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Colorings of (r, r)-Uniform, Complete, Circular, Mixed Hypergraphs

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Abstract: In colorings of some block designs, the vertices of blocks can be thought of as hyperedges of a hypergraph $\mathcal{H}$ that can be placed on a circle and colored according to some rules that are related to colorings of circular mixed hypergraphs. A mixed hypergraph $\mathcal{H}$ is called circular if there exists a host cycle on the vertex set X such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces a connected subgraph of this cycle. We propose an algorithm to color the (r,r)-uniform, complete, circular, mixed hypergraphs for all feasible values with no gaps. In doing so, we show $\chi(\mathcal{H}) = 2$ and $\bar{\chi}(\mathcal{H}) = n - s$ or $n - s - 1$ where s is the sieve number.

Keywords: coloring; circular; mixed hypergraphs

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1. Introduction

Initially, this paper was inspired by coloring star systems investigated in [1]. Darijani and Pike colored e-stars systems of the complete graph. As with the colorings of our family of graphs, each e-star could be considered a block and one would want to avoid a monochromatic coloring of any block.

In this paper, we work on a coloring problem that avoids monochromatic and rainbow colorings of the blocks with the additional structure that there must be a sequential ordering of vertices and each block consists of sequential vertices. We also investigate how many colors can be used. That is, we determine the range of color classes. It follows that these colorings can be modeled by circular mixed hypergraphs. The concept of circular mixed hypergraphs was introduced and studied in [2,3], and continued in [4].

In the traditional theory of coloring graphs and hypergraphs [5–7], we seek colorings of the vertices so that each edge has at least two vertices of different colors. Usually, the minimum number of colors required is sought. One can also seek the dual question to color the vertices so that each edge requires at least two vertices of the same color and ask for the maximum number of colors needed. In the case of mixed hypergraphs, we ask the combination of the above two questions [7–9].

In the present paper we deal with such a combination of constraints on colorings and use the terminology of Voloshin in [7]. We begin with the Preliminaries, including definitions, background, and motivation. We then introduce a coloring algorithm that will properly color the class of mixed hypergraphs. We then show that the algorithm properly colors all values from the lower chromatic number to the upper chromatic number with no gaps. In doing so, we provide the feasible set as Bujtás and Tuza did in [10] for interval hypergraphs and hypertrees.

2. Preliminaries

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$ and $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. Each element of $\mathcal{C} \cup \mathcal{D}$ is of a size of at least 2. In a mixed hypergraph, if a subset of vertices is a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time, then it is a bi-edge. In the results where we strictly look at bi-edges, we will simply use the term edge. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a bihypergraph if $\mathcal{C} = \mathcal{D}$. A proper k-coloring of a mixed hypergraph is a mapping from the vertex set to a set of k colors so that each $\mathcal{C}$-edge has two vertices with a $\mathcal{C}$ommon color and each $\mathcal{D}$-edge has two vertices with $\mathcal{D}$ifferent colors. A mixed hypergraph is k-colorable (uncolorable) if it has a proper coloring with at most k colors (admits no proper colorings). A strict k-coloring is a proper coloring using all k colors. The minimum number of colors in a proper coloring of $\mathcal{H}$ is the lower chromatic number $\chi(\mathcal{H})$; the maximum number of colors in a strict coloring is the upper chromatic number $\bar{\chi}(\mathcal{H})$. We use $c(x)$ for the color of vertex x. The set of values k such that $\mathcal{H}$ has a strict k-coloring is called the feasible set of $\mathcal{H}$, denoted by $F(\mathcal{H})$. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ has a gap at k if $F(\mathcal{H})$ contains elements larger and smaller than k, but omits k. $F(\mathcal{H})$ is called gap-free if it has no gaps. A mixed hypergraph $\mathcal{H}$ is called circular if there exists a host cycle on the vertex set X such that every $\mathcal{C}$-edge and every $\mathcal{D}$-edge induces a connected subgraph of the host cycle.

In other words, for a circular mixed hypergraph there exists a circular ordering of the vertex set X, say, $X = \{x_0, x_1, \ldots, x_{n-1}, x_0\}$ such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces an interval in this ordering. In [2], the lower chromatic number was investigated for the colorability and unique colorability of classical circular mixed hypergraphs, while the upper chromatic number was investigated in [3]. The generalizations of circular mixed hypergraphs have been investigated in [11]. In particular, it was shown that the feasible set of any mixed strong hypercactus is gap-free, and there are infinitely many mixed weak hypercacti such that the feasible set of any of them contains a gap. In this paper, we focus on a particular family of hypergraphs, the (r, r)-uniform complete circular hypergraphs.

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a circular mixed hypergraph. If every k consecutive vertices of X form a $\mathcal{C}$-edge, then we denote $\mathcal{C} = \mathcal{C}_k$. Similarly, if every l consecutive vertices of X form a $\mathcal{D}$-edge, then we denote $\mathcal{D} = \mathcal{D}_l$. The circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}_k, \mathcal{D}_l)$ is called (k, l)-uniform and is denoted by $\mathcal{KC}(n; k, l)$, where $n := |X|$. A (3,2)-uniform circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}_3, \mathcal{D}_2) = \mathcal{KC}(n; 3, 2)$ is a complete circular mixed hypergraph where each consecutive triple must have two colors in common and every consecutive pair must be colored differently.

In a mixed hypergraph $\mathcal{H}$, the subfamily of $\mathcal{C}$-edges $\Sigma \subseteq \mathcal{C}$ is a sieve [12], if for every pair of vertices $x, y \in X$ and every pair of different $\mathcal{C}-$edges $C, C' \in \Sigma$ the following implication holds:

$$\{x, y\} \in C \cap C' \Rightarrow \{x, y\} \in \mathcal{D}.$$

The maximum cardinality of a sieve of a hypergraph $\mathcal{H}$ is the sieve-number $s(\mathcal{H})$.

The authors proved for $\mathcal{H} = \mathcal{KC}(n; 3, 3)$, $\chi(\mathcal{H}) = 2$, and $\bar{\chi}(\mathcal{H}) = n/2$ when n is even and $\bar{\chi}(\mathcal{H}) = \frac{n-1}{2}$ when n is odd and produces colorings for all feasible values with no gaps [4,10].

It should be noted that $\bar{\chi}(H) = n - s$ or $n - s - 1$ satisfies the constraints of Theorem 1 below found in [3] since the sieve-number for $s(\mathcal{KC}(n; r, r))$ is $\lfloor \frac{n}{r-1} \rfloor$.

Theorem 1. *If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a circular mixed hypergraph, then $n - s(\mathcal{H}) - 2 \leq \bar{\chi}(\mathcal{H}) \leq n - s(\mathcal{H}) + 2$.*

Moreover, the following [3] implies the graphs in question have $\bar{\chi}(\mathcal{H}) \leq n - s - 1$ or $\bar{\chi}(\mathcal{H}) \leq n - s$.

Theorem 2. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a reduced colorable circular mixed hypergraph with n vertices and sieve number s. Then $\bar{\chi}(\mathcal{H}) \leq n - s$, if $0 \leq n - s$; $\bar{\chi}(\mathcal{H}) \leq n - s + 1$, if $s = 3$; and $\bar{\chi}(\mathcal{H}) \leq n - s + 2$, if $s \geq 4$. The upper bound is sharp for $0 \leq s \leq 3$.*

Corollary 1. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a circular mixed hypergraph with n vertices and sieve number s, $s \geq 3$. Let $\Sigma = \{C_1, C_2, ..., C_s\}$ be a sieve of $\mathcal{H}$. It follows: If $C_s \cap C_1 = \emptyset$ then $\bar{\chi}(\mathcal{H}) \leq n - s$, and if $C_{s-1} \cap C_1 = \emptyset$ then $\bar{\chi}(\mathcal{H})) \leq n - s + 1$, otherwise $\bar{\chi}(\mathcal{H})) \leq n - s$.*

Theorem 3. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a reduced circular mixed hypergraph with n vertices, $|C| \geq 5$ for all $C \in \mathcal{C}$. Then $\bar{\chi}(\mathcal{H}) \leq n - s + 1$ if and only if $\mathcal{C} = \Sigma \cup \mathcal{C}' = \emptyset$, where:*

(i) *$\Sigma = C_1, C_2, \ldots, C_s$ is a maximum sieve such that $1 \leq |C_i \cap C_{i+1}| \leq 2$ (and $C_i \cap C_{i+2} = \emptyset$) for all $1 \leq i \leq s, s \geq 3$ (indices mod s), and*

(ii) *Each C-edge $C \in \mathcal{C}'$ has the property: There exist two C-edges $C_i, C_{i+1} \in \Sigma$ with two common vertices, say u and u^+, such that either $u^+ \notin C$ and $C_i \setminus u^+ \subset C$ or $u \notin C$ and $C_{i+1} \setminus u \subset C$, and there is no other C-edge of $\mathcal{C}'$ containing precisely one of the vertices u and u^+.*

For our hypergraphs (ii) fails with $r \geq 5$.

3. Coloring Algorithm

The algorithm below will properly color the vertices of $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r)$. Within the algorithm, a sieve Σ is constructed of a circular mixed hypergraph. In our family of circular mixed hypergraphs, the intersection of two different C-edges of a sieve can only be empty, a common vertex, or two common vertices forming a D-edge. Hence, the intersection of two arbitrary C-edges is called good if and only if this intersection is empty, or a single vertex, or two vertices forming a D-edge. Otherwise, the intersection of two arbitrary C-edges is called bad. Due to our particular family of graphs, we will always have a good intersection with a single vertex.

Algorithm (Simplified and modified from Procedure M from [3])
INPUT: $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r) = \mathcal{KC}(n; r, r), n = |X|, n > r, X = \{x_0, x_1, \ldots, x_{n-1}\}$
OUTPUT: A proper coloring $c = (c(x_0), c(x_1), \ldots, c(x_{n-1}))$ of $\mathcal{KC}(n; r, r)$
Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a circular mixed hypergraph and C_0 one of its $\mathcal{C}$-edges.

1. *Construction of a maximum sieve Σ through C_0. Let $\Sigma = \{C_0\}$ where $C_0 = \{x_0, x_1, \ldots, x_{r-1}\}$.* Choose the $\mathcal{C}$-edge C_1 nearest to C_0, having a "good intersection." So, $C_1 = \{x_{r-1}, \ldots, x_{2r-2}\}$. Let C_2 have smallest distance from C_1 (measured in the cyclic order of the host cycle). Choose C_3 nearest to C_2, etc. Thus a maximum sieve $\Sigma = \{C_0, C_1, \ldots, C_{s-1}\}$ is obtained so that no new C_s can be found.
2. *Assigning colors to some vertices of* Σ. The vertices of the $\mathcal{C}$-edge C_i with r vertices are denoted by $x_0^i, x_1^i, \ldots, x_{r-1}^i$ according to the cyclic order of the host cycle. Next we assign colors to some vertices of the $\mathcal{C}$-edges of Σ in the following way, where the coloring is denoted by c. Assign color 1 to the last two vertices x_{r-1}^0 and x_{r-2}^0. We then color some vertices of $C_1, \ldots, C_{s-1}$ in the following way: Color x_{r-2}^i and x_{r-1}^i a new color $c(x_{r-1}^{i-1}) + 1$.
3. Fixing the vertices to satisfy the $\mathcal{C}$ condition. In step 2, we start coloring with the vertex $x_{r-1}^0 \in C_0$, proceed along the host cycle and end with coloring of the vertices of C_{s-1}.
   (a) If $\Sigma = X$ and $x_{r-1}^{s-1} = x_0^0$, all $\mathcal{C}$-conditions are satisfied.
   (b) If $\Sigma = X$ and $x_{r-1}^{s-1} \neq x_0^0$, assign $c(x_0^0 = x_0) = 1$ and then all $\mathcal{C}$-conditions are satisfied.
   (c) If $\Sigma \neq X$ and $s - 1 = 0$, assign 1 to x_0; otherwise, assign $c(x_{r-1}^{s-1}) + 1$ to x_0 and x_{n-1}.
4. Assign pairwise different colors to the remaining vertices of X. The graph is now colored with $\bar{\chi}(H) = n - s$ or $n - s - 1$ as we will see in the following proofs.
5. Downshifting colors: For any number of colors $2 \leq k < \bar{\chi}(H)$ do the following:
   (a) If more than $s + 1$ colors are used, recolor all vertices with the largest color, a color 1 less.
   (b) Repeat the above until the desired number of colors is used until the number of colors used is $s + 1$.

(c) If fewer than $s+1$ colors is desired, begin with C_1 and color x^1_{r-1} and x^1_{r-2} both 1. Then for all vertices assigned $s+1$, recolor these vertices 2. s colors are now used.

(d) If fewer than s colors is desired, begin with C_2 and color x^2_{r-1} and x^2_{r-2} both 1. Then for all vertices not assigned 1 or 2, recolor these vertices 1 less than the color already assigned. Repeat this process with C_3, C_4, and so on until the desired number of colors is met.

6. End.

Lemma 1. *The coloring algorithm produces a proper coloring for $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r) = \mathcal{KC}(n; r, r)$, $n > r > 3$.*

Proof. For any edge containing vertices between x^0_{r-1} and x^{s-1}_{r-2} (step 2) it is easily seen these edges are properly colored with a common color. Any sequential vertices colored this way (e.g., x^i_{r-2} and x^i_{r-1}) will properly color all edges extending a distance of $r-2$ in either direction with a common color. Steps 3b and 3c ensure any remaining edges satisfy the $\mathcal{C}$-condition since otherwise the sieve would not be maximum. As $r \geq 4$, there is at least one vertex between x^i_{r-1} and x^{i+1}_{r-2} colored a different color satisfying the $\mathcal{D}$-condition. □

Lemma 2. *For $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r)$, $r > 3$, the sieve number $s(\mathcal{H}) = \lfloor \frac{n}{r-1} \rfloor$.*

Proof. As there are no $\mathcal{D}$-edges of size 2, the intersection of consecutive $\mathcal{C}$-edges can be chosen to be 0 or 1 vertex. To maximize the number of edges in Σ we choose the intersection with 1 vertex each time. Thus, there are $r-1$ vertices in C_{i-1} that are not in C_i. This creates a partition of the vertices in Σ each of size $r-1$. Any sieve that contains intermediate $\mathcal{C}$-edges with an empty intersection must contain the same or fewer $\mathcal{C}$-edges. With this construction, there must be fewer than $r-2$ vertices not in Σ so $s(\mathcal{H}) = \lfloor \frac{n}{r-1} \rfloor$. □

Lemma 3. *If $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r) = \mathcal{KC}(n; r, r)$, $n > r > 3$ then $\bar{\chi}(H) = n - s$ if $\Sigma = X$ and $C_{s-1} \cap C_0 = \{x_0\}$ and $\bar{\chi}(\mathcal{H}) = n - s - 1$ if $C_{s-1} \cap C_0 = \emptyset$.*

Proof. Let $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r) = \mathcal{KC}(n; r, r)$ with vertices $x_0, x_1, \ldots, x_{n-1}$. Using the coloring algorithm, if $C_{s-1} \cap C_0 = \emptyset$ each element of the sieve has a pair of vertices that share a color. Additionally, a color needs to be repeated to get the edges that are not properly colored (in Step 3). Thus $\bar{\chi}(H) = n - s - 1$. If $C_s \cap C_1 = \{x_0\}$, then each element of the sieve has a pair of vertices that share a color. □

Theorem 4. *If $\mathcal{H} = (X, \mathcal{C}_r, \mathcal{D}_r) = \mathcal{KC}(n; r, r)$, $n > r > 3$, the feasible set is $F(\mathcal{H}) = \{2, \ldots, \bar{\chi}(H)\}$ where $\bar{\chi}(H) = n - s$ or $\bar{\chi}(H) = n - s - 1$.*

Proof. Together the lemmas and algorithm prove our theorem as we are able to color the graph any amount of colors from 2 to $\bar{\chi}(H)$. □

Additionally, it should be noted that the algorithm has a linear runtime $\mathcal{O}(n)$ as the worst-case scenario will have a run-time of kn where k is an integer dependent on the number of colors required.

4. Conclusions

Originally, the problems in [4] and this paper were to be tools to work toward coloring star systems. For the most part, Darijani and Pike [1] have answered this question. We now hope these results can be used for further research. Of particular interest would be coloring $\mathcal{KC}(n; r, 2)$, the class of complete, circular, mixed hypergraphs with $\mathcal{C}$-edges of size r, and $\mathcal{D}$-edges of size 2. This class of graphs would disallow the sequential coloring that was so beneficial in the coloring algorithm presented in this paper.

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