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Boundary Value Problems for ψ-Hilfer Type Sequential Fractional Differential Equations and Inclusions with Integral Multi-Point Boundary Conditions

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Abstract: In the present article, we study a new class of sequential boundary value problems of fractional order differential equations and inclusions involving ψ-Hilfer fractional derivatives, supplemented with integral multi-point boundary conditions. The main results are obtained by employing tools from fixed point theory. Thus, in the single-valued case, the existence of a unique solution is proved by using the classical Banach fixed point theorem while an existence result is established via Krasnosel'skiĭ's fixed point theorem. The Leray–Schauder nonlinear alternative for multi-valued maps is the basic tool to prove an existence result in the multi-valued case. Finally, our results are well illustrated by numerical examples.

Keywords: fractional differential equations; fractional differential inclusions; Hilfer fractional derivative; Riemann–Liouville fractional derivative; Caputo fractional derivative; boundary value problems; existence and uniqueness; fixed point theory

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1. Introduction

Fractional-order differentiations and integrations are more accurate tools in expressing real-world problems as compared to integer-order differentiations and integrations. Thus, the theory of fractional differential equations has attracted a lot of attention from many researchers for their wide applications to various fields, such as in physics, bioengineering, electrochemistry, and so on; see [1–3] and related references therein. The interested reader is referred to the monographs [4–11] for the basic theory of fractional calculus and fractional differential equations.

In the literature, there are several definitions of derivatives and integrals of arbitrary orders. For instance, Kilbas et al. in [5] introduced fractional integrals and fractional derivatives concerning another function. In a recent paper, Almeida [12] introduced the so-called ψ-Caputo fractional operator. Numerous interesting results concerning the existence, uniqueness, and stability of initial value problems and boundary value problems for fractional differential equations with ψ-Caputo fractional derivatives by applying different types of fixed-point techniques were obtained in [13–15].

Hilfer in [16] generalized both Riemann–Liouville and Caputo fractional derivatives, known as *the Hilfer fractional derivative*. We refer to [17,18], and references cited therein, for some properties and applications of the Hilfer fractional derivative and to [19–21] for initial value problems involving Hilfer fractional derivatives.

Fractional differential equations involving Hilfer derivative have many applications, and we refer to [22] and the references cited therein. There are actual world occurrences with uncharacteristic dynamics such as atmospheric diffusion of pollution, signal transmissions through strong magnetic fields, the effect of the theory of the profitability of stocks in economic markets, the theoretical simulation of dielectric relaxation in glass forming materials, network traffic, and so on. See [23,24] and references cited therein.

In [25], the authors initiated the study of nonlocal boundary value problems for the Hilfer fractional derivative, by studying the boundary value problem of Hilfer-type fractional differential equations with nonlocal integral boundary conditions

$$ {}^{H}\mathfrak{D}^{\alpha,\beta}x(t) = f(t,x(t)), \qquad t\in[a,b],\ \ 1<\alpha<2,\ \ 0\leq\beta\leq1, \tag{1} $$

$$ x(a)=0, \qquad x(b)=\sum_{i=1}^{m}\delta_i\mathcal{I}^{\varphi_i}x(\xi_i), \qquad \varphi_i>0,\ \delta_i\in\mathbb{R},\ \xi_i\in[a,b], \tag{2} $$

where ${}^{H}\mathfrak{D}^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, $1<\alpha<2$ and parameter β, $0\leq\beta\leq1$, $\mathcal{I}^{\varphi_i}$ is the Riemann–Liouville fractional integral of order $\varphi_i>0$, $\xi_i\in[a,b]$, $a\geq0$ and $\delta_i\in\mathbb{R}$. Several existence and uniqueness results were proved by using a variety of fixed point theorems.

In a series of papers [26–30], nonlocal boundary value problems involving Hilfer fractional derivatives were studied, with a variety of boundary conditions. Thus, the authors in [26] studied Hilfer Langevin three-point fractional boundary value problems, the authors in [27] studied pantograph Hilfer fractional boundary value problems with nonlocal integral boundary conditions, the authors in [28] studied Hilfer fractional boundary value problems with nonlocal integral integro-multipoint boundary conditions, the authors in [29] studied Hilfer fractional boundary value problems with nonlocal multipoint, fractional derivative multi-order, and fractional integral multi-order boundary conditions, and the authors in [30] studied sequential Hilfer fractional boundary value problems with nonlocal integro-multipoint boundary conditions.

Systems of Hilfer–Hadamard sequential fractional differential equations were studied in [31].

In the present paper, motivated by the research going on in this direction, we study a new class of boundary value problems of sequential Hilfer-type fractional differential equations involving integral multi-point boundary conditions of the form

$$ \begin{cases} \left({}^{H}D^{\alpha,\beta;\psi}+k\,{}^{H}D^{\alpha-1,\beta;\psi}\right)x(t)=f(t,x(t)), \quad t\in[a,b], \\ x(a)=0, \quad x(b)=\displaystyle\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)x(s)ds+\sum_{j=1}^{m}\theta_j x(\xi_j). \end{cases} \tag{3} $$

Here, ${}^{H}D^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative operator of order α, $1<\alpha<2$ and parameter β, $0\leq\beta\leq1$, $k\in\mathbb{R}$, $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a\geq0$, $\mu_i,\theta_j\in\mathbb{R}$, $\eta_i,\xi_j\in(a,b]$, $i=1,2,\ldots,n$, $j=1,2,\ldots m$ and ψ is a positive increasing function on $(a,b]$, which has a continuous derivative $\psi'(t)$ on (a,b).

We also cover the multi-valued case of the problem (3) by considering the following inclusion problem:

$$ \begin{cases} \left({}^{H}D^{\alpha,\beta;\psi}+k\,{}^{H}D^{\alpha-1,\beta;\psi}\right)x(t)\in F(t,x(t)), \quad t\in[a,b], \\ x(a)=0, \quad x(b)=\displaystyle\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)x(s)ds+\sum_{j=1}^{m}\theta_j x(\xi_j), \end{cases} \tag{4} $$

where $F:[a,b]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multi valued function, and $(\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R})$.

At the end of this section, we mention that the remaining part of the paper will be organized as follows. In Section 2, we recall some basic concepts of fractional calculus. In Section 3, we prove first a lemma relating a linear variant of the problem in (3) with an integral equation. Moreover, the existence and uniqueness results are established, in the single valued case, by using fixed point theorems. We obtain the existence of a unique solution via Banach's contraction mapping principle, while Krasnosel'skiĭ's fixed point theorem is applied to obtain the existence result for the sequential Hilfer fractional boundary value problem (3). In Section 4, an existence result is proved for the sequential Hilfer inclusion boundary value problem (4), via Leray–Schauder nonlinear alternative for multi-valued maps. Illustrative examples for the main results are provided.

2. Preliminaries

This section is assigned to recall some notation in relation to fractional calculus.

Throughout the paper, $C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [a,b]\}$. We denote by $AC^n([a,b],\mathbb{R})$ the n-times absolutely continuous functions given by

$$AC^n([a,b],\mathbb{R}) = \{f : [a,b] \to \mathbb{R}; f^{(n-1)} \in AC([a,b],\mathbb{R})\}.$$

We recall here that a function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{j=1}^{N}(y_j - x_j) \le \delta$ implies $\sum_{j=1}^{N} |f(y_j) - f(x_j)| \le \varepsilon$ for all mutually disjoint intervals $(x_j, y_j), 1 \le j \le N$, in $[a,b]$. A function $f \in AC([a,b])$ if and only if f is Lebesgue almost everywhere differentiable with derivative $g = f'$ which belongs to $L^1([a,b])$ such that $f(y) - f(x) = \int_x^y g(t)dt$ for all $a \le x < y \le b$.

Definition 1 ([5])**.** *Let (a,b), $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $(0,\infty)$ and $\alpha > 0$. In addition, let $\psi(t)$ be a positive increasing function on $(a,b]$, which has a continuous derivative $\psi'(t)$ on (a,b). The ψ-Riemann–Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by*

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s)ds, \quad t > a > 0, \tag{5}$$

where $\Gamma(\cdot)$ represents the Gamma function.

Definition 2 ([5])**.** *Let $\psi'(t) \neq 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann–Liouville derivatives of a function f with respect to another function ψ of order α correspondent to the Riemann–Liouville is defined by*

$$D_{a^+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t) \tag{6}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f(s)ds, \tag{7}$$

where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number α. This is the greatest integer n such that $n \le \alpha$.

Definition 3 ([32])**.** *Let $n - 1 < \alpha < n$ with $n \in \mathbb{N}$, $[a,b]$ is the interval such that $-\infty \le a < b \le \infty$ and $f, \psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0 \le \beta \le 1$ is defined by*

$${}^H D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t) = I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f(t), \tag{8}$$

where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number α with $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 1 ([5]). *Let $\alpha, \rho > 0$. Then, we have the following semigroup property given by*

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\rho;\psi} f(t) = I_{a^+}^{\alpha+\rho;\psi} f(t), \quad t > a. \tag{9}$$

Next, we present the ψ-fractional integral and derivatives of a power function.

Proposition 1 ([5,32]). *Let $\alpha \geq 0$, $v > 0$ and $t > a$. Then, ψ-fractional integral and derivative of a power function are given by*

(i) $I_{a^+}^{\alpha;\psi}(\psi(s) - \psi(a))^{v-1}(t) = \dfrac{\Gamma(v)}{\Gamma(v+\alpha)}(\psi(t) - \psi(a))^{v+\alpha-1}.$

(ii) ${}^{H}D_{a^+}^{\alpha,\beta;\psi}(\psi(s) - \psi(a))^{v-1}(t) = \dfrac{\Gamma(v)}{\Gamma(v-\alpha)}(\psi(t) - \psi(a))^{v-\alpha-1}$, $n-1 < \alpha < n$, $v > n$.

Lemma 2 ([32]). *If $f \in C^n(J, \mathbb{R})$, $n - 1 < \alpha < n$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta(n - \alpha)$, then*

$$I_{a^+}^{\alpha;\psi}\left({}^{H}D_{a^+}^{\alpha,\beta;\psi} f\right)(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} \nabla_{\psi}^{[n-k]} I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(a), \tag{10}$$

for all $t \in J$, where $\nabla_{\psi}^{[n]} f(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$.

3. Existence and Uniqueness Results for Problem (3)

The following auxiliary lemma concerning a linear variant of the sequential Hilfer boundary value problem (3) plays a fundamental role in establishing the existence and uniqueness results for the given nonlinear problem.

Lemma 3. *Let $a \geq 0$, $1 < \alpha < 2$, $0 \leq \beta \leq 1$, $\gamma = \alpha + 2\beta - \alpha\beta$ be given constants and*

$$\Lambda := (\psi(b) - \psi(a))^{\gamma-1} - \frac{1}{\gamma}\sum_{i=1}^{n} \mu_i (\psi(\eta_i) - \psi(a))^{\gamma} - \sum_{j=1}^{m} \theta_j (\psi(\xi_j) - \psi(a))^{\gamma-1} \neq 0. \tag{11}$$

For a given $h \in C([a, b], \mathbb{R})$, the unique solution of the sequential Hilfer linear fractional boundary value problem

$$\left({}^{H}D^{\alpha,\beta;\psi} + k\, {}^{H}D^{\alpha-1,\beta;\psi}\right)x(t) = h(t), \ \ t \in [a, b], \tag{12}$$

$$x(a) = 0, \quad x(b) = \sum_{i=1}^{n} \mu_i \int_a^{\eta_i} \psi'(s)x(s)ds + \sum_{j=1}^{m} \theta_j x(\xi_j), \tag{13}$$

is given by

$$\begin{aligned} x(t) \quad = \quad & I_{a+}^{\alpha;\psi} h(t) - k\int_a^t \psi'(s)x(s)ds + \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{n} \mu_i \int_a^{\eta_i} \psi'(s) I_{a+}^{\alpha;\psi} h(s)ds \\ & -k\sum_{i=1}^{n} \mu_i \int_a^{\eta_i} \psi'(s)\int_a^s \psi'(u)x(u)duds - k\sum_{j=1}^{m} \theta_j \int_a^{\xi_i} \psi'(s)x(s)ds \\ & +\sum_{j=1}^{m} \theta_j I_{a+}^{\alpha;\psi} h(\xi_i) + k\int_a^b \psi'(s)x(s)ds - I_{a+}^{\alpha;\psi} h(b)\Bigg]. \end{aligned} \tag{14}$$

Proof. Applying the operator $I_{a+}^{\alpha;\psi}$ on both sides of Equation (12) and using Lemma 2, there exist real numbers c_0 and c_1 such that

$$x(t) \quad = \quad c_0 \frac{(\psi(t) - \psi(a))^{-(2-\alpha)(1-\beta)}}{\Gamma(1 - (2-\alpha)(1-\beta))} + c_1 \frac{(\psi(t) - \psi(a))^{1-(2-\alpha)(1-\beta)}}{\Gamma(2 - (2-\alpha)(1-\beta))}$$

$$
\begin{aligned}
& -k\int_a^t \psi'(s)x(s)ds + I_{a+}^{\alpha;\psi}h(t) \\
= \;& c_0\frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)} + c_1\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\
& -k\int_a^t \psi'(s)x(s)ds + I_{a+}^{\alpha;\psi}h(t),
\end{aligned}
$$

since $(1-\beta)(2-\alpha) = 2-\gamma$.

From the boundary condition $x(a) = 0$, we see $c_0 = 0$. Then, we get

$$
x(t) = c_1\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} - k\int_a^t \psi'(s)x(s)ds + I_{a+}^{\alpha;\psi}h(t), \;\; t\in[a,b]. \tag{15}
$$

From $x(b) = \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)x(s)ds + \sum_{j=1}^{m}\theta_j x(\xi_i)$, we obtain

$$
\begin{aligned}
c_1 \;=\; & \frac{\Gamma(\gamma)}{\Lambda}\Big[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds + \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}h(s)ds \\
& -k\sum_{j=1}^{m}\theta_j\int_a^{\xi_i}\psi'(s)x(s)ds + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}h(\xi_i) + k\int_a^b\psi'(s)x(s)ds - I_{a+}^{\alpha;\psi}h(b)\Big].
\end{aligned}
$$

Substituting the values of c_1 in (15), we obtain the solution (14). That the function $x(t)$, as defined in formula (14), solves the boundary value problem in (12), (13) can be proved by direct computation. This finishes the proof of Lemma 3. □

Remark 1. *If $\psi(t) = t$ and $\beta = 0$, then (12) reduces to*

$$
\left({}^{RL}D^{\alpha} + k\,{}^{RL}D^{\alpha-1}\right)x(t) = h(t),
$$

which is the Riemann–Liouville fractional differential equation, where

$$
{}^{RL}D^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\left(\frac{d}{dt}\right)^2\int_a^t (t-s)^{1-\alpha}x(s)ds.
$$

If $\psi(t) = \log_e t$ and $\beta = 0$, then (12) is transformed to the Hadamard fractional differential equation of the form:

$$
\left({}^{Had}D^{\alpha} + k\,{}^{Had}D^{\alpha-1}\right)x(t) = h(t),
$$

where

$$
{}^{Had}D^{\alpha}x(t) = \frac{1}{\Gamma(2-\alpha)}\left(t\frac{d}{dt}\right)^2\int_a^t (\log_e t - \log_e s)^{1-\alpha}x(s)\frac{ds}{s}.
$$

Next, in view of Lemma 3, we define an operator $\mathcal{A} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$
\begin{aligned}
(\mathcal{A}x)(t) \;=\; & I_{a+}^{\alpha;\psi}f(t,x(t)) - k\int_a^t\psi'(s)x(s)ds \\
& + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds \\
& + \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}f(s,x(s))ds - k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds \\
& + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}f(\xi_j,x(\xi_j)) + k\int_a^b\psi'(s)x(s)ds - I_{a+}^{\alpha;\psi}f(b,x(b))\Bigg].
\end{aligned} \tag{16}
$$

The continuity of f shows that $\mathcal{A}$ is well defined and fixed points of the operator equation $x = \mathcal{A}x$ are solutions of the integral Equation (14) in Lemma 3.

In the sequel, we use the following abbreviations:

$$\begin{aligned}\Omega &= \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)}\right. \\ &\quad \left. + \sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right], \end{aligned} \tag{17}$$

and

$$\begin{aligned}\Omega_1 &= |k|(\psi(b)-\psi(a)) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\left[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2\right. \\ &\quad \left. + |k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a)) + |k|(\psi(b)-\psi(a))\right]. \end{aligned} \tag{18}$$

By using classical fixed point theorems, we establish in the following subsections existence, as well as existence and uniqueness results, for the sequential ψ-Hilfer fractional boundary value problem (3).

In our first result, we prove the existence of a unique solution of the sequential ψ-Hilfer fractional boundary value problem (3) based on Banach's fixed point theorem [33].

Theorem 1. *Assume that:*

(H_1) *There exists a finite number $L > 0$ such that, for all $t \in [a, b]$ and for all $x, y \in \mathbb{R}$, the following inequality is valid:*

$$|f(t, x) - f(t, y)| \le L|x - y|.$$

Then, the sequential ψ-Hilfer fractional boundary value problem (3) has a unique solution on $[a, b]$ provided that

$$L\Omega + \Omega_1 < 1, \tag{19}$$

where Ω and Ω_1 are defined by (17) and (18), respectively.

Proof. With the help of the operator $\mathcal{A}$ defined in (16), we transform the sequential ψ-Hilfer fractional boundary value problem (3) into a fixed point problem, $x = \mathcal{A}x$. By applying the Banach contraction mapping principle, we shall show that $\mathcal{A}$ has a unique fixed point.

We put $\sup_{t\in[a,b]} |f(t, 0, 0)| = M < \infty$, and choose $r > 0$ such that

$$r \ge \frac{M\Omega}{1 - L\Omega - \Omega_1}. \tag{20}$$

Let $B_r = \{x \in C([a, b], \mathbb{R}) : \|x\| \le r\}$. We show that $\mathcal{A}B_r \subset B_r$.

For any $x \in B_r$, we have

$$\begin{aligned}&|(\mathcal{A}x)(t)| \\ \le\ & I_{a+}^{\alpha;\psi}|f(t, x(t))| + |k|\int_a^t \psi'(s)|x(s)|ds \\ &+ \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\left[|k|\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)|x(u)|duds\right. \\ &+ \sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}|f(s, x(s))|ds + |k|\sum_{j=1}^{m}|\theta_j|\int_a^{\xi_j}\psi'(s)|x(s)|ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}|\theta_j| I_{a+}^{\alpha;\psi}|f(\xi_j,x(\xi_j))| + |k|\int_a^b \psi'(s)|x(s)|ds + I_{a+}^{\alpha;\psi}|f(b,x(b))|\Bigg] \\
\leq\ & I_{a+}^{\alpha;\psi}(|f(t,x(t)) - f(t,0)| + |f(t,0)|) + |k|\int_a^t \psi'(s)|x(s)|ds \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)|x(u)|duds \\
&+\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s) I_{a+}^{\alpha;\psi}[|f(s,x(s)) - f(s,0)| + f(s,0)|]ds \\
&+|k|\sum_{j=1}^{m}|\theta_j|\int_a^{\xi_j}\psi'(s)|x(s)|ds + \sum_{j=1}^{m}|\theta_j| I_{a+}^{\alpha;\psi}[|f(\xi_j,x(\xi_j)) - f(\xi_j,0)| + |f(\xi_j,0)|] \\
&+|k|\int_a^b \psi'(s)|x(s)|ds + I_{a+}^{\alpha;\psi}[|f(b,x(b)) - f(b,0)| + |f(b,0)|]\Bigg] \\
\leq\ & \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
&+\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}(L\|x\| + M) \\
&+\Bigg\{|k|(\psi(b)-\psi(a)) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2 \\
&+|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a)) + |k|(\psi(b)-\psi(a))\Big]\Bigg\}\|x\| \\
\leq\ & (Lr+M)\Omega + \Omega_1 r \leq r,
\end{aligned}
$$

and consequently $\|\mathcal{A}x\| \leq r$, which implies that $\mathcal{A}B_r \subset B_r$.

Next, we show that $\mathcal{A}$ is a contraction. Let $x, y \in C([a,b],\mathbb{R})$. Then, for $t \in [a,b]$, we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| \\
\leq\ & I_{a+}^{\alpha;\psi}|f(t,x(t)) - f(t,y(t))| + |k|\int_a^t \psi'(s)|x(s)-y(s)|ds \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)|x(u)-y(u)|duds \\
&+\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s) I_{a+}^{\alpha;\psi}|f(t,x(s)) - f(t,y(s))|ds \\
&+|k|\sum_{j=1}^{m}|\theta_j|\int_a^{\xi_j}\psi'(s)|x(s)-y(s)|ds + \sum_{j=1}^{m}|\theta_j| I_{a+}^{\alpha;\psi}|f(\xi_j,x(\xi_j)) - f(\xi_j,y(\xi_j))| \\
&+|k|\int_a^b \psi'(s)|x(s)-y(s)|ds + I_{a+}^{\alpha;\psi}|f(b,x(b)) - f(b,y(b))|\Bigg] \\
\leq\ & \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
&+\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}L\|x-y\| + \Bigg\{|k|(\psi(b)-\psi(a)) \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2 + |k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))
\end{aligned}
$$

$$
\begin{aligned}
&\left.+|k|(\psi(b)-\psi(a))\Big]\right\}\|x-y\| \\
=\;& (L\Omega+\Omega_1)\|x-y\|,
\end{aligned}
$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\| \le (L\Omega+\Omega_1)\|x-y\|$. As $L\Omega+\Omega_1<1$, $\mathcal{A}$ is a contraction. Therefore, by the Banach's contraction mapping principle, we deduce that $\mathcal{A}$ has a fixed point. Obviously, this is the unique solution of the sequential ψ-Hilfer fractional boundary value problem (3). The proof is complete now. □

The next existence result is based on the a classical fixed point theorem due to Krasnosel'skiĭ's [34].

Theorem 2. *Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that:*
(H_2) $|f(t,x)|\le\varphi(t)$, $\forall(t,x)\in[a,b]\times\mathbb{R}$, and $\varphi\in C([a,b],\mathbb{R}^+)$.
Then, the sequential ψ-Hilfer fractional boundary value problem (3) *has at least one solution on $[a,b]$ provided that $\Omega_1<1$, where Ω_1 is defined in (18).*

Proof. We consider $B_\rho=\{x\in C([a,b],\mathbb{R}):\|x\|\le\rho\}$, where $\rho>0$ such that $\rho\ge\dfrac{\|\varphi\|\Omega}{1-\Omega_1}$, and $\sup_{t\in[a,b]}\varphi(t)=\|\varphi\|$. We define the operators $\mathcal{A}_1$, $\mathcal{A}_2$ on B_ρ by

$$
\begin{aligned}
\mathcal{A}_1x(t) \;=\;& I_{a+}^{\alpha;\psi}f(t,x(t))+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}f(s,x(s))ds \\
&+\sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}f(\xi_j,x(\xi_j))-I_{a+}^{\alpha;\psi}f(b,x(b))\Bigg],\quad t\in[a,b],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_2x(t) \;=\;& -k\int_a^t\psi'(s)x(s)ds \\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds \\
&-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds+k\int_a^b\psi'(s)x(s)ds\Bigg],\quad t\in[a,b].
\end{aligned}
$$

For any $x,y\in B_\rho$, we have

$$
\begin{aligned}
&|(\mathcal{A}_1x)(t)+(\mathcal{A}_2y)(t)| \\
\le\;& I_{a+}^{\alpha;\psi}|f(t,x(t))|+|k|\int_a^t\psi'(s)|y(s)|ds \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)|y(u)|duds \\
&+\sum_{i=1}^{n}|\mu_i|\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}|f(s,x(s))|ds+|k|\sum_{j=1}^{m}|\theta_j|\int_a^{\xi_j}\psi'(s)|y(s)|ds \\
&+\sum_{j=1}^{m}|\theta_j|I_{a+}^{\alpha;\psi}|f(\xi_j,x(\xi_j))|+|k|\int_a^b\psi'(s)|y(s)|ds+I_{a+}^{\alpha;\psi}|f(b,x(b))|\Bigg] \\
\le\;& \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|\phi\| \\
& +\Bigg\{|k|(\psi(b)-\psi(a))+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2 \\
& +|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Big]\Bigg\}\|x\| \\
\leq\ & \|\varphi\|\Omega+\Omega_1\rho\leq\rho.
\end{aligned}
$$

Therefore, $\|\mathcal{A}_1x+\mathcal{A}_2y\|\leq\rho$, which shows that $\mathcal{A}_1x+\mathcal{A}_2y\in B_\rho$. It is easy to see, using the condition $\Omega_1<1$, that $\mathcal{A}_2$ is a contraction mapping.

The operator $\mathcal{A}_1$ is continuous because f is continuous. In addition, $\mathcal{A}_1$ is uniformly bounded on B_ρ because we have

$$
\begin{aligned}
\|\mathcal{A}_1x\| \leq & \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
& +\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|\phi\|.
\end{aligned}
$$

The compactness of the operator $\mathcal{A}_1$ is proved now. Let $t_1,t_2\in[a,b]$ with $t_1<t_2$. Then, we have

$$
\begin{aligned}
& |(\mathcal{A}_1x)(t_2)-(\mathcal{A}_1x)(t_1)| \\
=\ & \frac{1}{\Gamma(\alpha)}\Bigg|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}]f(s,x(s))ds \\
& +\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}f(s,x(s))ds\Bigg| \\
& +\frac{|(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}|}{|\Lambda|}\Bigg|\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}f(s,x(s))ds \\
& +\sum_{j=1}^{m}\theta_jI_{a+}^{\alpha;\psi}f(\xi_j,x(\xi_j))-I_{a+}^{\alpha;\psi}f(b,x(b))\Bigg| \\
\leq\ & \frac{\|\phi\|}{\Gamma(\alpha+1)}[2(\psi(t_2)-\psi(t_1))^{\alpha}+|(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha}|] \\
& +\frac{|(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}|}{|\Lambda|}\|\phi\|\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
& +\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\to 0,
\end{aligned}
$$

as $t_2-t_1\to 0$, and is independent of x. Thus, $\mathcal{A}_1(B_\rho)$ is an equicontinuous set. Thus, $\mathcal{A}_1$ is relatively compact on B_ρ. By the Arzelá–Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Applying the Krasnosel'skiĭ's fixed point theorem, the sequential ψ-Hilfer fractional boundary value problem (3) has at least one solution on $[a,b]$. The proof is completed. □

Example 1. *Considering the boundary value problems for ψ-Hilfer type sequential fractional differential equation with integral multi-point boundary conditions,*

$$\begin{cases} \left({}^{H}D^{\frac{3}{2},\frac{4}{5};t^2e^t} + \frac{1}{250}\,{}^{H}D^{\frac{1}{2},\frac{4}{5};t^2e^t}\right)x(t) = f(t,x(t)), \quad t \in \left[\frac{1}{4},\frac{11}{4}\right], \\ x\left(\frac{1}{4}\right) = 0, \quad x\left(\frac{11}{4}\right) = \frac{1}{11}\int_{\frac{1}{4}}^{\frac{3}{4}} g(s)x(s)ds + \frac{3}{22}\int_{\frac{1}{4}}^{\frac{3}{2}} g(s)x(s)ds \\ +\frac{5}{33}\int_{\frac{1}{4}}^{\frac{9}{4}} g(s)x(s)ds + \frac{7}{44}x\left(\frac{1}{2}\right) + \frac{9}{55}x\left(\frac{5}{4}\right) + \frac{13}{66}x\left(\frac{7}{4}\right) + \frac{15}{77}x\left(\frac{5}{2}\right), \end{cases} \tag{21}$$

where $g(s) = (s^2e^s)' = se^s(s+2)$.

In problem (21), specific constants can be chosen: $\alpha = 3/2$, $\beta = 4/5$, $\psi(t) = t^2e^t$, $k = 1/250$, $a = 1/4$, $b = 11/4$, $n = 3$, $\mu_1 = 1/11$, $\mu_2 = 3/22$, $\mu_3 = 5/33$, $\eta_1 = 3/4$, $\eta_2 = 3/2$, $\eta_3 = 9/4$, $m = 4$, $\theta_1 = 7/44$, $\theta_2 = 9/55$, $\theta_3 = 13/66$, $\theta_4 = 15/77$, $\xi_1 = 1/2$, $\xi_2 = 5/4$, $\xi_3 = 7/4$, $\xi_4 = 5/2$. Such choices lead to constants as $\gamma = 19/10$, $\Lambda \approx 8528.189896$, $\Omega \approx 982.5219141$ and $\Omega_1 \approx 0.4838262092$.

(i) Let the function $f(t,x)$ be given by

$$f(t,x) = \frac{8e^{1-4t}}{5(4t+79)^2}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{1}{4}. \tag{22}$$

Then, we can check the Lipchitz condition of $f(t,x)$ as

$$\begin{aligned} |f(t,x) - f(t,y)| &= \left|\frac{8e^{1-4t}}{5(4t+79)^2}\right|\left|\frac{x^2+2|x|}{1+|x|} - \frac{y^2+2|y|}{1+|y|}\right| \\ &\leq \frac{1}{2000}|x-y|, \end{aligned}$$

for all $x, y \in \mathbb{R}$, $t \in [1/4, 11/4]$. By setting a constant $L = 1/2000$, we obtain

$$L\Omega + \Omega_1 \approx 0.9750871662 < 1,$$

which claims that inequality (19) is fulfilled. Therefore, by an application of Theorem 1, the boundary value problem for ψ-Hilfer type sequential fractional differential equation with integral multi-point boundary conditions (21) with (22) has a unique solution on $[1/4, 11/4]$. Note that the Theorem 2 can not be applied to this problem because the given function in (22) is unbounded because $\lim_{x\to\infty}|f(t,x)| = \infty$.

(ii) Let the function $f(t,x)$ be defined, for $M \in \mathbb{R}$, by

$$f(t,x) = M\cos^2\left(\frac{t^2-3t+1}{x^2+3}\right) + \frac{4}{4t+119}\sin\left(\frac{|x|}{1+|x|}\right) + \frac{1}{2}. \tag{23}$$

It is obvious that the function $f(t,x)$ is bounded by

$$|f(t,x)| \leq |M| + \frac{4}{4t+119} + \frac{1}{2} := \varphi(t),$$

which satisfy condition (H_2) in Theorem 2. Since $\Omega_1 < 1$, the conclusion of Theorem 2 yields that the problem (21) with (22) has at least one solution on $[1/4, 11/4]$. If $M = 0$, then (23) is reduced to

$$f(t,x) = \frac{4}{4t+119}\sin\left(\frac{|x|}{1+|x|}\right) + \frac{1}{2},$$

which satisfies the Lipchitz condition $|f(t,x)-f(t,y)| \leq (1/30)|x-y|$, $L = 1/30$, for all $t \in [1/4, 11/4]$, $x, y \in \mathbb{R}$. Theorem 1 cannot be used in this case because $L\Omega + \Omega_1 \approx 33.23455668 > 1$.

4. Existence Results for Problem (4)

For details in multi-valued theory, we refer to [35–37].

Definition 4. *A function $x \in C^2([a,b],\mathbb{R})$ is a solution of the problem (4) if $x(a) = 0$, $x(b) = \sum_{i=1}^{n} \mu_i \int_a^{\eta_i} \psi'(s)x(s)ds + \sum_{j=1}^{m} \theta_j x(\xi_j)$, and there exists a function $v \in L^1([a,b],\mathbb{R})$ such that $v(t) \in F(t, x(t))$ a.e. on $[a,b]$ and*

$$\begin{aligned} x(t) &= I_{a+}^{\alpha;\psi} v(t) - k\int_a^t \psi'(s)x(s)ds \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i \int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)x(u)duds \\ &\quad + \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v(s)ds - k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}v(\xi_j) \\ &\quad + k\int_a^b \psi'(s)x(s)ds - I_{a+}^{\alpha;\psi}v(b)\Bigg]. \end{aligned} \tag{24}$$

In the next theorem, we prove the existence of solutions of the sequential Hilfer inclusion fractional boundary value problem (4) when the multi-valued map F has convex values assuming that it is L^1-Carathéodory, that is, (i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$; (ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [a,b]$; (iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([a,b],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \alpha$ and for a.e. $t \in [a,b]$.

For each $x \in C([a,b],\mathbb{R})$, denote the set of selections of F by

$$S_{F,x} := \{v \in L^1([a,b],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in [a,b]\}.$$

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. The following lemma is used in the sequel.

Lemma 4 ([38])**.** *Let $F : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then, the operator*

$$\Theta \circ S_F : C([a,b],\mathbb{R}) \to \mathcal{P}_{cp,c}(C([a,b],\mathbb{R})), \;\; x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R})$.

Theorem 3. *Assume that $\Omega_1 < 1$. In addition, we suppose that:*

(A_1) *$F : [a,b] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multi-valued map;*

(A_2) *there exists a nondecreasing and continuous function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([a,b],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\Phi(\|x\|) \;\; \textit{for each} \;\; (t,x) \in [a,b] \times \mathbb{R};$$

(A_3) *there exists a number $M > 0$ such that*

$$\frac{M}{\|p\|\Phi(M)\Omega} > \frac{1}{1-\Omega_1}, \tag{25}$$

where Ω and Ω_1 are given in (17) and (18), respectively.

Then, the sequential Hilfer inclusion fractional boundary value problem (4) has at least one solution on $[a,b]$.

Proof. We define an operator $\mathcal{N} : C([a,b],\mathbb{R}) \longrightarrow \mathcal{P}(C([a,b],\mathbb{R}))$ by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([a,b],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} I_{a+}^{\alpha;\psi} v(t) - k\int_a^t \psi'(s)x(s)ds \\ + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[-k\sum_{i=1}^{n}\mu_i \int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)x(u)duds \\ + \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v(s)ds - k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds \\ + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}v(\xi_j) + k\int_a^b \psi'(s)x(s)ds - I_{a+}^{\alpha;\psi}v(b)\Big], \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$, in order to transform the problem (4) into a fixed point problem. Clearly, the solutions of the boundary value problem (4) are fixed points of $\mathcal{N}$.

Our proof strategy is to show that all conditions of Leray–Schauder nonlinear alternative for multi-valued maps [39] are satisfied and, consequently, we conclude that sequential Hilfer inclusion fractional boundary value problem (4) has at least one solution on $[a,b]$. We will give the proof in several steps.

Step 1: *$\mathcal{N}(x)$ is convex for all $x \in C([a,b],\mathbb{R})$.*

For $z_1, z_2 \in \mathcal{B}(x)$,, there exist $v_1, v_2 \in S_{F,x}$ such that

$$\begin{aligned} z_i(t) \;=\; & I_{a+}^{\alpha;\psi} v_i(t) - k\int_a^t \psi'(s)x(s)ds \\ & + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i \int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)x(u)duds \\ & + \sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v_i(s)ds - k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds \\ & + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}v_i(\xi_j) + k\int_a^b \psi'(s)x(s)ds - I_{a+}^{\alpha;\psi}v_i(b)\Bigg], \;\; i = 1,2, \end{aligned}$$

for almost all $t \in [a,b]$. Let $0 \le \omega \le 1$. Then, we have

$$\begin{aligned} & [\omega z_1 + (1-\omega)z_2](t) \\ = \; & I_{a+}^{\alpha;\psi}[\omega v_1 + (1-\omega)v_2](t) - k\int_a^t \psi'(s)x(s)ds \\ & + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}[\omega v_1 + (1-\omega)v_2](s)ds \\ & + \sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}[\omega v_1 + (1-\omega)v_2](\xi_j) - I_{a+}^{\alpha;\psi}[\omega v_1 + (1-\omega)v_2](b) \\ & - k\sum_{i=1}^{n}\mu_i \int_a^{\eta_i}\psi'(s)\int_a^s \psi'(u)x(u)duds - k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds \end{aligned}$$

$$+k\int_a^b \psi'(s)x(s)ds\Big].$$

F has convex values and thus $S_{F,x}$ is convex and $\omega v_1(s)+(1-\omega)v_2(s)\in S_{F,x}$. Consequently, $\omega z_1+(1-\omega)z_2\in\mathcal{N}(x)$, which proves that $\mathcal{N}$ is convex-valued.

Step 2: *Bounded sets are mapped by $\mathcal{N}$ into bounded sets in $C([a,b],\mathbb{R})$.*

Let $B_r=\{x\in C([a,b],\mathbb{R}):\|x\|\le r\}, r>0$. For each $h\in\mathcal{N}(x), x\in B_r$, there exists $v\in S_{F,x}$ such that

$$\begin{aligned}
h(t) \;=\;& I_{a+}^{\alpha;\psi}v(t)-k\int_a^t \psi'(s)x(s)ds\\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^n\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds\\
&+\sum_{i=1}^n\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v(s)ds-k\sum_{j=1}^m\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds+\sum_{j=1}^m\theta_j I_{a+}^{\alpha;\psi}v(\xi_j)\\
&+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v(b)\Bigg],\ t\in[a,b].
\end{aligned}$$

Then, for $t\in[a,b]$, we have

$$\begin{aligned}
|h(t)| \;\le\;& I_{a+}^{\alpha;\psi}|v(t)|+|k|\int_a^t\psi'(s)|x(s)|ds\\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|k|\sum_{i=1}^n|\mu_i|\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)|x(u)|duds\\
&+\sum_{i=1}^n|\mu_i|\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}|v(s)|ds+|k|\sum_{j=1}^m|\theta_j|\int_a^{\xi_j}\psi'(s)|x(s)|ds\\
&+\sum_{j=1}^m|\theta_j|I_{a+}^{\alpha;\psi}|v(\xi_j)|+|k|\int_a^b\psi'(s)|x(s)|ds+I_{a+}^{\alpha;\psi}|v(b)|\Bigg]\\
\le\;& \|p\|\Phi(\|x\|)\Bigg\{\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^n|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)}\\
&+\sum_{j=1}^m|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\}+\|x\|\Bigg\{|k|(\psi(b)-\psi(a))\\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\frac{1}{2}|k|\sum_{i=1}^n|\mu_i|(\psi(\eta_i)-\psi(a))^2\\
&+|k|\sum_{j=1}^m|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Big]\Bigg\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|h\| \;\le\;& \|p\|\Phi(r)\Bigg\{\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^n|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)}\\
&+\sum_{j=1}^m|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\}+r\Bigg\{|k|(\psi(b)-\psi(a))\\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\frac{1}{2}|k|\sum_{i=1}^n|\mu_i|(\psi(\eta_i)-\psi(a))^2
\end{aligned}$$

$$+|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Big]\Bigg\}.$$

Step 3: *Bounded sets are mapped by $\mathcal{N}$ into equicontinuous sets.*

Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $x \in B_r$. For each $h \in \mathcal{N}(x)$, we obtain

$$\begin{aligned}
& |h(t_2)-h(t_1)| \\
= \; & \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}]v(s)ds\right. \\
& \left.+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}v(s)ds\right|+|k|r(\psi(t_2)-\psi(t_1)) \\
& +\frac{(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}\|p\|\Phi(r)\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
& +\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg] \\
& +\frac{(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}r\Big[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2 \\
& +|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Big] \\
\leq \; & \frac{\|p\|\Phi(r)}{\Gamma(\alpha+1)}[2(\psi(t_2)-\psi(t_1))^{\alpha}+|(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha}|] \\
& +\frac{(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}\|p\|\Phi(r)\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)} \\
& +\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]+|k|r(\psi(t_2)-\psi(t_1)) \\
& +\frac{(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}r\Big[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2 \\
& +|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Big]\to 0,
\end{aligned}$$

as $t_2 - t_1 \to 0$, and is independent of $x \in B_r$. By the Arzelá–Ascoli theorem, it follows that $\mathcal{N} : C([a,b],\mathbb{R}) \to \mathcal{P}(C([a,b],\mathbb{R}))$ is completely continuous.

In the next step, we will prove that $\mathcal{N}$ is upper semicontinuous. In order to reach the desired conclusion, we have to recall from [35], Proposition 1.2 that *a completely continuous operator is upper semicontinuous if it has a closed graph.* Therefore, we will show the following result.

Step 4: *$\mathcal{N}$ has a closed graph.*

Consider $x_n \to x_*, h_n \in \mathcal{N}(x_n)$ and $h_n \to h_*$. Then, we will show that $h_* \in \mathcal{N}(x_*)$. From $h_n \in \mathcal{N}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [a,b]$,

$$\begin{aligned}
h_n(t) \; = \; & I_{a+}^{\alpha;\psi}v_n(t)-k\int_a^t\psi'(s)x(s)ds \\
& +\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds
\end{aligned}$$

$$+\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v_n(s)ds-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds$$
$$\left.+\sum_{j=1}^{m}\theta_jI_{a+}^{\alpha;\psi}v_n(\xi_j)+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v_n(b)\right],\ t\in[a,b].$$

We must show that there exists $v_*\in S_{F,x_*}$ such that, for each $t\in[a,b]$,

$$\begin{aligned}h_*(t) &= I_{a+}^{\alpha;\psi}v_*(t)-k\int_a^t\psi'(s)x(s)ds\\&\quad+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds\right.\\&\quad+\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v_*(s)ds-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds\\&\quad\left.+\sum_{j=1}^{m}\theta_jI_{a+}^{\alpha;\psi}v_*(\xi_j)+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v_*(b)\right],\ t\in[a,b].\end{aligned}$$

Consider the linear operator $\Theta:L^1([a,b],\mathbb{R})\to C([a,b],\mathbb{R})$ given by

$$\begin{aligned}v\mapsto\Theta(v)(t) &= I_{a+}^{\alpha;\psi}v(t)-k\int_a^t\psi'(s)x(s)ds\\&\quad+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds\right.\\&\quad+\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v(s)ds-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds\\&\quad\left.+\sum_{j=1}^{m}\theta_jI_{a+}^{\alpha;\psi}v(\xi_j)+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v(b)\right],\ t\in[a,b].\end{aligned}$$

Observe that $\|h_n(t)-h_*(t)\|\to0$ as $n\to\infty$. By Lemma 4 that $\Theta\circ S_F$ is a closed graph operator. Moreover, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, we have that

$$\begin{aligned}h_*(t) &= I_{a+}^{\alpha;\psi}v_*(t)-k\int_a^t\psi'(s)x(s)ds\\&\quad+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds\right.\\&\quad+\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v_*(s)ds-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds\\&\quad\left.+\sum_{j=1}^{m}\theta_jI_{a+}^{\alpha;\psi}v_*(\xi_j)+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v_*(b)\right],\ t\in[a,b]\end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Step 5: *We show that there exists an open set $U\subseteq C([a,b],\mathbb{R})$ with $x\notin\theta\mathcal{N}(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$.*

Let $x\in\theta\mathcal{N}(x)$ for some $\theta\in(0,1)$. Then, there exists $v\in L^1([a,b],\mathbb{R})$ with $v\in S_{F,x}$ such that, for $t\in[a,b]$, we have

$$x(t) = \theta I_{a+}^{\alpha;\psi}v(t)-k\theta\int_a^t\psi'(s)x(s)ds$$

$$
\begin{aligned}
&+\theta\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Bigg[-k\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)\int_a^s\psi'(u)x(u)duds\\
&+\sum_{i=1}^{n}\mu_i\int_a^{\eta_i}\psi'(s)I_{a+}^{\alpha;\psi}v(s)ds-k\sum_{j=1}^{m}\theta_j\int_a^{\xi_j}\psi'(s)x(s)ds\\
&+\sum_{j=1}^{m}\theta_j I_{a+}^{\alpha;\psi}v(\xi_j)+k\int_a^b\psi'(s)x(s)ds-I_{a+}^{\alpha;\psi}v(b)\Bigg].
\end{aligned}
$$

Following the computation as in Step 2, we have for each $t\in[a,b]$,

$$
\begin{aligned}
|x(t)| &\le \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a))^{\alpha+1}}{\Gamma(\alpha+2)}\\
&\quad+\sum_{j=1}^{m}|\theta_j|\frac{(\psi(\xi_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|p\|\Phi(\|x\|)\\
&\quad+\Bigg\{|k|(\psi(b)-\psi(a))+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\frac{1}{2}|k|\sum_{i=1}^{n}|\mu_i|(\psi(\eta_i)-\psi(a))^2\\
&\quad+|k|\sum_{j=1}^{m}|\theta_j|(\psi(\xi_j)-\psi(a))+|k|(\psi(b)-\psi(a))\Bigg]\Bigg\}\|x\|\\
&= \|p\|\psi(\|x\|)\Omega+\Omega_1\|x\|.
\end{aligned}
$$

Thus,

$$
(1-\Omega_1)\|x\|\le\|p\|\Phi(\|x\|)\Omega,
$$

or

$$
\frac{M}{\|p\|\Phi(M)\Omega}\le\frac{1}{1-\Omega_1}. \tag{26}
$$

In view of (A_3), there exists M such that $\|x\|\neq M$. Consider

$$
U=\{x\in C([a,b],\mathbb{R}):\|x\|<M\}.
$$

Note that $\mathcal{N}:\overline{U}\to\mathcal{P}(C([a,b],\mathbb{R}))$ is a compact, upper semicontinuous multi-valued map with convex closed values, and there is no $x\in\partial U$ such that $x\in\theta\mathcal{N}(x)$ for some $\theta\in(0,1)$, from the choice of U. By the Leray–Schauder nonlinear alternative for multivalued maps [39], we deduce that $\mathcal{N}$ has a fixed point $x\in\overline{U}$, which is a solution of the sequential Hilfer inclusion fractional boundary value problem (4). This completes the proof. □

Example 2. *Consider the boundary value problem for Hilfer type sequential fractional differential inclusion involving integral multi-point boundary conditions*

$$
\begin{cases}
\left({}^{H}D^{\frac{7}{4},\frac{1}{3};\log_e(t^2+1)}+\frac{1}{6}\,{}^{H}D^{\frac{3}{4},\frac{1}{3};\log_e(t^2+1)}\right)x(t)=F(t,x(t)),\quad t\in\left[\frac{1}{8},\frac{5}{4}\right],\\
x\left(\frac{1}{8}\right)=0,\quad x\left(\frac{5}{4}\right)=\frac{1}{11}\int_{\frac{1}{8}}^{\frac{1}{4}}g(s)x(s)ds+\frac{2}{13}\int_{\frac{1}{8}}^{\frac{1}{2}}g(s)x(s)ds\\
+\frac{3}{17}\int_{\frac{1}{8}}^{\frac{3}{4}}g(s)x(s)ds+\frac{4}{19}\int_{\frac{1}{8}}^{\frac{9}{8}}g(s)x(s)ds+\frac{6}{29}x\left(\frac{3}{8}\right)+\frac{7}{31}x\left(\frac{5}{8}\right)+\frac{8}{37}x\left(\frac{7}{8}\right),
\end{cases} \tag{27}
$$

where the set $F(t,x)$ is defined by

$$
F(t,x)=\left[0,\frac{8}{8t+63}\left(\frac{x^{16}}{1+x^{14}}+\frac{2|x|^9}{3(1+|x|^9)}+\frac{1}{3}e^{-x^2}\right)\right],
$$

and the function $g(s) = (\log_e(s^2+1))' = 2s/(s^2+1)$.

Choosing constants $\alpha = 7/4$, $\beta = 1/3$, $\psi(t) = \log_e(t^2+1)$, $k = 1/6$, $a = 1/8$, $b = 5/4$, $n = 4$, $\mu_1 = 1/11$, $\mu_2 = 2/13$, $\mu_3 = 3/17$, $\mu_4 = 4/19$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $\eta_3 = 3/4$, $\eta_4 = 9/8$, $m = 3$, $\theta_1 = 6/29$, $\theta_2 = 7/31$, $\theta_3 = 8/37$, $\xi_1 = 3/8$, $\xi_2 = 5/8$, $\xi_3 = 7/8$. With the given data, it can be computed that $\gamma = 11/6$, $\Lambda \approx 0.5829695974$, $\Omega \approx 1.576164146$, $\Omega_1 \approx 0.4832654105$. It is obvious that the set $F(t,x)$ satisfies condition (A_1) in Theorem 3. In addition, from

$$\|F(t,x)\|_{\mathcal{P}} \leq \frac{8}{8t+63}\left(x^2+1\right),$$

we choose functions $p(t) = 8/(8t+63)$ and $\Phi : [0,\infty) \to (0,\infty)$ by $\Phi(u) = u^2+1$. Then, $\|p\| = 1/8$ and there exists a constant $M \in (0.4630224201, 2.159722634)$ satisfying inequality (25) in (A_3) of Theorem 3. Thus, we can conclude that the boundary value problem for Hilfer type sequential fractional differential inclusion involving integral multi-point boundary conditions (27) has at least one solution on $[1/8, 5/4]$.

5. Conclusions

In this work, we studied a new class of ψ-Hilfer sequential boundary value problems of fractional order, supplemented with integral multi-point boundary conditions. Fractional differential equations and inclusions are considered. Existence and uniqueness results are established in the single-valued case, by using the classical Banach and Krasnosel'skiĭ fixed point theorems. In the multi-valued case, an existence result is proved by using Leray–Schauder nonlinear alternative for multi-valued maps. Illustrative examples are presented to show the validity of our main results. The present work is innovative and interesting, and significantly contributes to the available material on ψ-Hilfer fractional differential equations and inclusions.

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