Risks **2014**, *2*, 434-455; doi:10.3390/risks2040434

OPEN ACCESS
risks
ISSN 2227-9091
www.mdpi.com/journal/risks

Article

A Markov Chain Model for Contagion

Angelos Dassios [1,*] and Hongbiao Zhao [2,*]

[1] Department of Statistics, London School of Economics, Houghton Street, London WC2A 2AE, UK
[2] Department of Finance, School of Economics & Wang Yanan Institute for Studies in Economics, Xiamen University, Xiamen, Fujian 361005, China

* Authors to whom correspondence should be addressed; E-Mails: a.dassios@lse.ac.uk (A.D.); hongbiao.z@gmail.com (H.Z.); Tel.: +44-2079557749 (A.D.).

External Editor: Dr. Montserrat Guillén

Received: 22 September 2014; in revised form: 27 October 2014 / Accepted: 29 October 2014 / Published: 5 November 2014

Abstract: We introduce a bivariate Markov chain counting process with contagion for modelling the clustering arrival of loss claims with delayed settlement for an insurance company. It is a general continuous-time model framework that also has the potential to be applicable to modelling the clustering arrival of events, such as jumps, bankruptcies, crises and catastrophes in finance, insurance and economics with both internal contagion risk and external common risk. Key distributional properties, such as the moments and probability generating functions, for this process are derived. Some special cases with explicit results and numerical examples and the motivation for further actuarial applications are also discussed. The model can be considered a generalisation of the dynamic contagion process introduced by Dassios and Zhao (2011).

Keywords: risk model; contagion risk; bivariate point process; Markov chain model; discretised dynamic contagion process; dynamic contagion process

---

1. Introduction

A self-exciting point process introduced earlier by Hawkes [1] and Hawkes [2] and later named as a Hawkes process, nowadays becomes a viable mathematical tool for modelling contagion risk and the clustering arrival of events in finance, insurance and economics; see Errais *et al.* [3], Embrechts *et al.* [4], Chavez-Demoulin and McGill [5], Bacry *et al.* [6] and Aït-Sahalia *et al.* [7]. More recently, Dassios and Zhao [8] introduced a more generalised self-exciting point process, named the dynamic contagion process (DCP), by extending the Hawkes process and the Cox process with exponentially decaying shot-noise intensity; the intensity process includes two types of random jumps—the self-excited and externally-excited jumps—which could be used to model the dynamics of the contagion impact from both the endogenous and exogenous factors of the underlying system in a single consistent framework.

In this paper, we introduce a new bivariate point process named the discretised dynamic contagion process (DDCP) for modelling the clustering arrival of loss claims with delayed settlement for an insurance company. This process in fact generalises the zero-reversion dynamic contagion process (ZDCP), an important special case of DCP with zero-reversion intensity (see Definition A.1). DDCP is a piecewise deterministic Markov process, and some key distributional properties, such as the moments and probability generating functions, have been derived. We also find interesting explicit results for some special cases. By comparing their infinitesimal generators and distribution functions, the transformation formulas between DDCP and ZDCP are obtained, and we find that the two processes are analogous and share some key distributional properties.

This new point process provides a general Markov chain framework. It has the potential to be applicable to modelling the clustering arrival of events such as jumps, bankruptcies, crises, catastrophes in finance, insurance and economics with both internal contagion risk and external common risk. Dassios and Zhao [9] studied the ruin problem for a special case of this model. This was a simple risk model with delayed claims. The claims arrive following a Poisson process, and each of the claims would be settled in an exponentially delayed period of time. Our paper extends this risk model to involve multiple arrivals and delayed settlements of claims with contagion.

The paper is organised as follows. Section 2 describes our model framework and gives a mathematical definition of the associated risk process. In Section 3, we derive the main results: distributional properties of the process, such as the moments and the probability generating functions. Some special cases with explicit results and numerical examples are also discussed in Section 4. The comparison analysis and transformation formulas between DDCP and ZDCP are presented in Appendix A.

2. Model Framework

For an insurance company at any time $t > 0$, suppose N_t is the number of cumulative settled claims within the time interval $[0, t]$ and M_t is the number of cumulative unsettled claims within the same time interval $[0, t]$. We assume that the claims arrive in clusters. Multiple claims may arrive simultaneously at the same time point. The clusters follow a Poisson process of constant rate ρ. They contain a random number K_P of claims with the associated probability function p_k. Each of the claims then will be settled with exponential delay of constant rate δ. We further assume that at each of the settlement times, only one claim can be settled. In practice, this settlement is partial, as a random number K_Q of new claims with the associated probability q_k are revealed and need further to be settled in the future. For a practical point of view, the assumption that only one claim can be settled appears restrictive, but this can be addressed by adjusting the rate of settlement and the distribution of new claims revealed. The assumption is common in the literature; see Yuen *et al.* [10] and the references therein.

The joint stochastic process $\{(N_t, M_t)\}_{t\geq 0}$ is a bivariate continuous-time Markov chain point process on state space $\mathbb{N}_0 \times \mathbb{N}_0$ with intensity of N_t given by ρp_k for a transition from state (i,j) to $(i+k,j)$ and intensity of M_t given by $\delta j q_k$ for a transition from state (i,j) to $(i+k-1, j+1)$, *i.e.*, the joint increment distribution of this process is specified by:

$$\begin{aligned} P\{M_{t+\Delta t} - M_t = k, N_{t+\Delta t} - N_t = 0 \mid \mathcal{F}_t\} &= \rho p_k \Delta t + o(\Delta t), \quad k = 1, 2..., \\ P\{M_{t+\Delta t} - M_t = k-1, N_{t+\Delta t} - N_t = 1 \mid \mathcal{F}_t\} &= \delta M_t q_k \Delta t + o(\Delta t), \quad k = 0, 1..., \\ P\{M_{t+\Delta t} - M_t = 0, N_{t+\Delta t} - N_t = 0 \mid \mathcal{F}_t\} &= 1 - \big(\rho(1-p_0) + \delta M_t\big)\Delta t + o(\Delta t), \\ P\{\text{Others} \mid \mathcal{F}_t\} &= o(\Delta t) \end{aligned}$$

where:

- $\delta, \rho > 0$ are constants;
- Δt is a sufficient, small time interval and $o(\Delta t)/\Delta t \to 0$ when $\Delta t \to 0$;
- K_P and K_Q follow the probability distributions on $\mathbb{N}_0$ by:

$$p_k =: P\{K_P = k\}, \quad q_k =: P\{K_Q = k\}, \quad k = 0, 1...$$

- $\mathcal{F}_t$ is the filtration generated by the joint process $\{(N_s, M_s)\}_{0\leq s\leq t}$.

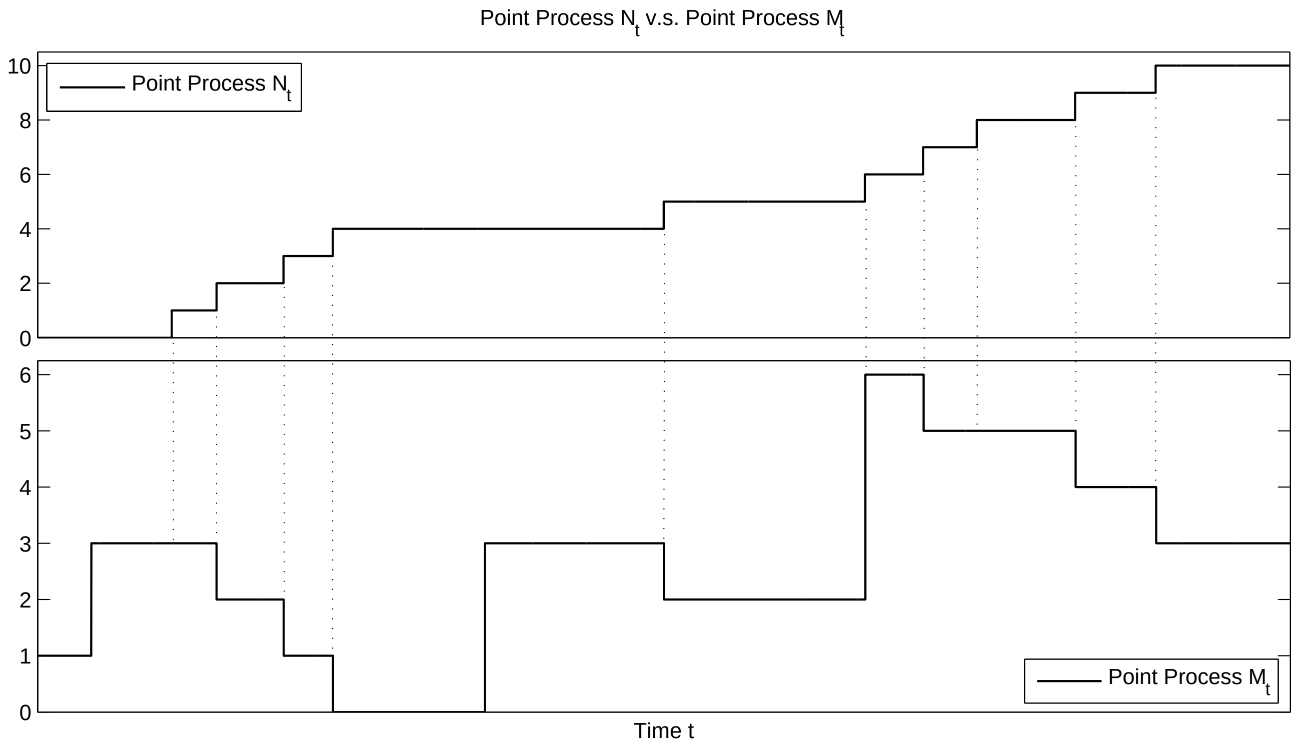
K_P and K_Q are two types of batches of jumps in point process M_t: K_P jumps independently of N_t, whereas K_Q jumps simultaneously with N_t. The first moments and probability generating functions of K_P and K_Q are denoted respectively by:

$$\mu_{1_P} =: \sum_{k=0}^{\infty} k p_k, \quad \mu_{1_Q} =: \sum_{k=0}^{\infty} k q_k; \quad \hat{p}(\theta) =: \sum_{k=0}^{\infty} \theta^k p_k, \quad \hat{q}(\theta) =: \sum_{k=0}^{\infty} \theta^k q_k$$

We can find that, by transformation, $\{(N_t, M_t)\}_{t\geq 0}$ is the generalisation of a special case of the dynamic contagion process [8], and hence, we name this process as a discretised dynamic contagion process. To understand this new process intuitively, a sample path of $\{(N_t, M_t)\}_{t\geq 0}$ is provided in Figure 1.

The process $\{(N_t, M_t)\}_{t\geq 0}$ could be a useful risk model for modelling the interim payments (claims) in insurance, such as cases of IBNR (incurred, but not reported) and IBNS (incurred, but not settled). This general framework can be also considered as the generalisation of a simpler risk model with delayed settlement used by Dassios and Zhao [9] where they assume that the arrival of claims follows a Poisson process of rate ρ, and each of the claims will be settled with an exponential delay of rate δ; however, there is no cluster arrival of claims nor any new claim revealed. The literature on delayed claims in insurance can also be found in Yuen *et al.* [10] for instance.

Figure 1. Point process N_t*vs*. Point process M_t.



Note that, the point process $\{M_t\}_{t\geq 0}$ is a non-negative process, as if $M_t = 0$, there is no joint jump and M_t cannot be brought downward further by one step or more; if $M_t = 1, 2, ...$, M_t is possible downward movement with a maximum of one step. The discrete piecewise non-negative process $\{\delta M_t\}_{t\geq 0}$, in fact, can be considered as the intensity process of the point process N_t (proven later by Equation (4)).

3. Distributional Properties

The infinitesimal generator of a discretised dynamic contagion process (M_t, N_t, t) acting on a function $f(m, n, t) \in \Omega(\mathcal{A})$ is given by:

$$\begin{aligned}\mathcal{A}f(m,n,t) &= \frac{\partial f}{\partial t} + \rho\left(\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right) \\ &\quad +\delta m\left(\sum_{k=0}^{\infty} f(m+k-1,n+1,t)q_k - f(m,n,t)\right) \qquad (1)\end{aligned}$$

where $\Omega(\mathcal{A})$ is the domain of the generator $\mathcal{A}$, such that $f(m, n, t)$ is differentiable with respect to t, and for all m, n and t,

$$\begin{aligned}\left|\sum_{k=0}^{\infty} f(m+k,n,t)p_k - f(m,n,t)\right| &< \infty \\ \left|\sum_{k=0}^{\infty} f(m+k-1,n+1,t)q_k - f(m,n,t)\right| &< \infty\end{aligned}$$

Following the methods adopted by Dassios and Embrechts [11] and later by Dassios and Jang [12] and Dassios and Zhao [8], we will use this generator Equation (1) with the aid of some properly selected martingales to find key distributional properties of (N_t, M_t) as below.

3.1. Moments of M_t and N_t

We derive the first moments of M_t and N_t by solving systems of ODEs and also discuss the stationarity condition for the process M_t.

Theorem 3.1. *The expectation of M_t conditional on M_0 is given by:*

$$\mathbb{E}[M_t \mid M_0] = \begin{cases} \frac{\mu_{1_P}\rho}{\kappa} + \left(M_0 - \frac{\mu_{1_P}\rho}{\kappa}\right) e^{-\kappa t}, & \kappa \neq 0 \\ M_0 + \mu_{1_P}\rho t, & \kappa = 0 \end{cases} \tag{2}$$

where $\kappa = \delta(1 - \mu_{1_Q})$.

Proof. Set $f(m, n, t) = m$ and plug into generator Equation (1); we have:

$$\mathcal{A}m = \rho\mu_{1_P} + \delta m \left(\mu_{1_Q} - 1\right)$$

or $\mathcal{A}m = -\kappa m + \mu_{1_P}\rho$. Since $M_t - M_0 - \int_0^t \mathcal{A}M_s \mathrm{d}s$ is a martingale, then,

$$\mathbb{E}[M_t|M_0] = M_0 + \mathbb{E}\left[\int_0^t \mathcal{A}M_s \mathrm{d}s \middle| M_0\right] = M_0 - \kappa \int_0^t \mathbb{E}[M_s|M_0]\mathrm{d}s + \mu_{1_P}\rho t$$

and we can derive the expectation via the ODE:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\kappa u(t) + \mu_{1_P}\rho$$

where $u(t) = \mathbb{E}[M_t|M_0]$ with the initial condition: $u(0) = M_0$ □

Remark 3.2. *The stationarity condition of process M_t is:*

$$\mu_{1_Q} < 1 \tag{3}$$

Corollary 3.3. *The expectation of N_t conditional on M_0 is given by:*

$$\mathbb{E}[N_t \mid M_0] = \begin{cases} \frac{\delta}{\kappa}\left[\mu_{1_P}\rho t + \left(M_0 - \frac{\mu_{1_P}\rho}{\kappa}\right)\left(1 - e^{-\kappa t}\right)\right], & \kappa \neq 0 \\ \delta\left(M_0 t + \frac{1}{2}\mu_{1_P}\rho t^2\right), & \kappa = 0 \end{cases}$$

Proof. Set $f(m, n, t) = n$ and plug into generator Equation (1); we have $\mathcal{A}n = \delta m$. Since $N_t - N_0 - \int_0^t \mathcal{A}N_s \mathrm{d}s$ is a martingale, then,

$$\mathbb{E}[N_t|M_0] = N_0 + \mathbb{E}\left[\int_0^t \mathcal{A}N_s \mathrm{d}s \middle| M_0\right] = \delta \int_0^t \mathbb{E}[M_s|M_0]\mathrm{d}s \tag{4}$$

where $\mathbb{E}[M_t|M_0]$ is given by Equation (2). □

Higher moments of M_t and N_t can also be obtained similarly by this ODE method, and we omit them here.

3.2. Joint Probability Generating Function of (M_T, N_T)

Theorem 3.4. *For constants* $0 \leq \theta, \varphi \leq 1$ *and time* $0 \leq t \leq T$*, we have the joint probability generating function of* (M_T, N_T)*,*

$$\mathbb{E}\left[\theta^{(N_T - N_t)}\varphi^{M_T} \mid \mathcal{F}_t\right] = e^{-(c(T)-c(t))}[A(t)]^{M_t} \tag{5}$$

where $A(t)$ *is determined by the non-linear ODE:*

$$A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0$$

with boundary condition $A(T) = \varphi$*; and* $c(t)$ *is determined by:*

$$c(t) = \rho \int_0^t [1 - \hat{p}(A(s))] \,\mathrm{d}s$$

Proof. Assume the exponential affine form:

$$f(m, n, t) = [A(t)]^m \theta^n e^{c(t)}$$

and set $\mathcal{A}f(m, n, t) = 0$ in generator Equation (1); then, we have:

$$\begin{cases} A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0 \\ c'(t) = \rho[1 - \hat{p}(A(t))] \end{cases}$$

Since $[A(t)]^{M_t}\theta^{N_t}e^{c(t)}$ is a martingale, we have:

$$\mathbb{E}\left[[A(T)]^{M_T}\theta^{N_T}e^{c(T)}\middle|\mathcal{F}_t\right] = [A(t)]^{M_t}\theta^{N_t}e^{c(t)}$$

with boundary conditions $A(T) = \varphi$. $\square$

3.3. Probability Generating Function of M_T

Theorem 3.5. *If* $\mu_{1_Q} < 1$*, the probability generating function of* M_T *conditional on* M_0 *is given by:*

$$\mathbb{E}[\varphi^{M_T} \mid M_0] = \exp\left(-\int_{\varphi}^{\mathcal{Q}^{-1}_{\varphi,1}(T)} \frac{\rho[1 - \hat{p}(u)]}{\delta\hat{q}(u) - \delta u}\mathrm{d}u\right) \times \left[\mathcal{Q}^{-1}_{\varphi,1}(T)\right]^{M_0}$$

where:

$$\mathcal{Q}_{\varphi,1}(L) =: \int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u) - \delta u} \tag{6}$$

Proof. Set $t = 0$, $\theta = 1$ and assume $N_0 = 0$ in Theorem 3.4, and we have:

$$\mathbb{E}\left[\varphi^{M_T}\middle|M_0\right] = e^{-c(T)}[A(0)]^{M_0} \tag{7}$$

where $A(0)$ is uniquely determined by the non-linear ODE:

$$A'(t) + \delta\hat{q}(A(t)) - \delta A(t) = 0$$

with boundary condition $A(T) = \varphi$. Under the condition $\mu_{1_Q} < 1$, it can be solved by the following steps:

1. Set $A(t) = L(T-t)$ and $\tau = T - t$; this is equivalent to the initial value problem:
$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = \delta\hat{q}(L(\tau)) - \delta L(\tau) =: f_1(L)$$
with initial condition $L(0) = \varphi$; we define the right-hand side as the function $f_1(L)$.
2. Since $\mu_{1_Q} < 1$, we have:
$$\frac{\mathrm{d}f_1(L)}{\mathrm{d}L} = \delta\left(\sum_{k=0}^{\infty} kL^{k-1}p_k - 1\right) \le \delta\left(\sum_{k=0}^{\infty} kp_k - 1\right) = \delta\left(\mu_{1_Q} - 1\right) < 0, \qquad 0 < L \le 1$$
then, $f_1(L) > 0$ for $0 < L < 1$, since $f_1(1) = 0$.
3. Rewrite as:
$$\frac{\mathrm{d}L}{\delta\hat{q}(L) - \delta L} = \mathrm{d}\tau$$
by integrating both sides from time zero to τ with initial condition $L(0) = \varphi > 0$; we have:
$$\int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u) - \delta u} = \tau$$
where $0 < L \le 1$. We define the function on left-hand side as:
$$\mathcal{Q}_{\varphi,1}(L) =: \int_{\varphi}^{L} \frac{\mathrm{d}u}{\delta\hat{q}(u) - \delta u}$$
then, $\mathcal{Q}_{\varphi,1}(L) = \tau$. Obviously, $L \to \varphi$ when $\tau \to 0$; by convergence test,
$$\lim_{u\to 1} \frac{\frac{1}{1-u}}{\frac{1}{\delta\hat{q}(u)-\delta u}} = \delta \lim_{u\to 1} \frac{\hat{q}(u) - u}{1-u} = \delta \lim_{u\to 1} \frac{\left(\hat{q}(u) - u\right)'}{(1-u)'} = 1 - \mu_{1_Q} > 0$$
and we know that $\int_{\varphi}^{1} \frac{1}{1-u}\mathrm{d}u = \infty$; then,
$$\int_{\varphi}^{1} \frac{\mathrm{d}u}{\delta\hat{q}(u) - \delta u} = \infty$$
Hence, $L \to 1$ when $\tau \to \infty$; the integrand is positive in the domain $u \in [\varphi, 1)$ and also $\mathcal{Q}_{\varphi,1}(L)$ is a strictly increasing function; therefore, $\mathcal{Q}_{\varphi,1}(L) : [\varphi, 1) \to [0, \infty)$ is a well-defined (monotone) function, and its inverse function $\mathcal{Q}_{\varphi,1}^{-1}(\tau) : [0, \infty) \to [\varphi, 1)$ exists.
4. The unique solution is found by:
$$L(\tau) = \mathcal{Q}_{\varphi,1}^{-1}(\tau), \quad \text{or,} \quad A(t) = \mathcal{Q}_{\varphi,1}^{-1}(T-t)$$
5. $A(0)$ is obtained,
$$A(0) = L(T) = \mathcal{Q}_{\varphi,1}^{-1}(T)$$

Then, $c(T)$ is determined by:
$$c(T) = \rho \int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{\varphi,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau$$
by the change of variable $\mathcal{Q}_{\varphi,1}^{-1}(\tau) = u$; we have $\tau = \mathcal{Q}_{\varphi,1}(u)$, and:
$$\int_0^T \left[1 - \hat{h}\left(\mathcal{Q}_{\varphi,1}^{-1}(\tau)\right)\right] \mathrm{d}\tau = \int_{\mathcal{Q}_{\varphi,1}^{-1}(0)}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} [1 - \hat{p}(u)] \frac{\partial\tau}{\partial u} \mathrm{d}u = \int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{1 - \hat{p}(u)}{\delta\hat{q}(u) - \delta u} \mathrm{d}u$$
□

Theorem 3.6. *If $\mu_{1_Q} < 1$, the probability generating function of the asymptotic distribution of M_T is given by:*

$$\lim_{T\to\infty} \mathbb{E}[\varphi^{M_T} \mid M_0] = \exp\left(-\int_{\varphi}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u}\mathrm{d}u\right) \tag{8}$$

and this is also the probability generating function of the stationary distribution of process $\{M_t\}_{t\geq 0}$.

Proof. Since $\lim_{T\to\infty} \mathcal{Q}_{\varphi,1}^{-1}(T) = 1$, and by Theorem 3.5, we have the probability generating function of the asymptotic distribution of M_T immediately.

To further prove the stationarity, by Proposition 9.2 of Ethier and Kurtz [13], we need to prove that, for any function $f \in \Omega(\mathcal{A})$, we have:

$$\sum_{m=0}^{\infty} \mathcal{A}f(m)\aleph_m = 0 \tag{9}$$

where $\mathcal{A}f(m)$ is the infinitesimal generator of the discretised dynamic contagion process acting on $f(m)$, *i.e.*,

$$\mathcal{A}f(m) = \rho\left(\sum_{k=0}^{\infty} f(m+k)p_k - f(m)\right) + \delta m\left(\sum_{k=0}^{\infty} f(m+k-1)q_k - f(m)\right) \tag{10}$$

and $\{\aleph_m\}_{k=0,1,2,...}$ are the probabilities of m with the probability generating function given by Equation (8). Now, we try to solve Equation (9).

For the first term of Equation (9), we have:

$$\begin{aligned} & \sum_{m=0}^{\infty}\left[\rho\left(\sum_{k=0}^{\infty} f(m+k)p_k\right)\right]\aleph_m \\ = \;& \rho\sum_{m=0}^{\infty}\aleph_m\sum_{k=0}^{\infty} f(m+k)p_k \qquad (j=m+k) \\ = \;& \rho\sum_{j=0}^{\infty} f(j)\sum_{k=0}^{j}\aleph_{j-k}p_k \\ = \;& \rho\sum_{m=0}^{\infty} f(m)\sum_{k=0}^{m}\aleph_{m-k}p_k \end{aligned}$$

For the second term of Equation (9), we have:

$$\begin{aligned} & \sum_{m=0}^{\infty}\left[\delta m\left(\sum_{k=0}^{\infty} f(m+k-1)q_k\right)\right]\aleph_m \\ = \;& \delta\sum_{m=0}^{\infty} m\aleph_m\sum_{k=0}^{\infty} f(m+k-1)q_k \\ = \;& \delta\sum_{m=-1}^{\infty}(m+1)\aleph_{m+1}\sum_{k=0}^{\infty} f(m+k)q_k \\ = \;& \delta\sum_{m=0}^{\infty}(m+1)\aleph_{m+1}\sum_{k=0}^{\infty} f(m+k)q_k \qquad (j=m+k) \\ = \;& \delta\sum_{j=0}^{\infty} f(j)\sum_{k=0}^{j}(j-k+1)\aleph_{j-k+1}q_k \\ = \;& \delta\sum_{m=0}^{\infty} f(m)\sum_{k=0}^{m}(m-k+1)\aleph_{m-k+1}q_k \end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{m=0}^{\infty} \mathcal{A}f(m)\aleph_m \\
= & \sum_{m=0}^{\infty} f(m) \left[\rho \left(\sum_{k=0}^{m} \aleph_{m-k} p_k - \aleph_m \right) + \delta \left(\sum_{k=0}^{m} (m-k+1)\aleph_{m-k+1} q_k - m\aleph_m \right) \right] \\
= & \ 0
\end{aligned}$$

for any function $f(m) \in \Omega(\mathcal{A})$; then, we have recursive equation:

$$\rho \left(\sum_{k=0}^{m} \aleph_{m-k} p_k - \aleph_m \right) + \delta \left(\sum_{k=0}^{m} (m-k+1)\aleph_{m-k+1} q_k - m\aleph_m \right) = 0$$

and:

$$\sum_{m=0}^{\infty} \varphi^m \times \left[\rho \left(\sum_{k=0}^{m} \aleph_{m-k} p_k - \aleph_m \right) + \delta \left(\sum_{k=0}^{m} (m-k+1)\aleph_{m-k+1} q_k - m\aleph_m \right) \right] = 0$$

By the Laplace transform:

$$\hat{\aleph}(\varphi) =: \mathcal{L}\{\aleph_m\} = \sum_{m=0}^{\infty} \aleph_m \varphi^m$$

since:

$$\begin{aligned}
& \sum_{m=0}^{\infty} \varphi^m \sum_{k=0}^{m} \aleph_{m-k} p_k \\
= & \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \varphi^{m-k} \aleph_{m-k} \varphi^k p_k \qquad \left(\sum_{m=k}^{\infty} \varphi^{m-k} \aleph_{m-k} = \hat{\aleph}(\varphi) \right) \\
= & \ \hat{\aleph}(\varphi)\hat{p}(\varphi)
\end{aligned}$$

$$\begin{aligned}
& \sum_{m=0}^{\infty} \varphi^m \sum_{k=0}^{m} (m-k+1)\aleph_{m-k+1} q_k \\
= & \sum_{m=0}^{\infty} \varphi^m \sum_{j=1}^{m+1} j\aleph_j q_{m+1-j} \qquad (j = m-k+1) \\
= & \ \frac{1}{\varphi} \sum_{m=0}^{\infty} \varphi^j \sum_{j=1}^{m+1} j\aleph_j q_{m+1-j} \varphi^{m+1-j} \qquad (i = m+1) \\
= & \ \frac{1}{\varphi} \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} q_{i-j} \varphi^{i-j} \varphi^j j\aleph_j \qquad \left(\sum_{i=j}^{\infty} q_{i-j} \varphi^{i-j} = \hat{q}(\varphi) \right) \\
= & \ \hat{q}(\varphi) \sum_{j=0}^{\infty} j\varphi^{j-1}\aleph_j \\
= & \ \hat{q}(\varphi)\hat{\aleph}'(\varphi)
\end{aligned}$$

and:

$$\sum_{m=0}^{\infty} \varphi^m m\aleph_m = \varphi \sum_{m=0}^{\infty} m\aleph_m \varphi^{m-1} = \varphi\hat{\aleph}'(\varphi)$$

we have the ODE of $\hat{\aleph}(\varphi)$,

$$\rho\Big(\hat{\aleph}(\varphi)\hat{p}(\varphi) - \hat{\aleph}(\varphi)\Big) + \delta\Big(\hat{q}(\varphi)\hat{\aleph}'(\varphi) - \varphi\hat{\aleph}'(\varphi)\Big) = 0$$

then,

$$\hat{\aleph}(\varphi) = \hat{\aleph}(1) \exp\left(-\int_{\varphi}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u} \mathrm{d}u\right)$$

with the initial condition $\hat{\aleph}(1) = 1$; hence, we have the unique solution:

$$\hat{\aleph}(\varphi) = \exp\left(-\int_{\varphi}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u} \mathrm{d}u\right)$$

which is exactly given by Equation (8).

Since the distribution $\aleph$ is the unique solution to Equation (9), we have the stationarity for the process $\{M_t\}_{t\geq 0}$. $\square$

Remark 3.7. *If $\mu_{1_Q} < 1$, $M_0 \sim \aleph$, then $M_T \sim \aleph$, since by Theorem 3.6 and Theorem 3.5, we have:*

$$\begin{aligned}
& \mathbb{E}\big[\psi^{M_T}\big] \\
= \; & \mathbb{E}\left[\left[\psi^{M_T} \middle| M_0\right]\right] \\
= \; & \exp\left(-\int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u} \mathrm{d}u\right) \times \mathbb{E}\left[\left[\mathcal{Q}_{\varphi,1}^{-1}(T)\right]^{M_0}\right] \\
= \; & \exp\left(-\int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u} \mathrm{d}u\right) \times \exp\left(-\int_{\mathcal{Q}_{\varphi,1}^{-1}(T)}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u)-\delta u} \mathrm{d}u\right) \\
= \; & \hat{\aleph}(\varphi)
\end{aligned}$$

which also reflects the stationarity of process $\{M_t\}_{t\geq 0}$.

3.4. Probability Generating Function of N_T

Theorem 3.8. *Suppose $\mu_{1_Q} < 1$ and $N_0 = 0$; the probability generating function of N_T conditional on M_0 is given by:*

$$\mathbb{E}\big[\theta^{N_T} \mid M_0\big] = \exp\left(-\int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^{1} \frac{\rho[1-\hat{p}(u)]}{\delta u - \delta\theta\hat{q}(u)} \mathrm{d}u\right) \times \left[\mathcal{Q}_{0,\theta}^{-1}(T)\right]^{M_0}$$

where:

$$\mathcal{Q}_{0,\theta}(L) =: \int_{L}^{1} \frac{\mathrm{d}u}{\delta u - \delta\theta\hat{q}(u)}, \quad 0 \leq \theta < 1 \tag{11}$$

Proof. By setting $t = 0$, $\varphi = 1$ and assuming $N_0 = 0$ in Theorem 3.4, we have:

$$\mathbb{E}\big[\theta^{N_T} \mid M_0\big] = e^{-c(T)}[A(0)]^{M_0} \tag{12}$$

where $A(0)$ is uniquely determined by the non-linear ODE:

$$A'(t) + \delta\theta\hat{q}(A(t)) - \delta A(t) = 0$$

with boundary condition $A(T) = 1$. It can be solved, under the condition $\mu_{1_Q} < 1$, by the following steps:

1. Set $A(t) = L(T-t)$ and $\tau = T-t$,
$$\frac{\mathrm{d}L(\tau)}{\mathrm{d}\tau} = \delta\theta\hat{q}(L(\tau)) - \delta L(\tau) =: f_2(L), \quad 0 \le \theta < 1 \tag{13}$$
with initial condition $L(0) = 1$; we define the right-hand side as the function $f_2(L)$.
2. There is only one positive singular point in the interval $[0,1]$, denoted by:
$$0 \le \varphi^* \le 1 \tag{14}$$
by solving the equation $f_2(L) = 0$. This is because, for the case $0 < \theta < 1$, the equation $f_2(L) = 0$ is equivalent to:
$$\hat{q}(u) = \frac{1}{\theta}u, \quad 0 < \theta < 1$$
note that $\hat{q}(\cdot)$ is a convex function, then it is clear that there is only one positive solution within $[0,1]$ to this equation; in particularly when $\theta \to 0$, $\varphi^* \to 0$. Then, we have $f_2(L) < 0$ for $\varphi^* < L \le 1$.
3. Rewrite Equation (13) as:
$$\frac{\mathrm{d}L}{\delta L - \delta\theta\hat{q}(L)} = -\mathrm{d}\tau$$
and integrate,
$$\int_L^1 \frac{\mathrm{d}u}{\delta u - \delta\theta\hat{q}(u)} = \tau$$
where $\varphi^* < L \le 1$; we define the function on left-hand side as:
$$\mathcal{Q}_{0,\theta}(L) =: \int_L^1 \frac{\mathrm{d}u}{\delta u - \delta\theta\hat{q}(u)}$$
then, $\mathcal{Q}_{0,\theta}(L) = \tau$, as $L \to 1$ when $\tau \to 0$ and $L \to \varphi^*$ when $\tau \to \infty$; the integrand is positive in the domain $u \in (\varphi^*, 1]$ and $L \ge 0$, $\mathcal{Q}_{0,\theta}(L)$ is a strictly decreasing function. Therefore, $\mathcal{Q}_{0,\theta}(L) : (\varphi^*, 1] \to [0,\infty)$ is a well-defined function, and its inverse function $\mathcal{Q}_{0,\theta}^{-1}(\tau) : [0,\infty) \to (\varphi^*, 1]$ exists.
4. The unique solution is found by $L(\tau) = \mathcal{Q}_{0,\theta}^{-1}(\tau)$, or, $A(t) = \mathcal{Q}_{0,\theta}^{-1}(T-t)$.
5. $A(0)$ is obtained,
$$A(0) = L(T) = \mathcal{Q}_{0,\theta}^{-1}(T)$$

Then, $c(T)$ is determined by:
$$c(T) = \rho\int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{0,\theta}^{-1}(\tau)\right)\right]\mathrm{d}\tau$$
where, by the change of variable,
$$\int_0^T \left[1 - \hat{p}\left(\mathcal{Q}_{0,\theta}^{-1}(\tau)\right)\right]\mathrm{d}\tau = \int_{\mathcal{Q}_{0,\theta}^{-1}(T)}^1 \frac{1-\hat{p}(u)}{\delta u - \delta\theta\hat{q}(u)}\mathrm{d}u$$

□

4. Special Cases

In this section, we focus on three important special cases where more explicit results for the distributional properties of the numbers of settled and unsettled claims $\{(N_t, M_t)\}_{t\geq 0}$ can be derived, and the associated numerical examples are also provided.

4.1. Case $p_1 = 1$

Case $p_1 = 1$ is defined as the special case of a discretised dynamic contagion process when:

$$p_1 = 1, \quad \{p_k\}_{k\neq 1} = 0; \quad q_0 = 1, \quad \{q_k\}_{k\neq 0} = 0 \tag{15}$$

This simple case could be applied, for instance, to model the delaying arrival of claims in the ruin problem for an insurance company; see more details in Dassios and Zhao [9].

Theorem 4.1. *For any time $t_2 > t_1 \geq 0$, if $M_{t_1} \sim \mathrm{Poisson}(\upsilon)$, $\upsilon \geq 0$, then,*

$$\begin{aligned} M_{t_2} &\sim \mathrm{Poisson}\left(\upsilon e^{-\delta(t_2-t_1)} + \rho\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right) \\ N_{t_2} - N_{t_1} &\sim \mathrm{Poisson}\left(\upsilon\left(1-e^{-\delta(t_2-t_1)}\right) + \rho\left((t_2-t_1) - \frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)\right) \end{aligned}$$

and they are independent.

Proof. By setting $T = t_2, t = t_1$ in Theorem 3.4, we have:

$$\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}} \mid M_{t_1}\right] = [A(t_1)]^{M_{t_1}} e^{-\left(c(t_2)-c(t_1)\right)}$$

where $A(t)$ and $c(t)$ can be solved explicitly as:

$$\begin{aligned} A(t) &= (\varphi-\theta)e^{-\delta(t_2-t)} + \theta, \\ c(t_2) - c(t_1) &= \rho\left((1-\theta)(t_2-t_1) - (\varphi-\theta)\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right) \end{aligned}$$

The joint probability generating function of M_{t_2} and $N_{t_2} - N_{t_1}$ is given by:

$$\begin{aligned} &\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}}\right] \\ = \; &\mathbb{E}\left[\mathbb{E}\left[\varphi^{M_{t_2}}\theta^{N_{t_2}-N_{t_1}} \mid M_{t_1}\right]\right] \\ = \; &\mathbb{E}\left[[A(t_1)]^{M_{t_1}}\right] e^{-\left(c(t_2)-c(t_1)\right)} \\ = \; &e^{-\upsilon(1-A(t_1))} e^{-\left(c(t_2)-c(t_1)\right)} \\ = \; &\exp\left(-\upsilon\left[(1-\theta) - (\varphi-\theta)e^{-\delta(t_2-t_1)}\right] - \rho\left[(1-\theta)(t_2-t_1) - (\varphi-\theta)\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right]\right) \end{aligned}$$

Set $\theta = 1$ and $\varphi = 1$, respectively; we have Poisson marginal distributions of M_{t_2} and $N_{t_2} - N_{t_1}$, since:

$$\begin{aligned} \mathbb{E}\left[\varphi^{M_{t_2}}\right] &= \exp\left(-(1-\varphi)\left[\upsilon e^{-\delta(t_2-t_1)} + \rho\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right]\right), \\ \mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right] &= \exp\left(-(1-\theta)\left[\upsilon\left(1-e^{-\delta(t_2-t_1)}\right) + \rho\left((t_2-t_1) - \frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)\right]\right) \end{aligned} \tag{16}$$

Obviously, they are also independent as:

$$\mathbb{E}\left[\varphi^{M_{t_2}} \times \theta^{N_{t_2}-N_{t_1}}\right] = \mathbb{E}\left[\varphi^{M_{t_2}}\right] \times \mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right]$$

$\square$

Corollary 4.2. *If* $M_0 \sim \text{Poisson}(\zeta)$, $\zeta \geq 0$, *then:*

$$\begin{aligned} M_t &\sim \text{Poisson}\left(\zeta e^{-\delta t} + \rho \frac{1-e^{-\delta t}}{\delta}\right) \\ N_t &\sim \text{Poisson}\left(\zeta\left(1-e^{-\delta t}\right) + \rho\left(t - \frac{1-e^{-\delta t}}{\delta}\right)\right) \end{aligned}$$

and they are independent.

Proof. Set $t_1 = 0$, $t_2 = t > 0$ and $\upsilon = \zeta$ in Theorem 4.1; the results follow immediately. $\square$

Corollary 4.3. *If* $M_0 \sim \text{Poisson}(\zeta)$, *then* N_t *is a non-homogeneous Poisson process of rate* $\rho + (\zeta\delta - \rho)\, e^{-\delta t}$.

Proof. For any time $t_2 > t_1 \geq 0$, by Corollary 4.2, we have:

$$M_{t_1} \sim \text{Poisson}\left(\zeta e^{-\delta t_1} + \rho \frac{1-e^{-\delta t_1}}{\delta}\right)$$

By Theorem 4.1, set $\upsilon = \zeta e^{-\delta t_1} + \rho \frac{1-e^{-\delta t_1}}{\delta}$ in Equation (16), then,

$$\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\right] = \exp\left(-(1-\theta)\int_{t_1}^{t_2}\left[\zeta\delta e^{-\delta s} + \rho\left(1-e^{-\delta s}\right)\right] \mathrm{d}s\right)$$

hence, the increments of N_t follow a Poisson distribution,

$$N_{t_2} - N_{t_1} \sim \text{Poisson}\left(\int_{t_1}^{t_2}\left[\zeta\delta e^{-\delta s} + \rho\left(1-e^{-\delta s}\right)\right] \mathrm{d}s\right)$$

Based on Theorem 4.1 and Corollary 4.2, we observe that M_{t_2} and $N_{t_2} - N_{t_1}$ are both Poisson distributed and independent. Because of the Markov property, all of the future increments after N_{t_2} only depend on M_{t_2}; they are independent of $N_{t_2} - N_{t_1}$, as well, *i.e.*, for any random variable $X \in \sigma\{\mathcal{N}_s : N_s - N_{t_2}, s \geq t_2\}$, we have:

$$\begin{aligned} \mathbb{E}\left[X\theta^{N_{t_2}-N_{t_1}}\right] &= \mathbb{E}\left[\mathbb{E}\left[X\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X\middle|M_{t_2}\right] \times \mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X\middle|M_{t_2}\right]\right] \times \mathbb{E}\left[\mathbb{E}\left[\theta^{N_{t_2}-N_{t_1}}\middle|M_{t_2}\right]\right] \\ &= \mathbb{E}\left[X\right]\theta^{N_{t_2}-N_{t_1}} \end{aligned}$$

The increments of the point process N_t follow a Poisson distribution and also they are independent; therefore, N_t is a non-homogeneous Poisson process of rate $\zeta\delta e^{-\delta t} + \rho\left(1-e^{-\delta t}\right)$. $\square$

In particular, if and only if $\zeta = \frac{\rho}{\delta}$, N_t is a Poisson process with a rate of ρ. Corollary 4.3, in fact, recovers the result obtained earlier by Mirasol [14], *i.e.*, a delayed (or displaced) Poisson process is still a (non-homogeneous) Poisson process; see also Newell [15] and Lawrance and Lewis [16].

4.2. Case $q_1 = q$

Case $q_1 = q$ is defined as the special case of a discretised dynamic contagion process when:

$$p_1 = 1, \quad \{p_k\}_{k\neq 1} = 0; \quad q_0 = 1 - q, \quad q_1 = q, \quad \{q_k\}_{k=2,3,...} = 0; \quad 0 \leq q < 1 \tag{17}$$

Corollary 4.4. *The stationary distribution of M_t is a Poisson distribution specified by:*

$$\{M_t\}_{t\geq 0} \sim \text{Poisson}\left(\frac{\rho}{\delta(1-q)}\right)$$

Proof. The stationarity condition holds as $\mu_{1_Q} = q < 1$; then, by Theorem 3.6, we have:

$$\hat{\aleph}(\varphi) = e^{-\frac{\rho}{\delta(1-q)}(1-\varphi)}$$

which is the probability generating function of a Poisson distribution with constant intensity $\frac{\rho}{\delta(1-q)}$. □

Corollary 4.5. *The probability generating function of N_T is given by:*

$$\mathbb{E}\big[\theta^{N_T} \mid M_0\big] = \exp\left(-\frac{\rho}{\delta}\frac{1-\theta}{1-\theta q}\left[\delta T - \frac{1-e^{-(1-\theta q)\delta T}}{1-\theta q}\right]\right)\left[\frac{\theta(1-q)+(1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q}\right]^{M_0} \tag{18}$$

if $M_0 \sim$ Poisson$\left(\frac{\rho}{\delta(1-q)}\right)$, then,

$$\mathbb{E}\big[\theta^{N_T}\big] = \exp\left(-\rho T\left(1-\frac{1-q}{1-\theta q}\theta\right)\right)\exp\left(-\frac{\rho}{\delta}\frac{q}{1-q}\left(1-\frac{1-q}{1-\theta q}\theta\right)^2\left(1-e^{-(1-\theta q)\delta T}\right)\right) \tag{19}$$

Proof. The stationarity condition holds as $\mu_{1_Q} = q < 1$. By Theorem 3.8, the results follow, since:

$$\mathcal{Q}^{-1}_{0,\theta}(T) = \frac{\theta(1-q)+(1-\theta)e^{-(1-\theta q)\delta T}}{1-\theta q}, \quad 0 \leq \theta < 1$$

□

Note that the first term of $\mathbb{E}[\theta^{N_T}]$ of Equation (19) is the probability generating function of a compound Poisson distribution N_1 with point $\mathring{N}_T \sim$ Poisson(ρT) and underlying $X_1 \sim$ Geometric$(1-q)$ where:

$$P\{X_1 = j\} = q^{j-1}(1-q), \quad j = 1, 2, ..., \quad \mathbb{E}[\theta^{X_1}] = \frac{1-q}{1-\theta q}\theta$$

The second term is the the probability generating function of a proper random variable $\tilde{O}$. Hence, $N_T = N_1 + \tilde{O}$, and N_T is stochastically larger than N_1, *i.e.*, $N_T \succ N_1$.

Given the probability generating function of N_T in Corollary 4.5, the probability distribution of the number of the cumulative settled claims at time T can be obtained explicitly by the basic property:

$$P\{N_T = n \mid M_0\} = \frac{\partial n}{\partial \theta^n}\mathbb{E}\big[\theta^{N_T} \mid M_0\big]\Big|_{\theta=0}$$

Numerical examples with the specified parameters $(\rho, \delta, q) = (1, 1, 0.5)$ are provided in Table 1.

Table 1. Numerical examples for case $q_1 = q$ based on Corollary 4.5: the probability distribution of the number of cumulative settled claims at time T with parameters $(\rho, \delta, q) = (1, 1, 0.5)$.

	$P\{N_T = n \mid M_0 = 0\}$ (%)			$P\{N_T = n \mid M_0 = 5\}$ (%)		
n	$T = 1$	$T = 2$	$T = 5$	$T = 1$	$T = 2$	$T = 5$
0	69.2201	32.1314	1.8193	0.4664	0.0015	0.0000
1	21.8777	27.7829	4.5175	3.3169	0.0319	0.0000
2	6.6404	18.9572	7.3929	10.3724	0.2956	0.0000
3	1.7365	10.9172	9.7336	18.9468	1.5260	0.0001
4	0.4113	5.6055	11.1254	22.8201	4.8543	0.0047
5	0.0906	2.6432	11.4776	19.6109	10.1416	0.0852
6	0.0188	1.1655	10.9392	12.8517	14.9512	0.3778
7	0.0037	0.4865	9.7807	6.8021	17.0146	1.0198
8	0.0007	0.1939	8.2921	3.0379	15.9371	2.0730
9	0.0001	0.0742	6.7191	1.1825	12.8316	3.4803
10	0.0000	0.0275	5.2352	0.4109	9.1512	5.0745
Sum	100.0000	99.9850	87.0325	99.8186	86.7366	12.1154

4.3. Case $q_0 = 1$

Case $q_0 = 1$ is defined as the special case of a discretised dynamic contagion process when:

$$q_0 = 1, \quad \{q_k\}_{k \neq 0} = 0 \tag{20}$$

Indeed, this is a special case which corresponds to a Cox process with shot-noise intensity via transformation, as given by Appendix A.

Corollary 4.6. *If* $\{p_k\}_{k=0,1,2,\ldots} \sim \text{Geometric}(p)$, $0 \leq p < 1$, *then, the stationary distribution of* M_t *is given:*

$$\{M_t\}_{t \geq 0} \sim \text{NegBin}\left(\frac{\rho}{\delta}, 1 - p\right)$$

Proof. If $\{p_k\}_{k=0,1,2,\ldots} \sim \text{Geometric}(p)$, then,

$$\hat{p}(u) = \frac{p}{1 - (1 - p)u} \tag{21}$$

The stationarity condition holds as $\mu_{1_Q} = 0 < 1$; then, by Theorem 3.6, we have:

$$\hat{\aleph}(\varphi) = \left(\frac{p}{1 - (1 - p)\varphi}\right)^{\frac{\rho}{\delta}}$$

which is the probability generating function of a negative binomial distribution with the parameters $\left(\frac{\rho}{\delta}, 1 - p\right)$. □

Corollary 4.7. *If* $\{p_k\}_{k=0,1,2,\ldots} \sim \text{Geometric}(p)$, *then,*

$$\mathbb{E}\left[\theta^{N_T} \mid M_0\right] = e^{-\rho T(1 - \hat{p}(\theta))} \left(\frac{p_T}{1 - (1 - p_T\theta)}\right)^{-\frac{\rho}{\delta}\hat{p}(\theta)} \left[(1 - \theta)e^{-\delta T} + \theta\right]^{M_0}$$

where $\hat{p}(u)$ is specified by Equation (21) and

$$p_T =: \frac{p}{1-(1-p)e^{-\delta T}}$$

if $M_0 \sim \mathrm{NegBin}\left(\frac{\rho}{\delta}, 1-p\right)$, then,

$$\mathbb{E}\left[\theta^{N_T}\right] = e^{-\rho T(1-\hat{p}(\theta))}\left(\frac{p_T}{1-(1-p_T\theta)}\right)^{\frac{\rho}{\delta}(1-\hat{p}(\theta))} \tag{22}$$

Proof. By Theorem 3.8, the results follow, since:

$$\mathcal{Q}_{0,\theta}^{-1}(T) = (1-\theta)e^{-\delta T} + \theta, \quad 0 \le \theta < 1 \tag{23}$$

□

Note that the first term of $\mathbb{E}[\theta^{N_T}]$ of Equation (22) is the probability generating function of a compound Poisson distribution N_2 with point $\mathring{N}_T \sim \mathrm{Poisson}(\rho T)$ and underlying $X_2 \sim \mathrm{Geometric}(p)$, where:

$$P\{X_2 = j\} = (1-p)^j p, \quad j = 0, 1, 2, ..., \quad \mathbb{E}[\theta^{X_2}] = \frac{p}{1-(1-p)\theta}$$

the second term of Equation (22) is the probability generating function of a proper random variable $\tilde{O}$. Hence, we have $N_T = N_2 + \tilde{O}$, and N_T is stochastically larger than N_2, *i.e.*, $N_T \succ N_2$.

Given the probability generating function of N_T in Corollary 4.7, the probability distribution of the number of the cumulative settled claims at time T can be obtained explicitly by expansion, and numerical examples with the specified parameters $(\rho, \delta, p) = (1, 0.5, 0.5)$ are provided in Table 2.

Table 2. Numerical examples for case $q_0 = 1$ based on Corollary 4.7: the probability distribution of the number of cumulative settled claims at time T with parameters $(\rho, \delta, p) = (1, 0.5, 0.5)$

	$P\{N_T = n \mid M_0 = 0\}$ (%)			$P\{N_T = n \mid M_0 = 5\}$ (%)		
n	$T=1$	$T=2$	$T=5$	$T=1$	$T=2$	$T=5$
0	84.5182	60.0424	15.7432	6.9377	0.4046	0.0001
1	11.2858	21.4735	17.2706	23.4295	3.6204	0.0033
2	3.0271	10.0735	16.3053	32.4498	13.2556	0.0770
3	0.8397	4.6412	13.8000	23.7139	25.4106	0.9043
4	0.2360	2.1013	10.8740	9.9613	27.2604	5.5660
5	0.0668	0.9376	8.1400	2.6542	16.8608	16.2074
6	0.0189	0.4134	5.8596	0.6155	7.1872	16.7767
7	0.0054	0.1805	4.0889	0.1710	3.3097	15.2520
8	0.0015	0.0781	2.7816	0.0481	1.4993	12.6134
9	0.0004	0.0336	1.8522	0.0136	0.6694	9.7828
10	0.0001	0.0143	1.2111	0.0039	0.2953	7.2381
Sum	100.0000	99.9895	97.9265	99.9985	99.7733	84.4212

Acknowledgments

Hongbiao Zhao acknowledges the financial support from the National Natural Science Foundation of China (#71401147) and the research funds provided by Xiamen University.

Author Contributions

The two authors contributed equally to all aspects of this work.

A. Comparison with the Zero-Reversion Dynamic Contagion Process

A.1. Zero-Reversion Dynamic Contagion Process

This section is to demonstrate an alternative representation of the dynamic contagion process [8] with zero-reversion intensity (as defined by Definition A.1). We find later in Theorem A.6 that this process is the special case of a discretised dynamic contagion process when both K_P and K_Q follow mixed Poisson distributions.

Definition A.1 (Zero-reversion dynamic contagion process)**.** *The zero-reversion dynamic contagion process is a point process* $N_t^* \equiv \left\{T_k^{(2)}\right\}_{k\geq 1}$ *with non-negative* $\mathcal{F}_t-$*stochastic intensity process:*

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{0\leq T_i^{(1)}<t} Y_i^{(1)} e^{-\delta\left(t-T_i^{(1)}\right)} + \sum_{0\leq T_k^{(2)}<t} Y_k^{(2)} e^{-\delta\left(t-T_k^{(2)}\right)} \tag{24}$$

where:

- $\{\mathcal{F}_t\}_{t\geq 0}$ *is a history of the process* N_t^**, with respect to which* $\{\lambda_t\}_{t\geq 0}$ *is adapted;*
- $\lambda_0 > 0$ *is a constant as the initial value of* λ_t *at time* $t = 0$*;*
- $\delta > 0$ *is the constant rate of exponential decay;*
- $\left\{Y_i^{(1)}\right\}_{i=1,2,...}$ *is a sequence of i.i.d. positive (externally-excited) jumps with distribution function* $H(y), y > 0$*, at the corresponding random times* $\left\{T_i^{(1)}\right\}_{i=1,2,...}$ *following a Poisson process of rate* $\rho > 0$*;*
- $\left\{Y_k^{(2)}\right\}_{k=1,2,...}$ *is a sequence of i.i.d. positive (self-excited) jumps with distribution function* $G(y), y > 0$*, at the corresponding random times* $\left\{T_k^{(2)}\right\}_{k=1,2,...}$*;*
- *the sequences* $\left\{Y_i^{(1)}\right\}_{i=1,2,...}$*,* $\left\{T_i^{(1)}\right\}_{i=1,2,...}$ *and* $\left\{Y_k^{(2)}\right\}_{k=1,2,...}$ *are assumed to be independent of each other.*

The first moments and Laplace transforms of two types of jumps $Y_i^{(1)}$ and $Y_i^{(2)}$ are denoted respectively by:

$$\mu_{1_H} =: \int_0^\infty y \mathrm{d}H(y), \quad \mu_{1_G} =: \int_0^\infty y \mathrm{d}G(y); \quad \hat{h}(u) =: \int_0^\infty e^{-uy} \mathrm{d}H(y), \quad \hat{g}(u) =: \int_0^\infty e^{-uy} \mathrm{d}G(y)$$

The generator of a zero-reversion dynamic contagion process (λ_t, N_t^*, t) acting on a function $f(m,n,t)$ is given by:

$$\begin{aligned}\mathcal{A}f(\lambda,n,t) &= \frac{\partial f}{\partial t} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho\Big(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\Big) \\ &\quad +\lambda\bigg(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\bigg) \end{aligned} \tag{25}$$

Key distributional properties, which will be used later, are listed as below; see the proofs in Dassios and Zhao [8].

Proposition A.2. *The stationarity condition of intensity process λ_t is $\delta > \mu_{1_G}$.*

Theorem A.3. *If $\delta > \mu_{1_G}$, the Laplace transform λ_T conditional λ_0 for a fixed time T is given by:*

$$\mathbb{E}\left[e^{-v\lambda_T} \mid \lambda_0\right] = \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{\rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \times e^{-\mathcal{G}_{v,1}^{-1}(T)\lambda_0} \tag{26}$$

where:

$$\mathcal{G}_{v,1}(L) =: \int_L^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1}$$

Theorem A.4. *If $\delta > \mu_{1_G}$, the Laplace transform of the asymptotic distribution of λ_t is given by:*

$$\hat{\Pi}(v) =: \lim_{t\to\infty} \mathbb{E}\left[e^{-v\lambda_t} \mid \lambda_0\right] = \exp\left(-\int_0^v \frac{\rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \tag{27}$$

and Equation (27) is also the Laplace transform of the stationary distribution of process $\{\lambda_t\}_{t\geq 0}$.

Dassios and Zhao [17] further apply the counting process N_t^* to model the arrival of insurance claims for the ruin problem via efficient Monte Carlo simulation. One of the advantages of the model using the discretised dynamic contagion process in this paper is that we would be able to investigate the properties of the unsettled number of claims (*i.e.*, M_t) itself explicitly, whereas this quantity is not explicit in Dassios and Zhao [17].

A.2. Transformations between Two Processes

We explore the analogy between (N_t^*, λ_t) and (N_t, M_t) via distributional transformations.

Lemma A.5. *If:*

$$\hat{p}(u) = \hat{h}\left(\frac{1-u}{\delta}\right), \quad \hat{q}(u) = \hat{g}\left(\frac{1-u}{\delta}\right) \tag{28}$$

then, the joint Laplace transform probability generating function of (N_T^*, λ_T) is given by:

$$\mathbb{E}\left[\theta^{(N_T^*-N_t^*)}e^{-v\lambda_T} \mid \mathcal{F}_t\right] = e^{-(D(T)-D(t))}e^{-B(t)\lambda_t}, \quad v \geq 0 \tag{29}$$

where:

$$B(t) = \frac{1-A(t)}{\delta}, \quad D(t) = c(t) \tag{30}$$

with boundary condition $B(T) = v$, and $A(t)$, $c(t)$ are given by Equation (5).

Proof. Similar to the proof for Theorem 3.4, for a zero-reversion dynamic contagion process (N_t^*, λ_t), assume the form $f(\lambda, n, t) = e^{-B(t)\lambda}\theta^n e^{D(t)}$, and set $\mathcal{A}f(\lambda, n, t) = 0$ in generator Equation (25); we have martingale $e^{-B(t)\lambda_t}\theta^{N_t^*}e^{D(t)}$ where:

$$\begin{cases} B'(t) = \theta\hat{g}(B(t)) + \delta B(t) - 1 \\ D'(t) = \rho[1 - \hat{h}(B(t))] \end{cases} \tag{31}$$

With boundary condition $B(T) = v$, and by the martingale property, we have Equation (29).

On the other hand, for the joint probability generating function of a discretised dynamic contagion process (M_t, N_t) as given by Theorem 3.4, we have:

$$\begin{cases} A'(t) = -\delta\theta\hat{q}(A(t)) + \delta A(t) \\ c'(t) = \rho[1 - \hat{p}(A(t))] \end{cases} \tag{32}$$

The analogy between (N_t^*, λ_t) and (N_t, M_t) is linked by Equation (32) and Equation (31): without solving the equations explicitly, if we set Equation (28) and Equation (30), then Equations (31) and (32) are equivalent. □

We can prove in Theorem A.6 that, via distributional transformations, the increments of N_t and N_t^* have the same distribution, and the finite-dimensional distributions of N_t and N_t^* are the same.

Theorem A.6. *If* $M_0 \sim \text{Poisson}\left(\frac{\lambda_0}{\delta}\right)$ *and:*

$$K_P \sim \textit{Mixed Poisson}\left(\frac{Y}{\delta}\,\middle|\, Y \sim H\right), \quad K_Q \sim \textit{Mixed Poisson}\left(\frac{Y}{\delta}\,\middle|\, Y \sim G\right) \tag{33}$$

i.e.,

$$p_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}H(y), \quad q_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}G(y)$$

then,

$$\mathbb{E}\left[\theta^{N_T^*} \mid \lambda_0\right] = \mathbb{E}\left[\theta^{N_T}\right]$$

Proof. If $M_0 \sim \text{Poisson}\left(\frac{\lambda_0}{\delta}\right)$, then $\mathbb{E}[\psi^{M_0}] = e^{-\frac{1-\psi}{\delta}\lambda_0}$. The condition Equation (28) is equivalent to Equation (33) since:

$$\begin{aligned} \sum_{k=0}^{\infty} u^k p_k &= \hat{p}(u) = \hat{h}\left(\frac{1-u}{\delta}\right) = \mathbb{E}\left[e^{-\frac{Y}{\delta}(1-u)}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[u^{K_P}\middle| K_P \sim \text{Poisson}\left(\frac{Y}{\delta}\right)\right]\right] \\ &= \sum_{k=0}^{\infty} u^k \mathbb{E}\left[P\left\{K_P = k\middle| K_P \sim \text{Poisson}\left(\frac{Y}{\delta}\right)\right\}\right] \\ &= \sum_{k=0}^{\infty} u^k P\left\{K_P = k\right\} \end{aligned}$$

and similarly, for K_Q.

Set $t = 0$, $\varphi = 1$ in Equation (5) and $t = 0$, $v = 0$ in Equation (29); then, by Equation (30) of Lemma A.5, we have:

$$\begin{aligned} \mathbb{E}\left[\theta^{N_T^*} \mid \lambda_0\right] &= e^{-(D(T)-D(0))}e^{-B(0)\lambda_0} = e^{-(c(T)-c(0))}e^{-B(0)\lambda_0}, \\ \mathbb{E}\left[\theta^{N_T}\right] &= \mathbb{E}\left[\mathbb{E}[\theta^{N_T} \mid M_0]\right] = e^{-(c(T)-c(0))}e^{-\frac{1-A(0)}{\delta}\lambda_0} = e^{-(c(T)-c(0))}e^{-B(0)\lambda_0} \end{aligned}$$

□

Corollary A.7. *If $H \sim \text{Exp}(\alpha)$ and $G \sim \text{Exp}(\beta)$, $\alpha, \beta > 0$, then the transformations are given by:*

$$\begin{aligned}\{p_k\}_{k=0,1,2\ldots} &\sim \text{Geometric}(p), \quad p =: \frac{\delta\alpha}{\delta\alpha+1} \\ \{q_k\}_{k=0,1,2\ldots} &\sim \text{Geometric}(\acute{q}), \quad \acute{q} =: \frac{\delta\beta}{\delta\beta+1}\end{aligned}$$

Proof. If $H \sim \text{Exp}(\alpha)$, then, by Equation (33) or Equation (28), we have:

$$\hat{p}(u) = \hat{h}\left(\frac{1-u}{\delta}\right) = \frac{\alpha}{\alpha + \frac{1-u}{\delta}} = \frac{p}{1-(1-p)u}$$

and similarly, for $G \sim \text{Exp}(\beta)$. □

Remark A.8. *The stationarity condition of process M_t given by Equation (3) can be alternatively derived via the transformation by Theorem A.6 from the stationarity condition $\delta > \mu_{1_G}$ for process λ_t by Proposition A.2, i.e.,*

$$\mu_{1_Q} = \mathbb{E}[K_Q] = \mathbb{E}\left[\frac{Y}{\delta}\,\middle|\, Y \sim G\right] = \frac{\mu_{1_G}}{\delta} < 1$$

In particular, if $K_Q \sim \text{Geometric}(\acute{q})$, then the stationarity condition is $\acute{q} > \frac{1}{2}$.

Corollary A.9. *If $M_0 \sim \text{Poisson}\left(\frac{\lambda_0}{\delta}\right)$ and:*

$$K_P \sim \textit{Mixed Poisson}\left(\frac{Y}{\delta}\,\middle|\, Y \sim H\right), \quad K_Q \sim \textit{Mixed Poisson}\left(\frac{Y}{\delta}\,\middle|\, Y \sim G\right)$$

then,

$$\mathbb{E}[\varphi^{M_T}] = \mathbb{E}\left[e^{-v\lambda_T} \mid \lambda_0\right]$$

Proof. By transformation Equation (30) of Lemma A.5, we have:

$$v = \frac{1-\varphi}{\delta}, \quad \mathcal{G}_{v,1}^{-1}(T) = \frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}, \quad A(0) = 1 - \delta B(0)$$

and:

$$\mathcal{G}_{v,1}^{-1}(T) = B(0), \quad \mathcal{Q}_{\varphi,1}^{-1}(T) = A(0)$$

Then, by comparing Theorem 3.4 and Theorem A.3, we have:

$$\begin{aligned}&\mathbb{E}[\varphi^{M_T}] \\ =\ & \mathbb{E}\left[\mathbb{E}[\varphi^{M_T} \mid M_0]\right] \\ =\ & \exp\left(-\int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{\rho[1-\hat{p}(u)]}{\delta\hat{q}(u) - \delta u}\mathrm{d}u\right) \mathbb{E}\left[\left[\mathcal{Q}_{\varphi,1}^{-1}(T)\right]^{M_0}\right] \\ =\ & \exp\left(-\int_{\varphi}^{\mathcal{Q}_{\varphi,1}^{-1}(T)} \frac{\rho\left[1-\hat{h}\left(\frac{1-u}{\delta}\right)\right]}{\delta\hat{g}\left(\frac{1-u}{\delta}\right) - \delta u}\mathrm{d}u\right) e^{-\frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}\lambda_0} \qquad \left(s = \frac{1-u}{\delta}\right) \\ =\ & \exp\left(-\int_{\frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}}^{\frac{1-\varphi}{\delta}} \frac{\rho[1-\hat{h}(s)]}{\delta s + \hat{g}(s) - 1}\mathrm{d}s\right) e^{-\frac{1-\mathcal{Q}_{\varphi,1}^{-1}(T)}{\delta}\lambda_0} \\ =\ & \mathbb{E}\left[e^{-v\lambda_T} \mid \lambda_0\right]\end{aligned}$$

□

Corollary A.10. *If* $K_P \sim$ *Mixed Poisson* $\left(\frac{Y}{\delta} \mid Y \sim H\right)$, $K_Q \sim$ *Mixed Poisson* $\left(\frac{Y}{\delta} \mid Y \sim G\right)$, *then,*

$$\hat{\aleph}(\varphi) = \hat{\Pi}(v)$$

Proof. By Theorem 3.6 and Theorem A.4, we have:

$$\begin{aligned}\hat{\Pi}(v) &= \exp\left(-\int_0^v \frac{\rho[1-\hat{p}(1-\delta u)]}{\delta u + \hat{q}(1-\delta u) - 1} \mathrm{d}u\right) \qquad (s = 1-\delta u) \\ &= \hat{\aleph}(1-\delta v) = \hat{\aleph}(\varphi)\end{aligned}$$

□

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Hawkes, A.G. Point spectra of some mutually exciting point processes. *J. R. Stat. Soc. Ser. B* **1971**, *33*, 438–443.
2. Hawkes, A.G. Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **1971**, *58*, 83–90.
3. Errais, E.; Giesecke, K.; Goldberg, L.R. Affine point processes and portfolio credit risk. *SIAM J. Financ. Math.* **2010**, *1*, 642–665.
4. Embrechts, P.; Liniger, T.; Lin, L. Multivariate Hawkes processes: An application to financial data. *J. Appl. Probab.* **2011**, *48A*, 367–378.
5. Chavez-Demoulin, V.; McGill, J. High-frequency financial data modeling using Hawkes processes. *J. Bank. Financ.* **2012**, *36*, 3415–3426.
6. Bacry, E.; Delattre, S.; Hoffmann, M.; Muzy, J.F. Modelling microstructure noise with mutually exciting point processes. *Quant. Financ.* **2013**, *13*, 65–77.
7. Aït-Sahalia, Y.; Cacho-Diaz, J.; Laeven, R.J. Modeling financial contagion using mutually exciting jump processes. *J. Financ. Econ.* **2014**, doi:10.3386/w15850.
8. Dassios, A.; Zhao, H. A dynamic contagion process. *Adv. Appl. Probab.* **2011**, *43*, 814–846.
9. Dassios, A.; Zhao, H. A risk model with delayed claims. *J. Appl. Probab.* **2013**, *50*, 686–702.
10. Yuen, K.C.; Guo, J.; Ng, K.W. On ultimate ruin in a delayed-claims risk model. *J. Appl. Probab.* **2005**, *42*, 163–174.
11. Dassios, A.; Embrechts, P. Martingales and insurance risk. *Stoch. Models* **1989**, *5*, 181–217.
12. Dassios, A.; Jang, J. Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity. *Financ. Stoch.* **2003**, *7*, 73–95.
13. Ethier, S.N.; Kurtz, T.G. *Markov Processes: Characterization and Convergence*; Wiley: Hoboken, NJ, USA, 1986.
14. Mirasol, N.M. The output of an M/G/∞ queuing system is poisson. *Oper. Res.* **1963**, *11*, 282–284.
15. Newell, G. The M/G/∞ Queue. *SIAM J. Appl. Math.* **1966**, *14*, 86–88.

16. Lawrance, A.; Lewis, P.A. Properties of the bivariate delayed Poisson process. *J. Appl. Probab.* **1975**, *12*, 257–268.
17. Dassios, A.; Zhao, H. Ruin by dynamic contagion claims. *Insur. Math. Econ.* **2012**, *51*, 93–106.

© 2014 by the authors; licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).