Review

Real-Valued Systemic Risk Measures

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Abstract: We describe the axiomatic approach to real-valued Systemic Risk Measures, which is a natural counterpart to the nowadays classical univariate theory initiated by Artzner et al. in the seminal paper "Coherent measures of risk", Math. Finance, (1999). In particular, we direct our attention towards Systemic Risk Measures of shortfall type with random allocations, which consider as eligible, for securing the system, those positions whose aggregated expected utility is above a given threshold. We present duality results, which allow us to motivate why this particular risk measurement regime is fair for both the single agents and the whole system at the same time. We relate Systemic Risk Measures of shortfall type to an equilibrium concept, namely a Systemic Optimal Risk Transfer Equilibrium, which conjugates Bühlmann's Risk Exchange Equilibrium with a capital allocation problem at an initial time. We conclude by presenting extensions to the conditional, dynamic framework. The latter is the suitable setup when additional information is available at an initial time.

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MSC: 91G45; 91G15; 91B05; 91G05; 60G99

1. Introduction

Both the financial crisis started in 2007 and the dramatic economic shocks related to the COVID-19 pandemic have brutally proved the partial inadequacy of past approaches to the management of risk for complex systems of interacting entities. It has become more and more evident how deep interconnectedness played a major role in propagation of risk at all levels, from smaller businesses to large scale multinational entities. The literature on systemic risk has widely focused on modeling the structure of financial networks, aiming at portraying the spread of shocks (both exogenous and endogenous) in the systems. These models assume a more or less detailed knowledge of balance sheets of institutions (interbank network and exposures, recovery rate at default, liquidation policy, bankruptcy costs, cross-holdings, leverage structures, fire sales, and liquidity freezes). As pointed out in [1], however, once such models have been implemented, "one still has to understand how to compare the possible final outcomes in a reasonable way or, in other words, how to measure the risk carried by the global financial system."We will adopt here an axiomatic approach, which addresses specifically this problem. Essentially, this axiomatic approach identifies suitable maps associated with a risky position for the system (i.e., a vector of financial positions) a single amount (namely, a real number). Such an amount, intuitively speaking, summarizes the overall risk of the system. The literature on systemic risk is nowadays very vast and growing. For empirical studies on banking networks, one might look at [2–4]. The works [5–9] study interbank lending with a mean field approach and using interacting diffusions. Concerning systemic risk modeling, we mention among the many contributions [10] for a classical contagion model, ref. [11] for a default model, ref. [12–14] for illiquidity cascade models, ref. [15,16] for an asset fire sale cascade model, and [17] for a model which includes cross-holdings. Further works on network modeling

are [18–24]. We refer the reader to [25,26] for a detailed overview on the literature on systemic risk.

In this work, we will mostly focus on real-valued Systemic Risk Measures, which have been studied in recent works inspired by the axiomatic approach to univariate monetary Risk Measures of [27]. An alternative, set-valued approach has also gained significant attention in the literature, as testified, for example, by the recent "Special Issue on Vector- and Set-Valued Methods in Stochastic Finance and Related Areas" (Finance Stoch. Volume 25, issue 1, January 2021).

By its own very nature, the classical framework of [27] allows for explicitly modeling the mechanism of injecting capital in order to make risky positions acceptable and thus seems a natural environment for treating the systemic case. The classical setup is static, with deterministic amounts exchanged at initial times and random events that take place at a terminal time T. To simplify the presentation, we will assume a zero interest rate, e.g., the time T amounts are already in discounted terms. The presence of additional information can be modeled by replacing the initial trivial sigma algebra, of the static case, with a general one. Thus, we will depict the extension of the theory of Systemic Risk Measures to a dynamic, multiperiod framework. Before discussing this topic, we will elaborate some details on the static case, which help with developing the intuition of the main concepts with less technicalities.

2. Univariate Monetary Risk Measures

We denote with $L^0(\Omega, \mathcal{F}, P)$ the set of (equivalence classes of) $P-$a.s. finite random variables on the probability space $(\Omega, \mathcal{F}, P)$.

The framework of [27], which is now textbook material and is excellently exposed in [28], consists of three essential ingredients. First, the financial positions whose risk one wants to quantify are represented by random variables $X \in L^0(\Omega, \mathcal{F}, P)$, where the amount $X(\omega)$ is interpreted as a gain if positive, loss if negative. Second, it is assumed that there exists a subset $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, P)$ of random variables that are considered *acceptable* by the agent or the financial regulator. The set $\mathbb{A}$ is assumed to be monotone that is $X \geq Y$, $Y \in \mathbb{A} \Rightarrow X \in \mathbb{A}$. Finally, the *univariate monetary Risk Measure η*, which measures the risk of the positions $X \in L^0(\Omega, \mathcal{F}, P)$, is defined as the *minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting (discounted) payoff acceptable* that is

$$X \mapsto \eta(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathbb{A}\}. \tag{1}$$

A key feature of such a map is the cash additivity property:

$$\eta(X + m) = \eta(X) - m, \text{ for all } m \in \mathbb{R}.$$

As this property points out, it is necessary (and meaningful) to express X, m and $\eta(X)$ in the same monetary unit, and this allows for the monetary interpretation of $\eta(X)$ as a value which can be effectively added to X. Assuming that the set $\mathbb{A}$ is convex (respectively is a convex cone) the map in (1) is convex (respectively convex and positively homogeneous) and is called *convex* (respectively *coherent*) *Risk Measure*, see [29,30]. A rather conservative example of coherent Risk Measure is the worst case Risk Measure $\rho_W : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R} \cup \{\pm\infty\}$ associated with the cone $\mathbb{A}^W := L^0_+(\Omega, \mathcal{F}, P)$ of non-negative random variables

$$\rho_W(X) := \inf\left\{m \in \mathbb{R} \mid X + m \in \mathbb{A}^W\right\} = -ess\inf(X). \tag{2}$$

Convexity is meant to express mathematically the principle of diversification, which is summarized in the famous saying "don't put all your eggs in one basket". Diversification might also be expressed with the weaker condition of quasiconvexity

$$\eta(\lambda X + (1 - \lambda)Y) \leq \max\{\eta(X), \eta(Y)\}, \ \lambda \in [0, 1],$$

which accurately express the principle that diversification can not increase the risk. As a result, in [31,32], *quasi-convex Risk Measure* is only assumed to satisfy monotonicity and quasi-convexity. Any quasi-convex Risk Measures can be written in the form

$$\eta(X) = \inf\{m \in \mathbb{R} \mid X \in \mathbb{A}^m\} \tag{3}$$

where each set $\mathbb{A}^m \subseteq L^0(\Omega, \mathcal{F}, P)$ is monotone and convex, for each m. The set $\mathbb{A}^m$ models the class of payoffs carrying the same risk level m. In the quasi-convex case, several degrees of acceptability, expressed via the risk level m (see also [33]), are admitted. This marks a difference with the convex and cash additive case, in which each financial position is either acceptable or not acceptable.

3. Systemic Risk Measures

Let us now consider a financial system of N agents or institutions, portrayed by the exposures $\mathbf{X} = [X^1, \ldots, X^N] \in (L^0(\Omega, \mathcal{F}, P))^N$. Given the theory of univariate Risk Measures, a natural first step to design the measurement of the risk of the system consists in applying (possibly different) Risk Measures η^n to X^n, $n = 1, \ldots, N$, and then aggregate the resulting amounts as $\rho(\mathbf{X}) := \sum_{n=1}^N \eta^n(X^n)$. This approach is clearly quite naïf, since it almost completely ignores the multivariate nature of the exposures $\mathbf{X}$ (if e.g., η^n is law invariant for each n, the amount $\rho(\mathbf{X})$ only depends on the marginals of the vector $\mathbf{X}$) and, more generally, it does not take into account the fact that securing each component of $\mathbf{X}$ is in principle not enough to secure the system as a whole.

It is worth noticing that, in the above mentioned classical theory of univariate Risk Measures, the valuation of the risk is obtained by aggregating the scenario-dependent exposures into a deterministic amount, namely $\eta(X)$, securing the financial positions. This rather trivial remark, however, stresses the fact that, for the systemic case, one needs to identify a second aggregation mechanism over the components of the system. This aggregation over agents in the system can be obtained in several different ways (both conceptually and mathematically), as we now explain.

3.1. First Aggregate, Then Allocate

In the axiomatic approach to Systemic Risk Measures, the objective is thus the identification of a functional $\rho : (L^0(\Omega, \mathcal{F}, P))^N \to \mathbb{R}$ that evaluates the overall risk $\rho(\mathbf{X})$ of the whole system $\mathbf{X} = [X^1, \ldots, X^N] \in (L^0(\Omega, \mathcal{F}, P))^N$. A significant part of recent literature has focused on Systemic Risk Measures in the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})) \tag{4}$$

where $\eta : L^0(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a univariate monetary Risk Measure and $\Lambda : \mathbb{R}^N \to \mathbb{R}$ expresses an aggregation rule which provides a univariate risk factor $\Lambda(\mathbf{X})$. We observe here that $\Lambda(\mathbf{X})$ stands for the random variable $\omega \mapsto \Lambda \circ \mathbf{X}(\omega)$. One of the most common and natural choices is $\Lambda(\mathbf{x}) = \sum_{j=1}^N x^j$, $\mathbf{x} = [x^1, ..., x^N] \in \mathbb{R}^N$, see, e.g., the *systemic Expected Shortfall* introduced in [34], or the *Contagion Value at Risk (CoVaR)* introduced in [35]. Among the drawbacks of this approach, we mention that this aggregation rule seems not entirely appropriate when modeling systems in which cross-subsidization between institutions might be rather unrealistic. Moreover, in this case, a more traditional approach consisting of applying a univariate (coherent) Risk Measure η to the single risky positions would be prudent enough. The latter claim is a consequence of the fact that, by sub-linearity of coherent Risk Measures, it holds that $\eta(\sum_{n=1}^N X^n) \leq \sum_{n=1}^N \eta(X^n)$.

A possible aggregation function that takes into account the lack of cross-subsidization between financial institutions is the summation of losses below a given threshold: $\Lambda(\mathbf{x}) = \sum_{n=1}^N -(x^n - d^n)^-$, where $x^- := -\min(x, 0)$. Such an approach is adopted for example in [36,37], and generalized in [38] where also the effect of gains (positive parts) is considered by using $\Lambda(\mathbf{x}) = -\sum_{n=1}^N \alpha_n(x^n)^- + \sum_{n=1}^N \beta_n(x^n - d^n)^+$. Contagion effects in the style of

the model in [10] can be also taken into account as in [39]. The aggregation rule is in this case given in the form

$$\Lambda_{CM}(\mathbf{x}) = \min_{y^i \geq x^i + \sum_{n=1}^{N} \Pi_{in} y^n, \forall i=1,\cdots,N, y \in \mathbb{R}_+^N} \left\{ \sum_{n=1}^{N} y^n \right\}.$$

The matrix $\Pi = (\Pi_{ij})_{i,j=1,\cdots,N}$ represents the relative liability matrix, i.e., firm i has to pay the proportion Π_{ij} of its total liabilities to firm j. It is of great importance to specify that, in this last case, the vector **X** is interpreted as future profits and losses *before* propagation of shocks and contagion take place. An alternative approach, which is adopted when working with aggregations of the form $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} x^n$ and $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -(x^n - d^n)^-$ assumes that the random vector **X** already incorporates the potential exposures due to contagion effects. Systemic Risk Measures of the form $\mathbf{X} \mapsto \eta(\Lambda(\mathbf{X}))$ are characterized axiomatically in [39] on a finite state space, in [40] on a general probability space, and in [41,42] in a conditional setting. These references also present further examples of possible aggregation functions. Whenever η in (4) is a monetary Risk Measure, the characterization (1) allows for writing

$$\rho(\mathbf{X}) := \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}. \tag{5}$$

Thus, systemic risk can again be interpreted as the *minimal cash amount that secures the system when it is added to the total aggregated system loss* $\Lambda(\mathbf{X})$. In the cases when $\Lambda(\mathbf{X})$ can not be directly interpreted as cash, then $\rho(\mathbf{X})$ has to be understood in a mild sense as some risk level of the system, rather than the more explicit and practical meaning as a capital requirement. As explained in [1], a considerable portion of existing Risk Measures in the literature are in the form (5): the DIP of [36,43], the CoVar ([35]), the MES ([34]), and many other examples that can be found in [44].

3.2. First Allocate, Then Aggregate

An alternative approach, introduced by [1], consists of measuring systemic risk as the minimal cash that secures the aggregated system by *adding the capital into the single institutions before aggregating their individual risks*. This is particularly meaningful when aiming at securing the whole system by intervention at the level of single agents. If again $\Lambda : \mathbb{R}^N \to \mathbb{R}$ denotes an aggregation function, this second approach can be expressed as

$$\rho(\mathbf{X}) := \inf\left\{ \sum_{n=1}^{N} m^n \mid \mathbf{m} = [m^1, ..., m^N] \in \mathbb{R}^N, \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A} \right\}. \tag{6}$$

Here, the amount m^n is added to the financial position X^n of institution $n \in \{1, ..., N\}$ *before* the corresponding aggregation $\Lambda(\mathbf{X} + \mathbf{m})$ takes place. This procedure is conceptually relevant once we realize that injecting cash first might prevent cascade effects and contagion to happen at all. The systemic risk is then measured as the minimal (total) amount $\sum_{n=1}^{N} m^n$ injected into the institutions to secure the system.

A related concept, also based on acceptance sets, is developed by [45] in the context of set-valued Systemic Risk Measures. The authors also admit capital injection before aggregating individual risks (this is called aggregation mechanism sensitive to capital levels in [45]). However, the latter approach and the one presented in (6) are hardly comparable due to the intrinsic conceptual differences in considering total amounts of capital to secure the system (as in (6)) versus set-valued approaches.

A risk measurement regime in the form (6) has a significant advantage with respect to the one described in Section 3.1. Indeed, assume for the sake of the discussion that (5) and (6) admit optima $m_{(5)} \in \mathbb{R}, \mathbf{m}_{(6)} \in \mathbb{R}^N$, respectively. In contrast to (5), we then see that (6) addresses not only the problem of determining a total amount $\rho(\mathbf{X})$ to secure the system as a whole, but also determines a way to decompose the total risk $\rho(\mathbf{X})$ into partial risk allocations $m_{(6)}^n \in \mathbb{R}$ to be attributed to each institution n: since $\rho(\mathbf{X}) = \sum_{n=1}^{N} m_{(6)}^n$, it is self evident that this procedure is consistent with the computation of the overall risk $\rho(\mathbf{X})$.

On the contrary, the amount $m_{(5)}$ secures the system in a less explicative way, especially from the point of view of a regulator imposing capital requirements to each institution. Once the amount $m_{(5)}$ is identified, it is yet to be specified if and how each component of the system participates in the benefits of its allocation.

Dual representation results have been obtained in [1] in the "first allocate, then aggregate" case and by [46] in both cases for Systemic Risk Measures based on acceptance sets. Duality results have also been studied for set-valued Systemic Risk Measures of the "first aggregate, then allocate type" in [47].

3.2.1. Multivariate Shortfall Risk Allocation

A particular case of Systemic Risk Measures of the type "first allocate, then aggregate"has been specifically studied in [48]. The authors consider a *loss function* $\ell : \mathbb{R}^N \to (-\infty, +\infty]$, which is nondecreasing in the componentwise order, convex, lower semicontinuous and such that $\inf \ell < 0$ and $\ell(\mathbf{x}) \geq \sum_{n=1}^{N} x^n - c$ for some constant c. Such a functional ℓ extends to the multivariate case the loss functions considered in [28]. The *multivariate shortfall risk* of the position X is then defined as

$$R(\mathbf{X}) := \inf\left\{ \sum_{n=1}^{N} m^n \mid \mathbf{m} \in \mathbb{R}^N, \mathbb{E}[\ell(\mathbf{X} - \mathbf{m})] \leq 0 \right\}.$$

A clever way to tackle this optimization problem and to get its dual representation is to apply the powerful theory of Orlicz spaces (for an overview of Orlicz space theory applied to utility maximization problem, we refer to [49]). Consider the multivariate Orlicz heart induced by the loss function ℓ, namely $M^\theta := \{ X \in (L^0(\Omega, \mathcal{F}, P))^N \mid \mathbb{E}[\theta(\lambda|X|)] < +\infty \, \forall \lambda > 0 \}$, where $\theta(x) := \ell(|x|)$ and $|\cdot|$ is taken componentwise.

By suitably restricting the domain of R to M^θ, the map R is found to be real-valued, convex, monotone, and translation invariant ($R(\mathbf{X} + \mathbf{m}) = R(\mathbf{X}) - \sum_{n=1}^{N} m^n$). As the topological dual of M^θ is the Orlicz space L^{θ^*} associated with the convex conjugate θ^* of θ, a dual representation of R based on the duality (M^θ, L^{θ^*}) holds true (see [48] Theorem 2.10). Assuming permutation invariance for ℓ, i.e., $\ell(\mathbf{x}) = \ell(\pi(\mathbf{x}))$ for every componentwise permutation π, optimal allocations are proved to exist and are characterized in terms of Lagrange multipliers ([48] Theorem 3.4).

If one selects loss functions of the form $\ell(\mathbf{x}) = \sum_{n=1}^{N} \ell_n(x^n)$ for univariate loss functions $\ell_1, \ldots, \ell_N$, the amount $R(\mathbf{X})$ only depends on the (one dimensional) marginals of the law $P_\mathbf{X}$ on $\mathbb{R}^N$. Thus, in order to capture the truly multivariate nature of the risk associated with the position $\mathbf{X}$ of the system, one needs to consider more sophisticated functions ℓ. The choice of a particular the loss function

$$\ell(\mathbf{x}) := \sum_{n=1}^{N} x^n + \frac{1}{2} \sum_{n=1}^{N} ((x^n)^+)^2 + \alpha \sum_{1 \leq k \leq j \leq N} (x^j)^+ (x^k)^+ - 1$$

allows for studying systemic sensitivity of shortfall risk and its allocation, impact of exogenous shocks, and computational aspects of the risk allocations, namely of the optimal allocations $\mathbf{m}_\mathbf{X}$ satisfying $\sum_{n=1}^{N} m_\mathbf{X}^n = R(\mathbf{X})$.

As we shall see in the following section, allowing for scenario-dependent allocations in place of the deterministic ones in [48] can prevent in general the aforementioned drawback of marginal dependencies, even in the case of aggregation of single institutions' loss functions $\sum_{n=1}^{N} \ell_n(x^n)$.

3.3. Scenario-Dependent Allocation

The approach described in Section 3.2 can be generalized replacing the deterministic amounts $\mathbf{m} \in \mathbb{R}^N$ with random vectors $\mathbf{Y} \in \mathcal{C} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$. Here, $\mathcal{C}$ stands for a set of admissible assets with possibly random, scenario-dependent payoffs. This approach, introduced by [1], raises the question on the *initial time evaluation* of those assets $\mathbf{Y}$ in $\mathcal{C}$: since

now the vector **m** will be substituted with a random vector **Y**, the simple componentwise addition $\sum_{n=1}^{N} m^n$ implemented in (6) has no natural counterpart.

Following [50], however, one can assign a measurement $\pi(\mathbf{Y})$ of the risk (or the cost) associated with $\mathbf{Y} \in \mathcal{C}$, by postulating the existence of a evaluation map

$$\pi : \mathcal{C} \to \mathbb{R},$$

which is assumed to be monotone increasing. Under these premises, (6) can be extended as

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\}. \tag{7}$$

For the sake of interpretation, $\mathcal{C}$ could be a set of (vectors of) admissible financial assets that can be used to secure the system. Adding **Y** to **X** componentwise yields the terminal time value $\mathbf{X} + \mathbf{Y}$. Aggregation via Λ allows for determining whether the addition of **Y** produces an acceptable position, and $\pi(\mathbf{Y})$ is the valuation of **Y**. An example for π and $\mathcal{C}$ which will play a major role is:

$$\mathcal{C} \subseteq \left\{ \mathbf{Y} \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{n=1}^{N} Y^n \in \mathbb{R} \right\} =: \mathcal{C}_{\mathbb{R}}, \tag{8}$$

and $\pi(\mathbf{Y}) = \sum_{n=1}^{N} Y^n$. Here, the notation $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ means that $\sum_{n=1}^{N} Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable. Then, as in (6), the Systemic Risk Measure

$$\rho(\mathbf{X}) := \inf\left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\}$$

can again be interpreted as the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed at an initial time to secure the system by distributing the cash at the future time T among the components of the risk vector **X**. However, as opposed to (6), in general, the allocation $Y^i(\omega)$ to institution i does not need to be known at an initial time, but depends instead on the scenario $\omega \in \Omega$ that has been realized at time T. As mentioned in [51], "this corresponds to the situation of a lender of last resort who is equipped with a certain amount of cash today and who will allocate it according to where it serves the most depending on the scenario that has been realized at T". Additional restrictions and constraints on the possible allocations of cash are given by the set $\mathcal{C}$. The Systemic Risk Measures with deterministic allocations presented in Section 3.1 can be absorbed in this setup by the extreme choice $\mathcal{C} = \mathbb{R}^N$.

3.4. Multidimensional Acceptance Set

The Systemic Risk Measure (7) is a particular instance of the wider class of Systemic Risk Measures defined through a general monotone multidimensional acceptance set $\mathcal{A} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A}\}. \tag{9}$$

Indeed, for given $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, P)$ and Λ and setting

$$\mathcal{A} := \left\{ \mathbf{Z} \in (L^0(\Omega, \mathcal{F}, P))^N \mid \Lambda(\mathbf{Z}) \in \mathbb{A} \right\}, \tag{10}$$

we have $\mathbf{X} + \mathbf{Y} \in \mathcal{A}$ iff $\Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}$. Obviously, not all multidimensional set $\mathcal{A}$ may be written in the form (10), as, for example, if $\mathcal{A} := \mathbb{A}_1 \times ... \times \mathbb{A}_N$ and $\mathbb{A}_n \subseteq L^0(\Omega, \mathcal{F}, P)$ for each n. Observe that, applying the definition (9), this latter acceptance set $\mathcal{A}$ generates the Risk Measure $\rho(\mathbf{X}) = \sum_{n=1}^{N} \eta_{\mathbb{A}_n}(X^n)$, where each univariate Risk Measure $\eta_{\mathbb{A}_n}$ is associated with the set $\mathbb{A}_n$.

In analogy to (3), another possible generalization of (7) is achieved, see [1], by allowing the acceptance set $\mathcal{A}^{\mathbf{Y}} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ to depend on the vector $\mathbf{Y} \in \mathcal{C}$ and by defining the quasi-convex map as:

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \ \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}. \tag{11}$$

In the remainder of this paper, however, we will focus only on real-valued convex Systemic Risk Measures defined via an aggregator functional and one dimensional acceptance set.

4. Axiomatic Definition of Systemic Risk Measures

As anticipated before, we present now the Systemic Risk Measures theory developed by [1], which parallels the classical axiomatic definition of univariate Risk Measures via acceptance sets. Again, $(L^0(\Omega, \mathcal{F}, P))^N$ is the set of (equivalence classes of) vectors of $P-$a.s. finite random variables. The space $(L^0(\Omega, \mathcal{F}, P))^N$ equipped with the classical componentwise P-a.s. order relation is a vector lattice. All inequalities between vectors of random variables are meant to hold the P-.a.s.. We recall that a map $f : (L^0(\Omega, \mathcal{F}, P))^N \to L^0(\Omega, \mathcal{F}, P)$ is monotone decreasing if $\mathbf{X}_2 \geq \mathbf{X}_1$ implies $f(\mathbf{X}_1) \geq f(\mathbf{X}_2)$. Analogously for functions $f : (L^0(\Omega, \mathcal{F}, P))^N \to [-\infty, \infty]$. As usual, a map $f : (L^0(\Omega, \mathcal{F}, P))^N \to [-\infty, \infty]$ is convex if

$$f(\lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2) \leq f(\lambda \mathbf{X}_1) + f((1 - \lambda)\mathbf{X}_2) \ \ \forall \lambda \in [0, 1].$$

A vector $\mathbf{X} = [X^1, \ldots, X^N] \in (L^0(\Omega, \mathcal{F}, P))^N$ denotes a configuration of risky factors at a future time T associated with a system of N entities. Let

$$\mathcal{C} \subseteq (L^0(\Omega, \mathcal{F}, P))^N \qquad \mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, P)$$

be the set of admissible allocations at terminal time and acceptable positions, and consider a map

$$\pi : \mathcal{C} \to \mathbb{R}$$

so that $\pi(\mathbf{Y})$ stands for the risk (or cost) associated with $\mathbf{Y}$. Finally let $\Theta : (L^0(\Omega, \mathcal{F}, P))^N \times \mathcal{C} \to L^0(\Omega, \mathcal{F}, P)$ denote some aggregation function, jointly in $\mathbf{X}$ and $\mathbf{Y}$.

Definition 1. *The Systemic Risk Measure associated with $\mathcal{C}$, $\mathbb{A}$, Θ and π is the map $\rho : (L^0(\Omega, \mathcal{F}, P))^N \to [-\infty, \infty]$ defined by*

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \ \Theta(\mathbf{X}, \mathbf{Y}) \in \mathbb{A}\}. \tag{12}$$

The map ρ is a Convex Systemic Risk Measure if it is monotone decreasing and convex on $\{\rho(\mathbf{X}) < +\infty\}$.

As is customary, we convene that $\inf\{\varnothing\} = +\infty$. This definition translates mathematically the idea that the systemic risk of a random vector $\mathbf{X}$ is measured by the minimal risk (cost) of those random vectors $\mathbf{Y}$ that make $\mathbf{X}$ acceptable if appropriately aggregated. Among the many advantages of the general formulation (12) is the possibility to design Systemic Risk Measures that include both the case "first allocate, then aggregate" as in (6) and (7) by putting $\Theta(\mathbf{X}, \mathbf{Y}) = \Lambda(\mathbf{X} + \mathbf{Y})$, and the case "first aggregate, then allocate" as in (5) by putting $\Theta(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_1 : (L^0(\Omega, \mathcal{F}, P))^N \to L^0(\Omega, \mathcal{F}, P)$ is an aggregation function and $\Lambda_2 : \mathcal{C} \to L^0(\Omega, \mathcal{F}, P)$ could be, for example, the total discounted cost of $\mathbf{Y}$.

Moreover, it is readily verified that the framework of Section 3.2.1 can be embedded in the one in (12). As shown in Proposition 4.1 [1], there are simple conditions ensuring that ρ in (12) is a convex Systemic Risk Measure.

Proposition 1. *Suppose that* $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, P)$ *and* $\mathcal{C} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ *are convex,* $\mathbb{A}$ *is monotone,* $\Theta : (L^0(\Omega, \mathcal{F}, P))^N \times \mathcal{C} \to L^0(\Omega, \mathcal{F}, P)$ *is concave and* $\Theta(\cdot, \mathbf{Y})$ *is increasing for all* $\mathbf{Y} \in \mathcal{C}$, *then* ρ *defined (12) is a convex Systemic Risk Measure.*

Remark 1. *For the sake of generality, the Definition 1 is given with no integrability assumptions on either* $\mathbf{X}$ *or* $\mathbf{Y}$ *and admitting the values* $+\infty$ *and* $-\infty$ *for the functional* ρ. *Nevertheless, if a more detailed analysis is to be carried over, a restriction to suitable spaces is in place. The motivations are analogous to those of the univariate case, where the most common restrictions are those to* $L^p(\Omega, \mathcal{F}, P)$ $p \in [1, \infty]$ *or to more general Orlicz spaces and Orlicz hearts. It is clear, and it was shown in detail [52], that finiteness of* ρ *can not be guaranteed if working on unrestricted vector spaces, as for example* $L^0(\Omega, \mathcal{F}, P)$.

Practice shows that selecting an appropriate environment for the variables $\mathbf{X}$ *in order to have* $\rho(\mathbf{X}) > -\infty$ *is best done on a case-by-case manner: in [48,53], finiteness is, for example, achieved by working on Orlicz hearts.*

Guaranteeing $\rho(\mathbf{X}) < +\infty$, *instead, lends itself to structural assumptions on* $\mathcal{C}$ *and* $\mathbb{A}$ *and* Θ. *For example, under the same assumptions of Proposition 1 , if additionally*

- $\left\{ m\mathbf{1} \in \mathbb{R}^{\mathbf{N}} \mid m \in \mathbb{R}_+, \mathbf{1} := [1, ..., 1] \right\} \subseteq \mathcal{C}$,
- $\Theta(-m\mathbf{1}, m\mathbf{1}) \in \mathbb{A}$ *for all* $m \in \mathbb{R}_+$,

Then, ρ *defined by (12) satisfies* $\rho(\mathbf{X}) < +\infty$ *for all* $\mathbf{X} \in (L^\infty(\Omega, \mathcal{F}, P))^N$. *Indeed, setting* $m := \max_j \| X^j \|_\infty$, *we deduce* $\Theta(\mathbf{X}, m\mathbf{1}) \geq \Theta(-m\mathbf{1}, m\mathbf{1}) \in \mathbb{A}$, *hence the monotonicity of* $\mathbb{A}$ *implies that also* $\Theta(\mathbf{X}, m\mathbf{1}) \in \mathbb{A}$.

Example 1. *We mention a few basic examples, taken from [1], to construct Risk Measures by using the acceptance set* $\mathbb{A}^W$ *associated with the worst case Risk Measure (see (2)) and the aggregation function*

$$\Lambda_d(\mathbf{X}) := \sum_{n=1}^{N} -(X^n - d^n)^-.$$

Possible candidates for the set of terminal time allocations $\mathcal{C}$ *are, on one hand, the deterministic allocations* $\mathcal{C} = \mathbb{R}^N$ *and, on the other hand, the family of so-called "constrained scenario-dependent cash allocations". Recall the definition of* $\mathcal{C}_{\mathbb{R}}$ *in (8) and consider the sets*

$$\mathcal{C}_\gamma := \left\{ \mathbf{Y} \in \mathcal{C}_{\mathbb{R}} \mid Y^i \geq \gamma^i \, i = 1, ...N \right\}$$

for $\gamma := [\gamma^1, ..., \gamma^N]$, $\gamma^i \in [-\infty, 0]$. *If* $\gamma := [-\infty, ..., -\infty]$ *this family of subsets includes* $\mathcal{C}_\infty = \mathcal{C}_{\mathbb{R}}$. *The valuation function is defined as*

$$\pi(\mathbf{Y}) := \sum_{n=1}^{N} Y^n.$$

Then, all the following can be expressed in the form described in Definition 1:

$$\rho^{ag}(\mathbf{X}) := \inf \left\{ y \in \mathbb{R} \mid \Lambda_0(\mathbf{X}) + y \in \mathbb{A}^W \right\} = \rho_W \left(\sum_{n=1}^{N} -(X^n)^- \right)$$

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf \left\{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathbb{R}^N \, \Lambda_0(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W \right\} = \sum_{n=1}^{N} \rho_W(X^n)$$

$$\rho^\gamma(\mathbf{X}) := \inf \left\{ \pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}_\gamma \, \Lambda_0(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W \right\} = \rho_W \left(\sum_{n=1}^{N} \left(X^n 1_{\{X^n \leq -\gamma^n\}} - \gamma 1_{\{X^n > -\gamma^n\}} \right) \right).$$

5. Risk Measures Associated with Utility Functions and Fairness Concepts

Given the general framework for measuring systemic risk presented in the previous sections, a particular choice $\pi, \mathcal{C}, \mathbb{A}$ clearly allows for a more detailed analysis. We here present some of the main findings in [53]. The starting point is fixing the aggregation function $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x^n)$ for utility functions u_n, $n = 1, \ldots, N$, representing preferences of the single agents in the system. We consider the acceptance set $\mathbb{A} = \{Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E}[Z] \geq B\}$ for a given constant B, a class of feasible allocation $\mathcal{C}$ such that

$$\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L},$$

where the set $\mathcal{C}_{\mathbb{R}}$ was introduced in (8), and the cost functional $\pi : \mathcal{C} \to \mathbb{R}$ defined by $\pi(\mathbf{Y}) = \sum_{n=1}^{N} Y^n$. The space $\mathcal{L} \subseteq (L^0(\Omega, \mathcal{F}, P))^N$ serves as an environment for the risky positions of the system $\mathbf{X}$ and describes possible integrability conditions. The Systemic Risk Measure introduced in Definition 1 then takes the form

$$\mathbf{X} \in \mathcal{L} \mapsto \rho_B(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}. \tag{13}$$

Thus, $\rho_B(\mathbf{X})$ represents the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed at an initial time to secure the system by distributing the cash at the future time T among the components of the risk vector $\mathbf{X}$. By considering scenario-dependent allocations, possible dependencies among the banks are taken into account, as the budget constraints in (13) will not depend only on the marginal distribution of $\mathbf{X}$. As already mentioned before, this would be the case for deterministic Y^n.

The choice of a space $\mathcal{L}$ allows for a more detailed study, e.g., for the problem of finiteness of the Systemic Risk Measures. In [53] the subspace $\mathcal{L}$ is determined by the Orlicz hearts associated with the utility functions $u_1, \ldots, u_N$. This specific choice of the underlying space has been popularized in several recent works in the non systemic case. Indeed, univariate convex Risk Measures on Orlicz spaces have been introduced by [49] and [54] and deeply analyzed in several works. Just to mention a few, we address the reader to [55–60]. If $u : \mathbb{R} \to \mathbb{R}$ is concave, increasing and satisfies $\lim_{x \to -\infty} \frac{u(x)}{x} = +\infty$ (say, for example, a utility function satisfying the Inada conditions), then the function $\phi(x) := -u(-|x|) + u(0)$ is a strict Young function that is: $\phi : \mathbb{R} \to [0, +\infty)$ is even and convex on $\mathbb{R}$, $\phi(0) = 0$ and $\lim_{x \to +\infty} \frac{\phi(x)}{x} = +\infty$.

The Orlicz space L^ϕ and Orlicz heart M^ϕ are then defined respectively as

$$L^\phi := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for some } \alpha > 0 \right\},$$

$$M^\phi := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for all } \alpha > 0 \right\},$$

and they are Banach spaces when endowed with the Luxemburg norm.

Once one realizes that requesting the integrability condition $\mathbb{E}[\phi(X)] < +\infty$ automatically yields $\mathbb{E}[u(X)] > -\infty$, it is easy to reach the conclusion that the Orlicz framework is a very natural setup when working with utility maximization problems. A detailed discussion on this can be found in [49]. Just to mention a few key properties that will help with understanding our discussion in the following, the topological dual of M^ϕ is the Orlicz space L^{ϕ^*}, where the convex conjugate ϕ^* of ϕ, defined by $\phi^*(y) := \sup_{x \in \mathbb{R}} \{xy - \phi(x)\}$, $y \in \mathbb{R}$, is also a strict Young function. It is also well known that $L^\infty(\Omega, \mathcal{F}, P) \subseteq M^\phi \subseteq L^\phi \subseteq L^1(\Omega, \mathcal{F}, P)$. In addition, for a probability measure $Q \ll \mathbb{P}$ such that $\frac{dQ}{dP} \in L^{\phi^*}$, the inclusion $L^\phi \subseteq L^1(\Omega, \mathcal{F}, Q)$ also holds true.

Given the utility functions $u_1, \ldots, u_N : \mathbb{R} \to \mathbb{R}$ with associated Young functions $\phi_1, \ldots, \phi_N$, the underlying space $\mathcal{L}$ is given by

$$\mathcal{L} = M^\Phi := M^{\phi_1} \times \cdots \times M^{\phi_N}.$$

Moreover, the space $L^{\Phi^*} := L^{\phi_1^*} \times \cdots \times L^{\phi_N^*}$, induced by the conjugates of the functions $\phi_1, \ldots, \phi_N$, is the topological dual space of M^Φ and the dual system (M^Φ, L^{Φ^*}) will be useful to obtain dual representation results. The following set of assumptions allows for several interesting findings on properties of the functional ρ_B. They are tacitly assumed in Section 5.

Assumption 1.

1. $\mathcal{C}_0 \subseteq \mathcal{C}_{\mathbb{R}}$ and $\mathcal{C} = \mathcal{C}_0 \cap M^\Phi$ is a convex cone satisfying $\mathbb{R}^N \subseteq \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$.
2. For all $n = 1, \ldots, N$, $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, strictly concave, differentiable, and satisfies the Inada conditions

$$u_n'(-\infty) := \lim_{x \to -\infty} u_n'(x) = +\infty, \ u_n'(+\infty) := \lim_{x \to +\infty} u_n'(x) = 0.$$

3. $B < \Lambda(+\infty)$, i.e., there exists $\mathbf{M} \in \mathbb{R}^N$ such that $\sum_{n=1}^N u_n(M^n) \geq B$.
4. For all $n = 1, \ldots, N$, it holds that, for any probability measure $Q \ll \mathbb{P}$

$$\mathbb{E}\left[v_n\left(\frac{dQ}{dP}\right)\right] < \infty \quad \text{iff} \quad \mathbb{E}\left[v_n\left(\lambda \frac{dQ}{dP}\right)\right] < \infty, \ \forall \lambda > 0,$$

where $v_n(y) := \sup_{x \in \mathbb{R}} \{u_n(x) - xy\}$ denotes the convex conjugate of u_n.

Under these assumptions, the functional ρ_B turns out to be finite-valued, monotone decreasing, convex, continuous, and subdifferentiable on $\mathcal{L}$ ([53] Proposition 2.4). The following results highlight a natural counterpart for the multivariate case of well-known ones for (univariate) Risk Measures. The proof is based on an automatic continuity result, namely on the extended Namioka–Klee Theorem 1 [61].

Proposition 2 ([53], Proposition 3.1). *For any* $\mathbf{X} \in M^\Phi$,

$$\rho_B(\mathbf{X}) = \max_{\mathbf{Q} \in \mathcal{D}} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[-X^n] - \alpha_B(\mathbf{Q}) \right\} = \sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}}^n}[-X^n] - \alpha_B(\mathbf{Q_X}), \ \mathbf{Q_X} \in \mathcal{D}, \quad (14)$$

where the penalty function is given by

$$\alpha_B(\mathbf{Q}) := \sup_{\mathbf{Z} \in \mathcal{A}} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[-Z^n] \right\},$$

with $\mathcal{A} := \{\mathbf{Z} \in M^\Phi \mid \sum_{n=1}^N \mathbb{E}[u_n(Z^n)] \geq B\}$ *and*

$$\mathcal{D} := dom(\alpha_B) \cap \left\{ \frac{d\mathbf{Q}}{dP} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1 \ \forall n \text{ and} \right.$$

$$\left. \sum_{n=1}^N (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \text{ for all } \mathbf{Y} \in \mathcal{C}_0 \cap M^\Phi \right\}, \quad (15)$$

where $dom(\alpha_B) := \{\mathbf{Q} = [Q^1, \ldots, Q^N] \mid Q^n \ll P \ \forall n \text{ and } \alpha_B(\mathbf{Q}) < +\infty\}$.

(i) *Suppose that, for some* $i, j \in \{1, \ldots, N\}$, $i \neq j$, *we have* $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* $A \in \mathcal{F}$, *where* $e_1, \ldots, e_N$ *are the elements of the canonical basis of* $\mathbb{R}^N$. *Then,*

$$\mathcal{D} = dom(\alpha_B) \cap \left\{ \frac{d\mathbf{Q}}{dP} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1 \ \forall n, \ Q^i = Q^j \text{ and} \right.$$

$$\left. \sum_{n=1}^N (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \text{ for all } \mathbf{Y} \in \mathcal{C} \right\}.$$

(ii)   *Suppose that* $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* i, j *and all* $A \in \mathcal{F}$. *Then,*

$$\mathcal{D} = dom(\alpha_B) \cap \left\{ \frac{d\mathbf{Q}}{dP} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1, \ Q^n = Q, \ \forall n \right\}.$$

A first relevant consequence of the dual representation result above is that it allows for establishing existence of allocations for the minimization problem expressed by ρ_B. It is remarkable that a solution is not known to exist in the space $\mathcal{L}$, thus an enlargement of the domain of optimization is needed (as customary, for example, in the classical utility maximization problems). An additional assumption is needed to establish existence.

Definition 2. *The set* $\mathcal{C}_0$ *is closed under truncation if, for each* $\mathbf{Y} \in \mathcal{C}_0$, *there exists* $m_Y \in \mathbb{N}$ *and* $\mathbf{c}_Y = [c_Y^1, \dots, c_Y^N] \in \mathbb{R}^N$ *such that*

$$\sum_{n=1}^N c_Y^n = \sum_{n=1}^N Y^n := c_Y \in \mathbb{R}$$

and for all $m \geq m_Y$

$$\mathbf{Y}_m := \mathbf{Y} 1_{\cap_{n=1}^N \{|Y^n| < m\}} + \mathbf{c}_Y 1_{\cup_{n=1}^N \{|Y^n| \geq m\}} \in \mathcal{C}_0.$$

It is easy to check that $\mathcal{C}_\mathbb{R}$ satisfies this property, and other examples are provided in Section 5.1. For the existence of the optimal allocations in ρ_B (in an extended sense), we will need the set

$$L^1(P; \mathbf{Q_X}) := (L^1(\Omega, \mathcal{F}, P))^N \cap L^1(\Omega, \mathcal{F}, Q_X^1) \times \cdots \times L^1(\Omega, \mathcal{F}, Q_X^N).$$

Theorem 1 ([53], Theorem 4.19). *Let* $\mathcal{C} = \mathcal{C}_0 \cap M^\Phi$ *and suppose that* $\mathcal{C}_0 \subseteq \mathcal{C}_\mathbb{R}$ *is closed for the convergence in probability and closed under truncation. For any* $\mathbf{X} \in M^\Phi$, *there exists* $\tilde{\mathbf{Y}}_{\mathbf{X}} \in \mathcal{C}_0 \cap L^1(P; \mathbf{Q_X})$ *such that*

$$\sum_{n=1}^N \tilde{Y}_{\mathbf{X}}^n \in \mathbb{R}, \quad \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \tilde{Y}_{\mathbf{X}}^n) \right] \geq B, \quad \sum_{n=1}^N \left(\mathbb{E}_{Q_X^n}[\tilde{Y}_{\mathbf{X}}^n] - \tilde{Y}_{\mathbf{X}}^n \right) = 0,$$

and

$$\begin{aligned}
\rho_B(\mathbf{X}) &= \inf\left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in \mathcal{C}_0 \cap M^\Phi, \ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \geq B \right\} = \sum_{n=1}^N \tilde{Y}_{\mathbf{X}}^n \\
&= \min\left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in \mathcal{C}_0 \cap L^1(P; \mathbf{Q_X}), \ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \geq B \right\} \\
&=: \tilde{\rho}_B(\mathbf{X}),
\end{aligned}$$

so that $\tilde{\mathbf{Y}}_{\mathbf{X}}$ *is the solution to the extended problem* $\tilde{\rho}_B(\mathbf{X})$.

The dual representation in Proposition 2 also allows for linking the Systemic Risk Measure based on random allocations (13) to one based on allocation of deterministic amounts. To this end, suppose that a probability vector $\mathbf{Q} = [Q^1, \dots, Q^N]$ is given. Treating the vector of probability measures $\mathbf{Q}$ as a vector of pricing measures, it is possible to introduce a Systemic Risk Measure that is quite naturally associated with $\mathbf{Q}$ by

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{L}} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \geq B \right\}. \tag{16}$$

The amount $\rho_B^{\mathbf{Q}}(\mathbf{X})$ represents the minimal systemic cost $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n]$ among all $\mathbf{Y} \in \mathcal{L}$ which fulfill the acceptability constraint $\mathbb{E}[\sum_{n=1}^{N} u_n(X^n + Y^n)] \geq B$.

Similarly, one may introduce *the systemic utility maximization problem* in the case when the valuation of the allocations is assigned by the expectation under $\mathbf{Q}$ that is:

$$\pi_A^{\mathbf{Q}}(\mathbf{X}) := \sup_{\mathbf{Y} \in \mathcal{L}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \mid \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \leq A \right\}. \tag{17}$$

One can also consider its counterpart

$$\pi_A(\mathbf{X}) := \sup \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \mid \mathbf{Y} \in \mathcal{C}, \sum_{j=1}^{N} Y^j \leq A \right\}.$$

Notice that, both in (16) and (17), the allocation $\mathbf{Y}$ belongs to a vector space $\mathcal{L}$ of random variables, without the requirement $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$. The problem $\pi_A^{\mathbf{Q}}(\mathbf{X})$ is a maximization of the expected systemic utility among all $\mathbf{Y} \in \mathcal{L}$ satisfying the budget constraint $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n] \leq A$.

The Systemic Risk Measures ρ_B and $\rho_B^{\mathbf{Q}}$ defined in (13) and (16) are a priori different objects: even though they both subsume the same systemic budget constraint (expected systemic utility above a certain threshold B), ρ_B is defined only through the cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$, while $\rho_B^{\mathbf{Q}}$ relies on the computation of the value (or the cost) of the random allocations $\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n]$. A similar comparison applies to π_A and $\pi_A^{\mathbf{Q}}$. In [53], it is shown that:

(i)　the optimizer $\mathbf{Q_X} = [Q_{\mathbf{X}}^1, \ldots, Q_{\mathbf{X}}^N]$ of the dual problem (14) satisfies

$$\rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}), \ \pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X});$$

(ii)　if $A := \rho_B(\mathbf{X})$, the four problems above share the same (unique) solution $\mathbf{Y_X}$;

(iii)　the dual optimizer $\mathbf{Q_X}$ also satisfies

$$\sum_{n=1}^{N} \mathbb{E}_{Q_{\mathbf{X}^n}}[Y_{\mathbf{X}}^n] = \rho_B(\mathbf{X}) \tag{18}$$

which can be interpreted saying that it determines a systemic risk allocation $[\mathbb{E}_{Q_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1], \ldots, \mathbb{E}_{Q_{\mathbf{X}}^N}[Y_{\mathbf{X}}^N]]$;

(iv)　$\rho_B(\mathbf{X}) = \max_{\mathbf{Q} \in \mathcal{D}} \rho_B^{\mathbf{Q}}(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X})$, for $\mathcal{D}$ defined in (15). Drawing a parallel between this property and related findings in utility maximization theory, one might say that the domain $\mathcal{D}$ plays the same role here of the set of martingale measures for the underlying stock in the classical theory.

Several conclusions can be drawn from the points above. Firstly, $\rho_B^{\mathbf{Q_X}}$ is a valid alternative to ρ_B (sharing same value and solution) and can then be used to compute the systemic risk. Additionally, (18) explains how the pricing operators given by $\mathbb{E}_{\mathbf{Q_X}}[\cdot]$ provide a valuation for the risk component $Y_{\mathbf{X}}^n$ of the optimal allocation which is consistent with ρ_B. Items (ii–iv) above also suggest how $\mathbf{Q_X}$ can be considered as pricing/valuation measure for allocating the amount $\rho_B(\mathbf{X})$ at initial time. More precisely:

Definition 3. *A vector $(\rho^n(\mathbf{X}))_{n=1,\ldots,N} \in \mathbb{R}^N$ is a* systemic risk allocation *of $\rho(\mathbf{X})$ if it fulfills $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X})$. The requirement $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X})$ is known as the "Full Allocation" property; see, for example, [38].*

In the case of deterministic allocations $\mathbf{Y} \in \mathbb{R}^N$, i.e., $\mathcal{C} = \mathbb{R}^N$, the optimal deterministic $\mathbf{Y_X}$ is a somehow canonical risk allocation $\rho^n(\mathbf{X}) := Y_{\mathbf{X}}^n \in \mathbb{R}$. For general (random) allocations $\mathbf{Y} \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}$, at first glance, there is no direct counterpart of such a natural systemic

risk allocation. However, as mentioned above, the properties listed above regarding $\rho_B(\mathbf{X})$ and $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ provide evidence of the fact that a natural choice is

$$\rho^n(\mathbf{X}) := \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n] \ \text{for } n = 1, \dots, N. \tag{19}$$

5.1. Interpretation and Implementation of $\rho(\mathbf{X})$

One main economic justification for the use of ρ_B as in (13) for systemic risk valuation is that the optimal allocation $\mathbf{Y_X}$ of $\rho_B(\mathbf{X})$ maximizes the expected systemic utility among all random allocations of cost less than or equal to $\rho_B(\mathbf{X})$ (problem π_A, using Item (ii) on page 12).

The class $\mathcal{C}$ determines the level of *risk sharing* between the banks, ranging from no risk sharing in the case of deterministic allocations $\mathcal{C} = \mathbb{R}^N$, to the case of full risk sharing $\mathcal{C} = \mathcal{C}_\mathbb{R}$. In between, several intermediate cases can be considered: fix $h \in \{1, \dots, N\}$ clusters of institutions, set $\mathbf{n} := [n^1, \dots, n^h] \in \mathbb{N}^h$, be the corresponding partition of $\{1, \cdots, N\}$ and let $I_m, m = 1, \dots, h$ denote the set of institutions belonging to the m-group. Then, the sets $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_0^{(\mathbf{n})} \cap \mathcal{L}$, where

$$\mathcal{C}_0^{(\mathbf{n})} = \{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \exists \, d = [d^1, \dots, d^h] \in \mathbb{R}^h :$$
$$\sum_{i \in I_m} Y^i = d^m \ \text{for } m = 1, \dots, h\} \subseteq \mathcal{C}_\mathbb{R}, \tag{20}$$

model a whole family of possible restrictions for allocations. Indeed, the conditions $\sum_{i \in I_m} Y^i = d^m$ model a constrained sharing and allocation procedure among the agents in the sole cluster I_m.

We conclude reporting some relevant remarks made in conclusion of the analysis in [53], explaining once again the relevance and potentiality of the family of Systemic Risk Measures in the form (13).

(a) "the mechanism can be described as a *default fund* as in the case of a CCP". Indeed, the properties of ρ_B inspire the following procedure: at time 0, according to some systemic risk allocation $\rho^n(\mathbf{X}) \, n = 1, \dots, N$, satisfying $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho_B(\mathbf{X})$, the amount $\rho_B(\mathbf{X})$ is collected. ρ^n could be determined consistently using (19). At terminal time, the amount $\rho_B(X)$ is distributed among the institutions according to $Y_{\mathbf{X}}^n$, the optimal scenario-dependent allocations satisfying $\sum_{n=1}^{N} Y_{\mathbf{X}}^n = \rho_B(\mathbf{X})$, so that "the fund acts as a clearing house".

(b) Alternatively, in terms of *capital requirements* and *risk sharing mechanism*, at time 0, $\rho^n(\mathbf{X})$ (a capital requirement) is associated with each institution $n = 1, \dots, N$ in the system. At terminal time, each bank provides (if negative) or collects (if positive) the amount $Y_{\mathbf{X}}^n - \rho^n(\mathbf{X})$. This means that, at terminal time, a risk sharing mechanism takes place. Observe that this sharing mechanism is possible given that

$$\sum_{n=1}^{N} (Y_{\mathbf{X}}^n - \rho^n(\mathbf{X})) = \sum_{n=1}^{N} Y_{\mathbf{X}}^n - \sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho_B(\mathbf{X}) - \rho_B(\mathbf{X}) = 0.$$

Remarkably, there is an incentive for a single bank to enter in such a mechanism, based on the principle of choosing a *fair risk allocation*, as explained below.

A fundamental issue for each financial institution is to decide whether its allocated share of the total systemic risk determined by the risk allocation $[\mathbb{E}_{Q_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1], \dots, \mathbb{E}_{Q_{\mathbf{X}}^N}[Y_{\mathbf{X}}^N]]$ is *fair*. With the choice $\mathbf{Q} = \mathbf{Q_X}$, it is possible to show ([53], Corollary 4.3, Lemma 4.5 and (4.11)) that

$$\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X}) = \max_{\sum_{n=1}^{N} a^n = A,} \sum_{n=1}^{N} \sup_{\mathbb{E}_{Q_{\mathbf{X}}^n}[Y^n] = a^n} \mathbb{E}[u_n(X^n + Y^n)]. \tag{21}$$

When the amount A is chosen to be equal to the risk measurement for the system that is $A = \rho_B(\mathbf{X})$, $\mathbf{Y_X}$ is the solution of $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$, $\mathbb{E}_{Q_{\mathbf{X}^n}}[Y_{\mathbf{X}}^n] = a_*^n$ maximizes (21), $\sum_{n=1}^{N} \mathbb{E}_{Q_{\mathbf{X}^n}}[Y_{\mathbf{X}}^n] = A$, and (21) can be rewritten as

$$\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X}) = \sum_{n=1}^{N} \sup_{\mathbb{E}_{Q_{\mathbf{X}}^n}[Y^n] = \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n]} \mathbb{E}[u_n(X^n + Y^n)].$$

Consequently, by exploiting $\mathbf{Q_X}$ for valuation, the problem of systemic utility maximization (17) reduces to individual utility maximization problems for the single banks:

$$\forall n, \quad \sup_{Y^n}\left\{ \mathbb{E}[u_n(X^n + Y^n)] \mid \mathbb{E}_{Q_{\mathbf{X}}^n}[Y^n] = \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n] \right\}. \tag{22}$$

The optimum $Y_{\mathbf{X}}^n$ and its value via the optimal measure $\mathbf{Q_X}$, namely $\mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n]$, are then *fair* for the nth bank in the system, as $Y_{\mathbf{X}}^n$ maximizes the *individual indirect utility* as in Equation (22). Notice, however, that this fairness principle for individual banks is conjugated with a systemic-regulatory mechanism that is expressed in (21) through the outer maximization over the allocations $a \in \mathbb{R}^N$. We will elaborate more on this feature when dealing with the equilibrium concept in Section 6.

5.2. The Exponential Case: Explicit Formulas

Explicit formulas can be found for the value of $\rho(\mathbf{X})$, $\mathbf{Y_X}$ and $\mathbf{Q_X}$ in the case of exponential utility functions and for $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$ (see (20)). To be more specific, these formulas are available for the choices $u_n(x) = -e^{-\alpha_n x}/\alpha_n$, $\alpha_n > 0$, $n = 1, \ldots, N$, and $B < \sum_{n=1}^{N} u_n(+\infty) = 0$.

Theorem 2 ([53], Theorem 6.2). *For $m = 1, \ldots, h$, and for $k \in I_m$, we have*

$$d^m = \beta_m \log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right),$$

$$Y_m^k = -X^k + \frac{1}{\beta_m \alpha_k}\overline{X}_m + \frac{1}{\beta_m \alpha_k}d^m, \tag{23}$$

where $\overline{X}_m = \sum_{k \in I_m} X^k$, $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$, $\beta = \sum_{i=1}^{N} \frac{1}{\alpha_i}$, and

$$\rho_B(\mathbf{X}) = \sum_{i=1}^{N} Y^i = \sum_{m=1}^{h} d^m.$$

The vector $\mathbf{Q_X}$ of probability measures with densities

$$\frac{dQ_{\mathbf{X}}^m}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta_m}\overline{X}_m}}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]} \quad m = 1, \ldots, h.$$

is the solution of the dual problem (14), i.e.,

$$\rho(\mathbf{X}) = \sum_{m=1}^{h} \mathbb{E}_{Q_{\mathbf{X}}^m}[-\overline{X}_m] - \alpha_B(\mathbf{Q_X}),$$

and $\mathbb{E}_{Q_{\mathbf{X}}^m}[Y_{\mathbf{X}}^n]$, $m = 1, \ldots, h$, $n \in I_m$, is a systemic risk allocation, as in Definition 3.

It is also possible to conduct a sensitivity analysis based on the explicit formulas above, and a study of monotonicity with respect to grouping (i.e., variation of the partition defining $\mathcal{C}^{(\mathbf{n})}$). We refer the reader to [53] Sections 6.1 and 6.2 for the details.

Remark 2. *Unlike in the general case, for exponential utility functions, the optimum* $\mathbf{Y_X}$ *in* (23) *actually belongs to the set* $\mathcal{L} = M^\Phi$, *thus being an optimum for* $\rho(\mathbf{X})$ *and* $\rho^{\mathbf{Q_x}}(\mathbf{X})$ *in the strict sense.*

6. On Systemic Optimal Risk Transfer Equilibrium

In order to introduce an equilibrium concept, we reformulate some of the results on the Systemic Risk Measure in (13), for the case $\mathcal{C} = \mathcal{C}_\mathbb{R} \cap M^\Phi$.

Proposition 3. *Suppose a proper optimum* $\mathbf{Y_X} \in M^\Phi$ *exists for* $\rho_B(\mathbf{X})$ *and* $\rho_B^{\mathbf{Q_x}}(\mathbf{X})$. *Then,*

$$\rho_B^{\mathbf{Q_x}}(\mathbf{X}) = \inf\left\{ \sum_{n=1}^N a^n \mid \mathbf{a} \in \mathbb{R}^N \text{ s.t. } \exists\,\mathbf{Z} \text{ with } \sum_{n=1}^N Z^n = 0,\ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n])\right] \geq B \right\}. \tag{24}$$

Proof of Proposition 3. Observe that

$$
\begin{aligned}
\rho_B^{\mathbf{Q_x}}(\mathbf{X}) &:= \inf_{\mathbf{Y}\in\mathcal{L}}\left\{ \sum_{n=1}^N \mathbb{E}_{Q_\mathbf{X}^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B \right\} \\
&= \inf_{\mathbf{Y}\in\mathcal{L}}\left\{ \sum_{n=1}^N \mathbb{E}_{Q_\mathbf{X}^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \mathbb{E}_{Q_\mathbf{X}^n}[Y^n] + Y^n - \mathbb{E}_{Q_\mathbf{X}^n}[Y^n])\right] \geq B \right\}.
\end{aligned}
\tag{25}
$$

Observe now that setting $a^n = \mathbb{E}_{Q_\mathbf{X}^n}[Y_\mathbf{X}^n]$ and $Z^n = Y_\mathbf{X}^n - \mathbb{E}_{Q_\mathbf{X}^n}[Y_{\mathbf{X}'}^n]$ we obtain that $\mathbb{E}_{Q_\mathbf{X}^n}[Z^n] = 0$ and $a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n] = Y_\mathbf{X}^n$. Additionally, $\sum_{n=1}^N Z^n = 0$ by items (i)-(iv) on page 12. Hence, $\mathbf{a}$ defined in this way satisfies the constraints in RHS of (24). We then get $\rho^{\mathbf{Q_x}}(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}[Y_\mathbf{X}^n] \geq$RHS of (24). Conversely, for any $\mathbf{a}, \mathbf{Z}$ satisfying the constraints of RHS of (24), setting $Y^n = a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n]$, one gets a vector fulfilling the constraints of (25) and such that $\mathbb{E}_{Q_\mathbf{X}^n}[Y^n] = a^n$, implying that also $\rho^{\mathbf{Q_x}}(\mathbf{X}) \leq$RHS of (24), which completes the proof. □

The formulation in (24) has an interesting consequence for the interpretation of $\rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_x}}(\mathbf{X})$. Consider indeed the following two-step procedure: each bank n pays or receives the amount a^n (according to whether this is positive or negative) at the initial time. At terminal time, a second, scenario-dependent allocation takes place: in exchange for the price $\mathbb{E}_{Q_\mathbf{X}^n}[Z^n]$, bank n pays or receives the additional amount Z^n. The terminal time allocation is a *reinsurance* exchange among the banks in the system, since the *clearing condition* $\sum_{n=1}^N Z^n = 0$ imposes a "conservation of capital" for the system as a whole. Now, this procedure yields a terminal time value equal to $a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n]$ which can be added to the initial position of $\mathbf{X}$, yielding the final position $X^n + a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n]$ for bank n. $\rho(\mathbf{X})$ is then the infimum total amount $\sum_{n=1}^N a^n$ which can be allocated at an initial time, in such a way that, for some reinsurance exchange, the overall satisfaction of the system ($\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + a^n + Z^n - \mathbb{E}_{Q_\mathbf{X}^n}[Z^n])\right]$) is above the threshold B.

A similar allocation and reinsurance procedure have been considered in [51], where the concept of Systemic Optimal Risk Transfer Equilibrium (SORTE) has been introduced as an equilibrium for a combination of capital allocation and reinsurance problems.

Equilibria related to exchange and allocation procedures, as well as risk sharing problems aimed at reducing the overall risk of a system via reallocation, have been widely studied in the literature. For a review on Arrow–Debreu Equilibrium, we refer to Section 3.6 of [28]. Following in the spirit the Arrow–Debreu Equilibrium theory and working in the framework of a pure exchange economy, the authors in [62,63] provide the existence of so-called risk exchange equilibria. The study of such equilibria, even though in different forms, had been initiated by the seminal papers of Borch (see [64]). In [65], inf-convolutions of convex Risk Measures have been successfully applied in the study of risk sharing, and these have been further analyzed in [66–71]. This area of research has been very active

and many other relevant contributions on risk sharing also appeared recently; in particular, we mention [72–79].

In addition, in this context, we consider a class of N agents, each having individual risky position or random endowments given by the components of the risk vector $\mathbf{X} := [X^1, ..., X^N]$. Instead of designing a measurement for quantifying at the initial time the risk for the system, the procedure we are to describe shows how to conjugate a capital allocation problem regarding an amount $A \in \mathbb{R}$ exogenously assigned to the system (say, a risk measurement obtained using a Systemic Risk Measure as described in previous sections) with a reinsurance mechanism. For additional possible interpretations of the amount A, we refer the reader to the related discussion in Section 5.2 of [53].

The SORTE combines a systemic optimal (deterministic) allocation with Bühlmann's Risk Exchange Equilibrium. In a one period framework $\{0, T\}$, each of the N agents is characterized by a strictly concave, strictly monotone utility function $u_n : \mathbb{R} \to \mathbb{R}$.

6.1. Systemic Optimal (Deterministic) Allocation

To describe a *systemic optimal allocation*, suppose that the amount A is allocated at an initial time among the agents in order to optimize the satisfaction/performance of the system as a whole. Denoting by $a^n \in \mathbb{R}$ the cash received (provided) if positive (resp., if negative) by agent n, the terminal time endowment at disposal of agent n will be given by $(X^n + a^n)$. The optimal allocation $\mathbf{a_X} \in \mathbb{R}^N$ can then be determined according to an optimization problem for the system's utility in the form

$$\sup \left\{ \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + a^n)] \mid \mathbf{a} \in \mathbb{R}^N \text{ s.t. } \sum_{n=1}^{N} a^n = A \right\}.$$

Assuming that the agents will retain a cooperative attitude also at a terminal time and that they will still believe in the overall reliability of the others, one can combine this initial time allocation with a reinsurance procedure inspired by [62,63]. The aim is twofold: on the one hand, this further increases the optimal total expected systemic utility. On the other hand, this guarantees that each participant will be maximizing his/her expected utility under budget constraints determined in an aggregated way by the system as a whole, thus adding a rationality component that takes into consideration also single agents' satisfaction.

6.2. Risk Transfer Equilibrium

In Bühlmann's Risk Exchange Equilibrium, each agent is entitled to risk exchange with the other participants. Agent n receives or provides the amount $\widetilde{Y}^n(\omega)$ at terminal time (with the same convention on signs as in the case of systemic optimal allocation above), in exchange for a price $E_Q[\widetilde{Y}^n]$ to be paid (if $E_Q[\widetilde{Y}^n] > 0$) or received (if $E_Q[\widetilde{Y}^n] < 0$) at the initial time. Q here is some pricing probability measure. Observe that $\widetilde{Y}^n$, being scenario-dependent, is a terminal time measurable random variable. The exchange variables $\widetilde{Y}^n$ have to satisfy the *clearing condition*:

$$\sum_{n=1}^{N} \widetilde{Y}^n = 0 \ \ P\text{-}a.s. \ .$$

The clearing condition expresses the fact that no capital is produced or lost in this reallocation via $\widetilde{\mathbf{Y}}$, so that the Risk Exchange Equilibrium also describes a reinsurance mechanism. The pair $(\widetilde{\mathbf{Y}}_{\mathbf{X}}, Q_{\mathbf{X}})$ is a *Risk Exchange Equilibrium* if (a) for each n, $\widetilde{Y}^n_{\mathbf{X}}$ maximizes

$$\mathbb{E}\left[u_n(X^n + \widetilde{Y}^n - E_{Q_{\mathbf{X}}}[\widetilde{Y}^n]) \right]$$

among all variables $\widetilde{Y}^n$ and (b) $\sum_{n=1}^{N} \widetilde{Y}^n_{\mathbf{X}} = 0 \ P-$a.s..

6.3. Systemic Optimal Risk Transfer Equilibrium

The two procedures can now be combined as follows. Given some amount a^n assigned to agent n, this agent may buy $\widetilde{Y}^n$ at the price $p^n(\widetilde{Y}^n)$ in order to optimize

$$\mathbb{E}\left[u_n(a^n + X^n + \widetilde{Y}^n - p^n(\widetilde{Y}^n))\right].$$

As in Bühlmann's definition, with initial endowments $X^n + a^n$ for each agent $n = 1, \ldots, N$, the pricing functionals p^n, $n = 1, ..., N$ have to be selected in such a way that the optimal solution verifies the clearing condition

$$\sum_{n=1}^{N} \widetilde{Y}^n = 0 \quad P\text{-a.s..}$$

However, a^n is not exogenously assigned to each agent, but only the total amount A is at the disposal of the whole system. Thus, the optimal way to allocate A among the agents is given by the solution $(\widetilde{Y}^n_{\mathbf{X}}, p^n_{\mathbf{X}}, a^n_{\mathbf{X}})$, $n = 1, \ldots, N$ of the following problem:

$$\sup_{\mathbf{a} \in \mathbb{R}^N} \left\{ \sum_{n=1}^{N} \sup_{\widetilde{Y}^n} \left\{ \mathbb{E}\left[u_n(a^n + X^n + \widetilde{Y}^n - p^n_{\mathbf{X}}(\widetilde{Y}^n))\right] \right\} \mid \sum_{n=1}^{N} a^n = A \right\},$$

$$\sum_{n=1}^{N} \widetilde{Y}^n_{\mathbf{X}} = 0 \ P - \text{a.s.} \ .$$

Among the main findings in [51], we mention existence, uniqueness, and Pareto optimality for SORTE. Additionally, explicit formulas are provided in the exponential case, i.e., when considering

$$u_n(x) := 1 - \exp(-\alpha_n x), \, n = 1, \ldots, N \quad \text{for} \quad \alpha_1, \ldots, \alpha_N > 0.$$

The SORTE concept can be extended, see [80], by replacing the utilitarian aggregation of single agents' utility functions, namely $\sum_{j=1}^{N} u_j(x^j)$, to a multivariate utility function $U : \mathbb{R}^N \to \mathbb{R}$ satisfying some regularity conditions including a multivariate formulation of the Inada conditions. As a particular example of such U, one may take $U(x) := \sum_{j=1}^{N} u_j(x^j) + \Gamma(x)$ for a concave increasing upper bounded function $\Gamma : \mathbb{R}^N \to \mathbb{R}$. As Γ is not required to be strictly increasing nor strictly concave, the special selection $\Gamma = 0$ leads to the original concept of SORTE. The additional term Γ could be imposed to the financial institutions in the system by some regulatory authority. One can show existence and uniqueness also for this multivariate extension of SORTE (called Multivariate Systemic Optimal Risk Transfer Equilibrium). In the multivariate extension, the preferences of each agent depend on the action of the other agents in the system. Hence, it is natural to formulate a Nash Equilibrium in this context. Remarkably, the optimal allocation $\widetilde{Y}^n_{\mathbf{X}}$ of the multivariate SORTE is indeed a Nash Equilibrium (see [80] for details).

7. Conditional Systemic Risk Measures

The setting in the approaches described in the previous sections is static, that is: the Systemic Risk Measures as described above do not leave room for incorporating dynamic elements (additional information or presence of intermediate payoffs and valuation, just to mention a few examples). It is essentially a one period setup where the initial sigma algebra is assumed to be the trivial one $\{\Omega, \varnothing\}$. In order to be able to consider the more realistic multiperiod setting, a further step consisting of selecting a general sub sigma algebra $\mathcal{G} \subseteq \mathcal{F}$ and quantifying the risk of a position $\mathbf{X}$ *given the information modeled by $\mathcal{G}$* is required. The resulting *conditional Risk Measure* will then be a map $\rho_{\mathcal{G}}$ having a range in $L^0(\Omega, \mathcal{G}, \mathbb{P})$. This well established technique in the literature has mostly been applied in the framework of *univariate* dynamic Risk Measures. Among the first contributions on conditional convex Risk Measures, we mention [81]. Since then, this approach has gained more and more

attention in the literature. We refer the reader to [82] for an overview on univariate dynamic Risk Measures. Several results have been obtained for the case of quasi-convex conditional maps and Risk Measures, see [32] and [83,84]. A vast amount of literature has focused on conditional counterparts to classical static results regarding dual representation and separation properties, using L^0-modules. Among the many contributions in this stream of research, we mention [85–89] and references therein. Overall, the fact that the natural conditional counterparts hold for static results is not so surprising. The two are intrinsically related by a Boolean Logic principle. As seen in [90], traditional Theorems carry over to the conditional setup assuming that suitable concatenation properties hold.

A Conditional Systemic Risk Measure is a map

$$\rho_{\mathcal{G}} : L_{\mathcal{F}} \to L^0(\Omega, \mathcal{G}, \mathbb{P})$$

that associates to a N-dimensional risk factor $X \in L_{\mathcal{F}} \subseteq (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ a $\mathcal{G}$-measurable random variable. This means that a conditional Systemic Risk Measure quantifies the risk of a given system taking into account the fact that more and more information accumulates over time. Conditional Systemic (multivariate) Risk Measures were first studied by [41,42]. In the latter, it is emphasized how additional information might also be of a spatial nature, namely concerning systemic relevant structures. The sigma algebra $\mathcal{G}$ might then incorporate information on the state of a subsystem, see [91] or [92].

The papers [41,42,93] consider only the conditional extension of (static) Systemic Risk Measures of the "first aggregate, then allocate" form. Multivariate and set-valued conditional Risk Measures, and related time consistency aspects, have also been analyzed in [94–98]. As can be easily guessed, a natural counterpart of Systemic Risk Measures of the form "first allocate, then aggregate" can also be defined in the conditional setting. This is the object of the analysis in [99], on which this section is based. Most properties of Shortfall Systemic Risk Measures, i.e., those in forms similar to (13), are seen to carry over to the conditional setting. Furthermore, the usual recursive property in the classical theory of time consistency for univariate Risk Measures can be replaced in the systemic conditional case by a new consistency property, this time of vector type, with respect to sub sigma algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Further details on this will be provided in the following: For conditional Systemic Risk Measures $\rho_{\mathcal{G}}$ in rather general form (conditionally convex and monetary, monotone maps), the following conditional dual representation holds:

$$\rho_{\mathcal{G}}(X) = \operatorname*{ess\,sup}_{\mathbf{Q} \in \mathcal{Q}_{\mathcal{G}}} \left(\sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n | \mathcal{G}] - \alpha(\mathbf{Q}) \right), \ \mathbf{X} \in L_{\mathcal{F}}, \tag{26}$$

where $\mathcal{Q}_{\mathcal{G}}$ is a set of vectors of probability measures and $\alpha(\mathbf{Q}) \in L^0(\Omega, \mathcal{G}, \mathbb{P})$ is a penalty function. A specific interesting case of such maps $\rho_{\mathcal{G}}$ is the class of *Conditional Shortfall Systemic Risk Measure*, associated with a multivariate utility functions U of the form $U(x) = \sum_{j=1}^{N} u_j(x^j) + \Gamma(x)$, already introduced in the previous section. These are inspired by the static ones (13), and defined by

$$\rho_{\mathcal{G}}(\mathbf{X}) := \operatorname{ess\,inf} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}_{\mathcal{G}}, \ \mathbb{E}[U(\mathbf{X} + \mathbf{Y}) | \mathcal{G}] \geq B \right\}. \tag{27}$$

Here, B is a bounded, $\mathcal{G}$-measurable random variable and the set of $\mathcal{G}$-admissible allocations is

$$\mathcal{C}_{\mathcal{G}} \subseteq \left\{ \mathbf{Y} \in (L^1(\Omega, \mathcal{F}, P))^N \text{ such that } \sum_{n=1}^{N} Y^n \in L^{\infty}(\Omega, \mathcal{G}, P) \right\}.$$

Observe that each Y^n is an $\mathcal{F}$-measurable random variable, but $\sum_{n=1}^{N} Y^n$ is required to be $\mathcal{G}$-measurable. These definitions clearly mimic those in (13) and (8). In the case of

$\mathcal{C}_{\mathcal{G}}$ and $\mathcal{C}_{\mathbb{R}}$, it is particularly evident how deterministic amounts have been substituted by $\mathcal{G}$-measurable ones, which are "known once the information in $\mathcal{G}$ is known" .

For the trivial selection $\mathcal{G} = \{\varnothing, \Omega\}$, the authors in [99] extend the results in [53] to the more general aggregator U.

Theorem 5.4 of [99] summarizes the fundamental properties of the Conditional Shortfall Systemic Risk Measure $\rho_{\mathcal{G}}$ and, in particular, shows that

(i) the functional $\rho_{\mathcal{G}}$ takes values in $L^{\infty}(\Omega, \mathcal{G}, P)$ and is both continuous from below and from above;

(ii) the essential infimum in (27) is actually a minimum, attained at the vector of allocations $\mathbf{Y}(\mathcal{G}, \mathbf{X}) = [Y^1(\mathcal{G}, \mathbf{X}), ..., Y^N(\mathcal{G}, \mathbf{X})] \in \mathcal{C}_{\mathcal{G}}$;

(iii) $\rho_{\mathcal{G}}$ admits a dual representation which specializes the one in (26) with a more explicit formulation of the penalty function α and of the set $\mathcal{Q}_{\mathcal{G}}$.

(iv) the supremum in the dual representation (26) of $\rho_{\mathcal{G}}$ is actually a maximum, with optimizer $\mathbf{Q}(\mathcal{G}, \mathbf{X}) = [Q^1(\mathcal{G}, \mathbf{X}), ..., Q^N(\mathcal{G}, \mathbf{X})]$ which is a vector of probability measures also satisfying:

$$\sum_{n=1}^{N} \mathbb{E}_{Q^n(\mathcal{G}, \mathbf{X})}[Y^n(\mathcal{G}, \mathbf{X}) \mid \mathcal{G}] = \sum_{n=1}^{N} Y^n(\mathcal{G}, \mathbf{X}) = \rho_{\mathcal{G}}(\mathbf{X}) \quad \mathbb{P} - \text{a.s.}.$$

Similarly to the static case described in Section 5.2, in the exponential case, it is possible to explicitly determine formulas for $\rho_{\mathcal{G}}(\mathbf{X})$, the value of the Conditional Shortfall Systemic Risk Measure , for the primal minimizer $\mathbf{Y}(\mathcal{G}, \mathbf{X})$ of $\rho_{\mathcal{G}}(\mathbf{X})$ and for the dual optimum vector of probability measures $\mathbf{Q}(\mathcal{G}, \mathbf{X})$. Such formulas closely resemble those in Theorem 2, with expectations replaced by conditional expectations w.r.t. $\mathcal{G}$.

As anticipated before, for given sub sigma algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, a new type of consistency property holds. Such a property seems to have no direct counterpart in the well-known univariate case. Indeed, the consistency $\rho_{\mathcal{H}}(-\rho_{\mathcal{G}}(\mathbf{X})) = \rho_{\mathcal{H}}(\mathbf{X})$, which is the usual time consistency property in the univariate case, is not even well defined in the systemic setting, as a Systemic Risk Measure maps a random vector into a single random variable. Nevertheless, time (or information) consistency properties turn out to be well defined for: (i) the vectors of primal minimizers $\mathbf{Y}(\mathcal{G}, \mathbf{X})$ (for $\rho_{\mathcal{G}}(\mathbf{X})$) and $\mathbf{Y}(\mathcal{H}, -\mathbf{Y}(\mathcal{G}, \mathbf{X}))$ (for $\rho_{\mathcal{H}}(-\mathbf{Y}(\mathcal{G}, \mathbf{X}))$); (ii) the initial time allocation vectors $(\mathbf{a}(\mathcal{G}, \mathbf{X}))_k := (\mathbb{E}_{Q^k(\mathcal{G}, \mathbf{X})}[Y^k(\mathcal{G}, \mathbf{X})|\mathcal{G}])_k$ (for $\rho_{\mathcal{G}}(\mathbf{X})$) and $\mathbf{a}(\mathcal{H}, -\mathbf{a}(\mathcal{G}, \mathbf{X}))$ (for $\rho_{\mathcal{H}}(-\mathbf{a}(\mathcal{G}, \mathbf{X}))$). Such consistency properties take the following form:

$$\mathbf{Y}(\mathcal{H}, -\mathbf{Y}(\mathcal{G}, \mathbf{X})) = \mathbf{Y}(\mathcal{H}, \mathbf{X}) + \mathbf{Y}(\mathcal{H}, \mathbf{0}),$$

$$\frac{d\widehat{Q}^k}{dP}(\mathcal{G}, \mathbf{X}) \frac{d\widehat{Q}^k}{dP}(\mathcal{H}, -\mathbf{Y}(\mathcal{G}, \mathbf{X})) = \frac{d\widehat{Q}^k}{dP}(\mathcal{G}, \mathbf{X}) \frac{d\widehat{Q}^k}{dP}(\mathcal{H}, -\mathbf{a}(\mathcal{G}, \mathbf{X})) = \frac{d\widehat{Q}^k}{dP}(\mathcal{H}, \mathbf{X}),$$

$$\mathbf{a}(\mathcal{H}, -\mathbf{a}(\mathcal{G}, \mathbf{X})) = \mathbf{a}(\mathcal{H}, \mathbf{X}) + \mathbf{a}(\mathcal{H}, \mathbf{0}).$$

It is also shown in [99] how the primal and dual optimal allocations for Shortfall Systemic Risk Measures, in both the static and dynamic cases, admit an interpretation in the sense of (a dynamic extension of) Multivariate Systemic Optimal Risk Transfer Equilibrium.

8. Conclusions

In this work, we presented an axiomatic theory for Systemic Risk Measures which extends the classical univariate framework. Such Systemic Risk Measures map the original risk carried by a system of financial institutions to real numbers that quantify the risk level for the system as a whole. The problem of determining such an amount is intimately related to allocation procedures, since it is implicit in these systemic risk measurement regimes that the total amount should serve to secure the system via allocation to each participant. We presented how different allocation procedures can be considered, before or after aggregation of individual risky positions on the one hand, at an initial or at terminal time on the other hand. In the case of terminal time random allocations and for Systemic

Risk Measures of shortfall type, we presented how allocation procedures can be viewed as fair and satisfactory for both the system as a whole and the single agents in it. This is intimately related to the fact that optimal allocations in this case are suitably defined equilibria for the system (SORTE). We also showed how possible additional information at an initial time can be incorporated in the axiomatic theory for the systemic case using a conditional approach inspired by the univariate case.

As briefly sketched in the previous sections, the theory of Systemic Risk Measures is rich and still has unexplored or partially unexplored ramifications.

In the univariate setting, the study of Risk Measures in *a continuous time framework* generated several theoretical developments in different areas. Just to mention a few examples, connections have been found with BSDEs ([82,100–102]) and Nonlinear Expectation ([103]).

We believe that also, in the systemic, multivariate case, the continuous time theory may herald new and interesting research topics.

Risk Measures have frequently been studied in the presence of a financial market. Indeed, given a securities stochastic market, let the subset $\mathscr{G} \subseteq L^0(\Omega, \mathcal{F}, P)$ describe the terminal time values of (self financing) trading strategies in the available securities. The *market adjusted Risk Measure* is then the functional $\rho : L^\infty(\Omega, \mathcal{F}, P) \to [-\infty, +\infty]$ defined by

$$\rho(X) := \inf\{m \in \mathbb{R} \mid \exists g \in \mathscr{G} : m + X + g \in \mathbb{A}\}. \tag{28}$$

In such a risk measurement regime, it is possible to trade in the underlying financial market in order to achieve an acceptable terminal position. A detailed study of these market adjusted Risk Measures and their relationship with the notion of no arbitrage can be found in [28] Section 4.8. Very recently, a similar approach has been developed for Systemic Risk Measures in [104], also in connection with some systemic notion of arbitrage. A deeper study of these connections, including equilibrium aspects, is still to be developed.

A further topic for future investigation concerns the *robust version of Systemic Risk Measures theory*. In the last decade, many papers were aimed at establishing the fundamental Theorems of asset pricing and the key pricing-hedging duality in a probability free setup, or in a non dominated setting. In the latter case, a class of—a priori non dominated—probability measures replaces the usual single reference probability P, leading to the theory of Quasi-Sure Stochastic Analysis ([105–109], just to mention only a few). In the former case, instead, a pathwise and more radical approach is preferred (see, for example, [110–112]).

The above-mentioned papers only treat the (robust) univariate case. A first attempt to examine Systemic Risk Measures in a probability free environment is elaborated by [104], and we foresee that a more detailed inspection on this subject could be very fruitful.

As is well known, Martingale Optimal Transport theory is an extremely powerful tool when aiming at pathwise pricing-hedging duality results ([113–118]). We observe that also in the recent theory of Entropy Martingale Optimal Transport (see [119]) the connections in the robust setup between Optimal Transport and Utility Functionals/Risk Measures arise quite naturally from the main duality relation.

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