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Refinements of Hermite–Hadamard Inequalities for Continuous Convex Functions via (p,q)-Calculus

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Abstract: In this paper, we present some new refinements of Hermite–Hadamard inequalities for continuous convex functions by using (p,q)-calculus. Moreover, we study some new (p,q)-Hermite–Hadamard inequalities for multiple integrals. Many results given in this paper provide extensions of others given in previous research.

Keywords: Hermite–Hadamard inequality; (p,q)-derivative; (p,q)-integral; convex functions

MSC: 05A30; 26A51; 26D10; 26D15; 81P68

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1. Introduction

Mathematical inequalities play important roles in the study of mathematics as well as in other areas of mathematics because of their wide applications in mathematics and physics; see [1–3] for more details. One of the most significant functions used to study many interesting inequalities is convex functions, which are defined as follows:

Let $I \subset \mathbb{R}$ be a non-empty interval. The function $f : I \to \mathbb{R}$ called as convex, if

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b)$$

holds for every $a, b \in I$ and $t \in [0,1]$.

In recent years, many researchers have been fascinated in the study of convex functions and, particularly, one of the well-known inequality for convex functions known as the Hermite–Hadamard inequality, which is defined as follows:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) was introduced by C. Hermite [4] and investigated by J. Hadamard [5] in 1893. So far, the Hermite–Hadamard inequality and a variety of refinements of Hermite–Hadamard inequalities have been extensively studied by many researchers; see [6–18] and the references therein for more details.

The study of calculus with no limits is called quantum calculus (in short, q-calculus). The main objective of studying q-calculus is to obtain the q-analoques of mathematical objects that can be recaptured by taking q tending toward 1. In the past few years, the topic of q-calculus has become an interesting topic for many researchers, and new results of q-calculus can be found in [19–41], and the references cited therein.

The generalization of q-calculus is post quantum calculus or, sometimes, is called (p,q)-calculus. (p,q)-calculus is known as two-parameter quantum calculus, for which applications plays significant roles in mathematics and physics such as combinatorics, fractals, special functions, number theory, dynamical systems, and mechanics, among others. In (p,q)-calculus, we obtain the q-calculus formula for the case $p = 1$ and obtain the original of mathematical formula when q tends towards 1.

Recently, Tunç and Göv [42–44] studied the concept of (p,q)-calculus over the intervals $[a,b]$ and gave some new definitions of (p,q)-derivatives and (p,q)-integrals. Moreover, they also derived some of its properties and many integral inequalities as in (1), which is called (p,q)-Hermite–Hadamard inequality, and some new results on (p,q)-calculus of several important integral inequalities. Next, Mehmet Kunt et al. [45] proved the left side of the (p,q)-Hermite–Hadamard inequality through (p,q)-differentiable convex and quasi-convex functions, and then, they had some new (p,q)-Hermite–Hadamard inequalities.

In 2019, Prabseang et al. [46] established some new (p,q)-calculus of Hermite–Hadamard inequalities for the double integral and refinements of the Hermite–Hadamard inequality for (p,q)-differentiable convex functions. In the last few years, the topic of (p,q)-calculus has been investigated extensively by many researchers, and a variety of new results can be found in the literature (see [47–64] and the references cited therein).

In 2020, Prabseang et al. [65] established some new refinement of quantum Hermite–Hadamard inequalities, which have been expanded to integration on a finite interval of an n-dimensional. Some new refinements of (p,q)-Hermite–Hadamard inequalities for convex functions are given.

In this paper, we aim to propose some new refinements of Hermite–Hadamard inequalities via (p,q)-calculus that have been expanded to integration on a finite interval of an n-dimensional. We obtain some new refinements of (p,q)-Hermite–Hadamard inequalities for convex functions and the results in special cases for $p = 1$ and $q \to 1$.

Before presenting our main results in Section 3, we introduce the definitions and results from (p,q)-calculus in Section 2. Finally, Section 4 concludes the paper by summarizing the results.

2. Preliminaries

In this section, the basic definitions used in our study are discussed. Throughout this paper, let $[a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < p \leq 1$ be constants. The following definitions for the (p,q)-derivative and (p,q)-integral were given in [42,43].

Definition 1. *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the (p,q)-derivative of function f at x is defined by*

$$\begin{aligned} {}_aD_{p,q}f(x) &= \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}, \quad x \neq a \\ {}_aD_{p,q}f(a) &= \lim_{x \to a} {}_aD_{p,q}f(x). \end{aligned} \tag{2}$$

If ${}_aD_{p,q}f(x)$ exists for all $x \in [a,b]$, then the function f is called (p,q)-differentiable on $[a,b]$.

In Definition 1, if $a = 0$, then ${}_0D_{p,q}f = D_{p,q}f$, which is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0. \tag{3}$$

In addition, if $p = 1$ in (3), then it reduces to D_qf, which is the q-derivative of the function f; see [32].

Example 1. *Define function $f : [a,b] \to \mathbb{R}$ by $f(x) = x^2 + x + C$, where $C \in \mathbb{R}$. Then, for $x \neq a$, we have*

$$\begin{aligned}
{}_aD_{p,q}(x^2+x+C) &= \frac{\left[(px+(1-p)a)^2+(px+(1-p)a)+C\right]}{(p-q)(x-a)} \\
&\quad - \frac{\left[(qx+(1-q)a)^2+(qx+(1-q)a)+C\right]}{(p-q)(x-a)} \\
&= \frac{(p+q)x^2+2ax[1-(p+q)]+a^2[(p+q)-2]+(x-a)}{(x-a)} \\
&= \frac{x(p+q)(x-a)-a(p+q)(x-a)+2a(x-a)+(x-a)}{(x-a)} \\
&= (p+q)(x-a)+2a+1.
\end{aligned} \tag{4}$$

Definition 2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, the (p,q)-integral on $[a,b]$ is defined by*

$$\int_a^x f(t)\ {}_ad_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right), \tag{5}$$

for $x \in [a,b]$. If $a = 0$ and $p = 1$ in (5), then we have the classical q-integral; see [32].

Example 2. *Define function $f : [a,b] \to \mathbb{R}$ by $f(x) = Ax + B$, where $A, B \in \mathbb{R}$. Then, we have*

$$\begin{aligned}
\int_a^b f(x)\ {}_ad_{p,q}x &= \int_a^b (Ax+B)\ {}_ad_{p,q}x \\
&= A(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right) \\
&\quad + B(p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} \\
&= \frac{A(b-a)(b-a(1-p-q))}{p+q}+B(b-a).
\end{aligned} \tag{6}$$

In addition, the following definition for the (p,q)-integral of the function of two variables can be defined; we referred to [47].

Definition 3. *Let $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, then the definite (p,q)-integral on $[a,b] \times [c,d]$ is defined by*

$$\begin{aligned}
\int_c^t \int_a^s f(x,y)\,{}_ad_{p,q}x\ {}_ad_{p,q}y =& (p-q)^2(s-a)(t-c) \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{m+n}}{p^{m+n+2}}f\left(\frac{q^n}{p^{n+1}}s+\left(1-\frac{q^n}{p^{n+1}}\right)a, \frac{q^m}{p^{m+1}}t+\left(1-\frac{q^m}{p^{m+1}}\right)a\right),
\end{aligned} \tag{7}$$

for $(s,t) \in [a,b] \times [c,d]$.

The proofs of the following theorems were given in [42,43].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, we have the following:*

(i) ${}_aD_{p,q}\int_a^x f(t)\ {}_ad_{p,q}t = f(x)$;

(ii) $\int_c^x {}_aD_{p,q}f(t)\ {}_ad_{p,q}t = f(x) - f(c)$ *for $c \in (a,x)$.*

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions and $\alpha \in \mathbb{R}$. Then, we have the following:*

(i) $\int_a^x [f(t) + g(t)] \,{}_a d_{p,q}t = \int_a^x f(t) \,{}_a d_{p,q}t + \int_a^x g(t) \,{}_a d_{p,q}t$;

(ii) $\int_a^x (\alpha f)(t)\,{}_a d_{p,q}t = \alpha \int_a^x f(t)\,{}_a d_{p,q}t$;

(iii) $\int_c^x f(pt + (1-p)a)\,{}_a D_{p,q}g(t) \,{}_a d_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\,{}_a D_{p,q}f(t) \,{}_a d_{p,q}t$ *for* $c \in (a, x)$.

3. Main Results

In this section, we present refinements of Hermite–Hadamard inequalities for continuous convex functions via (p, q)-calculus on the interval $J := [a, pb + (1-p)a]$.

Theorem 3. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, we have*

$$
\begin{aligned}
f\left(\frac{qa+pb}{p+q}\right) &\le \frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} f\left(\frac{x+y}{2}\right) {}_a d_{p,q}x\,{}_a d_{p,q}y \\
&\le \frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} \frac{1}{2}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right] {}_a d_{p,q}x\,{}_a d_{p,q}y \\
&\le \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)\,{}_a d_{p,q}x
\end{aligned}
\tag{8}
$$

for all $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$.

Proof. Since f is convex on J, for all $x, y \in J$ and $\alpha, \beta \ge 0$ with $\alpha + \beta > 0$, we have

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right] \\
&\le \frac{f(x)+f(y)}{2}.
\end{aligned}
\tag{9}
$$

Taking double (p, q)-integration on both sides of (9) on J^2, we obtain the second part of (8).

On the other hand, by using Jensen's inequality, we have

$$
\begin{aligned}
&f\left(\frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} \left(\frac{x+y}{2}\right) {}_a d_{p,q}x\,{}_a d_{p,q}y\right) \\
&\le \frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} f\left(\frac{x+y}{2}\right) {}_a d_{p,q}x\,{}_a d_{p,q}y.
\end{aligned}
$$

Since

$$
\frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} \left(\frac{x+y}{2}\right) {}_a d_{p,q}x\,{}_a d_{p,q}y = \frac{qa+pb}{p+q},
$$

this yields the first part of (8). This completes the proof. □

Remark 1. If $p = 1$, then (8) reduces to

$$\begin{aligned} f\left(\frac{qa+b}{1+q}\right) &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\,{}_ad_qx\,{}_ad_qy \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{1}{2}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]\,{}_ad_qx\,{}_ad_qy \\ &\leq \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx, \end{aligned} \tag{10}$$

see also [65]. Additionally, if $q \to 1$ in (10), then (10) reduces to

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{1}{2}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]dxdy \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx, \end{aligned}$$

which readily appeared in [66].

Theorem 4. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, we have*

$$\begin{aligned} f\left(\frac{qa+pb}{p+q}\right) &\leq \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\leq \frac{1}{p^{n-1}(b-a)^{n-1}}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_{n-1} \\ &\vdots \\ &\leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_ad_{p,q}x, \end{aligned} \tag{11}$$

for all $n \in \mathbb{N}$ with $n \geq 3$.

Proof. Since

$$\frac{x_1+\cdots+x_n}{n} = \frac{1}{n}\left[\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)+\left(\frac{x_2+\cdots+x_n}{n-1}\right)+\cdots+\left(\frac{x_n+\cdots+x_{n-2}}{n-1}\right)\right],$$

and by using Jensen's inequality, we have

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{1}{n}\left[f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)+f\left(\frac{x_2+\cdots+x_n}{n-1}\right)+\cdots+f\left(\frac{x_n+\cdots+x_{n-2}}{n-1}\right)\right].$$

Taking (p,q)-integration on both sides of the above inequality on J^n, we obtain

$$\begin{aligned} &\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\leq \frac{1}{n}\left[\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n+\cdots\right. \\ &\quad\left.+\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_n+\cdots+x_{n-2}}{n-1}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n\right]. \end{aligned}$$

On the other hand, we get

$$\begin{aligned}
&\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n \\
&= \int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_2+\cdots+x_{n}}{n-1}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n \\
&\vdots \\
&= \int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_n+\cdots+x_{n-2}}{n-1}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n \\
&= p(b-a)\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_{n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n}}{n}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n \\
&\leq \frac{1}{p^{n-1}(b-a)^{n-1}}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_{n-1},
\end{aligned}$$

which shows the middle part of (11).

On the other hand, by Jensen's inequality, we have

$$\begin{aligned}
&f\left(\frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} \left(\frac{x_1+\cdots+x_{n}}{n}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n\right) \\
&\leq \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_{n}}{n}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n.
\end{aligned}$$

Since

$$\frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} \left(\frac{x_1+\cdots+x_{n}}{n}\right)\, {}_a d_{p,q}x_1\cdots\, {}_a d_{p,q}x_n = \frac{qa+pb}{p+q},$$

this yields the first part of (11). This completes the proof. □

Remark 2. *If $p=1$, then* (11) *reduces to*

$$\begin{aligned}
f\left(\frac{qa+b}{1+q}\right) &\leq \frac{1}{(b-a)^n}\int_a^{b}\cdots\int_a^{b} f\left(\frac{x_1+\cdots+x_{n}}{n}\right)\, {}_a d_{q}x_1\cdots\, {}_a d_{q}x_n \\
&\leq \frac{1}{(b-a)^{n-1}}\int_a^{b}\cdots\int_a^{b} f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)\, {}_a d_{q}x_1\cdots\, {}_a d_{q}x_{n-1} \\
&\vdots \\
&\leq \frac{1}{b-a}\int_a^b f(x)\, {}_a d_q x;
\end{aligned} \tag{12}$$

see also [65]. In addition, if $q \to 1$ in (12), then (12) reduces to

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\cdots+x_n}{n}\right)dx_1 d\cdots dx_n \\
&\le \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)dx_1\cdots\, dx_{n-1} \\
&\vdots \\
&\le \frac{1}{b-a}\int_a^b f(x)dx,
\end{aligned}
$$

which readily appeared in [66].

Corollary 1. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, we have*

$$
\begin{aligned}
f\left(\frac{qa+pb}{p+q}\right) &\le \frac{1}{p^2(b-a)^2}\int_a^{pb+(1-p)a}\int_a^{pb+(1-p)a} f\left(\frac{x_1+x_2}{2}\right)\,{}_a d_{p,q}x_1\,{}_a d_{p,q}x_2 \\
&\le \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a d_{p,q}x.
\end{aligned} \tag{13}
$$

Remark 3. *If $p = 1$, then (13) reduces to*

$$
\begin{aligned}
f\left(\frac{qa+b}{1+q}\right) &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x_1+x_2}{2}\right)\,{}_a d_q x_1\,{}_a d_q x_2 \\
&\le \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x;
\end{aligned} \tag{14}
$$

see also [65]. In addition, if $q \to 1$ in (14), then (14) reduces to

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x_1+x_2}{2}\right)dx_1dx_2 \\
&\le \frac{1}{b-a}\int_a^b f(x)dx,
\end{aligned}
$$

which readily appeared in [67].

Theorem 5. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, we have*

$$
\begin{aligned}
f\left(\frac{qa+pb}{p+q}\right) &\le \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_a d_{p,q}x_1\cdots\,{}_a d_{p,q}x_n \\
&\le \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a d_q x,
\end{aligned} \tag{15}
$$

for all $t_i \ge 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} t_i = T_n > 0$ and $n \in \mathbb{N}$.

Proof. By Jensen's inequality, we have

$$
f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right) \le \frac{1}{T_n}[t_1f(x_1)+\cdots+t_nf(x_n)]
$$

for all $x_i \in J$ and $t_i \geq 0$, where $i = 1, 2, \ldots, n$. Taking (p,q)-integration on both sides of the above inequality on J^n, we obtain

$$\begin{aligned}
&\int_a^{pb+(1-p)a} \cdots \int_a^{pb+(1-p)a} f\left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) {}_ad_{p,q}x_1 \cdots {}_ad_{p,q}x_n \\
&\leq \frac{1}{T_n} \int_a^{pb+(1-p)a} \cdots \int_a^{pb+(1-p)a} [t_1 f(x_1) + \cdots + t_n f(x_n)] \, {}_ad_{p,q}x_1 \cdots {}_ad_{p,q}x_n \\
&= p^{n-1}(b-a)^{n-1} \int_a^{pb+(1-p)a} f(x) \, {}_ad_{p,q}x,
\end{aligned}$$

which yields the second part of (17).

On the other hand, by Jensen's inequality, we have

$$\begin{aligned}
&f\left(\frac{1}{p^n(b-a)^n} \int_a^{pb+(1-p)a} \cdots \int_a^{pb+(1-p)a} \left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) {}_ad_{p,q}x_1 \cdots {}_ad_{p,q}x_n\right) \\
&\leq \frac{1}{p^n(b-a)^n} \int_a^{pb+(1-p)a} \cdots \int_a^{pb+(1-p)a} f\left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) {}_ad_{p,q}x_1 \cdots {}_ad_{p,q}x_n.
\end{aligned}$$

Since

$$\frac{1}{p^n(b-a)^n} \int_a^{pb+(1-p)a} \cdots \int_a^{pb+(1-p)a} \left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) {}_ad_{p,q}x_1 \cdots {}_ad_{p,q}x_n = \frac{qa+pb}{p+q},$$

this yields the first part of (15). This completes the proof. □

Remark 4. *If $p = 1$, then (15) reduces to*

$$\begin{aligned}
f\left(\frac{qa+b}{1+q}\right) &\leq \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) {}_ad_qx_1 \cdots {}_ad_qx_n \\
&\leq \frac{1}{b-a} \int_a^b f(x) {}_ad_qx;
\end{aligned} \tag{16}$$

see also [65]. In addition, if $q \to 1$ in (16), then (16) reduces to

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f\left(\frac{t_1x_1 + \cdots + t_nx_n}{T_n}\right) dx_1 \cdots dx_n \\
&\leq \frac{1}{b-a} \int_a^b f(x) dx,
\end{aligned}$$

which readily appeared in [66].

Corollary 2. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, we have*

$$\begin{aligned}
f\left(\frac{qa+pb}{p+q}\right) &\leq \frac{1}{p^2(b-a)^2} \int_a^{pb+(1-p)a} \int_a^{pb+(1-p)a} f(t_1x_1 + t_2x_2) \, {}_ad_qx_1 \, {}_ad_{p,q}x_2 \\
&\leq \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x) \, {}_ad_{p,q}x.
\end{aligned} \tag{17}$$

Remark 5. *If $p = 1$, then (17) reduces to*

$$\begin{aligned}
f\left(\frac{qa+b}{1+q}\right) &\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f(t_1x_1 + t_2x_2) \, {}_ad_qx_1 \, {}_ad_qx_2 \\
&\leq \frac{1}{b-a} \int_a^b f(x) \, {}_ad_qx;
\end{aligned} \tag{18}$$

see also [65]. In addition, if $q \to 1$ in (18), then (18) reduces to

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f(t_1x_1+t_2x_2)dx_1dx_2$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which readily appeared in [67].

Theorem 6. *Let $f : J \to \mathbb{R}$ be a continuous convex function. Then, the following inequalities*

$$\begin{aligned} f\left(\frac{qa+pb}{p+q}\right) &\leq \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\leq \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_ad_{p,q}x \end{aligned} \tag{19}$$

are valid for all $t_i \geq 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} t_i = T_n > 0$ and $n \in \mathbb{N}$.

Proof. Since

$$\frac{x_1+\cdots+x_n}{n} = \frac{1}{n}\left[\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)+\cdots+\left(\frac{t_2x_1+\cdots+t_1x_n}{T_n}\right)\right],$$

we have

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{1}{n}\left[f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)+\cdots+f\left(\frac{t_2x_1+\cdots+t_1x_n}{T_n}\right)\right],$$

by using Jensen's inequality. Taking the (p,q)-integration on both sides of the above inequality on J^n, we obtain

$$\begin{aligned} &\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\leq \frac{1}{n}\left[\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n+\cdots\right. \\ &\quad\left.+\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_2x_1+\cdots+t_1x_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n\right]. \end{aligned}$$

Since

$$\begin{aligned} &\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &= \int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\ &\vdots \\ &= \int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_2x_1+\cdots+t_1x_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n. \end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n \\
&\leq \frac{1}{p^n(b-a)^n}\int_a^{pb+(1-p)a}\cdots\int_a^{pb+(1-p)a} f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_{p,q}x_1\cdots\,{}_ad_{p,q}x_n.
\end{aligned}$$

Using Theorems 4 and 5, we can obtain the desired result. □

Remark 6. *If $p=1$, then (19) reduces to*

$$\begin{aligned}
f\left(\frac{qa+b}{1+q}\right) &\leq \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_n}{n}\right)\,{}_ad_qx_1\cdots\,{}_ad_qx_n \\
&\leq \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)\,{}_ad_qx_1\cdots\,{}_ad_qx_n \\
&\leq \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx,
\end{aligned} \tag{20}$$

see also [65]. In addition, if $q\to 1$ in (20), then (20) reduces to

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{x_1+\cdots+x_n}{n}\right)dx_1\cdots dx_n \\
&\leq \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\frac{t_1x_1+\cdots+t_nx_n}{T_n}\right)dx_1\cdots dx_n \\
&\leq \frac{1}{b-a}\int_a^b f(x)dx,
\end{aligned}$$

which readily appeared in [68].

4. Conclusions

In the present paper, we used (p,q)-calculus to establish some new refinements of (p,q)-Hermite–Hadamard inequalities, which have been expanded to integration on an n-dimensional finite interval. Many existing results in the literature are deduced as special cases of our results for $p=1$ and $q\to 1$. The results of this paper are new and significantly contribute to the existing literature on the topic.

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