mathematics

Article

Qi Type Diamond-Alpha Integral Inequalities

Zhong-Xuan Mao, Ya-Ru Zhu *, Bao-Hua Guo, Fu-Hai Wang, Yu-Hua Yang and Hai-Qing Zhao

Department of Mathematics and Physics, North China Electric Power University, Yonghua Street 619, Baoding 071003, China; maozhongxuan@ncepu.edu.cn (Z.-X.M.); guobh@ncepu.edu.cn (B.-H.G.); wangfuhai@ncepu.edu.cn (F.-H.W.); yang_yuhua03@ncepu.edu.cn (Y.-H.Y.); zhaohaiqing@ncepu.edu.cn (H.-Q.Z.)
* Correspondence: zhuyaru1982@ncepu.edu.cn

Abstract: In this paper, we establish sufficient conditions for Qi type diamond-alpha integral inequalities and its generalized form on time scales.

Keywords: Qi type integral inequality; Jensen's inequality; diamond-α dynamic derivatives; analysis method; time scale

MSC: Primary 26E70; 26E25

check for updates

Citation: Mao, Z.-X.; Zhu, Y.-R.; Guo, B.-H.; Wang, F.-H.; Yang, Y.-H.; Zhao, H.-Q. Qi Type Diamond-Alpha Integral Inequalities. *Mathematics* **2021**, *9*, 449. https://doi.org/10.3390/math9040449

Academic Editor: Simeon Reich

Received: 26 January 2021
Accepted: 18 February 2021
Published: 23 February 2021

1. Introduction

In 2000, Qi provided an open problem following [1], "Qi type integral inequality" for short in this paper.

Theorem 1 (Open problem). *Under what conditions does the following inequality hold,*

$$\int_\alpha^\beta q^p(t)\mathrm{d}t \geq \Big(\int_\alpha^\beta q(t)\mathrm{d}t\Big)^{p-1},$$

for $p>1$.

Yu and Qi [2] deduced the following theorem via Jensen's inequality.

Theorem 2. *If $q \in C([\alpha,\beta])$, $\int_\alpha^\beta q(t)\mathrm{d}t \geq (\beta-\alpha)^{p-1}$ for given $p>1$, then*

$$\int_\alpha^\beta q^p(t)\mathrm{d}t \geq \Big(\int_\alpha^\beta q(t)\mathrm{d}t\Big)^{p-1}. \tag{1}$$

The open problem has attracted the interest of many authors [3–7]. The analytic method and employing the Jensen's inequality are two powerful methods for the study of Qi type integral inequality. Meanwhile, studies in the past two decades have provided some promotions of the inequality.

Pogány [3] posed the following inequality

$$\int_\alpha^\beta q^{p_1}(t)\mathrm{d}t \geq \Big(\int_\alpha^\beta q(t)\mathrm{d}t\Big)^{p_2}, \qquad p_2>0, p_1>\max\{p_2,1\}, \tag{2}$$

and gave a sufficient condition for (2) by using the Hölder inequality.

In [4], the authors proved the following results which strengthen the Qi type integral inequality.

Theorem 3. *If $q:[\alpha,\beta]\to\mathbb{R}$ is non-negative and increasing, $q'(t) > (p-2)(t-\alpha)^{p-3}$ for all $p>3$, then*

$$\int_\alpha^\beta q^p(t)\mathrm{d}t - \Big(\int_\alpha^\beta q(t)\mathrm{d}t\Big)^{p-1} \geq q^{p-1}(\alpha)\int_\alpha^\beta q(t)\mathrm{d}t.$$

Theorem 4. *If $q : [\alpha, \beta] \to \mathbb{R}$ is non-negative and increasing, $q'(t) > p((t-\alpha)/(\beta-\alpha))^{p-1}$ for given $p \geq 1$, then*

$$\int_{\alpha}^{\beta} q^{p+2}(t)\mathrm{d}t - \frac{1}{(b-a)^{p-1}}\Big(\int_{\alpha}^{\beta} q(t)\mathrm{d}t\Big)^{p+1} \geq q^{p+1}(\alpha)\int_{\alpha}^{\beta} q(t)\mathrm{d}t. \tag{3}$$

Since the theory of time scales was established by Hilger [8] in 1988, it has been used widely by many branches of sciences such as finance, statistics, physics. Moreover, many researches about the theory which unifies and gives a generalization of the discrete theory and the continuous theory have been published, such as [9–18].

As a generalization of the differential in calculus, Δ(delta) and ∇(nabla) dynamic derivatives play a foundational role in the time scales. Recently, researchers also have provided $\diamond_\alpha$ as a weighting between Δ and ∇ dynamic derivatives. It was defined as a linear combination of Δ and ∇ dynamic derivatives. Readers can consult [19] to find out more basic rules of $\diamond_\alpha$ dynamic derivatives.

Some works in recent years established the Qi type integral inequality on time scales [5,20].

Theorem 5 (**Qi type Δ-integral inequality**)**.** *([20]) If $p \geq 3$ and ϕ is a monotonic non-negative function defined on $[\alpha, \beta]_{\mathbb{T}}$ which satisfies*

$$\phi^{p-2}(s)\phi^{\Delta}(s) \geq \sigma^{\Delta}(s)(p-2)(\sigma^2(s)-\alpha)^{p-3}\phi^{p-2}(\sigma^2(s)),$$

for all $s \in [\alpha, \beta]_{\mathbb{T}}$. Then

$$\int_{\alpha}^{\beta} \phi^{p}(s)\Delta s \geq \Big(\int_{\alpha}^{\beta} \phi(s)\Delta s\Big)^{p-1}.$$

Theorem 6 (**Qi type ∇-integral inequality**)**.** *([20]) If $p \geq 3$ and ϕ is a monotonic non-negative function defined on $[\alpha, \beta]_{\mathbb{T}}$ which satisfies*

$$\phi^{\nabla}(s) \geq (p-2)(s-\alpha)^{p-3},$$

for all $s \in [\alpha, \beta]_{\mathbb{T}}$. Then

$$\int_{\alpha}^{\beta} \phi^{p}(s)\nabla s \geq \Big(\int_{\alpha}^{\beta} \phi(s)\nabla s\Big)^{p-1}.$$

Theorem 7. *([5]) If $p \geq 3$, $\phi : [\alpha, \beta]_{\mathbb{T}} \to [0, +\infty)$ is Δ-differential and increasing function satisfies*

$$\phi^{p-2}(s)\phi^{\Delta}(s) \geq (p-2)(\phi(\sigma^2(s)))^{p-2}(\sigma^2(s)-\alpha)^{p-3}\sigma^{\Delta}(s).$$

Then

$$\int_{\alpha}^{\beta} \phi^{p}(s)\Delta s - \Big(\int_{\alpha}^{\beta} \phi(s)\Delta s\Big)^{p-1} \geq \phi^{p-2}(\alpha)\Big(\phi(\alpha) - (p-1)\mu^{p-2}(\alpha)\Big)\int_{\alpha}^{\beta} \phi(s)\Delta s.$$

Theorem 8. *([5]) If $p \geq 3$, $\phi : [\alpha, \beta]_{\mathbb{T}} \to [0, +\infty)$ is ∇-differential and increasing function satisfies*

$$\Big(\phi(\rho(s))\Big)^{p-2}\phi^{\nabla}(s) \geq (p-2)\phi^{p-2}(s)(s-\alpha)^{p-3}.$$

Then

$$\int_{\alpha}^{\beta} \phi^{p}(s)\nabla s - \Big(\int_{\alpha}^{\beta} \phi(s)\nabla s\Big)^{p-1} \geq \phi^{p-1}(\alpha)\int_{\alpha}^{\beta} \phi(s)\Delta s.$$

However, generalizing the Qi type integral inequality to the diamond-alpha integral had been a largely under explored domain which none of works has been devoted to it.

The first aim of this paper is to determine a sufficient condition for inequality (5) via analytic method in Theorem 9.

Then we will consider the inequalities (2) and (3) generalized to diamond-alpha integral cases, that is, we will determine the sufficient conditions for inequalities (7) and (8) in Theorems 10 and 11.

Meanwhile we also consider a sufficient condition for the reverse of inequality (7) in Theorem 12.

Last but not least, we will give concise solutions of the open Problem 1 generalized on time scales via Jensen's inequalities. Meantime, we will consider the cases including n variables, more precisely, special cases $\alpha = 0, 1, \frac{1}{2}, \frac{1}{3}$ will be considered.

In the following part of this paper, some important and fundamental properties of time scales will be given in the Section 2. In Section 3 we will deduce Theorems 9–12 via analysis method. A concise method will be used to prove the Qi type high dimensional integral inequalities on time scales in Section 4.

2. Preliminaries

We introduce some definitions and algorithms of time scales in this section. Time scales is an arbitrary nonempty closed subset of the real number and we regard $[\alpha, \beta]_{\mathbb{T}}$ as $[\alpha, \beta] \cap \mathbb{T}$. In what follows, we always suppose $\alpha, \beta \in \mathbb{T}$. We refer the readers to [9] for more details.

Definition 1. *For any $s \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(s) = \inf\{t \in \mathbb{T} : t > s\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(s) = \sup\{t \in \mathbb{T} : t < s\}.$$

As a complement, set

$$\inf \emptyset = \sup \mathbb{T}, \qquad \sup \emptyset = \inf \mathbb{T}.$$

It is obvious that $\sigma(s) \geq s \geq \rho(s)$.

Definition 2. *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(s) = \sigma(s) - s.$$

Accordingly, $v : \mathbb{T} \to [0, \infty)$ is defined by

$$v(s) = s - \rho(s).$$

Definition 3. $\mathbb{T}_k$, $\mathbb{T}^k$ *is defined as follows:*

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T})), & \text{if} \inf \mathbb{T} > -\infty, \\ \mathbb{T}, & \text{if} \inf \mathbb{T} = -\infty. \end{cases}$$

$$\mathbb{T}_k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if} \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \sup \mathbb{T} = \infty. \end{cases}$$

Property 1. *If $q : \mathbb{T} \to \mathbb{R}$ is a continuous function.*
1. If $\sigma(s) > s$, then q is Δ-differentiable at $s \in \mathbb{T}^k$ and

$$q^{\Delta}(s) = \frac{q(\sigma(s)) - q(s)}{\mu(s)}.$$

2. If $\rho(s) < s$, then q is ∇-differentiable at $s \in \mathbb{T}_k$ and

$$q^{\nabla}(s) = \frac{q(s) - q(\rho(s))}{v(s)}.$$

3. If $\rho(s) = s = \sigma(s)$, then

$$q^{\nabla}(s) = q^{\Delta}(s) = q^{'}(s).$$

Property 2. *Suppose q_1, q_2 are differentiable at $s \in \mathbb{T}_k^k$. Then the following holds:*
1. The sum q_1+q_2 is differentiable at s and

$$(q_1 + q_2)^{\Delta}(s) = q_1^{\Delta}(s) + q_2^{\Delta}(s).$$

$$(q_1 + q_2)^{\nabla}(s) = q_1^{\nabla}(s) + q_2^{\nabla}(s).$$

2. For $\alpha \in \mathbb{R}$. αq is differentiable at s and

$$(\alpha q)^{\Delta}(s) = \alpha q^{\Delta}(s).$$

$$(\alpha q)^{\nabla}(s) = \alpha q^{\nabla}(s).$$

Definition 4. *Let $Q, q : \mathbb{T} \to \mathbb{R}$. If*

$$Q^{\Delta}(s) = q(s),$$

holds for any $s \in \mathbb{T}^k$. Then Q is called a delta antiderivative of q. Moreover

$$\int_a^s q(t)\Delta t = Q(s) - Q(a).$$

If for any $s \in \mathbb{T}_k$ satisfies

$$Q^{\nabla}(s) = q(s).$$

Then Q is called a nabla antiderivative of q. Moreover

$$\int_a^s q(t)\nabla t = Q(s) - Q(a).$$

Property 3. *If q_1, $q_2 : \mathbb{T} \to \mathbb{R}$ are integrable on* $[\alpha, \beta]$. Then
1. The sum q_1+q_2 is integrable on (α, β) and

$$\int_{\alpha}^{\beta} (q_1 + q_2)(s)\Delta s = \int_{\alpha}^{\beta} q_1(s)\Delta s + \int_{\alpha}^{\beta} q_2(s)\Delta s.$$

$$\int_{\alpha}^{\beta} (q_1 + q_2)(s)\nabla s = \int_{\alpha}^{\beta} q_1(s)\nabla s + \int_{\alpha}^{\beta} q_2(s)\nabla s.$$

2. For any const k, kq is integrable on (α, β) and

$$\int_{\alpha}^{\beta} kq(s)\Delta s = k\int_{\alpha}^{\beta} q(s)\Delta s.$$

$$\int_{\alpha}^{\beta} kq(s)\nabla s = k\int_{\alpha}^{\beta} q(s)\nabla s.$$

Property 4. *If $q_1, q_2, q : \mathbb{T} \to \mathbb{R}$ are integrable on* $[\alpha_1, \beta_1]$, then

$$\int_{\alpha_1}^{\beta_1} (q_1 + q_2)(s) \diamond_{\alpha} s = \int_{\alpha_1}^{\beta_1} q_1(s) \diamond_{\alpha} s + \int_{\alpha_1}^{\beta_1} q_2(s) \diamond_{\alpha} s,$$

and
$$\int_{\alpha_1}^{\beta_1} kq(s) \diamond_\alpha s = k \int_{\alpha_1}^{\beta_1} q(s) \diamond_\alpha s.$$

Next are two particularly useful formulas.

Property 5. *Let $s \in \mathbb{T}^k$.*
1. If $q \in C_{rd}$, then
$$\int_s^{\sigma(s)} q(t)\Delta t = \mu(s)q(s),$$
where $q \in C_{rd}$ mean q is continuous at right-dense points, and its left-sided limits exist at left-dense points.
2. If $q \in C_{ld}$, then
$$\int_{\rho(s)}^{s} q(s)\nabla s = v(s)q(s),$$
where $q \in C_{ld}$ mean f is continuous at left-dense points, and its right-sided limits exist at right-dense points.

Corollary 2.47 in [9] shows that there exists the relationship between monotonicity and the Δ-differential or the ∇-differential as follows.

Property 6. *If $q \in C([\alpha, \beta))$ are delta derivative at (α, β), then q is increasing (decreasing) if and only if $q^\Delta(s) \geq 0 (\leq 0)$ for all $s \in [\alpha, \beta)$.*

Property 7. *If $q \in C([\alpha, \beta))$ are nabla derivative at (α, β), then q is increasing (decreasing) if and only if $q^\nabla(s) \geq 0 (\leq 0)$ for all $s \in (\alpha, \beta]$.*

The following two propositions can be found in [19].

Property 8. *If $q : \mathbb{T} \to \mathbb{R}$ is continuous at t, $t \in [\alpha, \beta]_{\mathbb{T}_k}$, then*
$$\left(\int_\alpha^s q(t)\Delta t \right)^\nabla = q(\rho(s)).$$

Property 9. *If $q : \mathbb{T} \to \mathbb{R}$ is continuous at t, $t \in [\alpha, \beta]_{\mathbb{T}^k}$, then*
$$\left(\int_\alpha^s q(t)\nabla t \right)^\Delta = q(\sigma(s)).$$

Finally, we list some useful properties which can be found in [9].

Definition 5. *If g is Δ-integrable on*
$$R = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n],$$
then set
$$\int_R q(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \ldots \Delta_n v_n = \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} q(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \ldots \Delta_n v_n.$$

Property 10. *If q_1, q_2 are bounded Δ-integral over*
$$R = [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \times \cdots \times [\alpha_n, \beta_n)$$

and $k_1, k_2 \in \mathbb{R}$, then

$$\begin{aligned} &\int_R (k_1q_1 + k_2q_2)(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n \\ = \quad & k_1 \int_R q_1(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n + k_2 \int_R q_2(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n. \end{aligned}$$

Property 11. *If q_1 and q_2 are bounded functions that are Δ-integral over R with*

$$q_1(v_1, v_2, \ldots, v_n) \leq q_2(v_1, v_2, \ldots, v_n), \qquad \forall (v_1, v_2, \ldots, v_n) \in R,$$

then

$$\int_R q_1(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n \leq \int_R q_2(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n.$$

Remark 1. *In particular, if $q_1(v_1, v_2, \ldots, v_n) = 0$, then*

$$\int_R q_2(v_1, v_2, \ldots, v_n)\Delta_1 v_1 \Delta_2 v_2 \ldots \Delta_n v_n \geq 0.$$

3. Qi Type Diamond-Alpha Integral Integral and Its Generalized Form

In this section, analysis method will be used to deduce sufficient conditions for Qi type diamond-alpha integral inequalities and its generalized forms.

We need the following lemmas which give an estimation to the differential of the power of f.

Lemma 1. *([20]). Suppose $g : [\alpha, \beta]_{\mathbb{T}} \to [0, \infty)$ is a increasing function, and if $p \geq 1$, then*

$$pg^{p-1}(t)g^{\Delta}(t) \leq (g^p(t))^{\Delta} \leq pg^{p-1}(\sigma(t))g^{\Delta}(t),$$

where σ is forward jump operate.

Lemma 2. *([20]). Suppose $g : [\alpha, \beta]_{\mathbb{T}} \to [0, \infty)$ is a increasing function, and if $p \geq 1$, then*

$$pg^{p-1}(\rho(t))g^{\nabla}(t) \leq (g^p(t))^{\nabla} \leq pg^{p-1}(t)g^{\nabla}(t),$$

where ρ is backward jump operate.

Following we consider $G(h)$ as the difference between the left hand and their right hand side, and take its nabla differential. According to the Proposition 7, we can complete the proof with analysis.

Theorem 9. *If ϕ is a non-negative and continuous function defined on $[\beta, \gamma]_{\mathbb{T}}$, satisfies*

$$\phi^{p-2}(\rho^2(t))\phi^{\nabla}(t) \geq (p-2)(t-\beta)^{p-3}\phi^{p-3}(t)\Big(\alpha\phi(\rho(t)) + (1-\alpha)\phi(t)\Big), \tag{4}$$

for all $t \in [\beta, \gamma]$, where ρ is backward jump operator. Then for all $s \in [\beta, \gamma]_{\mathbb{T}_k^k}$ and $p \geq 3$, the following inequality holds,

$$\int_\beta^\gamma \phi^p(s) \diamond_\alpha s \geq \Big(\int_\beta^\gamma \phi(s) \diamond_\alpha s \Big)^{p-1}. \tag{5}$$

Proof. Set the difference

$$G(h) = \int_\beta^h \phi^p(s) \diamond_\alpha s - \Big(\int_\beta^h \phi(s) \diamond_\alpha s \Big)^{p-1},$$

and let

$$g(h) = \int_{\beta}^{h} \phi(s) \diamond_{\alpha} s.$$

It follows from Proposition 8 and Lemma 2 that

$$\begin{aligned}
G^{\nabla}(h) &= \Big(\alpha \int_{\beta}^{h} \phi^{p}(s)\Delta s + (1-\alpha)\int_{\beta}^{h} \phi^{p}(s)\nabla s\Big)^{\nabla} - (g^{p-1}(h))^{\nabla} \\
&\geq \alpha\phi^{p}(\rho(h)) + (1-\alpha)\phi^{p}(h) - (p-1)g^{p-2}(h)g^{\nabla}(h) \\
&= \alpha\phi^{p}(\rho(h)) + (1-\alpha)\phi^{p}(h) - (p-1)g^{p-2}(h)\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\
&= \alpha\phi(\rho(h))\Big(\phi^{p-1}(\rho(h)) - (p-1)g^{p-2}(h)\Big) \\
&+ (1-\alpha)\phi(h)\Big(\phi^{p-1}(h) - (p-1)g^{p-2}(h)\Big).
\end{aligned}$$

Since $\alpha, 1-\alpha, \phi(h)$ are non-negative, and $\phi^{p-1}(h) \geq \phi^{p-1}(\rho(h))$, it is sufficient to prove that $\phi^{p-1}(\rho(h)) - (p-1)g^{p-2}(h) \geq 0$ (define $\phi^{p-1}(\rho(h)) - (p-1)g^{p-2}(h)$ as $G_1(h)$). By Lemmas 1 and 2 again, we can get

$$\begin{aligned}
G_1^{\nabla}(h) &= \Big(\phi^{p-1}(\rho(h)) - (p-1)g^{p-2}(h)\Big)^{\nabla} \\
&\geq (p-1)\phi^{p-2}(\rho^{2}(h))\phi^{\nabla}(h) - (p-1)(p-2)g^{p-3}(h)g^{\nabla}(h) \\
&= (p-1)\phi^{p-2}(\rho^{2}(h))\phi^{\nabla}(h) - (p-1)(p-2)g^{p-3}(h)\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big).
\end{aligned}$$

Due to ϕ is increasing,

$$\begin{aligned}
g(h) &= \int_{\beta}^{h} \phi(s) \diamond_{\alpha} s \\
&= \alpha\int_{\beta}^{h} \phi(s)\Delta s + (1-\alpha)\int_{\beta}^{h} \phi(s)\nabla s \\
&\leq \alpha(h-\beta)\phi(h) + (1-\alpha)(h-\beta)\phi(h) \\
&= (h-\beta)\phi(h).
\end{aligned}$$

We immediately get

$$\begin{aligned}
G_1^{\nabla}(h) &= (p-1)\phi^{p-2}(\rho^{2}(h))\phi^{\nabla}(h) - (p-1)(p-2)g^{p-3}(h)\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\
&\geq (p-1)\phi^{p-2}(\rho^{2}(h))\phi^{\nabla}(h) \\
&\quad -(p-1)(p-2)(h-\beta)^{p-3}\phi^{p-3}(h)\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\
&\geq 0.
\end{aligned}$$

Clearly, $G_1(\beta) = \phi^{p-1}(\rho(\beta)) - (p-1)g^{p-2}(\beta) = \phi^{p-1}(\rho(\beta)) = \phi^{p-1}(\beta) \geq 0$, so $G_1(h) \geq 0$. Because $G(\beta) = 0$, we deduce $G(h) \geq 0$, thereby completes the proof. □

Theorem 10. *If ϕ is a non-negative and continuous function defined on $[\beta,\gamma]_{\mathbb{T}}$, satisfies*

$$(p_1-1)\phi^{p_1-2}(\rho^{2}(t))\phi^{\nabla}(t) \geq p_2(p_2-1)(t-\beta)^{p_2-2}\phi^{p_2-2}(t)\Big(\alpha\phi(\rho(t)) + (1-\alpha)\phi(t)\Big), \quad (6)$$

for all $t \in [\beta,\gamma]$, where ρ is backward jump operator. Then for all $s \in [\beta,\gamma]_{\mathbb{T}_k^k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_{\beta}^{\gamma} \phi^{p_1}(s) \diamond_{\alpha} s \geq \Big(\int_{\beta}^{\gamma} \phi(s) \diamond_{\alpha} s\Big)^{p_2}. \quad (7)$$

Proof. Set the difference

$$G(h)=\int_{\beta}^{h}\phi^{p_1}(s)\diamond_\alpha s-\Big(\int_{\beta}^{h}\phi(s)\diamond_\alpha s\Big)^{p_2},$$

and let

$$g(h)=\int_{\beta}^{h}\phi(s)\diamond_\alpha s.$$

It follows from Proposition 8 and Lemma 2 that

$$\begin{aligned}
G^{\nabla}(h) &= \Big(\alpha\int_{\beta}^{h}\phi^{p_1}(s)\Delta s+(1-\alpha)\int_{\beta}^{h}\phi^{p_1}(s)\nabla s\Big)^{\nabla}-(g^{p_2}(h))^{\nabla}\\
&\geq \alpha\phi^{p_1}(\rho(h))+(1-\alpha)\phi^{p_1}(h)-p_2g^{p_2-1}(h)g^{\nabla}(h)\\
&= \alpha\phi^{p_1}(\rho(h))+(1-\alpha)\phi^{p_1}(h)-p_2g^{p_2-1}(h)\Big(\alpha\phi(\rho(h))+(1-\alpha)\phi(h)\Big)\\
&= \alpha\phi(\rho(h))\Big(\phi^{p_1-1}(\rho(h))-p_2g^{p_2-1}(h)\Big)\\
&+ (1-\alpha)\phi(h)\Big(\phi^{p_1-1}(h)-p_2g^{p_2-1}(h)\Big).
\end{aligned}$$

Since $\alpha, 1-\alpha, \phi(h)$ are non-negative, and $\phi^{p_1-1}(h)\geq\phi^{p_1-1}(\rho(h))$, it is suffices to prove that $\phi^{p_1-1}(\rho(h))-p_2g^{p_2-1}(h)\geq 0$ (define $\phi^{p_1-1}(\rho(h))-p_2g^{p_2-1}(h)$ as $G_1(h)$). By Lemmas 1 and 2 again, we can get

$$\begin{aligned}
G_1^{\nabla}(h) &= \Big(\phi^{p_1-1}(\rho(h))-p_2g^{p_2-1}(h)\Big)^{\nabla}\\
&\geq (p_1-1)\phi^{p_1-2}(\rho^2(h))\phi^{\nabla}(h)-p_2(p_2-1)g^{p_2-2}(h)g^{\nabla}(h)\\
&= (p_1-1)\phi^{p_1-2}(\rho^2(h))\phi^{\nabla}(h)\\
&\quad -p_2(p_2-1)g^{p_2-2}(h)\Big(\alpha\phi(\rho(h))+(1-\alpha)\phi(h)\Big).
\end{aligned}$$

Due to ϕ is increasing,

$$\begin{aligned}
g(h) &= \int_{\beta}^{h}\phi(s)\diamond_\alpha s\\
&= \alpha\int_{\beta}^{h}\phi(s)\Delta s+(1-\alpha)\int_{\beta}^{h}\phi(s)\nabla s\\
&\leq \alpha(h-\beta)\phi(h)+(1-\alpha)(h-\beta)\phi(h)\\
&= (h-\beta)\phi(h).
\end{aligned}$$

We immediately get

$$\begin{aligned}
G_1^{\nabla}(h) &= (p_1-1)\phi^{p_1-2}(\rho^2(h))\phi^{\nabla}(h)\\
&\quad -p_2(p_2-1)g^{p_2-2}(h)\Big(\alpha\phi(\rho(h))+(1-\alpha)\phi(h)\Big)\\
&\geq (p_1-1)\phi^{p_1-2}(\rho^2(h))\phi^{\nabla}(h)\\
&\quad -p_2(p_2-1)(h-\beta)^{p_2-2}\phi^{p_2-2}(h)\Big(\alpha\phi(\rho(h))+(1-\alpha)\phi(h)\Big)\\
&\geq 0.
\end{aligned}$$

Clearly, $G_1(\beta)=\phi^{p_1-1}(\rho(\beta))-p_2g^{p_2-1}(\beta)=\phi^{p_1-1}(\rho(\beta))=\phi^{p_1-1}(\beta)\geq 0$, so $G_1(h)\geq 0$. According to that $G(\beta)=0$, we deduce $G(h)\geq 0$, thereby completes the proof. □

If take $p_2=p_1-1$ in Theorem 10, we can deduce Theorem 9 immediately.

By virtue Theorem 10, we obtain the following corollaries by setting $\alpha=0$ and 1.

Corollary 1. *If ϕ is a non-negative and continuous function defined on $[\beta,\gamma]_{\mathbb{T}}$, satisfies*

$$(p_1-1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2-1)(t-\beta)^{p_2-2}\phi^{p_2-1}(t),$$

for all $t \in [\beta,\gamma]$, where ρ is the backward jump operator. Then for all $s \in [\beta,\gamma]_{\mathbb{T}_k^k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s)\Delta s \geq \Big(\int_\beta^\gamma \phi(s)\Delta s\Big)^{p_2}.$$

Corollary 2. *If ϕ is a non-negative and continuous function defined on $[\beta,\gamma]_{\mathbb{T}}$, satisfies*

$$(p_1-1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2-1)(t-\beta)^{p_2-2}\phi^{p_2-2}(t)\phi(\rho(t)),$$

for all $t \in [\beta,\gamma]$, where ρ is backward jump operator. Then for all $s \in [\beta,\gamma]_{\mathbb{T}_k^k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s)\nabla s \geq \Big(\int_\beta^\gamma \phi(s)\nabla s\Big)^{p_2}.$$

Theorem 11. *Suppose ϕ is a non-negative and continuous function defined on $[\beta,\gamma]_{\mathbb{T}}$, satisfies*

$$(p_1-1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2-1)(t-\beta)^{p_2-2}\phi^{p_2-2}(t)\Big(\alpha\phi(\rho(t))+(1-\alpha)\phi(t)\Big),$$

for all $t \in [\beta,\gamma]$, where ρ is backward jump operator. Then for all $s \in [\beta,\gamma]_{\mathbb{T}_k^k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s)\diamond_\alpha s - \Big(\int_\beta^\gamma \phi(s)\diamond_\alpha s\Big)^{p_2} \geq \phi^{p_1-1}(\beta)\int_\beta^\gamma \phi(s)\diamond_\alpha s. \tag{8}$$

Proof. Set the difference

$$G(h) = \int_\beta^h \phi^{p_1}(s)\diamond_\alpha s - \Big(\int_\beta^h \phi(s)\diamond_\alpha s\Big)^{p_2} - C\int_\beta^h \phi(s)\diamond_\alpha s,$$

where $C = \phi^{p_1-1}(\beta)$, and let

$$g(h) = \int_\beta^h \phi(s)\diamond_\alpha s.$$

It follows from Proposition 8 and Lemma 2 that

$$\begin{aligned}
G^{\nabla}(h) &= \Big(\alpha\int_\beta^h \phi^{p_1}(s)\Delta s + (1-\alpha)\int_\beta^h \phi^{p_1}(s)\nabla s\Big)^{\nabla} - (g^{p_2}(h))^{\nabla} - Cg^{\nabla}(h)\\
&\geq \alpha\phi^{p_1}(\rho(h)) + (1-\alpha)\phi^{p_1}(h) - p_2 g^{p_2-1}(h)g^{\nabla}(h) - Cg^{\nabla}(h)\\
&= \alpha\phi^{p_1}(\rho(h)) + (1-\alpha)\phi^{p_1}(h) - \Big(p_2 g^{p_2-1}(h)+C\Big)\Big(\alpha\phi(\rho(h))+(1-\alpha)\phi(h)\Big)\\
&= \alpha\phi(\rho(h))\Big(\phi^{p_1-1}(\rho(h)) - p_2 g^{p_2-1}(h) - C\Big)\\
&+ (1-\alpha)\phi(h)\Big(\phi^{p_1-1}(h) - p_2 g^{p_2-1}(h) - C\Big).
\end{aligned}$$

Since $\alpha, 1-\alpha, \phi(h)$ are non-negative, and $\phi^{p_1-1}(h) \geq \phi^{p_1-1}(\rho(h))$, it is suffices to prove that

$$G_2(h) := \phi^{p_1-1}(\rho(h)) - p_2 g^{p_2-1}(h) - C \geq 0.$$

Note that $G_2(h)+C$ equal to $G_1(h)$, thus

$$G_2^{\nabla}(h) = G_1^{\nabla}(h) > 0.$$

In this sense,

$$\begin{aligned} G_1(\beta) &= \phi^{p_1-1}(\rho(\beta)) - p_2 g^{p_2-1}(\beta) - C \\ &= \phi^{p_1-1}(\rho(\beta)) - C \\ &= \phi^{p_1-1}(\beta) - C = 0, \end{aligned}$$

so $G_1(h) \geq 0$. Since $G(\beta) = 0$, we deduce $G(h) \geq 0$, thereby completes the proof. □

Some special cases, if take $\alpha = 0$ and 1, we obtain the following corollaries.

Corollary 3. *If ϕ is a non-negative and continuous function defined on $[\beta, \gamma]_{\mathbb{T}}$, satisfies*

$$(p_1 - 1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2 - 1)(t - \beta)^{p_2-2}\phi^{p_2-1}(t),$$

for all $t \in [\beta, \gamma]$. Then for all $s \in [\beta, \gamma]_{\mathbb{T}^k_k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s)\Delta s - \Big(\int_\beta^\gamma \phi(s)\Delta s\Big)^{p_2} \geq \phi^{p_1-1}(\beta)\int_\beta^\gamma \phi(s)\Delta s.$$

Corollary 4. *If ϕ is a non-negative and continuous function defined on $[\beta, \gamma]_{\mathbb{T}}$, satisfies*

$$(p_1 - 1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2 - 1)(h - \beta)^{p_2-2}\phi^{p_2-2}(t)\phi(\rho(t)),$$

for all $t \in [\beta, \gamma]$. Then for all $s \in [\beta, \gamma]_{\mathbb{T}^k_k}$ and $p_1, p_2 \geq 2$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s)\nabla s - \Big(\int_\beta^\gamma \phi(s)\nabla s\Big)^{p_2} \geq \phi^{p_1-1}(\beta)\int_\beta^\gamma \phi(s)\nabla s.$$

Theorem 12. *Suppose ϕ is a non-negative and continuous function defined on $[\beta, \gamma]_{\mathbb{T}}$, satisfies*

$$p_2(p_2 - 1)(t - \beta)^{p_2-2}\phi^{p_2-2}(\beta)\Big(\alpha\phi(\rho(t)) + (1 - \alpha)\phi(t)\Big) \geq (p_1 - 1)\phi^{p_1-2}(\rho(t))\phi^{\nabla}(t) \quad (9)$$

for all $t \in [\beta, \gamma]$, where ρ is backward jump operator. Then for all $s \in [\beta, \gamma]_{\mathbb{T}^k_k}$ and $p_1, p_2 \geq 3$, the following inequality holds,

$$\int_\beta^\gamma \phi^{p_1}(s) \diamond_\alpha s \leq \Big(\int_\beta^\gamma \phi(s) \diamond_\alpha s\Big)^{p_2}. \quad (10)$$

Proof. Set the difference

$$R(h) = \Big(\int_\beta^\gamma \phi(s) \diamond_\alpha s\Big)^{p_2} - \int_\beta^\gamma \phi^{p_1}(s) \diamond_\alpha s,$$

and

$$r(h) = \int_\beta^\gamma \phi(s) \diamond_\alpha s.$$

It follows from Proposition 8 and Lemma 2 that

$$\begin{aligned} R^{\nabla}(h) &= (r^{p_2}(h))^{\nabla} - \Big(\alpha\int_{\beta}^{h}\phi^{p_1}(s)\Delta s + (1-\alpha)\int_{\beta}^{h}\phi^{p_1}(s)\nabla s\Big)^{\nabla} \\ &\geq p_2 r^{p_2-1}(\rho(h))r^{\nabla}(h) - \alpha\phi^{p_1}(\rho(h)) - (1-\alpha)\phi^{p_1}(h) \\ &= p_2 r^{p_2-1}(\rho(h))\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\ &\quad -\alpha\phi^{p_1}(\rho(h)) - (1-\alpha)\phi^{p_1}(h) \\ &= \alpha\phi(\rho(h))\Big(p_2 r^{p_2-1}(\rho(h)) - \phi^{p_1-1}(\rho(h))\Big) \\ &\quad +(1-\alpha)\phi(h)\Big(p_2 r^{p_2-1}(\rho(h)) - \phi^{p_1-1}(h)\Big). \end{aligned}$$

Since $\alpha, 1-\alpha, \phi(h)$ are non-negative, and $\phi^{p_1-1}(h) \geq \phi^{p_1-1}(\rho(h))$, it is suffices to prove that $p_2 r^{p_2-1}(h) - \phi^{p_1-1}(\rho(h)) \geq 0$ (define $p_2 r^{p_2-1}(h) - \phi^{p_1-1}(\rho(h))$ as $R_1(h)$). By Lemmas 1 and 2 again, we can yield

$$\begin{aligned} R_1^{\nabla}(h) &= \Big(p_2 r^{p_2-1}(h) - \phi^{p_1-1}(\rho(h))\Big)^{\nabla} \\ &\geq p_2(p_2-1)r^{p_2-2}(\rho(h))r^{\nabla}(h) - (p_1-1)\phi^{p_1-2}(\rho(h))\phi^{\nabla}(h) \\ &= p_2(p_2-1)r^{p_2-2}(\rho(h))\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\ &\quad -(p_1-1)\phi^{p_1-2}(\rho(h))\phi^{\nabla}(h). \end{aligned}$$

According with monotony of ϕ,

$$r(h) \geq (h-\beta)\phi(\beta).$$

We immediately get

$$\begin{aligned} R_1^{\nabla}(h) &\geq p_2(p_2-1)(h-\beta)^{p_2-2}\phi^{p_2-2}(\beta)\Big(\alpha\phi(\rho(h)) + (1-\alpha)\phi(h)\Big) \\ &\quad -(p_1-1)\phi^{p_1-2}(\rho(h))\phi^{\nabla}(h) \\ &\geq 0. \end{aligned}$$

Taking $t = \beta$ in (9) we get

$$\begin{aligned} 0 &= p_2(p_2-1)(\beta-\beta)^{p_2-2}\phi^{p_2-2}(\beta)\Big(\alpha\phi(\rho(\beta)) + (1-\alpha)\phi(\beta)\Big) \\ &\leq (p_1-1)\phi^{p_1-2}(\rho(\beta))\phi^{\nabla}(\beta), \end{aligned}$$

hence we obtain

$$\phi(\rho(\beta)) = \phi(\beta) \leq 0.$$

Clearly,

$$\begin{aligned} R_1(\beta) &= p_2 r^{p_2-1}(\beta) - \phi^{p_1-1}(\rho(\beta)) \\ &= -\phi^{p_1-1}(\rho(\beta)) \geq 0, \end{aligned}$$

so $G_1(h) \geq 0$. According to that $G(\beta) = 0$, we deduce $G(h) \geq 0$, thereby completes the proof. $\square$

If we choose $\alpha = 0$ or 1, we obtain the following results.

Corollary 5. *If ϕ is a non-negative and continuous function defined on $[\beta,\gamma]_{\mathbb{T}}$, satisfies*

$$p_2(p_2-1)(t-\beta)^{p_2-2}\phi^{p_2-2}(\beta)\phi(t) \geq (p_1-1)\phi^{p_1-2}(\rho(t))\phi^{\nabla}(t)$$

for all $t \in [\beta, \gamma]$. *Then for all* $s \in [\beta, \gamma]_{\mathbb{T}_k^k}$ *and* $p_1, p_2 \geq 2$, *the following inequality holds,*

$$\int_\beta^\gamma \phi^{p_1}(s)\Delta s \leq \Big(\int_\beta^\gamma \phi(s)\Delta s \Big)^{p_2}.$$

Corollary 6. *If* ϕ *is a non-negative and continuous function defined on* $[\beta, \gamma]_{\mathbb{T}}$, *satisfies*

$$p_2(p_2 - 1)(t - \beta)^{p_2-2}\phi^{p_2-2}(\beta) \geq (p_1 - 1)\phi^{p_1-3}(\rho(t))\phi^{\nabla}(t)$$

for all $t \in [\beta, \gamma]$. *Then for all* $s \in [\beta, \gamma]_{\mathbb{T}_k^k}$ *and* $p_1, p_2 \geq 2$, *the following inequality holds,*

$$\int_\beta^\gamma \phi^{p_1}(s)\nabla s \leq \Big(\int_\beta^\gamma \phi(s)\nabla s \Big)^{p_2}.$$

Under the basic assumptions that $p_1, p_2 \geq 3$, ϕ is a non-negative and continuous function defined on $[\beta, \gamma]_{\mathbb{T}}$, based on Theorems 10 and 12, we obtain the following results for all $t \in [\beta, \gamma]$.

Remark 2.

$$(p_1 - 1)\phi^{p_1-2}(\rho^2(t))\phi^{\nabla}(t) \geq p_2(p_2 - 1)(h - \beta)^{p_2-2}\phi^{p_2-2}(t)\Big(\alpha\phi(\rho(t)) + (1 - \alpha)\phi(t)\Big),$$
$$p_2(p_2 - 1)(t - \beta)^{p_2-2}\phi^{p_2-2}(\beta)\Big(\alpha\phi(\rho(t)) + (1 - \alpha)\phi(t)\Big) \geq (p_1 - 1)\phi^{p_1-2}(\rho(t))\phi^{\nabla}(t).$$

If condition (11) holds, then (7) established; if condition (11) holds, then the reverse of (7) established.

If ϕ is decreasing, we can obtain the similar results using the same method. However, if ϕ satisfying neither (11) nor (11), whether inequality (7) or the reverse of inequality (7) holds needs further research.

4. Qi Type Integral Inequalities of N Variables

In Section 3, we use differential to observe the monotonicity of G. It is surely a useful method that can be used in Qi type integral of n variables. However, it will produce complex conditions. In fact, once we take the differential of one variable among n variables, in order to ensure it is greater than or equal to 0, we need a condition. Regardless of the initial conditions, it also need n conditions.

In this section, we use Jensen's inequalities on time scales to deduce concise condition for Qi type inequality. It's worth to point out that Jensen's inequalities and other related topics are still a research hot spot recent years [21–33].

4.1. Qi Type Integral Inequalities of One Variable on Time Scales

Firstly, we list three Jensen's inequalities of one variable on time scales, all of them have been given in [34–36].

Lemma 3. *([34]). If* $\phi \in C_{rd}([\alpha, \beta], (c, d))$ *and* q *is a continuous and convex function defined on* (c, d), *then*

$$q\Big(\frac{\int_\alpha^\beta \phi(s)\Delta s}{\beta - \alpha}\Big) \leq \frac{\int_\alpha^\beta q(\phi(s))\Delta s}{\beta - \alpha}.$$

Lemma 4. *([35]). If* $\phi \in C_{ld}([\alpha, \beta], (c, d))$ *and* q *is a continuous and convex function defined on* (c, d), *then*

$$q\Big(\frac{\int_\alpha^\beta \phi(s)\nabla s}{\beta - \alpha}\Big) \leq \frac{\int_\alpha^\beta q(\phi(s))\nabla s}{\beta - \alpha}.$$

Lemma 5. *([36]). Let $\alpha_1, \beta_1 \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $\phi \in C([\alpha_1, \beta_1]_{\mathbb{T}}, (c, d))$ and q is a continuous and convex function defined on (c, d), then*

$$q\Big(\frac{\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s}{\beta_1 - \alpha_1}\Big) \leq \frac{\int_{\alpha_1}^{\beta_1} q(\phi(s)) \diamond_\alpha s}{\beta_1 - \alpha_1}.$$

Using lemmas, we can get following three theorems.

Theorem 13 (Qi type Δ integral inequality). *If $\phi \in C_{rd}([\alpha, \beta]_{\mathbb{T}})$ is a non-negative function, and for given $p > 1$ satisfies*

$$\int_\alpha^\beta \phi(s)\Delta s \geq (\beta - \alpha)^{p-1}.$$

Then,

$$\int_\alpha^\beta \phi^p(s)\Delta s \geq \Big(\int_\alpha^\beta \phi(s)\Delta s\Big)^{p-1}.$$

Proof. Consider the function $q(x) = x^p$ defined on $[c, d] \subset \mathbb{R}$. If $x \geq 0$, we know that q is convex. Since ϕ is non-negative,

$$\frac{\int_\alpha^\beta \phi(s)\Delta s}{\beta - \alpha} \geq 0,$$

using Lemma 3 with $q(x) = x^p$, we have

$$\Big(\frac{\int_\alpha^\beta \phi(s)\Delta s}{\beta - \alpha}\Big)^p \leq \frac{\int_\alpha^\beta \phi^p(s)\Delta s}{\beta - \alpha}.$$

According to the condition,

$$\int_\alpha^\beta \phi(s)\Delta s \geq (\beta - \alpha)^{p-1}.$$

Thus,

$$\begin{aligned}
\int_\alpha^\beta \phi^p(s)\Delta s &\geq (\beta - \alpha)\Big(\frac{\int_\alpha^\beta \phi(s)\Delta s}{\beta - \alpha}\Big)^p \\
&= \frac{\Big(\int_\alpha^\beta \phi(s)\Delta s\Big)^p}{(\beta - \alpha)^{p-1}} \\
&= \Big(\int_\alpha^\beta \phi(s)\Delta s\Big)^{p-1} \frac{\int_\alpha^\beta \phi(s)\Delta s}{(\beta - \alpha)^{p-1}} \\
&\geq \Big(\int_\alpha^\beta \phi(s)\Delta s\Big)^{p-1}.
\end{aligned}$$

Thereby we complete the proof. □

In the same way, we can deduce other two inequalities.

Theorem 14 (Qi type ∇ integral inequality). *If $\phi \in C_{ld}([\alpha, \beta]_{\mathbb{T}})$ is a non-negative function, and for given $p > 1$ satisfies*

$$\int_\alpha^\beta \phi(s)\nabla s \geq (\beta - \alpha)^{p-1}.$$

Then

$$\int_\alpha^\beta \phi^p(s)\nabla s \geq \Big(\int_\alpha^\beta \phi(s)\nabla s\Big)^{p-1}.$$

Proof. We know that $q(x) = x^p$ where $p > 1$ is convex on $x \geq 0$. It is obvious that

$$\frac{\int_\alpha^\beta \phi(s)\nabla s}{\beta - \alpha} \geq 0,$$

using Lemma 4 with $q(x) = x^p$, we have

$$\Big(\frac{\int_\alpha^\beta \phi(s)\nabla s}{\beta - \alpha}\Big)^p \leq \frac{\int_\alpha^\beta \phi^p(s)\nabla s}{\beta - \alpha}. \tag{11}$$

According to the condition,

$$\int_\alpha^\beta \phi(s)\nabla s \geq (\beta - \alpha)^{p-1}. \tag{12}$$

Combining the inequalities (11) and (12), we complete the proof. □

Theorem 15 (**Qi type diamond-α integral inequality**)**.** *If $\phi \in C([\alpha_1, \beta_1]_{\mathbb{T}})$ is a non-negative function, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s \geq (\beta_1 - \alpha_1)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \phi^p(s) \diamond_\alpha s \geq \Big(\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s\Big)^{p-1}.$$

Proof. Based on that

$$\frac{\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s}{\beta_1 - \alpha_1} \geq 0,$$

and use Lemma 5 with $q(x) = x^p$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s}{\beta_1 - \alpha_1}\Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1} \phi^p(s) \diamond_\alpha s}{\beta_1 - \alpha_1}. \tag{13}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1} \phi(s) \diamond_\alpha s \geq (\beta_1 - \alpha_1)^{p-1}. \tag{14}$$

Combining the inequalities (13) and (14), we complete the proof. □

In particularly, if we let $\mathbb{T} = \mathbb{R}$ in arbitrary one among the above three theorems, then $\Delta = \nabla = \diamond_\alpha = \mathrm{d}$, it will deduce Theorem 2.

Comparing the proofs in Section 3 and this subsection, we find that Jensen's inequalities not only can simplify the condition, but also it can simplify the proof. Most importantly, it keeps the condition similar. This means that we can generalize it to the case of n dimensions.

4.2. Qi Type Integral Inequalities of Several Variables on Time Scales

In the same way, we generalize Qi type integral inequalities to higher dimensions. Firstly, we write down Jensen's inequalities of n variables as lemmas, them can be found in [37,38].

Lemma 6. *([37]). If ϕ: $R \to (m_1, m_2)$ is a non-negative function where n have been given, $n \geq 3$, and $q \in C((m_1, m_2), \mathbb{R})$ is convex, then*

$$q\Big(\frac{\int_R \phi(v_1, v2, \dots, v_n)\Delta_1 v_1 \dots \Delta_n v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)}\Big) \leq \frac{\int_R q(\phi(v_1, v2, \dots, v_n))\Delta_1 v_1 \dots \Delta_n v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)},$$

where R is $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n]$.

Lemma 7. *If ϕ: $R \to (m_1, m_2)$ is a non-negative function where n have been given, $n \geq 3$, and $q \in C((m_1, m_2), \mathbb{R})$ is convex, then*

$$q\Big(\frac{\int_R \phi(v_1, v_2, \dots, v_n)\nabla_1 v_1 \dots \nabla_n v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)}\Big) \leq \frac{\int_R q(\phi(v_1, v_2, \dots, v_n))\nabla_1 v_1 \dots \nabla_n v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)}.$$

Lemma 8. *([38]). If ϕ: $R \to (m_1, m_2)$ is a non-negative function where n have been given, $n \geq 3$, and $q \in C((m_1, m_2), \mathbb{R})$ is convex, then*

$$q\Big(\frac{\int_R \phi(v_1, v_2, \dots, v_n) \diamond_\alpha v_1 \cdots \diamond_\alpha v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)}\Big) \leq \frac{\int_R q(\phi(v_1, v_2, \dots, v_n)) \diamond_\alpha v_1 \cdots \diamond_\alpha v_n}{\prod_{i=1}^n(\beta_i - \alpha_i)}.$$

Next, we deduce Qi type Δ-integral inequalities of two, three, n variables.

Theorem 16 (Qi type Δ-integral inequalities of two variables). *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \to (m_1, m_2)$ is continuous with $m_1 \geq 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2 \geq (\beta_1 - \alpha_1)^{p-1}(\beta_2 - \alpha_2)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2 \geq \Big(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2\Big)^{p-1}.$$

Proof. Since $q(x) = x^p$ is convex and

$$\frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \geq 0,$$

using Lemma 6 with $q(x) = x^p$ and $n = 2$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}\Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}. \tag{15}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2 \geq \Big(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} \phi(s_1, s_2)\Delta_1 s_1 \Delta_2 s_2\Big)^{p-1}. \tag{16}$$

Combining the inequalities (15) and (16), we complete the proof. □

Theorem 17 (Qi type Δ-integral inequalities of three variables). *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times [\alpha_3, \beta_3]_{\mathbb{T}_3} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3)\Delta_1 s_1 \Delta_2 s_2 \Delta_3 s_3 \geq (\beta_1 - \alpha_1)^{p-1}(\beta_2 - \alpha_2)^{p-1}(\beta_3 - \alpha_3)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi^p(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3 s_3 \geq \Big(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3 s_3\Big)^{p-1}.$$

Proof. Since $q(x) = x^p$ is convex and

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3 \geq 0,$$

using Lemma 6 with $q(x) = x^p$ and $n = 3$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\beta_3-\alpha_3)}\Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi^p(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)(\beta_3-\alpha_3)}. \tag{17}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3} \phi(s_1,s_2,s_3)\Delta_1 s_1\Delta_2 s_2\Delta_3 s_3 \geq (\beta_1-\alpha_1)^{p-1}(\beta_2-\alpha_2)^{p-1}(\beta_3-\alpha_3)^{p-1}. \tag{18}$$

Combining the inequalities (17) and (18), we complete the proof. □

In the same way, we can generalize it to n dimensions.

Theorem 18 (**Qi type Δ-integral inequalities of *n* variables**)**.** *If* $\phi : [\alpha_1,\beta_1]_{\mathbb{T}_1}\times[\alpha_2,\beta_2]_{\mathbb{T}_2}\times\cdots\times[\alpha_n,\beta_n]_{\mathbb{T}_n}\to(m_1,m_2)$ *is continuous with* $m_1>0$*, and for given* $p>1$ *satisfies*

$$\int_R \phi(s_1,s_2,\ldots,s_n)\Delta s_1\Delta s_2\ldots\Delta s_n \geq \prod_{i=1}^n(\beta_i-\alpha_i)^{p-1}.$$

Then

$$\int_R \phi^p(s_1,s_2,\ldots,s_n)\Delta s_1\Delta s_2\ldots\Delta s_n \geq \Big(\int_R \phi(s_1,s_2,\ldots,s_n)\Delta s_1\Delta s_2\ldots\Delta s_n\Big)^{p-1}.$$

Proof. According to Remark 1,

$$\int_R \phi(s_1,s_2,\ldots,s_n)\Delta_1 s_1\ldots\Delta_n s_n \geq 0,$$

using Lemma 4 with $q(x) = x^p$, we have

$$\Big(\frac{\int_R \phi(s_1,s_2,\ldots,s_n)\Delta_1 s_1\ldots\Delta_n s_n}{\prod_{i=1}^n(\beta_i-\alpha_i)}\Big)^p \leq \frac{\int_R (\phi(s_1,s_2,\ldots,s_n))^p\Delta_1 s_1\ldots\Delta_n s_n}{\prod_{i=1}^n(\beta_i-\alpha_i)}.$$

Together with the condition

$$\int_R \phi(s_1,s_2,\ldots,s_n)\Delta_1 s_1\Delta_2 s_2\ldots\Delta_n s_n \geq \prod_{i=1}^n(\beta_i-\alpha_i)^{p-1}.$$

we can complete the proof. □

Next three inequalities is about Qi type ∇-integral inequalities of two, three, and n variables.

Theorem 19 (Qi type ∇-integral inequalities of two variables). *If ϕ: $[\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2 \geq \Big(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2 \Big)^{p-1}.$$

Proof. Since ϕ is non-negative, then

$$\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \geq 0,$$

using Lemma 7 with $q(x) = x^p$ and $n = 2$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}. \tag{19}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \nabla_1 s_1 \nabla_2 s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}. \tag{20}$$

Combining the inequalities (19) and (20), we complete the proof. □

Theorem 20 (Qi type ∇-integral inequalities of three variables). *If $\phi: [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times [\alpha_3, \beta_3]_{\mathbb{T}_3} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1} (\beta_3 - \alpha_3)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi^p(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3 \geq \Big(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3 \Big)^{p-1}.$$

Proof. Since ϕ is non-negative,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3 \geq 0,$$

using Lemma 7 where $q(x) = x^p$ and $n = 3$, for $q(x) = x^p$ is convex when $x \geq 0$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_3 - \alpha_3)} \Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi^p(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_3 - \alpha_3)}. \tag{21}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \nabla_1 s_1 \nabla_2 s_2 \nabla_3 s_3 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1} (\beta_3 - \alpha_3)^{p-1}. \tag{22}$$

Combining the inequalities (21) and (22), we complete the proof. □

We can get Qi type ∇-integral inequalities of n variables in the same way.

Theorem 21 (**Qi type ∇-integral inequalities of n variables**)**.** *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times \cdots \times [\alpha_n, \beta_n]_{\mathbb{T}_n} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_R \phi(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n \geq \prod_{i=1}^{n} (\beta_i - \alpha_i)^{p-1}.$$

Then

$$\int_R \phi^p(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n \geq \Big(\int_R \phi(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n \Big)^{p-1}.$$

Proof. In the same way, we find

$$\int_R \phi(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n \geq 0,$$

using Lemma 7 with $q(x) = x^p$, we have

$$\Big(\frac{\int_R \phi(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n}{\prod_{i=1}^{n} (\beta_i - \alpha_i)} \Big)^p \leq \frac{\int_R (\phi(s_1, s_2, \ldots, s_n))^p \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n}{\prod_{i=1}^{n} (\beta_i - \alpha_i)}.$$

Using now

$$\int_R \phi(s_1, s_2, \ldots, s_n) \nabla_1 s_1 \nabla_2 s_2 \ldots \nabla_n s_n \geq \prod_{i=1}^{n} (\beta_i - \alpha_i)^{p-1}, \tag{23}$$

the relation (23) is satisfied. □

Next three inequalities is about Qi type diamond-α integral inequalities of two, three, and n variables.

Theorem 22 (**Qi type diamond-α** integral inequalities of two variables)**.** *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2 \geq \Big(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2 \Big)^{p-1}.$$

Proof. We can find that

$$\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \geq 0.$$

In Lemma 8, take $q(x) = x^p$, we have

$$\Big(\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \Big)^p \leq \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}. \tag{24}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_\alpha s_1 \diamond_\alpha s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}. \tag{25}$$

Combining the inequalities (24) and (25), we complete the proof. □

If set α equal to $\frac{1}{2}$ or $\frac{1}{3}$, then we have following corollaries.

Corollary 7. *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}.$$

Then,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \geq \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \right)^{p-1}.$$

Corollary 8. *If$\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi^p(s_1, s_2) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \geq \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s_1, s_2) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \right)^{p-1}.$$

Theorem 23 (**Qi type diamond-$\boldsymbol{\alpha}$** integral inequalities of three variables)**.** *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times [\alpha_3, \beta_3]_{\mathbb{T}_3} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1} (\beta_3 - \alpha_3)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi^p(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3 \geq \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3 \right)^{p-1}.$$

Proof. In the same way, the following inequality holds.

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3 \geq 0,$$

using Lemma 8 with $q(x) = x^p$, we have

$$\left(\frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_3 - \alpha_3)} \right)^p \leq \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi^p(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\beta_3 - \alpha_3)}. \tag{26}$$

According to the condition,

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_\alpha s_1 \diamond_\alpha s_2 \diamond_\alpha s_3 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1} (\beta_3 - \alpha_3)^{p-1}. \tag{27}$$

Combining the inequalities (26) and (27), we complete the proof. □

In the same way, if set α equal to $\frac{1}{2}$ or $\frac{1}{3}$ in the theorem above, then following corollaries hold.

Corollary 9. *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times [\alpha_3, \beta_3]_{\mathbb{T}_3} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{\alpha_3}^{\beta_3} \phi(s_1, s_2, s_3) \diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \diamond_{\frac{1}{2}} s_3 \geq (\beta_1 - \alpha_1)^{p-1} (\beta_2 - \alpha_2)^{p-1} (\beta_3 - \alpha_3)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\phi^p(s_1,s_2,s_3)\diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \diamond_{\frac{1}{2}} s_3 \geq \Big(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\phi(s_1,s_2,s_3)\diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \diamond_{\frac{1}{2}} s_3\Big)^{p-1}.$$

Corollary 10. *If $\phi : [\alpha_1,\beta_1]_{\mathbb{T}_1} \times [\alpha_2,\beta_2]_{\mathbb{T}_2} \times [\alpha_3,\beta_3]_{\mathbb{T}_3} \to (m_1,m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\phi(s_1,s_2,s_3)\diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \diamond_{\frac{1}{3}} s_3 \geq (\beta_1-\alpha_1)^{p-1}(\beta_2-\alpha_2)^{p-1}(\beta_3-\alpha_3)^{p-1}.$$

Then

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\phi^p(s_1,s_2,s_3)\diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \diamond_{\frac{1}{3}} s_3 \geq \Big(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}\phi(s_1,s_2,s_3)\diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \diamond_{\frac{1}{3}} s_3\Big)^{p-1}.$$

Employing Lemma 8, we can deduce

Theorem 24 (Qi type diamond-$\boldsymbol{\alpha}$ integral inequalities of $\boldsymbol{n}$ variables)**.** *If $\phi : [\alpha_1,\beta_1]_{\mathbb{T}_1} \times [\alpha_2,\beta_2]_{\mathbb{T}_2} \times \cdots \times [\alpha_n,\beta_n]_{\mathbb{T}_n} \to (m_1,m_2)$ is a continuous function with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n \geq \prod_{i=1}^n(\beta_i-\alpha_i)^{p-1}. \tag{28}$$

Then

$$\int_R \phi^p(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n \geq \Big(\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n\Big)^{p-1}.$$

Proof. We can find that

$$\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n \geq 0,$$

using Lemma 8 with $q(x) = x^p$, we have

$$\Big(\frac{\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n}{\prod_{i=1}^n(\beta_i-\alpha_i)}\Big)^p \leq \frac{\int_R (\phi(s_1,s_2,\ldots,s_n))^p\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n}{\prod_{i=1}^n(\beta_i-\alpha_i)}.$$

Using now

$$\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_\alpha s_1 \diamond_\alpha s_2 \cdots \diamond_\alpha s_n \geq \prod_{i=1}^n(\beta_i-\alpha_i)^{p-1}.$$

We can complete the proof. □

If we consider $\alpha = \frac{1}{2}$, we obtain the following corollary.

Corollary 11. *If $\phi : [\alpha_1,\beta_1]_{\mathbb{T}_1} \times [\alpha_2,\beta_2]_{\mathbb{T}_2} \times \cdots \times [\alpha_n,\beta_n]_{\mathbb{T}_n} \to (m_1,m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \cdots \diamond_{\frac{1}{2}} s_n \geq \prod_{i=1}^n(\beta_i-\alpha_i)^{p-1}.$$

Then

$$\int_R \phi^p(s_1,s_2,\ldots,s_n)\diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \cdots \diamond_{\frac{1}{2}} s_n \geq \Big(\int_R \phi(s_1,s_2,\ldots,s_n)\diamond_{\frac{1}{2}} s_1 \diamond_{\frac{1}{2}} s_2 \cdots \diamond_{\frac{1}{2}} s_n\Big)^{p-1}.$$

If we consider $\alpha = \frac{1}{3}$, we obtain the following corollary.

Corollary 12. *If $\phi : [\alpha_1, \beta_1]_{\mathbb{T}_1} \times [\alpha_2, \beta_2]_{\mathbb{T}_2} \times \cdots \times [\alpha_n, \beta_n]_{\mathbb{T}_n} \to (m_1, m_2)$ is continuous with $m_1 > 0$, and for given $p > 1$ satisfies*

$$\int_R \phi(s_1, s_2, \ldots, s_n) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \cdots \diamond_{\frac{1}{3}} s_n \geq \prod_{i=1}^{n} (\beta_i - \alpha_i)^{p-1}. \tag{29}$$

Then

$$\int_R \phi^p(s_1, s_2, \ldots, s_n) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \cdots \diamond_{\frac{1}{3}} s_n \geq \Big(\int_R \phi(s_1, s_2, \ldots, s_n) \diamond_{\frac{1}{3}} s_1 \diamond_{\frac{1}{3}} s_2 \cdots \diamond_{\frac{1}{3}} s_n \Big)^{p-1}.$$

5. Examples

In this section, we give some examples which applied the conclusions in Sections 3 and 4.

Example 1. *Consider the inequality:*

$$\sum_{k=1}^{N-1} 2^{3k} + \sum_{k=2}^{N} 2^{3k} \geq \frac{1}{2} \Big(\sum_{k=1}^{N-1} 2^k + \sum_{k=2}^{N} 2^k \Big)^2, \tag{30}$$

where $N \in \mathbb{N} \geq 2$.

Proof. We take $\mathbb{T} = \mathbb{N}, \phi(t) = 2^t, p = 3$ and $\alpha = \frac{1}{2}$ in (4), then it transforms into

$$(2^{t-2})^{p-2} 2^{t-1} \geq \frac{3}{2} 2^{t(p-3)} 2^t,$$

it is always true when $t \geq 1$. Based on Theorem 9, we have

$$\int_1^N 2^{3t} \diamond_{\frac{1}{2}} t \geq \Big(\int_1^N 2^t \diamond_{\frac{1}{2}} t)^2, \qquad N \in \mathbb{N} \geq 2.$$

Noting that

$$\int_1^N f(t) \diamond_{\frac{1}{2}} t = \frac{1}{2} \int_1^N f(t) \Delta t + \frac{1}{2} \int_1^N f(t) \nabla t = \frac{1}{2} \sum_{k=1}^{N-1} f(k) + \frac{1}{2} \sum_{k=2}^{N} f(k).$$

Thereby we can arrive to inequality (30). □

Example 2. *Consider the inequality:*

$$\alpha \sum_{k=2}^{N-1} a^{p_1 k} + (1-\alpha) \sum_{k=3}^{N} a^{p_1 k} \geq \Big(\alpha \sum_{k=2}^{N-1} a^k + (1-\alpha) \sum_{k=3}^{N} a^k \Big)^2, \tag{31}$$

where $1 \geq \alpha \geq 0, a \geq 2, p_1 \geq 2$ and $N \geq 3$.

Proof. We take $\mathbb{T} = \mathbb{N}, \phi(t) = a^t, p_2 = 2$ in (6), then it transforms into

$$a^{(p_1-1)t - 2p_1 + 4} \geq \frac{2(\alpha + (1-\alpha)a)}{(p_1 - 1)(a-1)}, \tag{32}$$

noting that $a^{(p_1-1)t-2p_1+4} \geq a^2 \geq \frac{2a}{a-1} \geq \frac{2(\alpha+(1-\alpha)a)}{(p_1-1)(a-1)}$ for $t \geq 2$, so (32) holds for $t \geq 2$. Based on Theorem 10, we have

$$\int_2^N a^{p_1 t} \diamond_\alpha t \geq \Big(\int_2^N a^t \diamond_\alpha t)^2, \qquad N \in \mathbb{N} \geq 3.$$

Noting that

$$\int_2^N f(t)\diamond_\alpha t=\alpha\int_2^N f(t)\Delta t+(1-\alpha)\int_2^N f(t)\nabla t=\alpha\sum_{k=2}^{N-1}f(k)+(1-\alpha)\sum_{k=3}^{N}f(k).$$

Thereby we can arrive at inequality (31). □

Example 3. *Consider the inequality:*

$$\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}16^{(k_1+k_2+\cdots+k_n)}\geq\Big(\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}2^{(k_1+k_2+\cdots+k_n)}\Big)^3,$$

where $N\geq 10$.

Proof. Consider the condition (28) with $R=\mathbb{N}^n,\alpha=1,\phi(t_1,t_2,\cdots,t_n)=2^{(t_1+t_2+\cdots+t_n)}$, $p=4$ and $\alpha_i=1,\beta_i=N(i=1,2,\cdots,n)$, then it change into

$$\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}2^{(t_1+t_2+\cdots+t_n)}\geq(N-1)^{3n}. \tag{33}$$

Noting that $\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}2^{(t_1+t_2+\cdots+t_n)}=(2^N-2)^n$, so (33) is hold for $N\geq 10$. Based on Theorem 24, we get the desired inequality. □

Example 4. *Consider the inequalities:*
(i)

$$\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}t_1^{p^2}t_2^{p^2}\cdots t_n^{p^2}\geq\Big(\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}t_1^pt_2^p\cdots t_n^p\Big)^{p-1},$$

where $N\geq 2$.
(ii)

$$\begin{aligned}&\int_1^N\int_1^N\cdots\int_1^N t_1^{p^2}t_2^{p^2}\cdots t_n^{p^2}\diamond_{\frac{1}{3}}s_1\diamond_{\frac{1}{3}}s_2\cdots\diamond_{\frac{1}{3}}s_n\\ \geq&\Big(\int_1^N\int_1^N\cdots\int_1^N t_1^pt_2^p\cdots t_n^p\diamond_{\frac{1}{3}}s_1\diamond_{\frac{1}{3}}s_2\cdots\diamond_{\frac{1}{3}}s_n\Big)^{p-1},\end{aligned} \tag{34}$$

where $N\geq 2$ and $\mathbb{T}_i=\mathbb{N}$.

Proof. Consider the condition (28) with $R=\mathbb{N}^n,\phi(t_1,t_2,\cdots,t_n)=t_1^pt_2^p\cdots t_n^p$ and $\alpha_i=1,\beta_i=N(i=1,2,\cdots,n)$, then it change into

$$\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\int_{\alpha_3}^{\beta_3}t_1^pt_2^p\cdots t_n^p\diamond_\alpha t_1\diamond_\alpha t_2\cdots\diamond_\alpha t_n\geq(N-1)^{n(p-1)}. \tag{35}$$

1. If $\alpha=1$, then the left hand of (35) become

$$\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1}\cdots\sum_{k_n=1}^{N-1}k_1^pk_2^p\cdots k_n^p=\Big(\frac{N(N-1)}{2}\Big)^{np}, \tag{36}$$

thus (35) hold for $N\geq 2$. Based on Theorem 24, we have the first inequality.

2. If $\alpha=\frac{1}{3}$, then the left hand of (35) is bigger than (36), thus (35) hold for $N\geq 2$. Based on Theorem 24 or Corollary 12, we have the second inequality. □

Remark 3. *We take $n = 2$ in (34) for example, (34) will transforms into*

$$\begin{aligned}
&\frac{1}{9}\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1} k_1^{p^2}k_2^{p^2} + \frac{2}{9}\sum_{k_1=1}^{N-1}\sum_{k_2=2}^{N} k_1^{p^2}k_2^{p^2} + \frac{2}{9}\sum_{k_1=2}^{N}\sum_{k_2=1}^{N-1} k_1^{p^2}k_2^{p^2} + \frac{4}{9}\sum_{k_1=2}^{N}\sum_{k_2=2}^{N} k_1^{p^2}k_2^{p^2} \\
\geq\ &\Big(\frac{1}{9}\sum_{k_1=1}^{N-1}\sum_{k_2=1}^{N-1} k_1^{p}k_2^{p} + \frac{2}{9}\sum_{k_1=1}^{N-1}\sum_{k_2=2}^{N} k_1^{p}k_2^{p} + \frac{2}{9}\sum_{k_1=2}^{N}\sum_{k_2=1}^{N-1} k_1^{p}k_2^{p} + \frac{4}{9}\sum_{k_1=2}^{N}\sum_{k_2=2}^{N} k_1^{p}k_2^{p}\Big)^{p-1}.
\end{aligned}$$

6. Conclusions

In this paper, we greatly promoted the research of Qi type inequality on time scales theory. More completely, we generalize Qi type inequality and its two generalized forms through the Diamond-Alpha integral. The sufficient condition for the reverse of Qi type inequality is also considered. Then we generalize Qi type inequality to higher dimension via Jessen's inequality. In this method, concise conditions are deduced. Qi type high dimension inequality has been studied in great detail and some special cases are given as corollaries. Moreover, some examples are given to show our conclusions are useful in the end.

Author Contributions: Data curation, Z.-X.M.; funding acquisition, Y.-R.Z.; investigation, Z.-X.M.; methodology, Z.-X.M., B.-H.G., Y.-H.Y. and H.-Q.Z.; resources, Z.-X.M. and Y.-R.Z.; writing—original draft, Z.-X.M.; writing—review and editing, Y.-R.Z., B.-H.G., F.-H.W., Y.-H.Y. and H.-Q.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Fundamental Research Funds for the Central Universities under Grant MS117.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All data generated or analysed during this study are included in this published article.

Acknowledgments: The authors would like to express their sincere thanks to the anonymous referees for their great efforts to improve this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Qi, F. Several integral inequalities. *J. Inequal. Pure Appl. Math.* **2000** *1*, 19:1–19:3.
2. Yu, K.W.; Qi, F. A short note on an integral inequality. *RGMIA Res. Rep. Coll.* **2001**, *23*, 1–3.
3. Pogány, T.K. On an Open Problem of F. Qi. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 54:1–54:12.
4. Krasniqi, V. Some generalizations of Feng Qi type integral inequalities. *Octogon Math. Mag.* **2012**, *20*, 464–467.
5. Yin, L.; Krasniqi, V. Some generalizations of Feng Qi type integral inequalities on time scales. *Applied Math. ENote* **2016**, *16*, 231–243.
6. Xi, B.-Y.; Qi, F. Some inequalities of Qi type for double integrals. *J. Egyptian Math. Soc.* **2014**, *22*, 337–340. [CrossRef]
7. Sarikaya, M.Z.; Ozkan, U.M.; Yildirim, H. Time Scale Integral Inequalities Similar to Qi's inequality. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 128:1–128:15.
8. Hilger, S. Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universität Würzburg, Würzburg, Germany, 1988.
9. Bohner, M.; Georgiev, S.G. *Multivariable Dynamic Calculus on Time Scales*; Springer: Cham, Switzerland, 2016.
10. Tian, J.-F. Triple Diamond-Alpha integral and Hölder-type inequalities. *J. Inequal. Appl.* **2018**, *48*, 111:1–111:12. [CrossRef] [PubMed]
11. Hu, X.-M.; Tian, J.-F.; Chu, Y.-M. On Cauchy-Schwarz inequality for N-tuple Diamond-Alpha integral. *J. Inequal. Appl.* **2020**, *2020*, 8:1–8:15. [CrossRef]
12. Baric, J.; Bibi, R.; Bohner, M. *Jensen Inequalities and Their Applications on Time Scale*; Element: Zagreb, Croatia, 2015.
13. Bibi, R.; Nosheen, A.; Pecaric, J. Genaralization of Jensen-type linear functional on time scales via Lidstone polynomial. *Cogent Math.* **2017**, *4*, 1–14. [CrossRef]

14. Bibi, R.; Nosheen, A.; Pecaric, J. Jensen-Steffensen Inequality for Diamond Intergrals, Its Converse and Inprovment via Green Function and Taylors Formula. *Aequationes Math.* **2018**, *1*, 121:1–121:21.
15. Bibi, R.; Nosheen, A.; Pecaric, J.; Extended Jensen's Type Inequalities for Diamond Integrals via Taylors Formula. *Turkish J. Math.* **2019**, *3*, 7–18.
16. Ahmad, M.; Awan, K. M.; Hameed, S. Bivariate Montgomery identity for alpha diamond integrals. *Adv. Differ. Equ.* **2019**, *2019*, 314:1–314:13. [CrossRef]
17. Ahmad, W.; Khan, K.A.; Nosheen A. Copson, Leindler type inequalities for Function of several variables on Iime Scales. *Punjab Univ. J. Math.* **2019**, *51*, 157–168.
18. Nosheen, A.; Nawaz, A.; Khan, K.A. Multivariate Hardy and Littlewood Inequalities on Time Scales. *Arab J. Math. Sci.* **2019**, *2019*, 1–19. [CrossRef]
19. Sheng, Q.; Fadag, M.; Henderson, J. An exploration of combined dynamic derivatives on time scales and their applications. *Nonlinear Anal-Real* **2006**, *7*, 395–413. [CrossRef]
20. Fayyaz, T.; Irshad, N.; Khan, A.R. Generalized integral inequalities on time scales. *J. Inequal. Appl.* **2016**, *2016*, 235:1–235:12. [CrossRef]
21. Bibi, R. Jensen Inequality on Time Scales. Ph.D. Thesis, National University of Sciences and Technology, Islamabad, Pakistan, 2014.
22. Khan, M.A.; Hanif, M.; Khan, Z.A. Association of Jensen's inequality for s-convex function with Csiszár divergence. *J. Inequal. Appl.* **2019**, *2019*, 162:1–162:14.
23. Khan, M.A.; Pečarić, J.; Chu, Y.-M. Refinements of Jensen's and McShane's inequalities with applications. *AIMS Math.* **2020**, *5*, 4931–4945. [CrossRef]
24. Rafeeq, S.; Kalsoom, H.; Hussain, S. Delay dynamic double integral inequalities on time scales with applications. *Adv. Differ. Eq.* **2020**, *2020*, 40:1–40:32. [CrossRef]
25. Rashid, S.; Noor, M.A.; Noor, K.I. Hermite-Hadamrad type inequalities for the class of convex functions on time scale. *Mathematics* **2019**, *7*, 956. [CrossRef]
26. Zhao, T.-H.; Chu, Y.-M.; Wang, H. Logarithmically complete monotonicity properties relating to the gamma function. *Abstr. Appl. Anal.* **2011**, *2011*, 896483:1–896483:14. [CrossRef]
27. Chu, Y.-M.; Khan M.A.; Dragomir, A.T. Inequalities for α-fractional differentiable functions. *J. Inequal. Appl.* **2017**, *2017*, 93:1–93:12. [CrossRef] [PubMed]
28. Tian, J.-F.; Yang, Z.-H. Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders. *J. Math. Anal. Appl.* **2021**, *493*, 124545:1–124545:19. [CrossRef]
29. Yang, Z.-H.; Tian, J.-F.; Wang, M.-K. A positive answer to Bhatia-Li conjecture on the monotonicity for a new mean in its parameter. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2020**, *114*, 126:1–126:22.
30. Rashid, S.; Jarad, F.; Noor, M.A. Inequalities by means of generalized proportional fractional integral operators with respect to another function. *Mathematics* **2019**, *7*, 1225. [CrossRef]
31. Wang, M.-K.; Chu, Y.-M.; Li, Y.-M. Asymptotic expansion and bounds for complete elliptic integrals. *Math. Inequal. Appl.* **2020**, *23*, 821–841.
32. Wang, M.-K.; He, Z.-Y.; Chu, Y.-M. Sharp power mean inequalities for the generalized elliptic integral of the first Kind. *Comput. Methods Funct. Theory* **2019**, *20*, 111–124. [CrossRef]
33. Wang, M.-K.; Hong, M.-Y.; Xu, Y.-F. Inequalities for generalized trigonometric and hyperbolic functions with one parameter. *J. Math. Inequal.* **2020**, *14*, 1–21. [CrossRef]
34. Agarwal, R.P.; Bohner, M.; Peterson, A. Inequalities on time scales: A survey. *Math. Inequal. Appl.* **2001**, *7*, 535–557. [CrossRef]
35. Özkan, U.M.; Sarikaya, M.; Yildirim, H. Extensions of certain integral inequalities on time scales. *Appl. Math. Lett.* **2008**, *21*, 993–1000. [CrossRef]
36. Ammi, M.R.S.; Ferreira, R.A.C.; Torres, D.F.M. Diamond-α Jensen's inequality on time scales. *J. Inequal. Appl.* **2008**, *2008*, 576876:1–576876:13.
37. Anwar, M.; Bibi, R.; Bohner, M. Integral inequalities on time scales via the theory of isotonic linear functionals. *Abstr. Appl. Anal.* **2011**, *2011*, 483595:1–483595:17. [CrossRef]
38. Tian, J.-F.; Zhu, Y.-R.; Cheung, W.-S. N-tuple Diamond-Alpha integral and inequalities on time scales. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 2189–2200. [CrossRef]