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Fuzzy Inner Product Space: Literature Review and a New Approach

Lorena Popa and Lavinia Sida *

Department of Mathematics and Computer Science, Aurel Vlaicu University of Arad, Elena Dragoi 2, RO-310330 Arad, Romania; lorena.popa@uav.ro
* Correspondence: lavinia.sida@uav.ro

Abstract: The aim of this paper is to provide a suitable definition for the concept of fuzzy inner product space. In order to achieve this, we firstly focused on various approaches from the already-existent literature. Due to the emergence of various studies on fuzzy inner product spaces, it is necessary to make a comprehensive overview of the published papers on the aforementioned subject in order to facilitate subsequent research. Then we considered another approach to the notion of fuzzy inner product starting from P. Majundar and S.K. Samanta's definition. In fact, we changed their definition and we proved some new properties of the fuzzy inner product function. We also proved that this fuzzy inner product generates a fuzzy norm of the type Nădăban-Dzitac. Finally, some challenges are given.

Keywords: fuzzy Hilbert space; fuzzy inner product; fuzzy norm

MSC: 46A16; 46S40

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1. Introduction

The research papers of A.K. Katsaras [1,2] laid the foundations of the fuzzy functional analysis. Moreover, he was the first one who introduced the concept of a fuzzy norm. This concept has amassed great interest among mathematicians. Thus, in 1992, C. Felbin [3] introduced a new idea of a fuzzy norm in a linear space by associating a real fuzzy number to each element of the linear space. In 2003, T. Bag and S.K. Samanta [4] put forward a new concept of a fuzzy norm, which was a fuzzy set on $X \times \mathbb{R}$. New fuzzy norm concepts were later introduced by R. Saadati and S.M. Vaezpour [5], C. Alegre and S.T. Romaguera [6], R. Ameri [7], I. Goleţ [8], A.K. Mirmostafaee [9]. In this paper we use the definition introduced by S. Nădăban and I. Dzitac [10].

Hilbert spaces lay at the core of functional analysis. They frequently and naturally appear in the fields of Mathematics and Physics, being indispensable in the theory of differential equations, quantum mechanics, quantum logic, quantum computing, the Fourier analysis (with applications in the signal theory). Therefore, many mathematicians have focused on finding an adequate definition of the fuzzy inner product space. Although there are many research papers focused on the concept of a fuzzy norm and its diverse applications, there are few papers which study the concept of a fuzzy inner product.

Even though there are few results, we consider that every breakthrough has been an important one. Moreover, we are certain that there exists a correct definition of the fuzzy inner product that when discovered, will not only generate a worldwide consensus on this subject but trigger countless applications in various fields.

The importance of this subject has led us to write this paper and we have tried to further the knowledge on the matter.

Furthermore, we wish to mention the main results of the already existent literature.

R. Biswas in [11] defined the fuzzy inner product of elements in a linear space and two years later J.K. Kohli and R. Kumar altered the Biswas's definition of inner product space [12]. In fact, they showed that the definition of a fuzzy inner product space in terms of the conjugate of a vector is redundant and that those definitions are only restricted to the real linear spaces. They also introduced the fuzzy co-inner product spaces and the fuzzy co-norm functions in their paper.

Two years later, in 1995, Eui-Whan Cho, Young-Key Kim and Chae-Seob Shin introduced and defined in [13] a fuzzy semi-inner-product space and investigated some properties of this fuzzy semi inner product space, those definitions are not restricted to the real linear spaces.

In 2008, P. Majumdar and S.K. Samanta [14] succeeded in taking the first step towards finding a reliable definition of a fuzzy inner product space. From their definition we can identify a serious problem in regard to finding a new reliable definition for the fuzzy inner product. The classical inequality Cauchy-Schwartz cannot be obtained by applying the other axioms and thus had to be introduced itself as an axiom (axiom (FIP2)).

In 2009, M. Goudarzi, S.M. Vaezpour and R. Saadati [15] introduced the concept of intuitionistic fuzzy inner product space. In this context, the Cauchy-Schwartz inequality, the Pythagorean Theorem and some convergence theorems were established.

In the same year, M. Goudarzi and S.M. Vaezpour [16] alter the definition of the fuzzy inner product space and prove several interesting results which take place in each fuzzy inner product space. More specifically, they introduced the notion of a fuzzy Hilbert space and deduce a fuzzy version of Riesz representation theorem.

In 2013, S. Mukherjee and T.Bag [17] amends the definition put forward by M.Goudarzi and S.M. Vaezpour by discarding the (FI-6) condition and by enacting minor changes to the (FI-4) and (FI-5) conditions.

In 2010, A. Hasankhani, A. Nazari and M. Saheli [18] introduced a new concept of a fuzzy Hilbert space. This concept is entirely different from the previous ones as this fuzzy inner product generates a new fuzzy norm of type Felbin.

The disadvantage of this definition is that only linear spaces over $\mathbb{R}$ can be considered. Another disadvantage is the difficulty of working with real fuzzy numbers.

In the subsequent years, many papers addressing this theme were published (see [7,19–24]).

In 2016, M. Saheli and S.Khajepour Gelousalar [25] modified the definition of the fuzzy inner product space and proved some properties of the new fuzzy inner product space.

Also, in 2016, Z. Solimani and B. Daraby [26] slightly altered the definition of a fuzzy scalar product introduced in [18] by changing the (IP2) condition and merging the (IP4) and (IP5).

In 2017, E. Mostofian, M. Azhini and A. Bodaghi [27] presented two new concepts of fuzzy inner product spaces and investigated some of basic properties of these spaces.

This paper is organized as follows—in Section 2 we make a literature review. Such an approach is deemed useful for the readers as it would enable them to better understand the evolution of the fuzzy inner product space concepts. Thus, this section can constitute a starting point for other mathematicians interested in this subject. In Section 3 we introduce a new definition of the fuzzy inner product space starting from P. Majumdar and S.K. Samanta's definition [14]. In fact, we modified the P. Majumdar and S.K. Samanta's definition of inner product space and we introduced and proved some new properties of the fuzzy inner product function. This paper ends up with some conclusions and future works in Section 4.

2. Preliminaries

Definition 1. *[10] Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty]$ is called a fuzzy norm on X if it satisfies:*

(N1) $N(x, 0) = 0, (\forall)x \in X$;

(N2) $[N(x, t) = 1, (\forall)t > 0]$ *iff* $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall)x \in X, (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, t), (\forall)x, y \in X, (\forall)t, s \geq 0$;

(N5) *$(\forall)x \in X, N(x, \cdot)$ is left continuous and $\lim_{t\to\infty} N(x,t) = 1$.*

The triplet $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Definition 2. *[14] A fuzzy inner product space (FIP-space) is a pair (X, P), where X is a linear space over $\mathbb{C}$ and P is a fuzzy set in $X \times X \times \mathbb{C}$ s.t.*

(FIP1) For $s, t \in \mathbb{C}$, $P(x+y, z, |t|+|s|) \geq \min\{P(x,z,|t|), P(y,z,|s|)\}$;
(FIP2) For $s, t \in \mathbb{C}$, $P(x, y, |st|) \geq \min\{P(x,x,|s|^2), P(y,y,|t|^2)\}$;
(FIP3) For $t \in \mathbb{C}$, $P(x,y,t) = P(y,x,\bar{t})$;
(FIP4) $P(\alpha x, y, t) = P\left(x, y, \frac{t}{|\alpha|}\right), t \in \mathbb{C}, \alpha \in \mathbb{C}^*$;
(FIP5) $P(x,x,t) = 0, (\forall)t \in \mathbb{C} \setminus \mathbb{R}^+$;
(FIP6) $[P(x,x,t) = 1, (\forall)t > 0]$ iff $x = 0$;
(FIP7) $P(x,x,\cdot) : \mathbb{R} \to [0,1]$ is a monotonic non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} P(x,x,t) = 1$.

P will be called the fuzzy inner product on X.

Definition 3. *[15] A fuzzy inner product space (FIP-space) is a triplet $(X; P; *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set on $X^2 \times \mathbb{R}$ satisfying the following conditions for every $x; y; z \in X$ and $t \in \mathbb{R}$.*

(FIP1) $P(x,y,0) = 0$;
(FIP2) $P(x,y,t) = P(y,x,t)$;
(FIP3) $P(x,x,t) = H(t), \forall t \in \mathbb{R}$ iff $x = 0$, where $H(t) = \begin{cases} 1, & if\ t > 0 \\ 0, & if\ t \leq 0 \end{cases}$;
(FIP4) For any real number α, $P(\alpha x, y, t) = \begin{cases} P(x, y, \frac{t}{\alpha}), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P(x, y, \frac{t}{-\alpha}), & if\ \alpha < 0 \end{cases}$
(FIP5) $\sup_{s+r=t} \big(P(x,z,s) * P(y,z,r)\big) = P(x+y, z, t)$;
(FIP6) $P(x,y,\cdot) : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$;
(FIP7) $\lim_{t\to\infty} P(x,y,t) = 1$.

Definition 4. *[16] A fuzzy inner product space (FIP - space) is a triplet $(X, P, *)$, where X is a real linear space, $*$ is a continuous t-norm and P is a fuzzy set in $X \times X \times \mathbb{R}$ s.t. the following conditions hold for every $x, y, z \in X$ and $s, t, r \in \mathbb{R}$.*

(FI-1) $P(x,x,0) = 0$ and $P(x,x,t) > 0$, $(\forall)t > 0$;
(FI-2) $P(x,x,t) \neq H(t)$ for same $t \in \mathbb{R}$ iff $x \neq 0$;
(FI-3) $P(x,y,t) = P(y,x,t)$;
(FI-4) For any real number α, $P(\alpha x, y, t) = \begin{cases} P(x, y, \frac{t}{\alpha}), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P(x, y, \frac{t}{-\alpha}), & if\ \alpha < 0 \end{cases}$
(FI-5) $\sup_{s+r=t} \big(P(x,z,s) * P(y,z,r)\big) = P(x+y, z, t)$;
(FI-6) $P(x,y,\cdot) : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$;
(FI-7) $\lim_{t\to\infty} P(x,y,t) = 1$.

Definition 5. *[14] Let X be a linear space over $\mathbb{R}$. A fuzzy set P in $X \times X \times \mathbb{R}$ is called fuzzy real inner product on X if $(\forall)x, y, z \in X$ and $t \in \mathbb{R}$, the following conditions hold:*

(FI-1) $P(x,x,0) = 0$, $(\forall)t < 0$;
(FI-2) $\big[P(x,x,t) = 1, (\forall)t > 0\big]$ iff $x = 0$;
(FI-3) $P(x,y,t) = P(y,x,t)$;

(FI-4) $P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases}$

(FI-5) $P(x+y, z, t+s) \geq \min\{P(x,z,t), P(y,z,s)\}$;

(FI-6) $\lim\limits_{t\to\infty} P(x,y,t) = 1$.

The pair (X, P) is called fuzzy real inner space.

In order to present the definition of A. Hasankhani, A. Nazari and M. Saheli, we firstly need to define some concepts.

Definition 6. *[28] A fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0,1]$, with the following properties:*

(1) x is convex, that is, $x(t) \geq \min\{x(s), x(r)\}$ for $s \leq t \leq r$;

(2) x is normal, that is, $(\exists)t_0 \in \mathbb{R} : x(t_0) = 1$;

(3) x is upper semicontinuous, that is, $(\forall)t \in \mathbb{R}, (\forall)\alpha \in [0,1] : x(t) < \alpha, (\exists)\delta > 0\, s.t. |s-t| < \delta \Rightarrow x(s) < \alpha$

is called fuzzy real number. We denote by $\mathfrak{F}(\mathbb{R})$ the set of all fuzzy real numbers.

Definition 7. *[29] The arithmetic operation $+, -, \cdot, /$ on $\mathfrak{F}(\mathbb{R})$ are defined by:*

$$(x+y)(t) = \bigvee_{s\in\mathbb{R}} \min\left\{x(s), y(t-s)\right\},\ (\forall)t \in \mathbb{R};$$
$$(x-y)(t) = \bigvee_{s\in\mathbb{R}} \min\left\{x(s), y(s-t)\right\},\ (\forall)t \in \mathbb{R};$$
$$(xy)(t) = \bigvee_{s\in\mathbb{R}^*} \min\left\{x(s), y(t/s)\right\},\ (\forall)t \in \mathbb{R};$$
$$(x/y)(t) = \bigvee_{s\in\mathbb{R}} \min\left\{x(ts), y(s)\right\},\ (\forall)t \in \mathbb{R}.$$

Remark 1. *Let $x \in \mathfrak{F}(\mathbb{R})$ and $\alpha \in (0,1]$. The α-level sets $[x]_\alpha = \left\{t \in \mathbb{R} : x(t) \geq \alpha\right\}$ are closed intervals $\left[x_\alpha^-, x_\alpha^+\right]$.*

Definition 8. *[18] Let X be a linear space over $\mathbb{R}$. A fuzzy inner product on X is a mapping $< \cdot, \cdot >: X \times X \to \mathfrak{F}(\mathbb{R})$ s.t. $(\forall)x, y, z \in X, (\forall)r \in \mathbb{R}$, we have:*

(IP1) $< x+y, z > = < x, z > \oplus < y, z >$;

(IP2) $< rx, y > = \tilde{r} < x, y >$, *where* $\tilde{r} = \begin{cases} 1, & if\ t = r \\ 0, & if\ t \neq r \end{cases}$;

(IP3) $< x, y > = < y, x >$;

(IP4) $< x, x > \geq 0$;

(IP5) $\inf\limits_{\alpha\in(0,1]} < x, x >_\alpha^- = 0$ *if* $x \neq 0$;

(IP6) $< x, x > = \tilde{0}$ *iff* $x = 0$.

The pair $(X, \langle\cdot,\cdot\rangle)$ is called fuzzy inner product space.

Definition 9. *[25] A fuzzy inner product space is a triplet $(X, P, *)$, where X is a fuzzy set in $X \times X \times \mathbb{R}$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in \mathbb{R}$:*

(FI1) $P(x, y, 0) = 0$;

(FI2) $P(x, y, t) = P(y, x, t)$;

(FI3) $\left[P(x, x, t) = 1,\ (\forall)t > 0\right]$ *iff* $x = 0$;

(FI4) $(\forall)\alpha \in \mathbb{R}, t \neq 0, \quad P(\alpha x, y, t) = \begin{cases} P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha > 0 \\ H(t), & if\ \alpha = 0; \\ 1 - P\left(x, y, \frac{t}{\alpha}\right), & if\ \alpha < 0 \end{cases}$

(FI5) $P(x, z, t) * P(y, z, s) \leq P(x+y, z, t+s),\ (\forall)t, s > 0$;

(FI6) $\lim\limits_{t\to\infty} P(x, y, t) = 1$.

Definition 10. *[26] Let X be a linear space over $\mathbb{R}$. A fuzzy inner product on X is a mapping $< \cdot, \cdot >: X \times X \to \mathfrak{F}^*(\mathbb{R})$, where $\mathfrak{F}^*(\mathbb{R}) = \{\eta \in \mathfrak{F}(\mathbb{R}) : \eta(t) = 0 \text{ if } t < 0\}$, with the following properties $(\forall)x, y, z \in X, (\forall)r \in \mathbb{R}$:*

(FIP1) $< x + y, z >=< x, z > \oplus < y, z >$;
(FIP2) $< rx, y >=| \tilde{r} |< x, y >$;
(FIP3) $< x, y >=< y, x >$;
(FIP4) $x \neq 0 \Rightarrow < x, x > (t) = 0$, $(\forall)t < 0$;
(FIP5) $< x, x >= \tilde{0}$ *iff* $x = 0$.

The pair $(X, < \cdot, \cdot >)$ is called fuzzy inner product space.

3. A New Approach for Fuzzy Inner Product Space

We will denote by $\mathbb{C}$ the space of complex numbers and we will denote by $\mathbb{R}_+^*$ the set of all strict positive real numbers.

Definition 11. *Let H be a linear space over $\mathbb{C}$. A fuzzy set P in $H \times H \times \mathbb{C}$ is called a fuzzy inner product on H if it satisfies:*

(FIP1) $P(x, x, v) = 0, (\forall)x \in H, (\forall)v \in \mathbb{C} \setminus \mathbb{R}_+^*$;
(FIP2) $P(x, x, t) = 1, (\forall)t \in \mathbb{R}_+^*$ *if and only if* $x = 0$;
(FIP3) $P(\alpha x, y, v) = P\left(x, y, \frac{v}{|\alpha|}\right), (\forall)x, y \in H, (\forall)v \in \mathbb{C}, (\forall)\alpha \in \mathbb{C}^*$;
(FIP4) $P(x, y, v) = P(y, x, \overline{v}), (\forall)x, y \in H, (\forall)v \in \mathbb{C}$;
(FIP5) $P(x + y, z, v + w) \geq \min\{P(x, z, v), P(y, z, w)\}, (\forall)x, y, z \in H, (\forall)v, w \in \mathbb{C}$;
(FIP6) $P(x, x, \cdot) : \mathbb{R}_+ \to [0, 1], (\forall)x \in H$ *is left continuous and* $\lim_{t \to \infty} P(x, x, t) = 1$;
(FIP7) $P(x, y, st) \geq \min\{P(x, x, s^2), P(y, y, t^2)\}, (\forall)x, y \in H, (\forall)s, t \in \mathbb{R}_+^*$.

The pair (H, P) will be called fuzzy inner product space.

Example 1. *Let H be a linear space over $\mathbb{C}$ and $< \cdot, \cdot >: H \times H \to \mathbb{C}$ be an inner product. Then $P : H \times H \times \mathbb{C} \to [0, 1]$,*

$$P(x, y, s) = \begin{cases} \frac{s}{s + |<x,y>|}, & \text{if } s \in \mathbb{R}_+^* \\ 0, & \text{if } s \in \mathbb{C} \setminus \mathbb{R}_+^* \end{cases}$$

is a fuzzy inner product on H.

Let verify now the conditions from the definition.

(FIP1) $P(x, x, v) = 0, (\forall)x \in H, (\forall)v \in \mathbb{C} \setminus \mathbb{R}_+^*$ is is obvious from definition of P.
(FIP2) $P(x, x, t) = 1, (\forall)v \in \mathbb{R}_+^* \Leftrightarrow t + |< x, x >| = t, (\forall)t > 0 \Leftrightarrow |< x, x >| = 0 \Leftrightarrow x = 0$.
(FIP3) $P(\alpha x, y, v) = P\left(x, y, \frac{v}{|\alpha|}\right), (\forall)x, y \in H, (\forall)v \in \mathbb{C}, (\forall)\, \alpha \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, than

$$P(\alpha x, y, v) = \frac{v}{v + |< \alpha x, y >|} = \frac{v}{v + | \alpha | \cdot |< x, y >|} = \frac{\frac{v}{|\alpha|}}{\frac{v}{|\alpha|} + |< x, y >|} = P\left(x, y, \frac{v}{|\alpha|}\right).$$

(FIP4) $P(x, y, v) = P(y, x, \overline{v}), (\forall)x, y \in H, (\forall)v \in \mathbb{C}$ is obvious for $v \in \mathbb{C} \setminus \mathbb{R}_+^*$.
If $v \in \mathbb{R}_+^*$, then $v = \bar{v}$ and

$$P(x, y, v) = \frac{v}{v + |< x, y >|} = \frac{\overline{v}}{\overline{v} + |< y, x >|} = P(y, x, \overline{v})$$

(FIP5) $P(x + y, z, v + w) \geq \min\{P(x, z, v), P(y, z, w)\}, (\forall)x, y, z \in H, (\forall)v, w \in \mathbb{C}$.
If at least one of v and w is from $\mathbb{C} \setminus \mathbb{R}_+^*$, then the result is obvious.

If $v, w \in \mathbb{R}_+^*$, let us assume without loss of generality that $P(x,z,v) \leq P(y,z,w)$. Then

$$\frac{v}{v+ |< x,z >|} \leq \frac{w}{w+ |< y,z >|} \Rightarrow$$
$$\Rightarrow \frac{v+ |< x,z >|}{v} \geq \frac{w+ |< y,z >|}{w} \Rightarrow$$
$$\Rightarrow 1 + \frac{|< x,z >|}{v} \geq 1 + \frac{|< y,z >|}{w} \Rightarrow$$
$$\Rightarrow \frac{|< x,z >|}{v} \geq \frac{|< y,z >|}{w} \Rightarrow$$
$$\Rightarrow \frac{w}{v} |< x,z >| \geq |< y,z >| \Rightarrow$$
$$\Rightarrow |< x,z >| + \frac{w}{v} |< x,z >| \geq |< x,z >| + |< y,z >| \Rightarrow$$
$$\Rightarrow \frac{v+w}{v} |< x,z >| \geq |< x+y,z >| \Rightarrow$$
$$\Rightarrow \frac{|< x,z >|}{v} \geq \frac{|< x+y,z >|}{v+w} \Rightarrow$$
$$\Rightarrow \frac{|< x,z >|}{v} + 1 \geq \frac{|< x+y,z >|}{v+w} + 1 \Rightarrow$$
$$\Rightarrow \frac{v+ |< x,z >|}{v} \geq \frac{(v+w)+ |< x+y,z >|}{v+w} \Rightarrow$$
$$\Rightarrow \frac{v}{v+ |< x,z >|} \leq \frac{v+w}{(v+w)+ |< x+y,z >|} \Rightarrow$$
$$\Rightarrow P(x,z,v) \leq P(x+y,z,v+w)$$
$$\Rightarrow P(x+y,z,v+w) \geq \min\{P(x,z,v), P(y,z,w)\}, (\forall)x,y,z \in H, (\forall)v,w \in \mathbb{C}.$$

(FIP6) $P(x,x,\cdot) : \mathbb{R}_+ \to [0,1], (\forall)x \in H$ is left continuous function and $\lim_{t\to\infty} P(x,x,t) = 1$.

$$\lim_{t\to\infty} P(x,x,t) = \lim_{t\to\infty} \frac{t}{t+ |< x,x >|} = \lim_{t\to\infty} \frac{t}{t(1+\frac{|<x,x>|}{t})} = 1.$$

$F(x,x,\cdot)$ is left continuous in $t > 0$ follows from definition.

(FIP7) $P(x,y,st) \geq \min\{P(x,x,s^2), P(y,y,t^2)\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*$. If at least one of s and t is from $\mathbb{C} \setminus \mathbb{R}_+^*$, then the result is obvious.
If $s,t \in \mathbb{R}_+^*$, let us assume without loss of generality that $P(x,x,s^2) \leq P(y,y,t^2)$. Then

$$\frac{s^2}{s^2+ |< x,x >|} \leq \frac{t^2}{t^2+ |< y,y >|} \Leftrightarrow$$
$$t^2 |< x,x >| \geq s^2 |< y,y >| .$$

Thus by Cauchy–Schwartz inequality we obtain

$$s |< x,y >| \leq \sqrt{|< x,x >|} \cdot s\sqrt{|< y,y >|} \leq \sqrt{|< x,x >|} \cdot t\sqrt{|< x,x >|} = t |< x,x >| \Rightarrow$$
$$\Rightarrow s^2 |< x,y >| \leq st |< x,x >| \Rightarrow$$
$$\Rightarrow s^3 t + s^2 |< x,y >| \leq s^3 t + st |< x,x >| \Rightarrow$$
$$\Rightarrow s^2(st+ |< x,y >|) \leq st(s^2+ |< x,x >|) \Rightarrow$$
$$\frac{s^2}{s^2+ |< x,x >|} \leq \frac{st}{st+ |< x,y >|} \Rightarrow$$
$$\Rightarrow P(x,x,s^2) \leq P(x,y,st) \Rightarrow$$
$$\Rightarrow P(x,y,st) \geq \min\Big\{P(x,x,s^2), P(y,y,t^2)\Big\}, (\forall)x,y \in H, (\forall)s,t \in \mathbb{R}_+^*.$$

Proposition 1. *For $x, y \in H$, $v \in \mathbb{C}$ and $\alpha \in \mathbb{C}$ we have*

$$P(x, \alpha y, v) = P\left(x, y, \frac{v}{|\alpha|}\right).$$

Proof. From (FIP3) and (FIP4) it follows $P(x, \alpha y, v) = P(\alpha y, x, \overline{v}) = P\left(y, x, \frac{\overline{v}}{|\alpha|}\right) = P\left(x, y, \frac{\overline{\overline{v}}}{|\alpha|}\right) = P\left(x, y, \frac{v}{|\alpha|}\right)$. □

Proposition 2. *For $x \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(x, 0, v) = 1.$$

Proof. From (FIP3) and (FIP6) it follows

$$P(x, 0, v) = P(x, 0, 2nv) = P(x, x - x, nv + nv) \geq \min\{P(x, x, nv), P(x, x, nv)\} =$$

$$P(x, x, nv) \overset{n \to \infty}{\longrightarrow} 1$$

So $P(x, 0, v) = 1$. □

Proposition 3. *For $y \in H$, $v \in \mathbb{R}_+^*$ we have*

$$P(0, y, v) = 1.$$

Proposition 4. $P(x, y, \cdot) : \mathbb{R}_+ \to [0, 1]$ *is a monotonic non-decreasing function on $\mathbb{R}_+$, $(\forall) x, y \in H$.*

Proof. Let $s, t \in \mathbb{R}_+$, $s \leq t$. Then $(\exists) p$ such that $t = s + p$ and

$$P(x, y, t) = P(x + 0, y, s + p) \geq \min\{P(x, y, s), P(0, y, p)\} = P(x, y, s).$$

Hence $P(x, y, s) \leq P(x, y, t)$ for $s \leq t$. □

Corollary 1. $P(x, y, st) \geq \min\{P(x, y, s^2), P(x, y, t^2)\}, (\forall) x, y \in H, (\forall) s, t \in \mathbb{R}_+^*$.

Proof. Let $s, t \in \mathbb{R}_+$, $s \leq t$. Then

$$P(x, y, s^2) \leq P(x, y, st) \leq P(x, y, t^2).$$

Hence $P(x, y, st) \geq \min\{P(x, y, s^2), P(x, y, t^2)\}$.
Let now $s, t \in \mathbb{R}_+$, $t \leq s$. Then

$$P(x, y, t^2) \leq P(x, y, st) \leq P(x, y, s^2).$$

Hence $P(x, y, st) \geq \min\{P(x, y, s^2), P(x, y, t^2)\}$. □

Proposition 5. $P(x, y, v) \geq \min\{P(x, y - z, v), P(x, y + z, v)\}, (\forall) x, y, z \in H, (\forall) v \in \mathbb{C}$.

Proof.

$$P(x, y, v) = P(x, 2y, 2v) = P(x, y + z + y - z, v + v) \geq \min\{P(x, y + z, v), P(x, y - z, v)\}.$$

Hence $P(x, y, v) \geq \min\{P(x, y - z, v), P(x, y + z, v)\}$. □

Theorem 1. *If (H, P) be a fuzzy inner product space, then $N : X \times [0, \infty) \to [0, 1]$ defined by*

$$N(x, t) = P(x, x, t^2)$$

is a fuzzy norm on X.

Proof. ***(N1)*** $N(x,0) = P(x,x,0) = 0, (\forall)x \in H$ from **(FIP1)**;

(N2) $[N(x,t) = 1, (\forall)t > 0] \Leftrightarrow [P(x,x,t^2) = 1, (\forall)t > 0] \Leftrightarrow x = 0$ from **(FIP2)**;

(N3) $N(\lambda x,t) = P(\lambda x, \lambda x, t^2) = P\left(x, \lambda x, \frac{t^2}{|\lambda|}\right) = P\left(\lambda x, x, \frac{\bar{t^2}}{|\lambda|}\right) = P\left(\lambda x, x, \frac{t^2}{|\lambda|}\right) = P\left(x, x, \frac{t^2}{|\lambda|^2}\right) == N\left(x, \frac{t}{|\lambda|}\right), (\forall)t \geq 0, (\forall)\lambda \in \mathbb{K}^*$;

(N4) $N(x+t, t+s) \geq \min\{N(x,t), N(y,s)\}, (\forall)x,y \in H, (\forall)t,s \geq 0.$

If $t = 0$ or $s = 0$ the previous inequality is obvious. We asume that $t,s > 0$.

$$\begin{aligned}
N(x+y, t+s) &= P\left(x+y, x+y, (t+s)^2\right) = \\
&= P(x+y, x+y, t^2 + s^2 + ts + ts) \geq \\
&\geq P(x, x+y, t^2 + ts) \wedge P(y, x+y, s^2 + ts) \geq \\
&\geq P(x,x,t^2) \wedge P(x,y,ts) \wedge P(y,x,ts) \wedge P(y,y,s^2) = \\
&= P(x,x,t^2) \wedge P(y,y,s^2) = \\
&= \min\{N(x,t), N(y,s)\};
\end{aligned}$$

(N5) From **(FIP6)** it result that $N(x,\cdot)$ is left continuous and $\lim\limits_{t\to\infty} N(x,t) = 1$.

□

4. Conclusions and Future Works

In this paper, we wrote a literature review regarding the diverse approaches of fuzzy inner product space concept, but we also introduced a new approach.

We have thus laid the ground for further research on the problems within the fuzzy Hilbert space theory, searching for analogies in this fuzzy context for the Pythagorean theorem, for the parallelogram law, as well as for other orthogonality problems.

The following step would be to study the linear and bounded operators in a fuzzy Hilbert space. Recently, there have been many important results concerning the linear and bounded operators in a fuzzy Banach space (see [30–34]), fact which motivates us even further to try to achieve this goal.

This research will be followed by other papers in which we will firstly define the concept of an adjoint of a linear and bounded operator on a fuzzy Hilbert space. This concept will allow us to study important classes of operators such as self-adjoint operators, normal operators and unitary operators. We will then follow up with the spectral theory and we will also construct a analytic functional calculus.

Last, but not least, we will study the aforementioned orthogonality in the fuzzy Hilbert space. Thus, this will enable us to observe the properties of the self-adjoint projections and to undertake directly decompositions of the fuzzy Hilbert space.

This paper summarized the current research status of the fuzzy inner product spaces and can thus facilitate researchers in writing their future papers on this topic.

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