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Characterization of Frequency Domains of Bandlimited Frame Multiresolution Analysis

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Abstract: Framelets have been widely used in narrowband signal processing, data analysis, and sampling theory, due to their resilience to background noise, stability of sparse reconstruction, and ability to capture local time-frequency information. The well-known approach to construct framelets with useful properties is through frame multiresolution analysis (FMRA). In this article, we characterize the frequency domain of bandlimited FMRAs: there exists a bandlimited FMRA with the support of frequency domain G if and only if G satisfies $G \subset 2G$, $\bigcup_m 2^m G \cong \mathbb{R}^d$, and $\left(G \setminus \frac{G}{2}\right) \bigcap \left(\frac{G}{2} + 2\pi \boldsymbol{\nu}\right) \cong \emptyset$ $(\boldsymbol{\nu} \in \mathbb{Z}^d)$.

Keywords: framelets; bandlimited multiresolution analysis; frequency domain

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1. Introduction

Due to resilience to background noise, stability of sparse reconstruction, and the ability to capture local time-frequency information, framelets are becoming a powerful tool in applied analysis and data science. Let $\{h_n\}_1^\infty$ be a sequence in $L^2(\mathbb{R}^d)$. If there exist $A,\ B > 0$ such that

$$A \parallel f \parallel^2 \leq \sum_{n=1}^{\infty} |(f,\ h_n)|^2 \leq B \parallel f \parallel^2 \qquad \forall\ f \in L^2(\mathbb{R}^d),$$

then, $\{h_n\}_1^\infty$ is called a frame for $L^2(\mathbb{R}^d)$ with bounds A and B [1,2]. Let $\{\psi_\mu\}_1^l \subset L^2(\mathbb{R}^d)$ and

$$\psi_{\mu,j,k} := 2^{\frac{jd}{2}}\, \psi_\mu(2^j \cdot -k), \qquad \mu = 1, ..., l;\ \ j \in \mathbb{Z};\ \ k \in \mathbb{Z}^d.$$

If the affine system $\{\psi_{\mu,j,k}\}$ is a frame for $L^2(\mathbb{R}^d)$, then, the set $\{\psi_\mu\}_1^l$ is called a framelet [1,2].

The well-known approach to construct framelets is through frame multiresolution analysis (FMRA) [3–5]. In 1998, Benedetto and Li [3] introduced the theory of one-dimensional FMRA, which is a fundamental concept in framelet theory. In 2004, Mu et al. [4] further established the theory of high-dimensional FMRA. The structure of FMRA is as follows:

Let $\{V_m\}_{m\in\mathbb{Z}}$ be a sequence of subspaces of $L^2(\mathbb{R}^d)$ such that
(i) $V_m \subset V_{m+1}\ (m \in \mathbb{Z})$, $\quad \overline{\bigcup_{m\in\mathbb{Z}} V_m} = L^2(\mathbb{R}^d)$, $\quad \bigcap_m V_m = \{0\}$;
(ii) $f(\mathbf{t}) \in V_m$ if and only if $f(2t) \in V_{m+1}\ (m \in \mathbb{Z})$;
(iii) there exists a $\varphi(\mathbf{t}) \in V_0$ such that $\{\varphi(\mathbf{t} - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a frame for V_0.

Then $\{V_m\}$ is called a *frame multiresolution analysis* (FMRA) and φ is called a *frame scaling function*.

If the Fourier transform of a frame scaling function φ is compactly supported (i.e., supp $\widehat{\varphi} = G$), then φ is said to be bandlimited. Furthermore, $f \in V_0$, since $\{\varphi(\mathbf{t} - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a frame for V_0. When f is expanded into a frame series, $f(\mathbf{t}) = \sum_{\mathbf{n}} d_{\mathbf{n}} \varphi(\mathbf{t} - \mathbf{n})$; then, taking Fourier transform of both sides, it follows that

$$\widehat{f}(\boldsymbol{\omega}) = \left(\sum_{\mathbf{n}} d_{\mathbf{n}}\, e^{-i(\mathbf{n}\cdot\boldsymbol{\omega})} \right) \widehat{\varphi}(\boldsymbol{\omega}).$$

It means that $f \in V_0$ is bandlimited. Since $\varphi \in V_0$, the support of frequency domain of any function in V_0 is contained in G or equal to G. Therefore, such an FMRA is said to be a bandlimited FMRA with the support of frequency domain G.

Various framelets with nice properties are constructed by FMRAs [4–10]. Mu et al. [4] and Zhang [5] showed that the number of generators in framelets associated with FMRA is determined completely by the frequency domain of FMRAs. Zhang [6] discussed the convergence of a framelet series derived from FMRAs. Chui and He [7], Antolin and Zalin [8], and Atreasa et al. [9] constructed many examples on compactly supported framelets when FMRAs extend into general MRAs. Zhang [10] extended framelets derived by FMRAs into the case of framelet packets with finer time-frequency resolution.

Frequency domain of bandlimited FMRAs plays a key role when derived framelets are applied into narrowband signal processing and data analysis. A suitable frequency domain of bandlimited FMRAs can mitigate the effects of narrowband noises well, so the perfect reconstruction filter bank associated with a bandlimited FMRA can achieve quantization noise reduction simultaneously with reconstruction of a given narrowband signal [3]. This is a unique and key advantage of framelets over traditional wavelets [3]. However, until now, the structure of frequency domain of bandlimited FMRAs has not been investigated.

In this study, the frequency domain of bandlimited FMRA will be characterized. In Section 2, the necessary condition for G to be the support of frequency domain of bandlimited FMRA is given first. In Section 3, in order to obtain sufficient condition, a fine partition of any bounded region G is presented, satisfying

$$G \subset 2G, \qquad \bigcup_m 2^m G \cong \mathbb{R}^d, \quad \text{and} \quad \left(G \setminus \frac{G}{2}\right) \bigcap \left(\frac{G}{2} + 2\pi\boldsymbol{\nu}\right) \cong \emptyset \ (\boldsymbol{\nu} \in \mathbb{Z}^d).$$

Based on this partition, in Section 4, a bandlimited FMRA with the support of frequency domain G is directly constructed. With the help of it, for any given narrowband signal, one can choose the most suitable bandlimited FMRA to analyze it and, at the same time, mitigate noise effects.

2. Necessary Condition for the Support of Frequency Domain of Bandlimited FMRA

Let $\{V_m\}$ be an FMRA and φ be the corresponding frame scaling function. Since $\varphi \in V_0$, $\frac{1}{2^d}\varphi(\frac{t}{2}) \in V_1 \subset V_0$, and $\{\varphi(\mathbf{t} - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a frame for V_0, $\frac{1}{2^d}\varphi(\frac{\mathbf{t}}{2})$ can be expanded into a frame series, $\frac{1}{2^d}\varphi(\frac{\mathbf{t}}{2}) = \sum_{\mathbf{n}} c_{\mathbf{n}}\varphi(\mathbf{t} - \mathbf{n})$. Take the Fourier transform of both sides, it follows that

$$\widehat{\varphi}(2\boldsymbol{\omega}) = \left(\sum_{\mathbf{n}} c_{\mathbf{n}}\, e^{-i(\mathbf{n}\cdot\boldsymbol{\omega})}\right)\widehat{\varphi}(\boldsymbol{\omega}),$$

where $H(\boldsymbol{\omega}) = \sum_{\mathbf{n}} c_{\mathbf{n}} e^{-i(\mathbf{n}\cdot\boldsymbol{\omega})}$ is a $2\pi-$periodic function (i.e., $H \in L^2(T^d)$). From this, we have

$$\widehat{\varphi}(2\boldsymbol{\omega}) = H(\boldsymbol{\omega})\widehat{\varphi}(\boldsymbol{\omega}).$$

In this section, a necessary condition for the support of the frequency domain of bandlimited FMRA is given as follows.

Theorem 1. *Let G be a bounded closed set in $\mathbb{R}^d$. If G is the support of the frequency domain of a bandlimited FMRA, then*

(i) $G \subset 2G$, *(ii)* $\bigcup_m 2^m G \cong \mathbb{R}^d$, *and* *(iii)* $\left(G \setminus \frac{G}{2}\right) \bigcap \left(\frac{G}{2} + 2\pi\boldsymbol{\nu}\right) \cong \emptyset \ (\boldsymbol{\nu} \in \mathbb{Z}^d)$

hold.

Proof. Since G is the support of the frequency domain of a bandlimited FMRA, the corresponding frame scaling function φ satisfies supp $\widehat{\varphi} = G$. Let $V_m = \overline{\text{span}}\{\varphi(2^m\mathbf{t} - \mathbf{n}),\ \mathbf{n} \in \mathbb{Z}^d\}$. Then, $V_m \subset V_{m+1}$ $(m \in \mathbb{Z})$ and $\overline{\bigcup_m V_m} = L^2(\mathbb{R}^d)$, and $\{2^{\frac{md}{2}}\varphi(2^m\mathbf{t} - \mathbf{n})\}_{n\in\mathbb{Z}^d}$ is a frame of V_m. So, for any $f \in V_m$, f can be expanded into a frame series as follows:

$$f(\mathbf{t}) = \sum_{\mathbf{n}} d_{m,\mathbf{n}}\varphi(2^m\mathbf{t} - \mathbf{n}).$$

By taking the Fourier transform on both sides, it follows that

$$\widehat{f}(\boldsymbol{\omega}) = \left(\sum_{\mathbf{n}} 2^{-md} d_{m,\mathbf{n}} e^{-i\frac{(\mathbf{n}\cdot\boldsymbol{\omega})}{2^m}}\right)\widehat{\varphi}\Big(\frac{\boldsymbol{\omega}}{2^m}\Big).$$

Furthermore, $\mathrm{supp}\widehat{f} \subset 2^m \mathrm{supp}\widehat{\varphi}(\boldsymbol{\omega}) = 2^m G\,(f \in V_m)$. Noticing that $\varphi(2^m\mathbf{t}) \in V_m$, it is clear that

$$\bigcup_{f\in V_m} \mathrm{supp}\widehat{f} = 2^m G, \qquad \bigcup_{f\in V_{m+1}} \mathrm{supp}\widehat{f} = 2^{m+1}G.$$

Again by $V_m \subset V_{m+1}$, it follows that $2^m G \subset 2^{m+1}G$, i.e., $G \subset 2G$. Again, by $\overline{\bigcup}_m V_m = L^2(\mathbb{R}^d)$, it means that $\bigcup_m \bigcup_{f\in V_m} \mathrm{supp}\widehat{f} = \mathbb{R}^d$, and so, $\bigcup_m 2^m G = \mathbb{R}^d$.

For $\boldsymbol{\omega} \in G \setminus \frac{G}{2}$, $2\boldsymbol{\omega} \in 2G \setminus G$, and so, $\widehat{\varphi}(2\boldsymbol{\omega}) = 0$ and $\widehat{\varphi}(\boldsymbol{\omega}) \neq 0$. From this and $\widehat{\varphi}(2\boldsymbol{\omega}) = H(\boldsymbol{\omega})\widehat{\varphi}(\boldsymbol{\omega})$ $(H \in L^2(T^d))$, it follows that $H(\boldsymbol{\omega}) = 0$ for $\boldsymbol{\omega} \in G \setminus \frac{G}{2}$. On the other hand, for $\boldsymbol{\omega} \in \frac{G}{2}$, by $2\boldsymbol{\omega} \in G$ and $\frac{G}{2} \subset G$, it means that $\widehat{\varphi}(2\boldsymbol{\omega}) \neq 0$ and $\widehat{\varphi}(\boldsymbol{\omega}) \neq 0$. Again, by $\widehat{\varphi}(2\boldsymbol{\omega}) = H(\boldsymbol{\omega})\widehat{\varphi}(\boldsymbol{\omega})$, it follows that $H(\boldsymbol{\omega}) \neq 0\,(\boldsymbol{\omega} \in \frac{G}{2})$. Since $H(\boldsymbol{\omega})$ is a periodic function with period $2\pi\mathbb{Z}^d$, $H(\boldsymbol{\omega}) \neq 0\,(\boldsymbol{\omega} \in \frac{G}{2} + 2\pi\boldsymbol{\nu}\ (\boldsymbol{\nu} \in \mathbb{Z}^d))$. Combining these results, it means that $(G \setminus \frac{G}{2}) \bigcap (\frac{G}{2} + 2\pi\boldsymbol{\nu}) \cong \emptyset\,(\boldsymbol{\nu} \in \mathbb{Z}^d)$. $\square$

3. Partition of the Support of Frequency Domain

Let G be a bounded closed set in $\mathbb{R}^d$, satisfying Theorem 1(i)–(iii). A fine partition of G will be given in this section. This partition will be used to further prove that there exists a bandlimited FMRA with the support of frequency domain G in Section 4, i.e., the converse of Theorem 1 holds.

Some notations are needed as follows: For $m = 0, 1, \ldots$,

$$G_0^* = G \setminus \frac{G}{2}, \quad G_m^* = 2^{-m}G_0^*, \quad G_m = 2^{-m}G, \quad E_m^* = G_m^* + 2\pi\mathbb{Z}^d, \quad E_m = G_m + 2\pi\mathbb{Z}^d.$$

Noticing that G is bounded, one can choose a $k \in \mathbb{Z}_+$ such that

$$G \subset [-2^k\pi, 2^k\pi]^d. \tag{1}$$

Lemma 1. *If G satisfies (i) and (iii), then,*

(a) $(E_m^* \setminus G_m^*) \bigcap G \cong \emptyset \quad (m \geq 0)$; *(b)* $G \bigcap (G_k + 2\pi\boldsymbol{\nu}) \cong \emptyset \quad (\boldsymbol{\nu} \neq \mathbf{0})$.

Hereafter, $\boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ is written simply as $\boldsymbol{\nu} \neq \mathbf{0}$.

Proof. By (i), it follows that

$$G = \left(G \setminus \frac{G}{2}\right) \bigcup \frac{G}{2} = G_0^* \bigcup G_1^* \bigcup \frac{G_2}{4} = \cdots = \left(\bigcup_{m=0}^{k-1} G_m^*\right) \bigcup G_k \tag{2}$$

and clearly, this is an union of $k+1$ disjoint point sets. So, $\frac{G}{2} = (\bigcup_{m=1}^{k} G_m^*) \bigcup G_{k+1}$, and so,

$$\frac{G}{2} + 2\pi\mathbb{Z}^d = \left(\bigcup_{m=1}^{k} (G_m^* + 2\pi\mathbb{Z}^d)\right) \bigcup (G_{k+1} + 2\pi\mathbb{Z}^d) = \left(\bigcup_{m=1}^{k} E_m^*\right) \bigcup E_{k+1}.$$

For any m $(m \geq 0)$, by (iii), it follows that $2^{-m}G_0^* \bigcap (2^{-m-1}G + 2\pi 2^{-m}\boldsymbol{\nu}) = \emptyset\,(\boldsymbol{\nu} \in \mathbb{Z}^d)$. So,

$$G_m^* \bigcap (G_{m+1} + 2\pi 2^{-m}\boldsymbol{\nu}) \cong \emptyset \qquad (\boldsymbol{\nu} \in \mathbb{Z}^d).$$

Let $\boldsymbol{\nu} = 2^m \boldsymbol{\mu}\,(\boldsymbol{\mu} \in \mathbb{Z}^d)$. Then, $G_m^* \bigcap (G_{m+1} + 2\pi\boldsymbol{\mu}) \cong \emptyset\,(\boldsymbol{\mu} \in \mathbb{Z}^d)$. Again, by $G_n \subset G_{m+1}\,(m < n)$, it means that

$$(G_m^* + 2\pi\mathbb{Z}^d)\bigcap(G_n + 2\pi\mathbb{Z}^d) \cong \emptyset \qquad (m < n). \tag{3}$$

From this and $E_n^* \subset E_n$, it follows that

$$E_m^* \bigcap E_n \cong \emptyset \quad (m < n), \qquad E_m^* \bigcap E_n^* \cong \emptyset \quad (m \neq n,\ m, n \geq 0). \tag{4}$$

By $G_n \subset E_n$ and $G_n^* \subset E_n^*$, it follows that $(E_m^* \setminus G_m^*) \bigcap G_n^* \cong \emptyset\,(m, n \geq 0)$ and $(E_m^* \setminus G_m^*) \bigcap G_n \cong \emptyset\,(0 \leq m < n)$. Further, $(E_m^* \setminus G_m^*) \bigcap \left(\left(\bigcup_{n=0}^{l-1} G_n^* \right) \bigcup G_l \right) \cong \emptyset\ (0 \leq m < l)$. From this and (2), $\left(\bigcup_{n=0}^{l-1} G_n^* \right) \bigcup G_l = G$, and so,

$$(E_m^* \setminus G_m^*) \bigcap G \cong \emptyset \qquad (m \geq 0).$$

By (1), $G_k \subset [-\pi, \pi]^d$. So $G_k \bigcap (G_k + 2\pi\boldsymbol{\nu}) \cong \emptyset\,(\boldsymbol{\nu} \neq \mathbf{0})$. On the other hand, by (3), $G_m^* \bigcap (G_k + 2\pi\boldsymbol{\nu}) \cong \emptyset\,(m < k,\ \boldsymbol{\nu} \in \mathbb{Z}^d)$. So, $\left(\left(\bigcup_{m=0}^{k-1} G_m^* \right) \bigcup G_k \right) \bigcap (G_k + 2\pi\boldsymbol{\nu}) \cong \emptyset\,(\boldsymbol{\nu} \neq \mathbf{0})$. From this and (2), it fol'lows that

$$G \bigcap (G_k + 2\pi\boldsymbol{\nu}) \cong \emptyset \qquad (\boldsymbol{\nu} \neq \mathbf{0}).$$

Lemma 1 is proved. □

Next, a decomposition of each $G_m^*\,(m \geq 0)$ is given. For arbitrarily, finitely, many distinct points $\boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_r\ (\mathbf{0} \in \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r} \subset \mathbb{Z}^d)$, define a point set $F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r}$:

$$F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} := \{\boldsymbol{\omega} \in G_m^* : \boldsymbol{\omega} + \pi\boldsymbol{\nu} \in G_m^*\ (\boldsymbol{\nu} \in \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r}) \text{ and } \boldsymbol{\omega} + \pi\boldsymbol{\nu} \notin G_m^*\ (\boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r})\}. \tag{5}$$

Let $\boldsymbol{\omega} \in G_m^*$. Note that $G_m^* \subset G$. Since G is bounded, there only exist finitely many $\boldsymbol{\nu} \in \mathbb{Z}^d$ such that $\boldsymbol{\omega} + \pi\boldsymbol{\nu} \in G_m^*$. So, $\boldsymbol{\omega}$ must lie in some $F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r}$, and so,

$$G_m^* = \bigcup_{r=0}^{\infty} \bigcup_{\mathbf{0} \in \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r} \subset \mathbb{Z}^d} F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r}, \tag{6}$$

and the right-hand side of (6) is a disjoint union. Let

$$P^m =: \{F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} :\ \ \mathbf{0} \in \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r} \subset \mathbb{Z}^d,\ r \geq 0 \text{ and } F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} \neq \emptyset\}. \tag{7}$$

By (1) and $G_m^* \subset G$, if some $\boldsymbol{\nu}_l$ in $\boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_r\ (\mathbf{0} \in \{\boldsymbol{\nu}_l\}_{l=0,\ldots,r} \subset \mathbb{Z}^d)$ satisfies $|\boldsymbol{\nu}_l| > 2^{k+1}\sqrt{d}$, then, for any $\boldsymbol{\omega} \in G_m$, it follows that $\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l \notin G_m$. Again, by (5), it follows that $F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} = \emptyset$. Therefore, P^m only consists of finitely many point sets.

Lemma 2. *If $F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} \in P^m$, then, $F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} + \pi\boldsymbol{\nu}_s \in P^m\,(s = 0, \ldots, r)$ and these point sets are disjoint.*

Proof. For each s, $\boldsymbol{\omega} \in F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} + \pi\boldsymbol{\nu}_s$ is equivalent to $\boldsymbol{\omega} - \pi\boldsymbol{\nu}_s \in F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r}$. Let $\boldsymbol{\alpha}_{l,s} = \boldsymbol{\nu}_l - \boldsymbol{\nu}_s\,(l = 0, \ldots, r)$. Then, $\boldsymbol{\alpha}_{s,s} = \mathbf{0}$. So, $\mathbf{0} \in \{\boldsymbol{\alpha}_{l,s}\}_{l=0,\ldots,r} \subset \mathbb{Z}^d$. Again, by (5), it follows that $\boldsymbol{\omega} - \pi\boldsymbol{\nu}_s \in F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r}$ is equivalent to $\boldsymbol{\omega} \in G_m^*$ and $\boldsymbol{\omega} + \pi\boldsymbol{\nu} \in G_m^*\,(\boldsymbol{\nu} \in \{\boldsymbol{\alpha}_{l,s}\}_{l=0,\ldots,r})$, and $\boldsymbol{\omega} + \pi\boldsymbol{\nu} \notin G_m^*\,(\boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\boldsymbol{\alpha}_{l,s}\}_{l=0,\ldots,r})$. It means that $\boldsymbol{\omega} \in F^m_{\boldsymbol{\alpha}_{0,s},\ldots,\boldsymbol{\alpha}_{r,s}}$, i.e.,

$$F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} + \pi\boldsymbol{\nu}_s = F_{\boldsymbol{\alpha}_{0,s},\ldots,\boldsymbol{\alpha}_{r,s}} \in P^m.$$

Let $s_1 \neq s_2$. Then, $\boldsymbol{\nu}_{s_1} \neq \boldsymbol{\nu}_{s_2}$ and $\boldsymbol{\alpha}_{l,s_1} \neq \boldsymbol{\alpha}_{l,s_2}\ (l = 0, \ldots, r)$. This implies that

$$(F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} + \pi\boldsymbol{\nu}_{s_1})\bigcap(F^m_{\boldsymbol{\nu}_0,\ldots,\boldsymbol{\nu}_r} + \pi\boldsymbol{\nu}_{s_2}) == F_{\boldsymbol{\alpha}_{0,s_1},\ldots,\boldsymbol{\alpha}_{r,s_1}} \bigcap F_{\boldsymbol{\alpha}_{0,s_2},\ldots,\boldsymbol{\alpha}_{r,s_2}} = \emptyset.$$

Lemma 2 is proved. □

Lemma 3. *There exist finitely many point sets* $A_j^m := F^m_{\boldsymbol{\nu}_0^{(j)},...,\boldsymbol{\nu}_{r_j}^{(j)}}$ $(j = 1, ..., \lambda)$ *in* P^m*, where* λ *is some natural number, such that* G_m^* *is the following disjoint union*

$$G_m^* = \bigcup_{j=1}^{\lambda} \bigcup_{l=0}^{r_j} (A_j^m + \pi \boldsymbol{\nu}_l^{(j)}).$$

Proof. Take a point set $A_1^m = F^m_{\boldsymbol{\nu}_0^{(1)},...,\boldsymbol{\nu}_{r_1}^{(1)}} \in P^m$. By Lemma 2, the point sets $A_1^m + \pi \boldsymbol{\nu}_l^{(1)} \in P^m$ $(l = 0, ..., r_1)$ and these point sets are disjoint. Denote

$$S_1^m := \left\{ A_1^m + \pi \boldsymbol{\nu}_l^{(1)} \ \ (l = 0, ..., r_1) \right\}.$$

Let $P_1^m := P^m \setminus S_1^m$. After that, take a point set $A_2^m = F^m_{\boldsymbol{\nu}_0^{(2)},...,\boldsymbol{\nu}_{r_2}^{(2)}} \in P_1^m$. Denote

$$S_2^m = \left\{ A_2^m + \pi \boldsymbol{\nu}_l^{(2)} \ \ (l = 0, ..., r_2) \right\}.$$

Since P^m consists of finitely many nonempty point sets, repeating the above process, one can finally choose finitely many point sets $\{A_j^m\}_{j=1,...,\lambda} \subset P^m$ (λ is some natural number) such that

$$P^m = \left\{ A_j^m + \pi \boldsymbol{\nu}_l^{(j)} : \ l = 0, ..., r_j; \ j = 1, ..., \lambda \right\}.$$

By (6) and (7), Lemma 3 is proved. □

4. Sufficient Conditions for the Support of Frequency Domains of Bandlimited FMRA

In this section, the converse of Theorem 1 will be proved:

Theorem 2. *Let* G *be a bounded closed set in* $\mathbb{R}^d$ *satisfying Theorem 1(i)–(iii), then, there exists a bandlimited FMRA with the support of frequency domain* G*.*

Proof. Since G is bounded, one can choose a $k \in \mathbb{Z}_+$ such that $G = [-2^k\pi, 2^k\pi]^d$. Two functions $\varphi \in L^2(\mathbb{R}^d)$ and $H \in L^\infty(T^d)$ will be constructed such that $G = \operatorname{supp}\widehat{\varphi}$ and
(*) $\widehat{\varphi}(2\boldsymbol{\omega}) = H(\boldsymbol{\omega})\widehat{\varphi}(\boldsymbol{\omega})$ $(\boldsymbol{\omega} \in \mathbb{R}^d)$;
(**) $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = \chi_\Omega(\boldsymbol{\omega})$ $(\boldsymbol{\omega} \in \mathbb{R}^d)$, where $\Omega = G + 2\pi\mathbb{Z}^d$ and χ_Ω is the characteristic function of Ω. □

The process is divided into four steps.
Step 1. Define $\widehat{\varphi}(\boldsymbol{\omega}) = 0$ $(\boldsymbol{\omega} \notin G)$.
Step 2. Define $\widehat{\varphi}(\boldsymbol{\omega})$ on G_k^* and $H(\boldsymbol{\omega})$ on E_{k+1}^*, in detail;
Define $\widehat{\varphi}(\boldsymbol{\omega}) = 1$ $(\boldsymbol{\omega} \in G_k)$. For $\boldsymbol{\omega} \in G_k$, by Lemma 1(b), it follows that $\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \notin G$ $(\boldsymbol{\nu} \neq 0)$. So,

$$\widehat{\varphi}(\boldsymbol{\omega}) = \begin{cases} 0, & \boldsymbol{\omega} \in G_k + 2\pi\boldsymbol{\nu} \ \ (\boldsymbol{\nu} \neq \mathbf{0}), \\ 1, & \boldsymbol{\omega} \in G_k. \end{cases} \tag{8}$$

Furthermore, $\widehat{\varphi}$ has been defined on E_k and

$$\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = |\widehat{\varphi}(\boldsymbol{\omega})|^2 = 1 \qquad (\boldsymbol{\omega} \in G_k),$$

i.e., Formula (**) holds for $\boldsymbol{\omega} \in G_k$.

Define $H(\boldsymbol{\omega}) = 1(\boldsymbol{\omega} \in E_{k+1})$. If $\boldsymbol{\omega} \in G_{k+1}$, then, $2\boldsymbol{\omega} \in G_k$. By $G_{k+1} \subset G_k$ and (8), $\widehat{\varphi}(2\boldsymbol{\omega}) = \widehat{\varphi}(\boldsymbol{\omega}) = 1$. Hence, Formula (*) holds for $\boldsymbol{\omega} \in G_{k+1}$. If $\boldsymbol{\omega} \in G_{k+1} + 2\pi\boldsymbol{\nu}$ $(\boldsymbol{\nu} \neq \mathbf{0})$, then, $2\boldsymbol{\omega} \in G_k + 4\pi\boldsymbol{\nu}(\boldsymbol{\nu} \neq \mathbf{0})$. From this and (8), $\widehat{\varphi}(2\boldsymbol{\omega}) = \widehat{\varphi}(\boldsymbol{\omega}) = 0$, Clearly, Formula (*) holds for $\boldsymbol{\omega} \in G_{k+1} + 2\pi\boldsymbol{\nu}$ $(\boldsymbol{\nu} \neq \mathbf{0})$. Thus, Formula (*) holds for $\boldsymbol{\omega} \in E_{k+1}$.

Step 3. Based on Step 2, the idea of mathematics induction will be used. For $k \geq m > 0$, assume that $\widehat{\varphi}(\boldsymbol{\omega})$ is defined on G_m^* and $H(\boldsymbol{\omega})$ is defined on E_{m+1}^* such that Formula (*) holds on E_{m+1}^*, Formula (**) holds on G_m^*, and

$$0 < C_m \leq \widehat{\varphi}(\boldsymbol{\omega}) \leq D_m < \infty, \quad \boldsymbol{\omega} \in G_m^*,$$

$$0 < \widetilde{C}_m \leq H(\boldsymbol{\omega}) \leq \widetilde{D}_m < \infty, \quad \boldsymbol{\omega} \in E_{m+1}^*,$$

where $C_m, D_m, \widetilde{C}_m$, and $\widetilde{D}_m$ are constants.

Define $H(\boldsymbol{\omega})$ on G_m^*. By Lemma 3, one only needs to define $H(\boldsymbol{\omega})$ on each $A_j^m + \pi\boldsymbol{\nu}_l^{(j)}$.

Since $\widehat{\varphi}(\boldsymbol{\omega})$ has been defined on G_m^* and $C_m \leq \widehat{\varphi}(\boldsymbol{\omega}) \leq D_m$. Again, by Lemma 3:

$$G_m^* = \bigcup_{j=1}^{\lambda} \bigcup_{l=0}^{r_j} (A_j^m + \pi\boldsymbol{\nu}_l^{(j)}),$$

so, for each $\boldsymbol{\omega} \in A_j^m$ and $l = 0, ..., r_j$, the values of $\widehat{\varphi}(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)})$ $(j = 1, ..., \lambda)$ have been defined and $C_m \leq \widehat{\varphi}(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)}) \leq D_m$. Write

$$\boldsymbol{\nu}_l^{(j)} = 2\mathbf{u}_l^{(j)} + \boldsymbol{\alpha}_l^{(j)} \qquad (\mathbf{u}_l^{(j)} \in \mathbb{Z}^d;\ \boldsymbol{\alpha}_l^{(j)} \in \{0,1\}^d). \tag{9}$$

Noticing that the point sets $A_j^m + \pi\boldsymbol{\alpha}_l^{(j)}$ $(l = 0, 1, ..., r_j)$ are disjoint, one can define $H(\boldsymbol{\omega} + \pi\boldsymbol{\alpha}_l^{(j)})$ on A_j^m such that for $\boldsymbol{\omega} \in A_j^m$,

$$\sum_{l=0}^{r_j} b_l^{(j)}(\boldsymbol{\omega}) |H(\boldsymbol{\omega} + \pi\boldsymbol{\alpha}_l^{(j)})|^2 = 1, \tag{10}$$

where $b_l^{(j)}(\boldsymbol{\omega}) = |\widehat{\varphi}(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)})|^2$ and

$$0 < \widetilde{C}_{m-1}^{(j)} \leq H(\boldsymbol{\omega} + \pi\boldsymbol{\alpha}_l^{(j)}) \leq \widetilde{D}_{m-1}^{(j)} < \infty \qquad (l = 0, ..., r_j), \tag{11}$$

where $\widetilde{C}_{m-1}^{(j)}$ and $\widetilde{D}_{m-1}^{(j)}$ are constants.

By (9), for $j = 1, ..., \lambda$, define

$$H(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)}) = H(\boldsymbol{\omega} + \pi\boldsymbol{\alpha}_l^{(j)}) \qquad (\boldsymbol{\omega} \in A_j^m;\ l = 0, ..., r_j). \tag{12}$$

Due to the Lemma 3, $H(\boldsymbol{\omega})$ is defined on G_m^* and

$$\widetilde{C}_{m-1} \leq H(\boldsymbol{\omega}) \leq \widetilde{D}_{m-1} \qquad (\boldsymbol{\omega} \in G_m^*),$$

where $\widetilde{C}_{m-1} = \min_{1 \leq j \leq \lambda}\{\widetilde{C}_{m-1}^{(j)}\}$ and $\widetilde{D}_{m-1} = \max_{1 \leq j \leq \lambda}\{\widetilde{D}_{m-1}^{(j)}\}$.

Below, we will prove that if $\boldsymbol{\omega} \in G_m^*$ and $\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}} \in G_m^*$ for some $\widetilde{\boldsymbol{\nu}} \in \mathbb{Z}^d$, then $H(\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}}) = H(\boldsymbol{\omega})$.

By $\boldsymbol{\omega} \in G_m^*$ and Lemma 3, there exist j and n such that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \pi\boldsymbol{\nu}_n^{(j)}, \tag{13}$$

where $\boldsymbol{\omega}_0 \in A_j^m$. Since $\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}} = \boldsymbol{\omega}_0 + \pi(2\widetilde{\boldsymbol{\nu}} + \boldsymbol{\nu}_n^{(j)}) \in G_m^*$ and $\boldsymbol{\omega}_0 \in A_j^m$, there is some $s(0 \leq s \leq r_j)$ such that

$$2\widetilde{\boldsymbol{\nu}} + \boldsymbol{\nu}_n^{(j)} = \boldsymbol{\nu}_s^{(j)}. \tag{14}$$

By (12)–(14), it follows that

$$H(\boldsymbol{\omega}) = H(\boldsymbol{\omega}_0 + \pi \boldsymbol{\nu}_n^{(j)}) = H(\boldsymbol{\omega}_0 + \pi \boldsymbol{\alpha}_n^{(j)}),$$
$$H(\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}}) = H(\boldsymbol{\omega}_0 + \pi(2\widetilde{\boldsymbol{\nu}} + \boldsymbol{\nu}_n^{(j)})) = H(\boldsymbol{\omega}_0 + \pi\boldsymbol{\nu}_s^{(j)}) = H(\boldsymbol{\omega}_0 + \pi\boldsymbol{\alpha}_s^{(j)}). \tag{15}$$

By uniqueness of decomposition in (9), it follows that $\boldsymbol{\alpha}_n^{(j)} = \boldsymbol{\alpha}_s^{(j)}$. It means that $H(\boldsymbol{\omega}) = H(\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}})$ when $\boldsymbol{\omega} \in G_m^*$ and $\boldsymbol{\omega} + 2\pi\widetilde{\boldsymbol{\nu}} \in G_m^*$ for some $\widetilde{\boldsymbol{\nu}} \in \mathbb{Z}^d$.

Based on this fact, one can further define $H(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu}) = H(\boldsymbol{\omega})\,(\boldsymbol{\omega} \in G_m^*,\ \boldsymbol{\nu} \in \mathbb{Z}^d)$. So, $H(\boldsymbol{\omega})$ is well-defined on E_m^* and $\widetilde{C}_{m-1} \le H(\boldsymbol{\omega}) \le \widetilde{D}_{m-1}\,(\boldsymbol{\omega} \in E_m^*)$.

Now, define $\widehat{\varphi}(\boldsymbol{\omega})$ on G_{m-1}^*. Let

$$\widehat{\varphi}(2\boldsymbol{\omega}) = H(\boldsymbol{\omega})\widehat{\varphi}(\boldsymbol{\omega}) \qquad (\boldsymbol{\omega} \in G_m^*). \tag{16}$$

Then, $\widehat{\varphi}(\boldsymbol{\omega})$ is defined on G_{m-1}^* $(G_{m-1}^* = 2G_m^*)$ and

$$C_{m-1} \le \widehat{\varphi}(\boldsymbol{\omega}) \le D_{m-1}, \text{ where } C_{m-1} = C_m\widetilde{C}_{m-1} \text{ and } D_{m-1} = D_m\widetilde{D}_{m-1}.$$

By Step 1 and Lemma 1(a), it follows that $\widehat{\varphi}(\boldsymbol{\omega}) = 0\,(\boldsymbol{\omega} \in \boldsymbol{E_{m-1}^*} \setminus \boldsymbol{G_{m-1}^*})$. So, $\widehat{\varphi}(\boldsymbol{\omega})$ is defined on E_{m-1}^*.

This will prove that Formula (*) holds on E_m^*: If $\boldsymbol{\omega} \in G_m^*$, by (16), Formula (*) holds. If $\boldsymbol{\omega} \notin G_m^*$, then, $\boldsymbol{\omega} \in E_m^* \setminus G_m^*$. By Lemma 1(a), it follows that $\boldsymbol{\omega} \notin G$. By $G \subset 2G$, we get $2\boldsymbol{\omega} \notin G$. So, $\widehat{\varphi}(2\boldsymbol{\omega}) = \widehat{\varphi}(\boldsymbol{\omega}) = 0$. Formula (*) also holds.

Finally, it will prove that Formula (**) holds on G_{m-1}^*.

Let $\boldsymbol{\omega} \in A_j^m = F^m_{\boldsymbol{\nu}_0^{(j)},\dots,\boldsymbol{\nu}_{r_j}^{(j)}}$. Then,

$$\boldsymbol{\omega} \in G_m^*, \quad \boldsymbol{\omega} + \pi\boldsymbol{\nu} \in G_m^* \ (\boldsymbol{\nu} \in \{\boldsymbol{\nu}_l^{(j)}\}_{l=0,\dots,r_j}), \quad \boldsymbol{\omega} + \pi\boldsymbol{\nu} \notin G_m^* \ (\boldsymbol{\nu} \notin \{\boldsymbol{\nu}^{(j)}\}_{l=0,\dots,r_j}). \tag{17}$$

By $2G_m^* = G_{m-1}^*$, it follows that $2\boldsymbol{\omega} \in G_{m-1}^*$ and then, $2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \in E_{m-1}^*\,(\boldsymbol{\nu} \in \mathbb{Z}^d)$. Again, by (17), it follows that $2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \notin G_{m-1}^*\,(\boldsymbol{\nu} \notin \{\boldsymbol{\nu}_l^{(j)}\}_{l=0,\dots,r_j})$. So, $2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \in E_{m-1}^* \setminus G_{m-1}^*\,(\boldsymbol{\nu} \notin \{\boldsymbol{\nu}_l\}_{l=0,\dots,r_j})$. By Lemma 1(a), it means that $2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \notin G\,(\boldsymbol{\nu} \notin \{\boldsymbol{\nu}_l\}_{l=0,\dots,r_j})$. So, $\widehat{\varphi}(2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu}) = 0\,(\boldsymbol{\nu} \notin \{\boldsymbol{\nu}_l^{(j)}\}_{l=0,\dots,r_j})$, and so,

$$\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = \sum_{l=0}^{r_j} |\widehat{\varphi}(2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu}_l^{(j)})|^2. \tag{18}$$

By (9) and (16), it follows that

$$\widehat{\varphi}(2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu}_l^{(j)}) = H(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)})\widehat{\varphi}(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)}) = H(\boldsymbol{\omega} + \pi\boldsymbol{\alpha}_l^{(j)})\widehat{\varphi}(\boldsymbol{\omega} + \pi\boldsymbol{\nu}_l^{(j)}).$$

Again by (18) and (10), it follows that for $\boldsymbol{\omega} \in A_j^m$,

$$\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(2\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = 1. \tag{19}$$

Similar to the above process, it follows that (19) also holds for $\boldsymbol{\omega} \in A_j^m + \pi\boldsymbol{\nu}_l^{(j)}$. Again, by Lemma 3, (19) holds for $\boldsymbol{\omega} \in G_m^*$. Noticing that $2G_m^* = G_{m-1}^*$, it means that $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = 1$ holds on G_{m-1}^*.

Step 4. From Steps 1–3, $\widehat{\varphi}(\boldsymbol{\omega})$ is defined on G and $H(\boldsymbol{\omega})$ is defined on $\frac{1}{2}G + 2\pi\mathbb{Z}^d$. Now, define $H(\boldsymbol{\omega}) = 0\,(\boldsymbol{\omega} \in G_0^* + 2\pi\mathbb{Z}^d)$ and $H(\boldsymbol{\omega}) = 0\,(\boldsymbol{\omega} \in \mathbb{R}^d \setminus \Omega)$. By assumption (iii) in Theorem 2, it follows that

$$\begin{aligned} &(G_0^* + 2\pi\mathbb{Z}^d) \bigcup \left(\tfrac{G}{2} + 2\pi\mathbb{Z}^d\right) = G + 2\pi\mathbb{Z}^d \cong \Omega, \\ &(G_0^* + 2\pi\mathbb{Z}^d) \bigcap \left(\tfrac{G}{2} + 2\pi\mathbb{Z}^d\right) \cong \emptyset. \end{aligned} \tag{20}$$

It means that H has been well-defined on $\mathbb{R}^d$ and $H \in L^\infty(T^d)$.

From Steps 1–3, Formula (*) holds on $\frac{G}{2} + 2\pi\mathbb{Z}^d$. For $\boldsymbol{\omega} \in G_0^* + 2\pi\mathbb{Z}^d$, it follows that $2\boldsymbol{\omega} \in 2G_0^* + 4\pi\mathbb{Z}^d$. By (20), it follows that $(2G_0^* + 4\pi\mathbb{Z}^d) \bigcap G \cong \emptyset$. Hence, $\widehat{\varphi}(2\boldsymbol{\omega}) = 0$. Again, by $H(\boldsymbol{\omega}) = 0(\boldsymbol{\omega} \in G_0^* + 2\pi\mathbb{Z}^d)$, Formula (*) holds also on $G_0^* + 2\pi\mathbb{Z}^d$. Thus, by (20), Formula (*) holds for $\boldsymbol{\omega} \in \Omega$. When $\boldsymbol{\omega} \notin \Omega$, by assumption (i) in Theorem 2, it follows that $2\boldsymbol{\omega} \notin \Omega$, which means that Formula (*) also holds for $\boldsymbol{\omega} \notin \Omega$. Therefore, Formula (*) holds on $\mathbb{R}^d$.

From Steps 1–3, Formula (**) holds on G. Since the sum $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2$ is a $2\pi\mathbb{Z}^d-$periodic function, $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = 1$ holds on $G + 2\pi\mathbb{Z}^d = \Omega$. If $\boldsymbol{\omega} \notin \Omega$, then, for any $\boldsymbol{\nu} \in \mathbb{Z}^d$, it follows that $\boldsymbol{\omega} + 2\pi\boldsymbol{\nu} \notin G$. So, $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = 0$ on $\mathbb{R}^d \setminus \Omega$, and so, $\sum_{\boldsymbol{\nu}} |\widehat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{\nu})|^2 = \chi_\Omega(\boldsymbol{\omega})\,(\boldsymbol{\omega} \in \mathbb{R}^d)$, i.e., Formula (**) holds on $\mathbb{R}^d$.

Up to now, the constructed $\varphi \in L^2(\mathbb{R}^d)$ and $H \in L^\infty(T^d)$ satisfy Formulas (*) and (**). Let $V_m = \overline{\text{span}}\{\varphi(2^m\mathbf{t} - \mathbf{n}),\, \mathbf{n} \in \mathbb{Z}^d\}\ (m \in \mathbb{Z})$. Since H is a $2\pi\mathbb{Z}^d-$periodic bounded function, $H(\boldsymbol{\omega})$ can be expanded into a Fourier series: $H(\boldsymbol{\omega}) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{i(\mathbf{k}\cdot\boldsymbol{\omega})}$. By Formula (*), it follows that

$$\widehat{\varphi}(2\boldsymbol{\omega}) = \sum_{\mathbf{k}} c_{\mathbf{k}} \widehat{\varphi}(\boldsymbol{\omega}) e^{i(\mathbf{k}\cdot\boldsymbol{\omega})}.$$

Taking the inverse Fourier transform on both sides, we have

$$\frac{1}{2^d}\varphi\left(\frac{\mathbf{t}}{2}\right) = \sum_{\mathbf{k}} c_{\mathbf{k}} \varphi(\mathbf{t} - \mathbf{k}),$$

and so, $\varphi(2^m\mathbf{t} - \mathbf{n}) = 2^d \sum_{\mathbf{k}} c_{\mathbf{k}} \varphi(2^{m+1}\mathbf{t} - (2\mathbf{n} + \mathbf{k})) \subset V_{m+1}$. This implies that $V_m \subset V_{m+1}$.

By a known results in [1-2], Formula (**) implies the system $\{\varphi(\cdot - \mathbf{n})\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a frame for $V_0 = \overline{\text{span}}\{\varphi(\cdot - \mathbf{n}),\, \mathbf{n} \in \mathbb{Z}^d\}$. For any $f \in V_m$, it follows that $f(2^{-m}\cdot) \in V_0$. It can be expanded into a frame series with respect to $\{\varphi(\cdot - \mathbf{n})\}_{\mathbf{n}}$:

$$f(2^{-m}\mathbf{t}) = \sum_{\mathbf{n}} d_{\mathbf{n}} \varphi(\mathbf{t} - \mathbf{n}).$$

Taking the Fourier transform on both sides, we have

$$2^m \widehat{f}(2^m\boldsymbol{\omega}) = \left(\sum_{\mathbf{n}} d_{\mathbf{n}} e^{-i(\mathbf{n}\cdot\boldsymbol{\omega})}\right)\widehat{\varphi}(\boldsymbol{\omega}),$$

i.e., $\widehat{f}(\boldsymbol{\omega}) = \tau(\boldsymbol{\omega})\widehat{\varphi}(2^{-m}\boldsymbol{\omega})$, where $\tau(\boldsymbol{\omega}) = 2^{-m}\sum_{\mathbf{n}} d_{\mathbf{n}} e^{-i(\mathbf{n}\cdot\boldsymbol{\omega})/2^m}$. So, $\text{supp}\widehat{f} \subset \text{supp}\widehat{\varphi}(2^{-m}\cdot)$ for $f \in V_m$. Since $\varphi(2^m\cdot) \in V_m$, it implies that

$$\bigcup_{f\in V_m} \text{supp}\widehat{f} = \text{supp}\,\widehat{\varphi}(2^{-m}\cdot).$$

By Assumption (ii), it follows that

$$\bigcup_m \bigcup_{f\in V_m} \text{supp}\widehat{f} = \bigcup_m 2^m G = \mathbb{R}^d.$$

By a known result in [1,2], we have $\overline{\bigcup_m V_m} = L^2(\mathbb{R}^d)$. Therefore, $\{V_m\}$ is a bandlimited FMRA with frequency domain G.

Example 1. *Let $G = \Omega_1 \bigcup \Omega_2 \in \mathbb{R}^2$, where*

$$\Omega_1 = \{(x, y) : \ -\pi \le x \le \pi, \ \pi - \sqrt{\pi^2 - x^2} \le y \le \pi\}$$

$$\Omega_2 = \{(x, y) : \ -\pi \le x \le \pi, \ -\pi \le y \le -\pi + \sqrt{\pi^2 - x^2}\}.$$

It is very clear that G satisfies

$$\text{(i) } G \subset 2G, \qquad \text{(ii) } \bigcup_m 2^m G \cong \mathbb{R}^2, \quad \text{and} \quad \text{(iii) } \left(G \setminus \tfrac{G}{2}\right) \bigcap \left(\tfrac{G}{2} + 2\pi\boldsymbol{\nu}\right) \cong \emptyset \ (\boldsymbol{\nu} \in \mathbb{Z}^2).$$

Theorem 2 shows that there exists a bandlimited FMRA with frequency domain $G = \Omega_1 \bigcup \Omega_2$.

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