

Counting Hamiltonian Cycles in 2-Tiled Graphs

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Abstract: In 1930, Kuratowski showed that $K_{3,3}$ and K_5 are the only two minor-minimal nonplanar graphs. Robertson and Seymour extended finiteness of the set of forbidden minors for any surface. Širáň and Kochol showed that there are infinitely many k -crossing-critical graphs for any $k \geq 2$, even if restricted to simple 3-connected graphs. Recently, 2-crossing-critical graphs have been completely characterized by Bokal, Oporowski, Richter, and Salazar. We present a simplified description of large 2-crossing-critical graphs and use this simplification to count Hamiltonian cycles in such graphs. We generalize this approach to an algorithm counting Hamiltonian cycles in all 2-tiled graphs, thus extending the results of Bodroža-Pantić, Kwong, Doroslovački, and Pantić.

Keywords: crossing number; crossing-critical graph; Hamiltonian cycle

MSC: 05C30; 05C38



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1. Introduction

In 1930, Kuratowski characterized graphs that can be drawn in a plane with no crossings [1]. His research was an opening step for several directions characterizing graph families using forbidden substructures, such as extremal graph theory that forbids any subgraph isomorphic from being in a given graph [2], and the related structural graph theory emanating from forbidding induced subgraphs, for instance, several characterizations of Trotter and Moore [3] as well as perfect graph theorems [4,5]. Robertson, Seymour, and others extended these results to graph minor theory [6,7]. While far from complete, a more detailed review of the related research emanating from Kuratowski theorem is presented in the next section. In this introduction, we only focus on the two topics that are fundamental to the results of our paper, crossing-critical graphs and graph Hamiltonicity.

Observe (as is elaborated in the next section), that Kuratowski graphs can be interpreted as the only two 3-connected 1-crossing-critical graphs. A parallel theorem describing all 2-crossing-critical graphs was established by Bokal, Oporowski, Salazar, and Richter [8], who characterized the complete list of minimal forbidden subdivisions for a graph to be realizable in a plane with only one crossing. They exhibit a significantly richer structure: unlike just two 3-connected 1-crossing-critical graphs, the graphs realizable in a plane with at most one crossing already exhibit infinite families of topologically minimal obstruction graphs. Although it cannot be claimed that all these 2-crossing-critical graphs are Hamiltonian (with Petersen graph being the most known counterexample),

the claim is fairly easy to see for large such graphs using the aforementioned characterization of 2-crossing-critical graphs. Hence, almost all of these graphs are Hamiltonian. In this paper, we address a significantly more difficult problem of counting their different Hamiltonian cycles. The interest in understanding the number of different Hamiltonian cycles in various graph families originates from biochemical modelling of the polymers [9], where a collapsed polymer globule is modelled by a Hamiltonian cycle and the number of Hamiltonian cycles corresponds to the entropy of a polymer system in a collapsed but disordered phase. This shows an interesting intuitive duality to counting Eulerian cycles that showed relevance in constructing controlled, de novo protein structure folding [10,11]. In 1990, a characterization of Hamiltonian cycles of the Cartesian product $P_4 \square P_n$ was established [12]. In 1994, Kwong and Rogers developed a matrix method for counting Hamiltonian cycles in $P_m \square P_n$, obtaining exact results for $m = 4, 5$ [13]. Their method was extended to arbitrarily large grids by Bodroža-Pantić et al. [14] and by Stoyan and Strehl [15]. Later, Bodroža-Pantić et al. gave some explicit generating functions for the number of Hamiltonian cycles in graphs $P_m \square P_n$ and $C_m \square P_n$ [16,17]. Earlier, Saburo developed a field theoretic approximation of the number of Hamiltonian cycles in graphs $C_m \square C_n$ in [18] as well as in planar random lattices [19]. Fireze et al. considered generating and counting Hamiltonian cycles in random regular graphs [20]. Given these results, we note that our approach renders large 2-crossing-critical graphs to be the first nonplanar graph family for which the number of Hamiltonian cycles can be exactly determined. It may be relevant that the dissertation [21] similarly investigates links between 2-crossing-critical graphs, graph embeddings, and Hamiltonian cycles in higher surfaces.

In addition to an alphabetic description of large 2-crossing-critical graphs that allows for specifying the above formulae and may inspire further investigation of this graph class and ease access to graph-theoretic research building in this next step beyond the Kuratowski theorem, we hence extend this body of research on counting Hamiltonian cycles by going beyond Cartesian products of paths and cycles and apply the matrix method for counting Hamiltonian cycles to general 2-tiled graphs, which by the previously mentioned characterization theorem includes almost all 2-crossing-critical graphs. By allowing for nonplanar graphs in our approach, a new type of Hamiltonian cycle not observed previously appears. We complement the previous approaches of devising generating functions (which is feasible for well-structured graphs, such as aforementioned Cartesian products, or for the expected number of Hamiltonian cycles in random graphs) by an algorithm, which is in the case of 2-crossing-critical graphs implementable in linear time. For certain subfamilies of 2-crossing-critical graphs, the algorithm can even be simplified to a closed formula, using only the counts of specific letters in our alphabetic representation of the 2-crossing-critical graph, thus rendering these graphs the first nonplanar graphs for which an exact number of distinct Hamiltonian cycles is known. Specifically, we constructively prove the following theorems:

Theorem 1. *Let G be a 2-connected 2-crossing-critical graph containing a subdivision $H \simeq V_{10}$. There exists an algorithm of linear time complexity in the number of vertices of G that computes the number of Hamiltonian cycles in G .*

The theorem has the following easier-to-state corollary:

Corollary 1. *There exists an integer N such that any 2-connected 2-crossing-critical graph G with at least N vertices is Hamiltonian.*

As Petersen graph is a 3-connected 2-crossing-critical graph and is not Hamiltonian, containing a V_{10} subdivision (or, equivalently, being large) cannot simply be omitted for the above conclusions. For a picture of V_{10} , see Figure 1.

The algorithm in Theorem 1 is a special case of the general algorithm from the following theorem:

Theorem 2. Let \mathcal{T} be a finite family of 2-tiles, and let \mathcal{G} be a family of cyclizations of finite sequences of such tiles. There exists an algorithm that yields, for each graph $G \in \mathcal{G}$, the number of distinct Hamiltonian cycles in G . For a fixed set \mathcal{T} , the run time of the algorithm is quadratic in the number of tiles (and hence vertices) of G .

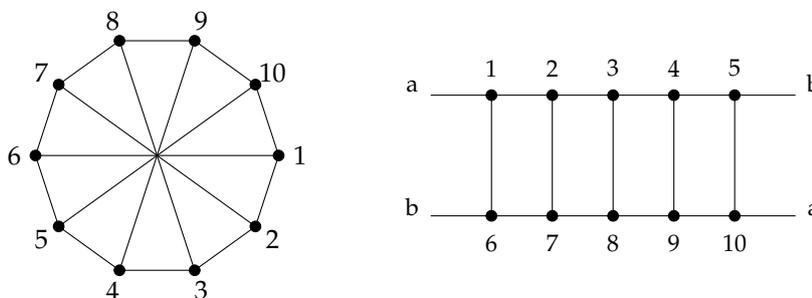


Figure 1. Two pictures of V_{10} . In general, V_{2n} is obtained from a $2n$ -cycle by adding the n main diagonals.

The rest of the paper is organized as follows. In the next section, we put the central concept of tiled graphs and crossing-critical graphs into the wider context of graph theory research. We introduce tiles and tiled graphs in Section 3, where a general algorithm for counting Hamiltonian cycles in 2-tiled graphs is presented. In Section 4, we introduce 2-crossing-critical graphs and the recent characterization of large such graphs as 2-tiled graphs. In Section 5, we combine the results by adapting the general counting algorithm to 2-crossing-critical graphs and constructively prove the above key results.

2. Related Research

Kuratowski theorem inspired several characterizations of graph families using forbidden subgraphs, which paved paths to significantly different areas of graph theory. Extremal graph theory is concerned with forbidding any subgraph isomorphic from being in a given graph [2] and maximizing the number of edges under this constraint. Significant structural theory was developed when forbidden subgraphs were replaced by forbidden induced subgraphs, for instance, several characterizations of Trotter and Moore [3] and the remarkable weak and strong perfect graph theorems [4,5].

Interest has also been shown in finding forbidden subgraphs that imply Hamiltonicity of graphs. In 1974, Goodman and Hedetniemi showed that a graph not containing induced $K_{1,3}$ and $K_{1,3} + e$, where e creates a 3-cycle, is Hamiltonian [22]. A series of several similar results was closed in 1997 by Faudree and Gould, who characterized all pairs of graphs such that forbidding their induced presence in a graph implies a graph’s Hamiltonicity [23]. This was via several papers extended to a complete characterization of triples of forbidden graphs implying Hamiltonicity, the final one being [24].

Graph minor theory extended the Kuratowski theorem to higher surfaces, showing that the set of graphs embeddable into any surface can be characterized by a finite set of forbidden minors [7]. The exact characterization was devised by Archdeacon for the projective plane [6], but already on the torus, the number of forbidden minors reached tens of thousands [25]. Mohar devised algorithms to embed graphs on surfaces [26], which was later improved by Kawarabayashi, Mohar, and Reed [27]. Characterizations of graph classes with subdivisions received somewhat less renowned attention. Early on the above path, Chartrand, Geller, and Hedetniemi pointed at some common generalizations of forbidding a small complete graph and a corresponding complete bipartite subgraph as a subdivision, resulting in empty graphs, trees, outerplanar graphs, and planar graphs [28]. That unifying approach apparently did not yield fruitful results, but more recently, Dvořák established a characterization of several graph classes using forbidden subdivisions [29], thus reaching even outside of topological graph theory.

The cornerstone of our contribution is yet another generalization of Kuratowski theorem. Note that the theorem elementarily implies that the two Kuratowski graphs K_5 and $K_{3,3}$ are the only 3-connected graphs that need at least one crossing to be drawn in a plane, but each their proper subgraph is planar and hence has a strictly smaller crossing number. Furthermore, all the graphs with the latter property can be obtained from them by subdividing their edges. Using the definition from the next section, Kuratowski theorem characterizes all 3-connected 1-crossing-critical graphs and consequently describes all 1-crossing critical graphs as their subdivisions.

It was already known that, contrary to fixed-genus-embeddable graphs, fixed-crossing-number realizable graphs exhibit a richer structure of topologically minimal obstruction graphs, as first demonstrated by Šiřan [30] constructing infinite families of c -crossing-critical graphs. Kochol extended this result to simple, 3-connected graphs [31].

A parallel to this interpretation of Kuratowski theorem but for 2-crossing-critical graphs was established recently by Bokal, Oporowski, Salazar, and Richter [8]. They characterized the complete list of minimal forbidden subdivisions for a graph to be realizable in a plane with only one crossing, i.e., 2-crossing-critical graphs. They showed that all such graphs are either small or 2-connected and obtained from 3-connected ones using subdivisions and similar operations, or 3-connected and obtained similarly as Kochol 2-crossing-critical graphs [31] but using 42 different structures (of which Kochol used just one). Later in this paper, we formalize these structures as tiles.

Tiles as a central tool in graph theory were first formally defined by Pinontoan and Richter [32], who extended the answer to Salazar's question on average degrees in infinite families of c -crossing-critical graphs [33]. Salazar's question was resolved by Bokal, again using tiles. They were instrumental in further studies on degrees in crossing critical graphs by Hliněný [34], whose most complete results so far are published in [35]. Most of this desire for understanding the degrees in crossing-critical graphs was inspired by a conjecture of Richter that c -crossing-critical graphs have their maximum degree bounded from above by a function of c . The conjecture was first existentially disproved by Dvořák and Mohar [36]. A constructive counterexample was obtained by Bokal et al. [37], who also showed that bounded degree conjecture holds precisely for $c \leq 12$. Instrumental in this result were wedges, a degenerate form of tiles that has yet to be formalized. Tiles, wedges, and planar belts were shown to be (in addition to a small connecting graph) the three key ingredients of large c -crossing-critical graphs by Dvořák, Hliněný, and Mohar [38].

In addition to applications of tiles for studying crossing-critical graphs, Pinontoan and Richter opened another direction of results. They studied the limit crossing number of tiled graphs, showing the existence of a limit crossing number for a periodic family of graphs (in the terms defined later, k -tiled graphs resulting from the cyclizations of repeated joins of a single tile). Richter later asked whether this limit is computable, which was proven by Dvořák and Mohar in [39]. Some further graph invariants on 2-crossing-critical graphs were obtained in parallel with our results and published in [40].

It may be interesting to note that tiles and their joins are a specific kind of labeled graphs and operations on them, as introduced in Lovász's seminal book on large networks and graph limits [41]. Hence, introductory understanding of the concepts presented here may motivate researchers in pursuit of that direction. We do not harmonize the notation here, but it may be feasible to do so when extending the theory of tiles to the theory of earlier mentioned wedges.

3. Hamiltonian Cycles in 2-Tiled Graphs

In this section, we introduce the concept of a tile that was introduced in [32] and later redesigned in [42] and applied in [8,35] and the k -tiled graphs. We use the notation from [8]. To facilitate brevity, we have prepared Table A1 with a summary of frequently used notation. It is located in the appendix.

Definition 1. A tile is a triple $T = (G, x, y)$, consisting of a connected graph G and two sequences $x = (x_1, x_2, \dots, x_k)$ (left wall) and $y = (y_1, y_2, \dots, y_l)$ (right wall) of distinct vertices of G , with no vertex of G appearing in both x and y . If $|x| = |y| = k$, we call T a k -tile.

We use the following notation when combining tiles:

Definition 2.

1. The tiles $T = (G, x, y)$ and $T' = (G', x', y')$ are compatible whenever $|y| = |x'|$.
2. A sequence $\mathcal{T} = (T_0, T_1, \dots, T_m)$ of tiles is compatible if, for each $i \in \{1, 2, \dots, m\}$, T_{i-1} is compatible with T_i .
3. The join of compatible tiles (G, x, y) and (G', x', y') is the tile $T = (G, x, y) \otimes (G', x', y')$ for which the graph is obtained from disjoint union of G and G' by identifying the sequence y term by term with the sequence x' .
4. The join of a compatible sequence $\mathcal{T} = (T_0, T_1, \dots, T_m)$ of tiles is defined as $\otimes \mathcal{T} = T_0 \otimes T_1 \otimes \dots \otimes T_m$.
5. A tile T is cyclically compatible if T is compatible with itself.
6. For a cyclically compatible tile $T = (G, x, y)$, the cyclization of T is the graph $\circ T$ obtained by identifying the respective vertices of x with y .
7. A cyclization of a cyclically compatible sequence of tiles \mathcal{T} is defined as $\circ \mathcal{T} = \circ(\otimes \mathcal{T})$.
8. A k -tiled graph is a cyclization of a sequence of at least $(k + 1)$ k -tiles.

Lemma 1. Let C be a Hamiltonian cycle in a 2-tiled graph $G = \circ(T_0, T_1, \dots, T_m)$. Then, we have the following:

1. $C = \bigcup_{i=0}^m (C \cap T_i)$.
2. $C \cap T_i$ is a union of paths and isolated vertices.
3. Let v be a vertex of a component of $C \cap T_i$. Then, v has degree 2 in $C \cap T_i$ or v is a wall vertex.
4. There are at most two distinct non-degenerate paths in $C \cap T_i$.
5. If $C \cap T_i$ consists of distinct non-degenerate paths P_1 and P_2 , then $C \cap T_i = P_1 \oplus P_2$.

Proof.

1. $C = C \cap G = C \cap \left(\bigcup_{i=0}^m T_i \right) \stackrel{\text{Distributive law}}{=} \bigcup_{i=0}^m (C \cap T_i)$.
2. Let K be a component of $C \cap T_i$. As C is a cycle, K is a connected subgraph of C . Then, K is either equal to C , a path, or a vertex. If $K = C$, then T_i contains all the vertices of G , a contradiction to $m \geq 2$ (in at least one tile, C does not contain all the vertices). The claim follows.
3. Let v be a vertex of $C \cap T_i$ of degree different from 2. As maximum degree in C is 2, v has degree 1 or 0. If v is an internal vertex of T_i , its degree in $C = \bigcup_{i=0}^m (C \cap T_i)$ is equal to its degree in $C \cap T_i$. This contradicts C being a cycle and the claim follows.
4. By Claim 3, paths start and end in a wall vertex. Each distinct non-degenerate path needs two unique wall vertices, and the claim follows.
5. By Claim 4, P_1 and P_2 contain all the wall vertices. By Claim 3, isolated vertices can only be wall vertices; hence, there are no isolated vertices and $C \cap T_i = P_1 \oplus P_2$. □

Corollary 2. Let C be a Hamiltonian cycle in a 2-tiled graph $G = \circ(T_0, T_1, \dots, T_m)$ and N_i be the set of all isolated vertices in $C \cap T_i$. Then, $(C \cap T_i) \setminus N_i$ is one of the following:

1. A path that begins in a vertex of the left wall, ends in a vertex of the right wall, and covers all internal vertices of T_i .
2. A pair of distinct paths for which each begins and ends at opposite walls, span T_i , and respect the vertex order of the walls.

3. A pair of distinct paths for which each begins and ends at opposite walls, span T_i , and invert the vertex order of the walls.
4. An empty set.
5. A pair of distinct paths for which each begins and ends at the same wall and span T_i .
6. A path that begins and ends at the same wall and covers all internal vertices of T_i .

We say that a path “traverses” a 2-tile, if it starts at the left wall and ends at the right wall of a 2-tile. Based on Corollary 2, we define three groups of C-types of tiles as follows:

Definition 3. Let C be a Hamiltonian cycle in a 2-tiled graph $G = \circ(T_0, T_1, \dots, T_m)$, $x = (x_1, x_2)$ be the left, and $y = (y_1, y_2)$ be the right wall of T_i .

1. If a cycle C traverses T_i with a single path and covers all internal vertices of T_i , then we say that T_i is of zigzagging C-type, of which there exist four kinds, relevant for completing the Hamiltonian cycles between the tiles.
 First, T_i is of zigzagging C-type $\{x_{3-j}\} - \{y_{3-k}\}$ if $C \cap T_i$ contains a single path P , the endvertices of P are vertices x_j and y_k of distinct walls of T_i , and P contains the non-endvertex wall vertices x_{3-j}, y_{3-k} . If either wall vertex that is not an endvertex of P is not contained in P , then $C \cap T_i$ contains a path P and these vertices as isolated vertex components. We denote such isolated components using an overline, leading to zigzagging C-types $\{x_{3-j}\} - \{\bar{y}_{3-k}\}, \{\bar{x}_{3-j}\} - \{y_{3-k}\}, \{\bar{x}_{3-j}\} - \{\bar{y}_{3-k}\}$.
2. If a cycle C traverses T_i with a pair of distinct traversing paths that span T_i , then we say that T_i is of traversing C-type.
 - (a) T_i is of aligned (Aligned pairs of traversing paths were introduced in [8].) traversing C-type $=$ if $C \cap T_i$ contains a pair of distinct paths P_1 and P_2 , the endvertices of P_1 are x_1 and y_1 , and the endvertices of P_2 are x_2 and y_2 .
 - (b) T_i is of twisted (Twisted pairs of traversing paths were introduced in [8].) traversing C-type \times if $C \cap T_i$ contains a pair of distinct paths P_1 and P_2 , the endvertices of P_1 are x_1 and y_2 , and the endvertices of P_2 are x_2 and y_1 .
3. If a cycle C does not traverse T_i , then we say that T_i is of flanking C-type.
 - (a) T_i is of flanking C-type \emptyset if $C \cap T_i$ is an empty graph spanned by $\{x_1, x_2, y_1, y_2\}$.
 - (b) T_i is of flanking C-type \parallel if $C \cap T_i$ contains a pair of distinct paths P_1 and P_2 , the endvertices of P_1 are x_1 and x_2 , and the endvertices of P_2 are y_1 and y_2 .
 - (c) T_i is of flanking C-type $|_{\{y_1, y_2\}}$ if $C \cap T_i$ contains a single path P , the endvertices of P are x_1, x_2 of the same wall of T_i , and P contains the non-endvertex wall vertices y_1, y_2 . If either wall vertex that is not endvertex of P is not contained in P , then $C \cap T_i$ contains a path P and these vertices as isolated components. We denote this using an overline, leading to flanking C-types $|_{\{\bar{y}_1, \bar{y}_2\}}, |_{\{y_1, \bar{y}_2\}}, |_{\{\bar{y}_1, \bar{y}_2\}}$. The respective notations for flanking C-types of P with endvertices y_1, y_2 are $|_{\{x_1, x_2\}}, |_{\{\bar{x}_1, x_2\}}, |_{\{x_1, \bar{x}_2\}}, |_{\{\bar{x}_1, \bar{x}_2\}}$.

We denote the set of all possible zigzagging C-types using Λ_z , the set of all possible traversing C-types using Λ_t , and the set of all possible flanking C-types using Λ_f . Finally, we set $\Lambda = \Lambda_z \cup \Lambda_t \cup \Lambda_f$.

We refer to C-types by their group name or by their notation. The first type of reference is used in the case of the reference to the whole group of C-types, the second one is used in the case of the reference to a specific C-type.

Lemma 2. Let C be a Hamiltonian cycle in a 2-tiled graph $G = \circ(T_0, T_1, \dots, T_m)$. Then, precisely one of the following holds:

1. $\forall i : T_i$ is of zigzagging C-type.
2. $\exists! i \in \{0, 1, \dots, m\} : T_i$ is of flanking C-type \parallel and $\forall j \in \{0, 1, \dots, m\} \setminus \{i\}, T_j$ is of traversing C-type.
3. $\exists! i \in \{0, 1, \dots, m\} : T_i$ and T_{i+1} are of compatible flanking C-type of form $|_{\{x, y\}}$ and $|_{\{z, w\}}$, respectively, and $\forall j \in \{0, 1, \dots, m\} \setminus \{i, i + 1\}, T_j$ is of traversing C-type.

4. $\exists! i \in \{0, 1, \dots, m\}$: T_i, T_{i+1} , and T_{i+2} are of flanking C-types $|\{y_1, y_2\}, \emptyset, \{x_1, x_2\}|$, respectively, and $\forall j \in \{0, 1, \dots, m\} \setminus \{i - 1, i, i + 1\}$, T_j is of traversing C-type.
5. $\forall i$: T_i is of traversing C-type, where the number of indices i of tiles of traversing C-type \times is odd.

Proof.

1. We prove that, if T_i is of zigzagging C-type, then the same holds for T_{i-1} and T_{i+1} . Suppose that T_i is of zigzagging C-type. Then, T_i has the property that exactly one of its left and one of its right wall vertices have a degree 1 in $C \cap T_i$ (the ones that are endvertices of the path from the left to right walls). Because the degree of every vertex in C is 2, the left wall vertex has degree 1 in $C \cap T_{i-1}$ and the right wall vertex has degree 1 in $C \cap T_{i+1}$. Because the degree of every vertex in C is 2, other wall vertices are either of degree 0 (isolated vertex) or 2 (vertex is part of a path) in $C \cap T_i$. If a wall vertex is of degree 0 (2) in $C \cap T_i$, then its degree in $C \cap T_{i-1}$ (left wall vertex of T_i) or in $C \cap T_{i+1}$ (right wall vertex of T_i) is 2 (0). Hence, based on Corollary 2, T_{i-1} and T_{i+1} are of zigzagging C-type. By extending the argument to their neighbors, we establish Claim 1 of Lemma 2. For the rest of the proof, we may therefore assume that none of the tiles are of zigzagging C-type.
2. Let T_i be of C-type \parallel . Let $P_1 \oplus P_2$ be paths in $C \cap T_i$. Assume without loss of generality that P_1 starts in x_1^i and ends in x_2^i and that P_2 starts in y_1^i and ends in y_2^i . Because $m \geq 2$, $C - P_1 \oplus P_2 = Q_1 \oplus Q_2$, where Q_1, Q_2 are paths in $T_{i+1} \otimes \dots \otimes T_m \otimes T_0 \otimes \dots \otimes T_{i-1}$, which start in y_1^i, y_2^i and end in x_1^i, x_2^i , respectively. Then, $\forall j \in \{0, 1, \dots, m\} \setminus \{i\}$, $C \cap T_j = (C - P_1 \oplus P_2) \cap T_j = (Q_1 \oplus Q_2) \cap T_j \stackrel{\text{Distributive law}}{=} (Q_1 \cap T_j) \oplus (Q_2 \cap T_j)$. For $k \in \{1, 2\}$, $Q_k \cap T_j$ is nontrivial and connected; otherwise, Q_k would not be connected. Hence, $(Q_1 \cap T_j), (Q_2 \cap T_j)$ are vertex disjoint paths in T_j that cover all internal vertices (C is a Hamiltonian cycle) with endvertices in the opposite walls. Therefore, T_j is of traversing C-type. We established Claim 2 of Lemma 2, and for the rest of the proof, we may assume that none of the tiles is of C-type \parallel .
3. Let T_i be of C-type $\lambda \in \{|\{y_1, y_2\}, |\{y_1, \bar{y}_2\}, |\{y_1, \bar{y}_2\}\}$. Then, $C \cap T_i$ consists of some isolated vertices (candidates are y_1^i, y_2^i) and path P_i . Because any isolated vertex of $C \cap T_i$ is part of a path of a neighbouring tile (in this case, T_{i+1}) and is not the endvertex of this path, there is a path P_{i+1} in $C \cap T_{i+1}$ for which the endvertices are y_1^{i+1}, y_2^{i+1} and covers possible isolated nodes $y_1^i = x_1^{i+1}, y_2^i = x_2^{i+1}$ of a tile T_i . Hence, T_{i+1} is of compatible C-type $\mu \in \{|\{x_1, \bar{x}_2\}, |\{\bar{x}_1, x_2\}, |\{x_1, x_2\}\}$. We now suppose that T_i is of C-type $|\{y_1, y_2\}$ and that T_{i+1} is not of C-type \emptyset . Then, $C \cap T_{i+1}$ consists of isolated nodes $y_1^i = x_1^{i+1}, y_2^i = x_2^{i+1}$, and path P_{i+1} , where P_{i+1} is a path for which the endvertices are y_1^{i+1}, y_2^{i+1} . Hence, T_{i+1} is of C-type $\{\bar{x}_1, \bar{x}_2\}$. Because $m \geq 2$, in both cases, $C - P_i \oplus P_{i+1} = Q_1 \oplus Q_2$, where Q_1, Q_2 are paths in $T_{i+2} \otimes \dots \otimes T_m \otimes T_0 \otimes \dots \otimes T_{i-1}$, which start in y_1^{i+1}, y_2^{i+1} and end in x_1^i, x_2^i . Then, $\forall j \in \{0, 1, \dots, m\} \setminus \{i, i + 1\}$, $C \cap T_j = (C - P_i \oplus P_{i+1}) \cap T_j = (Q_1 \oplus Q_2) \cap T_j \stackrel{\text{Distributive law}}{=} (Q_1 \cap T_j) \oplus (Q_2 \cap T_j)$ and (similarly as in Item 2 of the proof of Lemma 2) T_j is of traversing C-type. We established Claim 3 of Lemma 2, and for the rest of the proof, we may assume that each tile of C-type $|\{y_1, y_2\}$ has an adjacent tile of C-type \emptyset .
4. Let T_i be of C-type $|\{y_1, y_2\}$ and T_{i+1} be of C-type \emptyset . Then, there exists a path P_i in $C \cap T_i$ that covers y_1^i, y_2^i . Because y_1^{i+1}, y_2^{i+1} are isolated vertices in $C \cap T_{i+1}$, there is a path P_{i+2} in $C \cap T_{i+2}$ for which the endvertices are y_1^{i+2}, y_2^{i+2} and covers these isolated nodes $y_1^{i+1} = x_1^{i+2}, y_2^{i+1} = x_2^{i+2}$. Hence, T_{i+2} is of C-type $\{x_1, x_2\}$. Because $m \geq 2$, $C - P_i \oplus P_{i+2} = Q_1 \oplus Q_2$, where Q_1, Q_2 are paths in $T_{i+3} \otimes \dots \otimes T_m \otimes T_0 \otimes \dots \otimes T_{i-1}$, which start in y_1^{i+2}, y_2^{i+2} and end in x_1^i, x_2^i . Then, $\forall j \in \{0, 1, \dots, m\} \setminus \{i, i + 1, i + 2\}$, $C \cap T_j = (C - P_i \oplus P_{i+2}) \cap T_j = (Q_1 \oplus Q_2) \cap T_j \stackrel{\text{Distributive law}}{=} (Q_1 \cap T_j) \oplus (Q_2 \cap T_j)$ and (similarly as in Item 2 of the proof of Lemma 2) T_j is of traversing C-type. We

established Claim 4 of Lemma 2, and for the rest of the proof, we may assume that there are no tiles of C-type of form $|\{x,y\}$.

5. Assume now that there is a tile T_i of C-type of form $|\{z,w\}|$. Then, as we assumed that there are no tiles of C-type of form $|\{x,y\}$, a symmetric argument to Item 3 of the proof of Lemma 2 implies T_{i-1} is of C-type \emptyset (hence, T_i can only be of C-type $|\{x_1,x_2\}|$). However, then a symmetric argument to Item 4 of the proof of Lemma 2 implies that T_{i-2} is of C-type $|\{y_1,y_2\}$, a contradiction to the assumption that implies all tiles are either of C-type \emptyset or traversing C-type.

If there is at least one tile T_i of C-type \emptyset , then $C \cap T_{i+1} \otimes \dots \otimes T_m \otimes T_0 \otimes \dots \otimes T_{i-1}$ consists of at least two disconnected paths Q_1 and Q_2 . However, as C only intersects T_i in wall vertices, C is equal to $C \cap T_{i+1} \otimes \dots \otimes T_m \otimes T_0 \otimes \dots \otimes T_{i-1}$, a contradiction implying that all the tiles are of traversing C-types.

The remaining case is that $\forall i : T_i$ is of traversing C-type. Therefore, $\forall i : C \cap T_i = P_1^i \oplus P_2^i$, where each path starts in a left wall vertex and ends in a right wall vertex. Without loss of generality, we may assume that, $\forall k \in \{1,2\}$, $P_k^0, P_k^1, \dots, P_k^m$ are such that $\forall j \in \{0,1, \dots, m\}$, P_k^j ends in same vertex as P_k^{j+1} starts (if not, we can reindex them). Without loss of generality, we may assume that P_1^0 starts in x_1^0 and P_2^0 in x_2^0 .

Each tile of C-type \times implies that the path moves from the top left wall vertex to the bottom right wall vertex and from the bottom left wall vertex to the top right wall vertex. In case of an even number of tiles T_i of C-type \times , $x_1^0 P_1^0 P_1^1 \dots P_1^m x_1^0$ and $x_2^0 P_2^0 P_2^1 \dots P_2^m x_2^0$ are distinct cycles. When this number is odd, $x_1^0 P_1^0 P_1^1 \dots P_1^m x_2^0 P_2^0 P_2^1 \dots P_2^m x_1^0$ is a Hamiltonian cycle.

□

Hamiltonian cycles of types 2–4 from Lemma 2 are of similar construction, so we use the same name for all of them.

Definition 4. We define names for types of Hamiltonian cycle from Lemma 2: zigzagging Hamiltonian cycles (type 1), flanking Hamiltonian cycles (types 2–4), and traversing Hamiltonian cycles (type 5).

Definition 5. Let $\{T_i, T_{i+1}, \dots, T_j\}$ be a sequence of 2-tiles in a 2-tiled graph $G = \circ(T_0, T_1, \dots, T_m)$, where indices i and j are considered cyclically. Then,

$$K(\{T_i, T_{i+1}, \dots, T_j\}) = \{C \cap T_i \otimes T_{i+1} \otimes \dots \otimes T_j \mid C \text{ is a Hamiltonian cycle in } G\}.$$

Using Definition 5, we define as follows:

Definition 6. Let $G = \circ(T_0, T_1, \dots, T_m)$ be a 2-tiled graph. For $\lambda \in \Lambda$ and $i \in \{0,1, \dots, m\}$, let

$$a_\lambda^i = |\{C \cap T_i \in K(\{T_i\}) \mid T_i \text{ be of C-type } \lambda\}|.$$

We prove that the number of Hamiltonian cycles of each type can be counted efficiently. In the counting of Hamiltonian cycles that follows, index 0 is used for the starting condition of the recursive counting, i.e., when there are no tiles. We adjust to this notation by using a one based labelling for tiles throughout the rest of Section 2. By definition of cyclization, $T_{m+1} = T_1$.

3.1. Counting Traversing Hamiltonian Cycles

Lemma 3. Let \mathcal{T} be a fixed finite family of 2-tiles, and let $G = \circ(T_1, T_2, \dots, T_m)$, where $\forall i : T_i \in \mathcal{T}$. Traversing Hamiltonian cycles in G can be counted in time $\mathcal{O}(m)$.

Proof. For $i \in \{1, 2, \dots, m\}$, let

$$c_{even}^i = |\{C \cap T_1 \otimes T_2 \otimes \dots \otimes T_i \in K(\{T_1, T_2, \dots, T_i\}) \mid \text{even number of tiles of C-type } \times, \text{ all other of C-type } =\}|,$$

$$c_{odd}^i = |\{C \cap T_1 \otimes T_2 \otimes \dots \otimes T_i \in K(\{T_1, T_2, \dots, T_i\}) \mid \text{odd number of tiles of C-type } \times, \text{ all other of C-type } =\}|.$$

We define starting condition c^0 as

$$c^0 \stackrel{\text{def}}{=} [1 \ 0]^T, \tag{1}$$

because, in an empty graph, there are zero (even number) tiles of C-type \times . Then, for each $i \in \{1, 2, \dots, m\}$:

$$\begin{bmatrix} c_{even}^i \\ c_{odd}^i \end{bmatrix} = \begin{bmatrix} a_{=}^i & a_{\times}^i \\ a_{\times}^i & a_{=}^i \end{bmatrix} \begin{bmatrix} c_{even}^{i-1} \\ c_{odd}^{i-1} \end{bmatrix}, \tag{2}$$

$$c^i = R_i \cdot c^{i-1}.$$

Hence,

$$c^m = R_m \cdot R_{m-1} \dots R_1 \cdot c^0. \tag{3}$$

By Lemma 2, the number of traversing Hamiltonian cycles in G is equal to c_{odd}^m (the combination of tiles with an even number of tiles of C-type \times gives us two distinct cycles that contain all vertices of a 2-tiled graph). For a fixed family of 2-tiles, we precalculate matrices R_i . The time complexity to compute the product $R_m \cdot R_{m-1} \dots R_1$ and then the number c_{odd}^m is $\mathcal{O}(m)$. \square

3.2. Counting Flanking Hamiltonian Cycles

Definition 7. We say that a cycle turns around in a 2-tile if there exist two vertex disjoint paths, one with both endvertices in the left wall and the second one with both endvertices in the right wall, that cover all internal vertices of a 2-tile.

Lemma 4. Let \mathcal{T} be a fixed finite family of 2-tiles, and let $G = \circ(T_1, T_2, \dots, T_m)$, where $\forall i : T_i \in \mathcal{T}$, and $l \in \{0, 1, 2\}$. Flanking Hamiltonian cycles that turn around in the join of $(l + 1)$ consecutive tiles can be counted in time $\mathcal{O}(m^2)$. In the case where the corresponding matrices R_j , $j \in \{1, 2, \dots, m\}$, are invertible, we can count them in time $\mathcal{O}(m)$.

Proof. For $i \in \{1, 2, \dots, m\}$, let

- $T^{i,i+l} = T_i \otimes T_{i+1} \otimes \dots \otimes T_{i+l}$,
- $a^{i,i+l}$ be number of distinct possibilities for $T_i, T_{i+1}, \dots, T_{i+l}$ to be of compatible flanking C-types to turn around a cycle in $T^{i,i+l}$.

To get the number of flanking Hamiltonian cycles that turn around in $T^{i,i+l}$, $i \in \{1, \dots, m\}$, we do the following:

1. We calculate the value $a^{i,i+l}$.
2. Using the idea from the proof for traversing Hamiltonian cycles over the sequence $(T_{i+l+1}, \dots, T_m, T_1, \dots, T_{i-1})$, we get

$$c^{i+l+1,i-1} = R_{i-1} \dots R_1 \cdot R_m \dots R_{i+l+1} \cdot c^0,$$

where c^0 is as in (1). Then, $c_{even}^{i+l+1,i-1}$ presents the number of different combinations of $C \cap T_{i+l+1} \otimes \dots \otimes T_m \otimes T_1 \otimes \dots \otimes T_{i-1}$ with an even number of tiles of traversing C-type \times and $c_{odd}^{i+l+1,i-1}$ presents the number of different combinations of $C \cap T_{i+l+1} \otimes \dots \otimes T_m \otimes T_1 \otimes \dots \otimes T_{i-1}$ with an odd number of tiles of traversing C-type \times .

3. The number of Hamiltonian cycles turning around in $T^{i,i+l}$ is equal to

$$a^{i,i+l} \cdot (c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1}).$$

Hence, the total number of Hamiltonian cycles turning around in the join of $(l + 1)$ consecutive tiles in graph G is equal to

$$\sum_{i=1}^m a^{i,i+l} \cdot (c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1}).$$

Because there are finitely many different tiles and l is a constant, values $a^{i,i+l}$ and matrices R_i can be precalculated. The time complexity to compute the product $R_{i-1} \cdot \dots \cdot R_1 \cdot R_m \cdot \dots \cdot R_{i+l+1}$ and then the number $c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1}$ is $\mathcal{O}(m)$. Hence, to get the number $a^{i,i+l} \cdot (c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1})$, we need $\mathcal{O}(m)$ time. For m such numbers, the total time complexity is $\mathcal{O}(m^2)$.

Suppose that every matrix $R_j, j \in \{1, 2, \dots, m\}$, is invertible ($(a_{=}^j)^2 - (a_{\times}^j)^2 \neq 0$) and let

$$c^m = R_m \cdot \dots \cdot R_1 \cdot c^0$$

be as in (3). We can get the value $c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1}$ by solving the equation

$$c^m = R_{i+l} \cdot \dots \cdot R_{i+1} \cdot R_i \cdot c^{i+l+1,i-1}.$$

Because matrices $R_j, j \in \{1, 2, \dots, m\}$ are invertible, we get

$$c^{i+l+1,i-1} = R_i^{-1} \cdot R_{i+1}^{-1} \cdot \dots \cdot R_{i+l}^{-1} \cdot c^m.$$

In this case, we need $\mathcal{O}(m)$ time to get c^m . We need only $\mathcal{O}(1)$ additional time to compute each $c^{i+l+1,i-1}$ and then the number $c_{even}^{i+l+1,i-1} + c_{odd}^{i+l+1,i-1}$ and hence $\mathcal{O}(m)$ to compute them all. The total time complexity in this case is then $\mathcal{O}(m)$. \square

Lemma 5. Let \mathcal{T} be a fixed finite family of 2-tiles, and let $G = \circ(T_1, T_2, \dots, T_m)$, where $\forall i : T_i \in \mathcal{T}$. Flanking Hamiltonian cycles in G can be counted in time $\mathcal{O}(m^2)$. In the case where the corresponding matrices $R_j, j \in \{1, 2, \dots, m\}$, are invertible, we can count them in time $\mathcal{O}(m)$.

Proof. We can get flanking Hamiltonian cycles in three ways:

1. cycle turns around in one tile,
2. cycle turns around in two consecutive tiles, and
3. cycle turns around in three consecutive tiles.

1. *Counting flanking Hamiltonian cycles that turn around in one tile:*

Flanking Hamiltonian cycles that turn around in one tile consist of two parts. One tile is of C-type \parallel ; other tiles are of traversing C-type. Using Lemma 4 with $l = 0$ and $a^{i,i+l} = a_{\parallel}^i$, we get the desired result.

2. *Counting flanking Hamiltonian cycles that turn around in two consecutive tiles:*

In consecutive tiles T_i and T_{i+1} , we have $y_1^i = x_1^{i+1}$ and $y_2^i = x_2^{i+1}$. Let $a_{\parallel}^{i,i+1}$ denote the number of distinct possibilities for T_i and T_{i+1} to be of compatible flanking C-types of the forms $|\{x,y\}$ and $|\{z,w\}|$. Then,

$$a_{\parallel}^{i,i+1} = a_{|\{y_1,y_2\}}^i \cdot a_{|\{\bar{x}_1,\bar{x}_2\}}^{i+1} + a_{|\{y_1,\bar{y}_2\}}^i \cdot a_{|\{\bar{x}_1,x_2\}}^{i+1} + a_{|\{\bar{y}_1,y_2\}}^i \cdot a_{|\{x_1,\bar{x}_2\}}^{i+1} + a_{|\{\bar{y}_1,\bar{y}_2\}}^i \cdot a_{|\{x_1,x_2\}}^{i+1}.$$

Flanking Hamiltonian cycles that turn around in two consecutive tiles consist of two parts. In consecutive tiles, compatible C-types of forms $|\{x,y\}$ and $|\{z,w\}|$ are used and

other tiles are of traversing C-type. Using Lemma 4 with $l = 1$ and $a^{i,i+1} = a_{\lfloor}^{i,i+1}$, we get the desired result.

3. *Counting flanking Hamiltonian cycles that turn around in three consecutive tiles:*
 Considering three consecutive tiles T_i, T_{i+1} , and T_{i+2} implies $y_1^i = x_1^{i+1}, y_2^i = x_2^{i+1}, y_1^{i+1} = x_1^{i+2}$, and $y_2^{i+1} = x_2^{i+2}$. Let $a_{\lfloor}^{i,i+1,i+2}$ denote the number of distinct possibilities to turn around in three consecutive tiles T_i, T_{i+1} , and T_{i+2} . Then,

$$a_{\lfloor}^{i,i+1,i+2} = a_{\lfloor_{\{y_1,y_2\}}}^i \cdot a_{\emptyset}^{i+1} \cdot a_{\lfloor_{\{x_1,x_2\}}}^{i+2}$$

where

$$a_{\emptyset}^{i+1} = \begin{cases} 1; & \text{there is no internal vertex in the tile } T_{i+1}. \\ 0; & \text{there is an internal vertex in the tile } T_{i+1}. \end{cases}$$

Flanking Hamiltonian cycles that turn around in three consecutive tiles consist of two parts. In consecutive tiles T_i, T_{i+1} , and T_{i+2} , respectively, the C-types that are used are $\lfloor_{\{y_1,y_2\}}, \emptyset$, and $\lfloor_{\{x_1,x_2\}}$. Other tiles are of traversing C-type. Using Lemma 4 with $l = 2$ and $a^{i,i+1} = a_{\lfloor}^{i,i+1,i+2}$, we get the desired result. □

3.3. Counting Zigzagging Hamiltonian Cycles

Lemma 6. *Let \mathcal{T} be a fixed finite family of 2-tiles, and let $G = \circ(T_1, T_2, \dots, T_m)$, where $\forall i : T_i \in \mathcal{T}$. Zigzagging Hamiltonian cycles in G can be counted in time $\mathcal{O}(m)$.*

Proof. We observe that there exist four possibilities for covering wall vertices of the same wall in a 2-tile of zigzagging C-type from Definition 3:

1. x_1 is an endvertex of a path and x_2 is part of a path (notation (x_1, x_2)),
2. x_2 is an endvertex of a path and x_1 is part of a path (notation (x_2, x_1)),
3. x_1 is an endvertex of a path and x_2 is an isolated vertex (notation (x_1, \bar{x}_2)),
4. x_2 is an endvertex of a path and x_1 is an isolated vertex (notation (x_2, \bar{x}_1)).

For $i \in \{1, 2, \dots, m\}$ and $(k, l) \in \{(x_1, x_2), (x_2, x_1), (x_1, \bar{x}_2), (x_2, \bar{x}_1)\}$, let

$$c_{(k,l)}^i = |\{C \cap T_1 \otimes T_2 \otimes \dots \otimes T_i \in K(\{T_1, T_2, \dots, T_i\}) \mid T_i \text{ ends with type } (k, l)\}|.$$

In adjacent tiles T_i and T_{i+1} , we have $y_1^i = x_1^{i+1}$ and $y_2^i = x_2^{i+1}$. We define different starting conditions c^0 , dependent on the starting type (k, l) in the first tile (in this notation T_1):

- for $(k, l) = (x_1, x_2)$:

$$c(x_1, x_2)^0 = [1 \ 0 \ 0 \ 0]^T,$$
- for $(k, l) = (x_2, x_1)$:

$$c(x_2, x_1)^0 = [0 \ 1 \ 0 \ 0]^T,$$
- for $(k, l) = (x_1, \bar{x}_2)$:

$$c(x_1, \bar{x}_2)^0 = [0 \ 0 \ 1 \ 0]^T,$$
- for $(k, l) = (x_2, \bar{x}_1)$:

$$c(x_2, \bar{x}_1)^0 = [0 \ 0 \ 0 \ 1]^T.$$

For each $i \in \{1, 2, \dots, m\}$, we get

$$\begin{bmatrix} c^i_{(y_1, y_2)} \\ c^i_{(y_2, y_1)} \\ c^i_{(y_1, \bar{y}_2)} \\ c^i_{(y_2, \bar{y}_1)} \end{bmatrix} = \begin{bmatrix} a^i_{\{\bar{x}_2\} - \{y_2\}} & a^i_{\{\bar{x}_1\} - \{y_2\}} & a^i_{\{x_2\} - \{y_2\}} & a^i_{\{x_1\} - \{y_2\}} \\ a^i_{\{\bar{x}_2\} - \{y_1\}} & a^i_{\{\bar{x}_1\} - \{y_1\}} & a^i_{\{x_2\} - \{y_1\}} & a^i_{\{x_1\} - \{y_1\}} \\ a^i_{\{\bar{x}_2\} - \{\bar{y}_2\}} & a^i_{\{\bar{x}_1\} - \{\bar{y}_2\}} & a^i_{\{x_2\} - \{\bar{y}_2\}} & a^i_{\{x_1\} - \{\bar{y}_2\}} \\ a^i_{\{\bar{x}_2\} - \{\bar{y}_1\}} & a^i_{\{\bar{x}_1\} - \{\bar{y}_1\}} & a^i_{\{x_2\} - \{\bar{y}_1\}} & a^i_{\{x_1\} - \{\bar{y}_1\}} \end{bmatrix} \begin{bmatrix} c^{i-1}_{(x_1, x_2)} \\ c^{i-1}_{(x_2, x_1)} \\ c^{i-1}_{(x_1, \bar{x}_2)} \\ c^{i-1}_{(x_2, \bar{x}_1)} \end{bmatrix}, \tag{4}$$

$$c^i = Z_i \cdot c^{i-1}.$$

Then,

$$c^m = Z_m \cdot Z_{m-1} \cdots Z_1 \cdot c^0.$$

For each starting type (k, l) , we get the equation

$$c(k, l)^m = Z_m \cdot Z_{m-1} \cdots Z_1 \cdot c(k, l)^0.$$

Because of the definition of cyclization, we get zigzagging Hamiltonian cycles if we have a combination of tile types with compatible starting type in tile T_1 and ending type in tile T_m (we can combine them to cycles). Hence, the number of zigzagging Hamiltonian cycles in a graph G is equal to

$$c(x_1, x_2)^m_{(x_1, \bar{x}_2)} + c(x_2, x_1)^m_{(x_2, \bar{x}_1)} + c(x_1, \bar{x}_2)^m_{(x_1, x_2)} + c(x_2, \bar{x}_1)^m_{(x_2, x_1)},$$

which is equal to

$$tr(Z_m \cdot Z_{m-1} \cdots Z_1).$$

To compute this number, we have to efficiently calculate matrices Z_i . Because there is a finite number of different tiles, we can precompute them independently from m . The time complexity to compute the product $Z_m \cdot Z_{m-1} \cdots Z_1$ and then the number $tr(Z_m \cdot Z_{m-1} \cdots Z_1)$ is $\mathcal{O}(m)$. □

Theorem 2. *Let \mathcal{T} be a finite family of 2-tiles, and let \mathcal{G} be a family of cyclizations of finite sequences of such tiles. There exists an algorithm that yields, for each graph $G \in \mathcal{G}$, the number of distinct Hamiltonian cycles in G . For a fixed set \mathcal{T} , the running time of the algorithm is quadratic in the number of tiles (and hence vertices) of G .*

Proof. By Lemma 2, we know that there exist three types of Hamiltonian cycles in such a graph (traversing, flanking, and zigzagging). We proved that traversing and zigzagging Hamiltonian cycles can be counted in time $\mathcal{O}(m)$ (Lemma 3 and Lemma 6). Flanking Hamiltonian cycles can be counted in time $\mathcal{O}(m^2)$ (Lemma 5). For adding all three counters, we need $\mathcal{O}(1)$ additional time and the theorem holds. □

4. Large 2-Crossing-Critical Graphs as 2-Tiled Graphs

In this section, we introduce 2-crossing-critical graphs and their characterization from [8]. We continue with the introduction of an alphabet describing the tiles, which are the construction parts of large 2-crossing-critical graphs.

4.1. Characterization of 2-Crossing-Critical Graphs

Definition 8.

1. Crossing number $cr(G)$ of a graph G is the lowest number of edge crossings of a plane drawing of the graph G .
2. For a positive integer c , a graph G is c -crossing-critical if the crossing number $cr(G)$ is at least c but every proper subgraph H of G has $cr(H) < c$.

Theorem 3 ([8], Classification of 2-crossing-critical graphs). *Let G be a 2-crossing-critical graph with a minimum degree of at least 3. Then, one of the following holds:*

1. G is 3-connected, contains a subdivision of V_{10} , and has a very particular twisted Möbius band tile structure, with each tile isomorphic to one of 42 possibilities. All such structures are 3-connected and 2-crossing-critical.
2. G is 3-connected, does not have a subdivision of V_{10} , and has at most 3 million vertices.
3. G is not 3-connected and is one of 49 particular examples.
4. G is 2-but not 3-connected and is obtained from a 3-connected, 2-crossing-critical graph by replacing digons by digonal paths.

4.2. Construction of Large 2-Crossing-Critical Graphs

Definition 9.

1. For a sequence x , \bar{x} denotes the reversed sequence.
2. (a) The right-inverted tile of a tile $T = (G, x, y)$ is the tile $T^\uparrow = (G, x, \bar{y})$.
 (b) The left-inverted tile of a tile $T = (G, x, y)$ is the tile ${}^\downarrow T = (G, \bar{x}, y)$.
 (c) The inverted tile of a tile $T = (G, x, y)$ is the tile ${}^\uparrow T^\downarrow = (G, \bar{x}, \bar{y})$.
3. The set \mathcal{S} of tiles consists of those tiles obtained as combinations of two frames, shown in Figure 2, and 13 pictures, shown in Figure 3, in such a way that a picture is inserted into a frame by identifying the two geometric squares. (This typically involves subdividing the frame's square.) A given picture may be inserted into a frame either with the given orientation or with a 180° rotation.
4. The set $\mathcal{T}(\mathcal{S})$ consists of all graphs of the form $\circ((\otimes \mathcal{T})^\uparrow)$, where \mathcal{T} is a sequence $(T_0, {}^\uparrow T_1^\uparrow, T_2, \dots, {}^\uparrow T_{2m-1}^\uparrow, T_{2m})$ such that $m \geq 1$ and $\forall i : T_i \in \mathcal{S}$.



Figure 2. Two available frames.

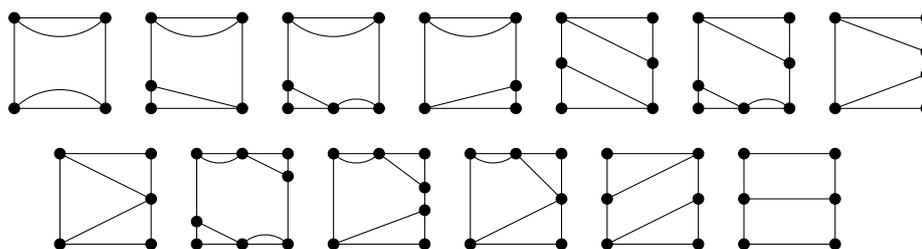


Figure 3. Thirteen available pictures to insert into a frame.

Large 2-crossing-critical graphs are described in Item 1 of Theorem 3. Item 3 of Definition 9 describes the set of tiles (see Figure 4 for example) used in the construction of large 2-crossing-critical graphs as described in Item 4 (see Figure 5 for example).

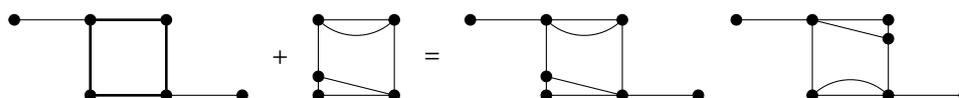


Figure 4. Demonstration of creation of tiles from \mathcal{S} .

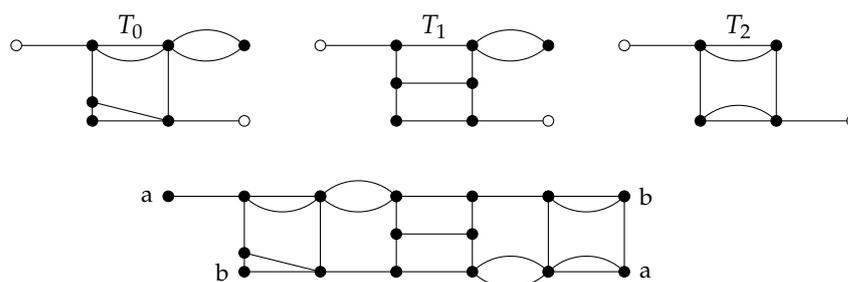


Figure 5. Example for $m = 1$. For $\mathcal{T} = (T_0, T_1, T_2)$, $G = \circ((\otimes \mathcal{T})^\dagger)$ is shown. When appropriate white vertices are identified, they are suppressed (see [8] for details).

4.3. The Alphabet Describing Tiles

In [43], the reader can find an alphabet to describe tiles in large 2-crossing-critical graphs. There are four attributes that describe a tile:

1. top path P_t : Describes the top path of the tile. $P_t \in \{D, A, V, B, H\}$ (see Figure 6 and 7).
2. identification Id : Describes if top and bottom paths of the tile intersect. $Id \in \{I, \emptyset\}$ (see Figure 8).
3. bottom path P_b : Describes the bottom path of the tile. $P_b \in \{D, A, V, B, \emptyset\}$ (see Figure 9).
4. frame Fr : Describes the frame used for the tile. $Fr \in \{L, dL\}$ (see Figure 10).

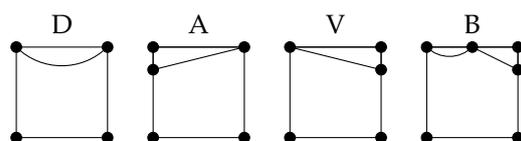


Figure 6. Alphabet letters describing top paths in tiles from \mathcal{S} .

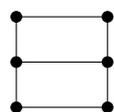


Figure 7. Additional letter H is used to describe one special picture. In this case, $P_t = H, Id = \emptyset, P_b = \emptyset$.

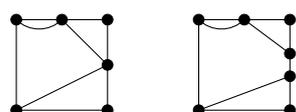


Figure 8. On the left side, $Id = I$; on the right side, $Id = \emptyset$.

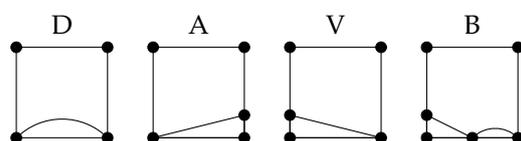


Figure 9. Alphabet letters describing bottom paths in tiles from \mathcal{S} .



Figure 10. Alphabet letters describing frames.

Using this notation, each tile $T \in \mathcal{S}$ has its own signature:

$$sig(T) = P_t Id P_b Fr.$$

If some attribute is equal to \emptyset , it is omitted in the signature. For a graph $G \in \mathcal{T}(\mathcal{S})$, $G = \circ((\otimes \mathcal{T})^\downarrow)$, where $\mathcal{T} = (T_0, \uparrow T_1^\uparrow, T_2, \dots, \uparrow T_{2m-1}^\uparrow, T_{2m})$, we introduce a signature in a natural way:

$$sig(G) = sig(T_0) sig(T_1) \dots sig(T_{2m-1}) sig(T_{2m}).$$

In connection with the introduced signature, we later use the following notation:

- for $X \in \{B, D, A, V, H, I, d\}$, $\#X$ is the number of occurrences of X in $sig(G)$;
- for $j \in \{0, 1, \dots, 2m\}$ and $X \in \{B, D, A, V, H, I, d\}$, $\#_j X$ is the number of occurrences of X in $sig(T_j)$; and
- for $j \in \{0, 1, \dots, 2m\}$, $p \in \{P_t, P_b\}$ and $X \in \{B, D, A, V\}$, $\#_j^p X$ is the number of occurrences of X as $sig(T_j)_p$.

5. Hamiltonian Cycles in Large 2-Crossing-Critical Graphs

In this section, we use the fact that large 2-crossing-critical graphs are a special case of 2-tiled graphs with a finite set of tiles to efficiently count Hamiltonian cycles with the use of algorithms from Section 3.

Remark 1. In the construction of large 2-crossing-critical graphs, degree one vertices of adjacent tiles that are to be identified are suppressed after identification so that there is no degree 2 vertex in G (see [8] for details and Figure 5 for example). Because of this, we define new types of frames, which are obtained from original frames by removing the tail of a frame (see Figure 11). We use these frames for constructing tiles in \mathcal{S} . Then, the cyclization of old tiles with additional suppression of a vertex is equivalent to the cyclization of new tiles (see Figure 12 for example). Note that all old graphs are the same as new ones, but the new tiles are not 2-degenerate; hence, for this method of construction of large 2-crossing-critical graphs, Theorem 2.18 from [8] does not yield 2-crossing-criticality. Each tile in a new set \mathcal{S} is a 2-tile, and large 2-crossing-critical graphs are obtained by cyclization of at least three such 2-tiles. Therefore, by Definition 2, they are 2-tiled graphs. Because of that, we can use the algorithms from Section 3 to count Hamiltonian cycles (efficiently).

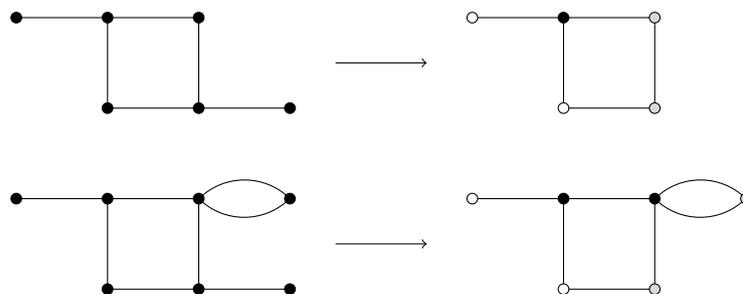


Figure 11. Transformation of frames. In transformed frames, white vertices are the left wall vertices and gray vertices are the right wall vertices of a 2-tile.

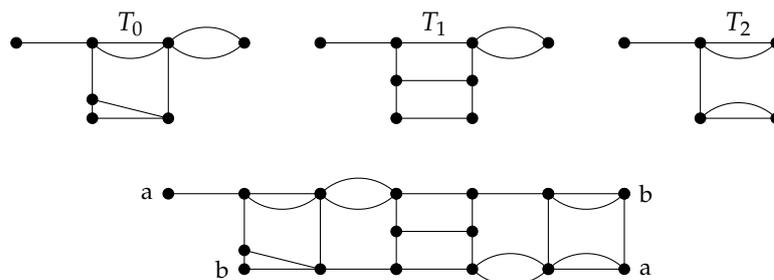


Figure 12. Graph G from Figure 5 can be obtained using modified tiles T_0, T_1, T_2 . The signature of this graph is $sig(G) = DVdL HdL DDL$. Later in Example 1, we show that the total number of Hamiltonian cycles in G is 224.

Let

$$R = \begin{bmatrix} a_{=} & a_{\times} \\ a_{\times} & a_{=} \end{bmatrix}$$

and

$$Z = \begin{bmatrix} a_{\{\bar{x}_2\}^{-}\{y_2\}} & a_{\{\bar{x}_1\}^{-}\{y_2\}} & a_{\{x_2\}^{-}\{y_2\}} & a_{\{x_1\}^{-}\{y_2\}} \\ a_{\{\bar{x}_2\}^{-}\{y_1\}} & a_{\{\bar{x}_1\}^{-}\{y_1\}} & a_{\{x_2\}^{-}\{y_1\}} & a_{\{x_1\}^{-}\{y_1\}} \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_2\}} & a_{\{\bar{x}_1\}^{-}\{\bar{y}_2\}} & a_{\{x_2\}^{-}\{\bar{y}_2\}} & a_{\{x_1\}^{-}\{\bar{y}_2\}} \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_1\}} & a_{\{\bar{x}_1\}^{-}\{\bar{y}_1\}} & a_{\{x_2\}^{-}\{\bar{y}_1\}} & a_{\{x_1\}^{-}\{\bar{y}_1\}} \end{bmatrix}$$

be matrices of tiles from \mathcal{S} (R is from Equation (2) and Z from Equation (4)).

Remark 2. In the construction of large 2-crossing-critical graphs, tiles at odd index (even index in algorithms) are inverted (see Item 4 of Definition 9). The matrices in the algorithms for such tiles (inverted ones) can be obtained from the original ones in time $\mathcal{O}(1)$:

$$\updownarrow R \updownarrow = \begin{bmatrix} \updownarrow a_{=} \updownarrow & \updownarrow a_{\times} \updownarrow \\ \updownarrow a_{\times} \updownarrow & \updownarrow a_{=} \updownarrow \end{bmatrix} = \begin{bmatrix} a_{=} & a_{\times} \\ a_{\times} & a_{=} \end{bmatrix} = R,$$

$$\begin{aligned} \updownarrow Z \updownarrow &= \begin{bmatrix} \updownarrow a_{\{\bar{x}_2\}^{-}\{y_2\}} \updownarrow & \updownarrow a_{\{\bar{x}_1\}^{-}\{y_2\}} \updownarrow & \updownarrow a_{\{x_2\}^{-}\{y_2\}} \updownarrow & \updownarrow a_{\{x_1\}^{-}\{y_2\}} \updownarrow \\ \updownarrow a_{\{\bar{x}_2\}^{-}\{y_1\}} \updownarrow & \updownarrow a_{\{\bar{x}_1\}^{-}\{y_1\}} \updownarrow & \updownarrow a_{\{x_2\}^{-}\{y_1\}} \updownarrow & \updownarrow a_{\{x_1\}^{-}\{y_1\}} \updownarrow \\ \updownarrow a_{\{\bar{x}_2\}^{-}\{\bar{y}_2\}} \updownarrow & \updownarrow a_{\{\bar{x}_1\}^{-}\{\bar{y}_2\}} \updownarrow & \updownarrow a_{\{x_2\}^{-}\{\bar{y}_2\}} \updownarrow & \updownarrow a_{\{x_1\}^{-}\{\bar{y}_2\}} \updownarrow \\ \updownarrow a_{\{\bar{x}_2\}^{-}\{\bar{y}_1\}} \updownarrow & \updownarrow a_{\{\bar{x}_1\}^{-}\{\bar{y}_1\}} \updownarrow & \updownarrow a_{\{x_2\}^{-}\{\bar{y}_1\}} \updownarrow & \updownarrow a_{\{x_1\}^{-}\{\bar{y}_1\}} \updownarrow \end{bmatrix} \\ &= \begin{bmatrix} a_{\{\bar{x}_1\}^{-}\{y_1\}} & a_{\{\bar{x}_2\}^{-}\{y_1\}} & a_{\{x_1\}^{-}\{y_1\}} & a_{\{x_2\}^{-}\{y_1\}} \\ a_{\{\bar{x}_1\}^{-}\{y_2\}} & a_{\{\bar{x}_2\}^{-}\{y_2\}} & a_{\{x_1\}^{-}\{y_2\}} & a_{\{x_2\}^{-}\{y_2\}} \\ a_{\{\bar{x}_1\}^{-}\{\bar{y}_1\}} & a_{\{\bar{x}_2\}^{-}\{\bar{y}_1\}} & a_{\{x_1\}^{-}\{\bar{y}_1\}} & a_{\{x_2\}^{-}\{\bar{y}_1\}} \\ a_{\{\bar{x}_1\}^{-}\{\bar{y}_2\}} & a_{\{\bar{x}_2\}^{-}\{\bar{y}_2\}} & a_{\{x_1\}^{-}\{\bar{y}_2\}} & a_{\{x_2\}^{-}\{\bar{y}_2\}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot Z \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= X \cdot Z \cdot X. \end{aligned}$$

Remark 3. In the construction of large 2-crossing-critical graphs, there is a twist in connecting the last and the first tiles (see Item 4 of Definition 9).

The matrices in the algorithms for the last tile (the right-inverted one) can be obtained from the original one in time $\mathcal{O}(1)$:

$$R \updownarrow = \begin{bmatrix} a_{=} \updownarrow & a_{\times} \updownarrow \\ a_{\times} \updownarrow & a_{=} \updownarrow \end{bmatrix} = \begin{bmatrix} a_{\times} & a_{=} \\ a_{=} & a_{\times} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot R,$$

$$\begin{aligned}
 Z^\updownarrow &= \begin{bmatrix} a_{\{\bar{x}_2\}^{-}\{y_2\}}^\updownarrow & a_{\{\bar{x}_1\}^{-}\{y_2\}}^\updownarrow & a_{\{x_2\}^{-}\{y_2\}}^\updownarrow & a_{\{x_1\}^{-}\{y_2\}}^\updownarrow \\ a_{\{\bar{x}_2\}^{-}\{y_1\}}^\updownarrow & a_{\{\bar{x}_1\}^{-}\{y_1\}}^\updownarrow & a_{\{x_2\}^{-}\{y_1\}}^\updownarrow & a_{\{x_1\}^{-}\{y_1\}}^\updownarrow \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_2\}}^\updownarrow & a_{\{\bar{x}_1\}^{-}\{\bar{y}_2\}}^\updownarrow & a_{\{x_2\}^{-}\{\bar{y}_2\}}^\updownarrow & a_{\{x_1\}^{-}\{\bar{y}_2\}}^\updownarrow \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_1\}}^\updownarrow & a_{\{\bar{x}_1\}^{-}\{\bar{y}_1\}}^\updownarrow & a_{\{x_2\}^{-}\{\bar{y}_1\}}^\updownarrow & a_{\{x_1\}^{-}\{\bar{y}_1\}}^\updownarrow \end{bmatrix} \\
 &= \begin{bmatrix} a_{\{\bar{x}_2\}^{-}\{y_1\}} & a_{\{\bar{x}_1\}^{-}\{y_1\}} & a_{\{x_2\}^{-}\{y_1\}} & a_{\{x_1\}^{-}\{y_1\}} \\ a_{\{\bar{x}_2\}^{-}\{y_2\}} & a_{\{\bar{x}_1\}^{-}\{y_2\}} & a_{\{x_2\}^{-}\{y_2\}} & a_{\{x_1\}^{-}\{y_2\}} \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_1\}} & a_{\{\bar{x}_1\}^{-}\{\bar{y}_1\}} & a_{\{x_2\}^{-}\{\bar{y}_1\}} & a_{\{x_1\}^{-}\{\bar{y}_1\}} \\ a_{\{\bar{x}_2\}^{-}\{\bar{y}_2\}} & a_{\{\bar{x}_1\}^{-}\{\bar{y}_2\}} & a_{\{x_2\}^{-}\{\bar{y}_2\}} & a_{\{x_1\}^{-}\{\bar{y}_2\}} \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot Z \\
 &= X \cdot Z.
 \end{aligned}$$

Remark 4. As the tiles in \mathcal{S} are planar, none of them contain an intertwined pair of disjoint paths; hence, none of the tiles from the set \mathcal{S} is of C -type \times . Using this observation with Remark 2 and Remark 3, we get that, for each tile in \mathcal{S} , the following holds:

$$R = \updownarrow R^\updownarrow = \begin{bmatrix} a_{=} & 0 \\ 0 & a_{=} \end{bmatrix} \text{ and } R^\updownarrow = \begin{bmatrix} 0 & a_{=} \\ a_{=} & 0 \end{bmatrix}.$$

Corollary 3. Let $G \in \mathcal{T}(\mathcal{S})$. The number of traversing Hamiltonian cycles in G is equal to

$$THC(G) = \prod_{i=1}^{2m+1} a_{=}^i,$$

where $a_{=}^i$ is the number of possibilities for T_i to be of C -type $=$.

Proof. Using Remark 4 in Equation (2), for $i \in \{1, 2, \dots, 2m\}$, we get

$$\begin{aligned}
 c^i &= a_{=}^i \cdot I \cdot c^{i-1} \\
 &= a_{=}^i \cdot c^{i-1}
 \end{aligned}$$

and

$$c^{2m+1} = a_{=}^{2m+1} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot c^{2m}.$$

Then,

$$c^{2m+1} = a_{=}^{2m+1} \cdot a_{=}^{2m} \dots a_{=}^1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \prod_{i=1}^{2m+1} a_{=}^i \end{bmatrix}.$$

Hence,

$$THC(G) = \prod_{i=1}^{2m+1} a_{=}^i.$$

□

Corollary 4. Let $G \in \mathcal{T}(\mathcal{S})$. The number of traversing Hamiltonian cycles in G is equal to

$$THC(G) = 2^{\#B+\#D+\#H+\#I+\#d}.$$

Proof. We have shown before that $THC(G) = \prod_{i=1}^{2m+1} a_{=}^i$. Using the alphabet defined above, we notice that

$$a_{=}^i = 2^{\#_i B + \#_i D + \#_i H + \#_i I + \#_i d}.$$

Then,

$$\begin{aligned} THC(G) &= \prod_{i=1}^{2m+1} 2^{\#_i B + \#_i D + \#_i H + \#_i I + \#_i d} \\ &= 2^{\left(\sum_{i=1}^{2m+1} \#_i B\right) + \left(\sum_{i=1}^{2m+1} \#_i D\right) + \left(\sum_{i=1}^{2m+1} \#_i H\right) + \left(\sum_{i=1}^{2m+1} \#_i I\right) + \left(\sum_{i=1}^{2m+1} \#_i d\right)} \\ &= 2^{\#B + \#D + \#H + \#I + \#d}. \end{aligned}$$

□

Corollary 5. Let $G \in \mathcal{T}(\mathcal{S})$. The number of flanking Hamiltonian cycles in G is equal to

$$FHC(G) = THC(G) \cdot \sum_{i=1}^{2m+1} \frac{a_{\parallel}^i \cdot a_{=}^{i+1} + a_{\parallel\parallel}^{i,i+1}}{a_{=}^i \cdot a_{=}^{i+1}},$$

where

- $THC(G)$ is the number of traversing Hamiltonian cycles in G ,
- $a_{=}^i$ is the number of possibilities for T_i to be of C-type =,
- a_{\parallel}^i is the number of possibilities for T_i to be of C-type ||,
- $a_{\parallel\parallel}^{i,i+1}$ is the number of distinct possibilities for T_i and T_{i+1} to be of compatible flanking C-types of forms $|\{x,y\}$ and $|\{z,w\}|$.

Proof. As shown in the proof of Lemma 5,

$$\begin{aligned} FHC(G) &= \sum_{i=1}^{2m+1} a_{\parallel}^i \cdot (c_{even}^{i+1,i-1} + c_{odd}^{i+1,i-1}) + \sum_{i=1}^{2m+1} a_{\parallel\parallel}^{i,i+1} \cdot (c_{even}^{i+2,i-1} + c_{odd}^{i+2,i-1}) \\ &\quad + \sum_{i=1}^{2m+1} a_{\parallel\parallel}^{i,i+1,i+2} \cdot (c_{even}^{i+3,i-1} + c_{odd}^{i+3,i-1}). \end{aligned}$$

Because each tile in \mathcal{S} contains an internal vertex, for each tile in \mathcal{S} , the value $a_{\emptyset} = 0$ (see the proof of Lemma 5). Hence, $\forall i \in \{1, 2, \dots, 2m + 1\}, a_{\parallel\parallel}^{i,i+1,i+2} = 0$. Using Remark 4 as in the proof of Corollary 3, we get that

$$\begin{bmatrix} i+1, i-1 \\ c_{even} \\ i+1, i-1 \\ c_{odd} \end{bmatrix} = \begin{bmatrix} 0 \\ \prod_{\substack{j=1 \\ j \neq i}}^{2m+1} a_{=}^j \end{bmatrix} \text{ and } \begin{bmatrix} i+2, i-1 \\ c_{even} \\ i+2, i-1 \\ c_{odd} \end{bmatrix} = \begin{bmatrix} 0 \\ \prod_{\substack{j=1 \\ j \notin \{i, i+1\}}}^{2m+1} a_{=}^j \end{bmatrix}.$$

It is easy to check that, for each tile in \mathcal{S} , the value $a_{=} > 0$. Using observations and the result from the proof of Corollary 3 that $THC(G) = \prod_{j=1}^{2m+1} a_{=}^j$, we get

$$c_{odd}^{i+1,i-1} = \prod_{\substack{j=1 \\ j \neq i}}^{2m+1} a_{=}^j = \frac{1}{a_{=}^i} \prod_{j=1}^{2m+1} a_{=}^j = \frac{1}{a_{=}^i} \cdot THC(G),$$

$$c_{odd}^{i+2,i-1} = \prod_{j \notin \{i,i+1\}}^{2m+1} a_{=}^j = \frac{1}{a_{=}^i \cdot a_{=}^{i+1}} \prod_{j=1}^{2m+1} a_{=}^j = \frac{1}{a_{=}^i \cdot a_{=}^{i+1}} \cdot THC(G).$$

Hence,

$$FHC(G) = THC(G) \cdot \sum_{i=1}^{2m+1} \frac{a_{=}^i \cdot a_{=}^{i+1} + a_{=}^{i+1}}{a_{=}^i \cdot a_{=}^{i+1}}.$$

□

Corollary 6. Let $G \in \mathcal{T}(\mathcal{S})$. The number of flanking Hamiltonian cycles in G is equal to

$$FHC(G) = THC(G) \cdot \sum_{i=1}^{2m+1} \frac{2^{\#_i d} \cdot 2^{\#_{i+1} B + \#_{i+1} D + \#_{i+1} H + \#_{i+1} I + \#_{i+1} d} + Y(T_i, T_{i+1})}{2^{\#_i B + \#_i D + \#_i H + \#_i I + \#_i d} \cdot 2^{\#_{i+1} B + \#_{i+1} D + \#_{i+1} H + \#_{i+1} I + \#_{i+1} d'}}$$

where

$$Y(T_i, T_{i+1}) = \begin{cases} (1 - \#_i H) \cdot 2^{\#_i B + \#_i D} \cdot (1 - \#_{i+1} H) \cdot 2^{\#_{i+1} B + \#_{i+1} D + \#_{i+1} d}, & \text{if } sig(T_i)_{Fr} = dL \\ (1 - \#_i H) \cdot 2^{\#_i B + \#_i D} \cdot (\#_{i+1}^{P_b} V \cdot 2^{\#_{i+1}^{P_t} B + \#_{i+1}^{P_t} D} + 2 \cdot \#_{i+1}^{P_b} B \cdot 2^{\#_{i+1}^{P_t} B + \#_{i+1}^{P_t} D} + \#_{i+1}^{P_t} V) \cdot 2^{\#_{i+1} d} + (1 - \#_{i+1}^{P_b} B) + \#_{i+1} H - \#_{i+1}^{P_b} V \cdot \#_{i+1}^{P_t} V) \cdot 2^{\#_{i+1} d} + (\#_i^{P_t} V \cdot 2^{\#_i^{P_b} B + \#_i^{P_b} D} + 2 \cdot \#_i^{P_t} B \cdot 2^{\#_i^{P_b} B + \#_i^{P_b} D} + \#_i^{P_b} V \cdot (1 - \#_i^{P_t} B) + \#_i H - \#_i^{P_t} V \cdot \#_i^{P_b} V) \cdot 2^{\#_i d} + (1 - \#_{i+1} H) \cdot 2^{\#_{i+1} B + \#_{i+1} D + \#_{i+1} d}, & \text{if } sig(T_i)_{Fr} = L \end{cases}$$

Note that the notation implies the number can be computed solely from the signature of a 2-crossing-critical graph. The structure of the equation is explained through the proof that follows.

Proof. We have shown in the proof of Corollary 4 that $a_{=}^i = 2^{\#_i D + \#_i B + \#_i H + \#_i I + \#_i d}$. It is easy to see that

$$a_{=}^i = \begin{cases} 2; & \text{if } sig(T_i)_{Fr} = dL \\ 1; & \text{if } sig(T_i)_{Fr} = L \end{cases} = 2^{\#_i d}.$$

It remains to show that $a_{=}^{i,i+1} = Y(T_i, T_{i+1})$:

- Let $sig(T_i)_{Fr} = dL$. Using Figure 13, it is easy to see that

$$a_{=}^{i,i+1} = a_{=}^i|_{\{\bar{y}_1, \bar{y}_2\}} \cdot a_{=}^{i+1}|_{\{x_1, \bar{x}_2\}}.$$

For $a_{=}^i|_{\{\bar{y}_1, \bar{y}_2\}}$, all pictures except H are valid, and paths B and D add a multiplier 2. Hence,

$$a_{=}^i|_{\{\bar{y}_1, \bar{y}_2\}} = (1 - \#_i H) \cdot 2^{\#_i B + \#_i D}.$$

For $a_{\{x_1, \bar{x}_2\}}^{i+1}$, all pictures except H are valid, and paths B and D , and the frame dL add a multiplier 2. Hence,

$$a_{\{x_1, \bar{x}_2\}}^{i+1} = (1 - \#_{i+1}H) \cdot 2^{\#_{i+1}B + \#_{i+1}D + \#_{i+1}d}.$$

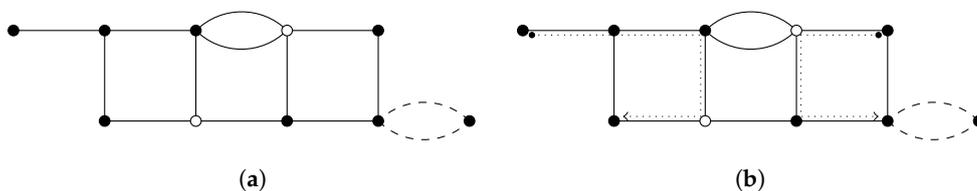


Figure 13. (a) Drawing of $T_i \otimes T_{i+1}^\uparrow$, where $sig(T_i)_{Fr} = dL$. White vertices are right wall vertices of tile T_i and left wall vertices of tile T_{i+1} . (b) Dotted arrows show the only possible combination for $a_{\{y_1, y_2\}}^{i+1}$.

2. Let $sig(T_i)_{Fr} = L$. Using Figure 14, it is easy to see that

$$a_{\{y_1, y_2\}}^{i+1} = a_{\{y_1, y_2\}}^i \cdot a_{\{\bar{x}_1, \bar{x}_2\}}^{i+1} + a_{\{\bar{y}_1, y_2\}}^i \cdot a_{\{x_1, \bar{x}_2\}}^{i+1}.$$

For $a_{\{y_1, y_2\}}^i$, all pictures except H are valid, and paths B and D add a multiplier 2. Hence,

$$a_{\{y_1, y_2\}}^i = (1 - \#_iH) \cdot 2^{\#_iB + \#_iD}.$$

For $a_{\{\bar{x}_1, \bar{x}_2\}}^{i+1}$, there are several options:

- The bottom path is V ; the top path is any of the possible ones; and top paths B and D , and the frame dL add a multiplier 2.
- The bottom path is B ; the top path is any of the possible ones; and the bottom path B , top paths B and D , and the frame dL add a multiplier 2.
- The top path is V ; the bottom path is any of the possible ones, except B ; and the frame dL adds a multiplier 2.
- The top path is H , and the frame dL adds a multiplier 2.

The first and the third options both cover the picture with the top path V and the bottom path V . Hence,

$$a_{\{\bar{x}_1, \bar{x}_2\}}^{i+1} = (\#_{i+1}^{P_b} V \cdot 2^{\#_{i+1}^{P_t} B + \#_{i+1}^{P_t} D} + 2 \cdot \#_{i+1}^{P_b} B \cdot 2^{\#_{i+1}^{P_t} B + \#_{i+1}^{P_t} D} + \#_{i+1}^{P_t} V \cdot (1 - \#_{i+1}^{P_b} B) + \#_{i+1}H - \#_{i+1}^{P_b} V \cdot \#_{i+1}^{P_t} V) \cdot 2^{\#_{i+1}d}.$$

For $a_{\{\bar{y}_1, y_2\}}^i$, there are several options:

- The top path is V ; the bottom path is any of the possible ones; and the bottom paths B and D add a multiplier 2.
- The top path is B ; the bottom path is any of the possible ones; and the top path B and bottom paths B and D add a multiplier 2.
- The bottom path is V , and the top path is any of the possible ones, except B .
- The top path is H .

The first and the third options both cover the picture with top path V and bottom path V . Hence,

$$a_{\{\bar{y}_1, y_2\}}^i = \#_i^{P_t} V \cdot 2^{\#_i^{P_b} B + \#_i^{P_b} D} + 2 \cdot \#_i^{P_t} B \cdot 2^{\#_i^{P_b} B + \#_i^{P_b} D} + \#_i^{P_b} V \cdot (1 - \#_i^{P_t} B) + \#_iH - \#_i^{P_t} V \cdot \#_i^{P_b} V.$$

For $a_{\{x_1, \bar{x}_2\}}^{i+1}$, all pictures except H are valid, and paths B and D , and the frame dL add a multiplier 2. Hence,

$$a_{\{x_1, \bar{x}_2\}}^{i+1} = (1 - \#_{i+1}H) \cdot 2^{\#_{i+1}B + \#_{i+1}D + \#_{i+1}d}.$$

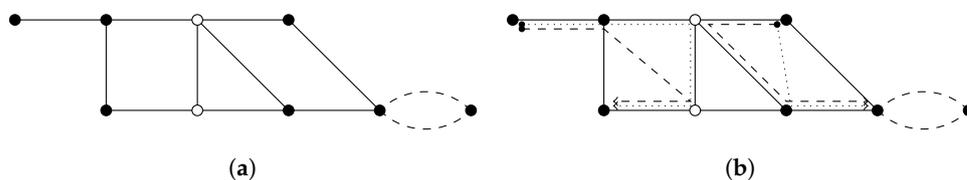


Figure 14. (a) Drawing of $T_i \otimes \uparrow T_{i+1}^\uparrow$, where $sig(T_i)_{Fr} = L$. White vertices are right wall vertices of tile T_i and left wall vertices of tile T_{i+1} . (b) Dotted and dashed arrows show two possible combinations for $a_{\{x_1, \bar{x}_2\}}^{i,i+1}$. □

Corollary 7. Let $G \in \mathcal{T}(S)$. The number of zigzagging Hamiltonian cycles in G is bounded by

$$0 \leq ZHC(G) \leq 8^{2m+1}.$$

Proof. To count zigzagging Hamiltonian cycles, the algorithm from the proof of Lemma 6 with a slight difference (explained in Remark 2 and Remark 3) is used:

$$\begin{aligned} ZHC(G) &= tr(Z_{2m+1}^\uparrow \cdot \downarrow Z_{2m}^\uparrow \cdot Z_{2m-1} \cdots Z_3 \cdot \downarrow Z_2^\uparrow \cdot Z_1) \\ &= tr((X \cdot Z_{2m+1}) \cdot (X \cdot Z_{2m} \cdot X) \cdot Z_{2m-1} \cdots Z_3 \cdot (X \cdot Z_2 \cdot X) \cdot Z_1) \\ &= tr((X \cdot Z_{2m+1}) \cdot (X \cdot Z_{2m}) \cdot (X \cdot Z_{2m-1}) \cdots (X \cdot Z_2) \cdot (X \cdot Z_1)), \end{aligned} \tag{5}$$

where X is the matrix from Remark 3.

The lower bound is achieved by a graph $G \in \mathcal{T}(S)$, where $\forall i \in \{1, 2, \dots, 2m + 1\} : sig(T_i) = DDdL$. In this case, the matrices Z_i are the following:

$$Z_i = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}.$$

Hence,

$$X \cdot Z_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{bmatrix}.$$

Then,

$$(X \cdot Z_{2m+1}) \cdot (X \cdot Z_{2m}) \cdot (X \cdot Z_{2m-1}) \cdots (X \cdot Z_2) \cdot (X \cdot Z_1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4 \cdot 8^m & 0 \\ 2 \cdot 8^m & 2 \cdot 8^m & 0 & 0 \\ 4 \cdot 8^m & 4 \cdot 8^m & 0 & 0 \end{bmatrix}$$

and

$$ZHC(G) = tr((X \cdot Z_{2m+1}) \cdot (X \cdot Z_{2m}) \cdot (X \cdot Z_{2m-1}) \cdots (X \cdot Z_2) \cdot (X \cdot Z_1)) = 0.$$

We now study the upper bound for the number of zigzagging Hamiltonian cycles.

Remark 5. For matrices $X, Y \in M_n(\mathbb{R}_0^+)$, let the coefficient $K(X, Y)$ be defined as

$$K(X, Y) = n - (\#_{\text{zero columns in } X} + \#_{\text{zero rows in } Y} - \#_{\text{of indices } i \text{ so that the } X^i \text{ column and } Y_i \text{ row are both zero}}).$$

If $UB(X)$ and $UB(Y)$ are the upper bounds for elements in matrices X and Y , then

$$UB(X \cdot Y) = K(X, Y) \cdot UB(X) \cdot UB(Y)$$

is an upper bound for elements in matrix $X \cdot Y$ (this is a direct corollary of the definition of matrix multiplication).

Based on the frame, we get the following two types of matrices:

- Tile with frame L :

$$Z_L = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix},$$

where $a_{ij} \leq 2$ (frame adds a factor 1 and pictures add a factor 2).

- Tile with frame dL :

$$Z_{dL} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix},$$

where $a_{ij} \leq 4$ (frame adds a factor 2 and pictures add a factor 2).

If we use the observation (5), we have two types of matrices in the product:

- Tile with frame L :

$$X \cdot Z_L = \begin{bmatrix} a_{21} & a_{22} & a_{23} & 0 \\ a_{11} & a_{22} & a_{13} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix},$$

where $a_{ij} \leq 2$ and so $UB(X \cdot Z_L) = 2$.

- Tile with frame dL :

$$X \cdot Z_{dL} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix},$$

where $a_{ij} \leq 4$ and so $UB(X \cdot Z_{dL}) = 4$.

We introduce two types of matrices:

$$R_1 = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{bmatrix} \text{ and } R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{bmatrix}.$$

Remark 6. It is obvious that $X \cdot Z_L$ is of type R_1 and that $X \cdot Z_{dL}$ is of type R_2 .

For their product, the following holds:

$$\begin{aligned}
 R_1 \cdot R_1 &\text{ is of type } R_1, \\
 R_1 \cdot R_2 &\text{ is of type } R_1, \\
 R_2 \cdot R_1 &\text{ is of type } R_2, \\
 R_2 \cdot R_2 &\text{ is of type } R_2.
 \end{aligned}
 \tag{6}$$

For coefficient defined in Remark 5, the following holds:

$$\begin{aligned}
 K(R_1, R_1) &= 3, \\
 K(R_1, R_2) &= 2, \\
 K(R_2, R_1) &= 3, \\
 K(R_2, R_2) &= 2.
 \end{aligned}
 \tag{7}$$

Using Remark 6 and observations (6) and (7), we get the following combinations:

$$\begin{aligned}
 UB((X \cdot Z_L) \cdot (X \cdot Z_L)) &= 3 \cdot 2 \cdot 2 = 12, \\
 UB((X \cdot Z_{dL}) \cdot (X \cdot Z_L)) &= 3 \cdot 4 \cdot 2 = 24, \\
 UB((X \cdot Z_L) \cdot (X \cdot Z_{dL})) &= 2 \cdot 2 \cdot 2 = 8, \\
 UB((X \cdot Z_{dL}) \cdot (X \cdot Z_{dL})) &= 2 \cdot 4 \cdot 4 = 32.
 \end{aligned}$$

It is easy to check that we get the largest bound by using a combination with all components of the product (5) equal to $Z = X \cdot Z_{dL}$. Then,

$$\begin{aligned}
 UB(Z^2) &= K(Z, Z) \cdot UB(Z) \cdot UB(Z) = 2 \cdot UB(Z)^2 \\
 UB(Z^3) &= K(Z, Z^2) \cdot UB(Z) \cdot UB(Z^2) = 2 \cdot UB(Z) \cdot 2 \cdot UB(Z)^2 = 2^2 \cdot UB(Z)^3 \\
 &\vdots \\
 UB(Z^{2m+1}) &= K(Z, Z^{2m}) \cdot UB(Z) \cdot UB(Z^{2m}) = 2 \cdot UB(Z) \cdot 2^{2m-1} \cdot UB(Z)^{2m} \\
 &= 2^{2m} \cdot UB(Z)^{2m+1}.
 \end{aligned}$$

Because $UB(Z) = 4$, we get

$$UB(Z^{2m+1}) = 2^{2m} \cdot 4^{2m+1}.$$

Because Z^{2m+1} is of type R_2 , we get that

$$ZHC(G) \leq ZHC(Z^{2m+1}) = tr(Z^{2m+1}) = 2 \cdot UB(Z^{2m+1}) = 2 \cdot 2^{2m} \cdot 4^{2m+1} = 8^{2m+1}.$$

□

Example 1. We now compute the number of Hamiltonian cycles for the graph from Figure 12:

$$\begin{aligned}
 THC(G) &= \prod_{i=0}^2 a_{=}^i \\
 &= 4 \cdot 4 \cdot 4 \\
 &= 64, \\
 FHC(G) &= THC(G) \cdot \sum_{i=0}^2 \frac{a_{=}^i \cdot a_{=}^{i+1} + a_{=}^{i,i+1}}{a_{=}^i \cdot a_{=}^{i+1}} \\
 &= 64 \cdot \left(\frac{2 \cdot 4 + 0}{4 \cdot 4} + \frac{2 \cdot 4 + 0}{4 \cdot 4} + \frac{1 \cdot 4 + 16}{4 \cdot 4} \right) \\
 &= 144, \\
 ZHC(G) &= tr((X \cdot Z_2) \cdot (X \cdot Z_1) \cdot (X \cdot Z_0)) \\
 &= 16,
 \end{aligned}$$

because

$$Z_0 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and so

$$(X \cdot Z_2) \cdot (X \cdot Z_1) \cdot (X \cdot Z_0) = \begin{bmatrix} 8 & 8 & 8 & 0 \\ 4 & 4 & 4 & 0 \\ 4 & 4 & 4 & 0 \\ 8 & 8 & 8 & 0 \end{bmatrix}.$$

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Appendix A

Table A1. Glossary of frequently used symbols.

Notation	Description
Λ_z	Set of all possible zigzagging C-types. $\Lambda_z = \{ \{x_1\} - \{y_1\}, \{x_1\} - \{\bar{y}_1\}, \{x_1\} - \{\bar{x}_1\} - \{y_1\}, \{x_1\} - \{\bar{y}_1\} - \{\bar{x}_1\} - \{y_1\},$ $\{x_1\} - \{y_2\}, \{x_1\} - \{\bar{y}_2\}, \{x_1\} - \{\bar{x}_1\} - \{y_2\}, \{x_1\} - \{\bar{y}_2\} - \{\bar{x}_1\} - \{y_2\},$ $\{x_2\} - \{y_1\}, \{x_2\} - \{\bar{y}_1\}, \{x_2\} - \{\bar{x}_2\} - \{y_1\}, \{x_2\} - \{\bar{y}_1\} - \{\bar{x}_2\} - \{y_1\},$ $\{x_2\} - \{y_2\}, \{x_2\} - \{\bar{y}_2\}, \{x_2\} - \{\bar{x}_2\} - \{y_2\}, \{x_2\} - \{\bar{y}_2\} - \{\bar{x}_2\} - \{y_2\} \}.$
Λ_t	Set of all possible traversing C-types. $\Lambda_t = \{ =, \times \}.$
Λ_f	Set of all possible flanking C-types. $\Lambda_f = \{ \emptyset, , _{\{y_1, y_2\}}, _{\{\bar{y}_1, y_2\}}, _{\{y_1, \bar{y}_2\}}, _{\{\bar{y}_1, \bar{y}_2\}}, _{\{x_1, x_2\}}, _{\{\bar{x}_1, x_2\}}, _{\{x_1, \bar{x}_2\}}, _{\{\bar{x}_1, \bar{x}_2\}} \}.$
Λ	Set of all possible C-types. $\Lambda = \Lambda_z \cup \Lambda_t \cup \Lambda_f.$
a_{λ}^i	Number of possibilities for tile T_i to be of C-type $\lambda \in \Lambda.$
c_{even}^i	Number of distinct possibilities in $C \cap T_1 \otimes T_2 \otimes \dots \otimes T_i$ with even number of tiles of C-type \times and all others of C-type $=.$
c_{odd}^i	Number of distinct possibilities in $C \cap T_1 \otimes T_2 \otimes \dots \otimes T_i$ with odd number of tiles of C-type \times and all other of C-type $=.$
R_i	Notation for matrix $\begin{bmatrix} a_{=}^i & a_{\times}^i \\ a_{\times}^i & a_{=}^i \end{bmatrix}.$
$T^{i,i+l}$	Represents the join $T_i \otimes T_{i+1} \otimes \dots \otimes T_{i+l}.$
$a^{i,i+l}$	Number of distinct possibilities for $T_i, T_{i+1}, \dots, T_{i+l}$ to be of compatible flanking C-types to turn around a cycle in $T^{i,i+l}, l \in \{0, 1, 2\}.$
$a_{ }^{i,i+1}$	Number of distinct possibilities for T_i and T_{i+1} to be of compatible flanking C-types of form $ _{\{x,y\}}$ and $ _{\{z,w\}}.$
$a_{ \emptyset }^{i,i+1,i+2}$	Number of distinct possibilities to turn around in three consecutive tiles T_i, T_{i+1} and $T_{i+2}.$
Z_i	Notation for matrix $\begin{bmatrix} a_{\{\bar{x}_2\} - \{y_2\}}^i & a_{\{\bar{x}_1\} - \{y_2\}}^i & a_{\{x_2\} - \{y_2\}}^i & a_{\{x_1\} - \{y_2\}}^i \\ a_{\{\bar{x}_2\} - \{y_1\}}^i & a_{\{\bar{x}_1\} - \{y_1\}}^i & a_{\{x_2\} - \{y_1\}}^i & a_{\{x_1\} - \{y_1\}}^i \\ a_{\{\bar{x}_2\} - \{\bar{y}_2\}}^i & a_{\{\bar{x}_1\} - \{\bar{y}_2\}}^i & a_{\{x_2\} - \{\bar{y}_2\}}^i & a_{\{x_1\} - \{\bar{y}_2\}}^i \\ a_{\{\bar{x}_2\} - \{\bar{y}_1\}}^i & a_{\{\bar{x}_1\} - \{\bar{y}_1\}}^i & a_{\{x_2\} - \{\bar{y}_1\}}^i & a_{\{x_1\} - \{\bar{y}_1\}}^i \end{bmatrix}.$
P_t	Describes the top path of the tile. $P_t \in \{A, V, D, B, H\}.$
Id	Describes if the top and bottom paths of the tile intersect. $Id \in \{I, \emptyset\}.$
P_b	Describes the bottom path of the tile. $P_b \in \{A, V, D, B, \emptyset\}.$
Fr	Describes the frame used for the tile. $Fr \in \{L, dL\}.$

Table A1. Cont.

Notation	Description
$\#X$	The number of occurrences of X in $\text{sig}(G)$.
$\#_j X$	The number of occurrences of X in $\text{sig}(T_j)$.
$\#_j^p X$	The number of occurrences of X in $\text{sig}(T_j)_p$.
$\text{THC}(G)$	The number of traversing Hamiltonian cycles in $G \in \mathcal{T}(\mathcal{S})$.
$\text{FHC}(G)$	The number of flanking Hamiltonian cycles in $G \in \mathcal{T}(\mathcal{S})$.
$\text{ZHC}(G)$	The number of zigzagging Hamiltonian cycles in $G \in \mathcal{T}(\mathcal{S})$.

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