^{-3}	2.6×10^{-4}	2.6×10^{-4}
1.5	0.75	RVCE	2.4×10^{-5}	7.4×10^{-5}	4.0×10^{-5}	2.3×10^{-5}
1.5	0.75	CE	5.6×10^{-4}	1.9×10^{-3}	3.3×10^{-4}	3.3×10^{-4}
1.5	1	RVCE	3.4×10^{-5}	9.8×10^{-5}	6.9×10^{-5}	3.2×10^{-5}
1.5	1	CE	5.9×10^{-4}	1.7×10^{-3}	3.9×10^{-4}	3.9×10^{-4}
1.75	0.1	RVCE	8.3×10^{-4}	4.7×10^{-3}	4.6×10^{-5}	4.6×10^{-5}
1.75	0.1	CE	4.5×10^{-2}	4.6×10^{-1}	1.2×10^{-3}	4.2×10^{-1}
1.75	0.25	RVCE	4.0×10^{-6}	6.3×10^{-5}	1.2×10^{-6}	6.3×10^{-6}
1.75	0.25	CE	8.0×10^{-4}	6.1×10^{-3}	7.1×10^{-5}	7.1×10^{-5}

Table B1. *Cont.*

α	β	Method	MSE ($\hat{\alpha}_n$)	MSE ($\hat{\beta}_n$)	MSE ($\hat{\gamma}_n$)	MSE ($\hat{\delta}_n$)
1.75	0.5	RVCE	3.0×10^{-6}	3.1×10^{-5}	1.0×10^{-6}	4.2×10^{-6}
1.75	0.5	CE	7.8×10^{-4}	7.1×10^{-3}	6.5×10^{-5}	6.5×10^{-5}
1.75	0.75	RVCE	5.7×10^{-6}	4.4×10^{-5}	2.0×10^{-6}	4.6×10^{-6}
1.75	0.75	CE	7.6×10^{-4}	7.4×10^{-3}	5.8×10^{-5}	5.8×10^{-5}
1.75	1	RVCE	8.1×10^{-6}	9.1×10^{-5}	3.4×10^{-6}	5.2×10^{-6}
1.75	1	CE	9.7×10^{-4}	9.4×10^{-3}	7.3×10^{-5}	7.3×10^{-5}

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