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# Stability Concepts of Riemann-Liouville Fractional-Order Delay Nonlinear Systems

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**Abstract:** First, we set up in an appropriate way the initial value problem for nonlinear delay differential equations with a Riemann-Liouville (RL) fractional derivative. We define stability in time and generalize Mittag-Leffler stability for RL fractional differential equations and we study stability properties by an appropriate modification of the Razumikhin method. Two different types of derivatives of Lyapunov functions are studied: the RL fractional derivative when the argument of the Lyapunov function is any solution of the studied problem and a special type of Dini fractional derivative among the studied problem.

**Keywords:** Riemann-Liouville fractional derivative; time-varying delay; stability; Lyapunov functions; fractional derivatives of Lyapunov functions; Razumikhin method



**Citation:** Agarwal, R.; Hristova, S.; O'Regan, D. Stability Concepts of Riemann-Liouville Fractional-Order Delay Nonlinear Systems. *Mathematics* **2021**, *9*, 435. https://doi.org/10.3390/math9040435

Academic Editor: Christopher Goodrich

Received: 4 January 2021
Accepted: 16 February 2021
Published: 22 February 2021



## 1. Introduction

Various processes with anomalous dynamics in science and engineering can be formulated mathematically using fractional differential operators ([1–3]). When the Riemann-Liouville (RL) fractional derivative is applied in differential equations, the statement of initial conditions is important. It is worth mentioning that the physical and geometric interpretations of operations of fractional integration and differentiation were suggested by Podlubny [4]. Fractional differential equations in terms of the RL derivative require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function ([5,6]). In [7], it was shown that the initial conditions for fractional differential equations with RL derivatives expressed in terms of fractional derivatives has physical meaning. In fact, it was shown that for any physically realistic model, zero initial conditions will be found for a continuous loading program or even in the case of a step discontinuity. Nonzero conditions will only be found in the case of an impulse, and this type of process can be found in physics, chemistry, engineering, biology and economics. In the case of zero initial conditions, the RL, Grünwald-Letnikov (GL) and Caputo fractional derivatives coincide ([4]). For this reason, some authors either study Caputo derivatives or use RL derivatives but avoid the problem of initial values of fractional derivatives by treating only the case of zero initial conditions. This leads to the consideration of mathematical correct problems, but without taking the physical nature of the described process into account. Sometimes, such as in the case of impulse response, nonzero initial conditions appear (see, for example, [7]).

In connection with the main idea of stability properties, we will consider in this paper nonzero initial conditions for RL fractional equations, and we will define in an appropriate way stability properties that are slightly different than those for Caputo fractional differential equations.

Note that stability properties of delay differential equations can be considered by an application of the Lyapunov-Krasovskii method by functionals or by the Razumikhin method by Lyapunov functions. It is worth mentioning that both mentioned methods are applied for stability study of Caputo fractional delay differential equations (see, for example, [8–11]).

In the case of delay fractional differential equations with the RL fractional derivative, following the idea of initial conditions in ordinary delay differential equations and the above-mentioned idea concerning the initial condition for RL fractional differential equations without any delay, we will set up initial conditions in an appropriate way. Note that any solution of the defined initial conditions with RL fractional derivatives is not continuous at zero (the initial point), which is the same as in the case without any delay. Delay RL fractional differential equations are set up and studied in [12], but the initial condition does not correspond to the idea of the case of delay differential equations with ordinary derivatives (the lower bound of the RL fractional derivative coincides with the left end side of the initial interval).

Asymptotic stability for RL fractional differential equations with delays is studied in [13–15], but only the autonomous case is considered. A Lyapunov functional and its integer order derivative is applied. This functional is similar to the one used in the theory of differential equations with ordinary derivative and delay. On one hand, the application of ordinary derivative of the Lyapunov functional is not similar to the used fractional derivatives in the equation; on the other hand, it leads to some restrictions on both the delay and the right side parts of the equation ([16]). Additionally, in [15], the initial condition is not adequately associated with the RL fractional derivative. RL fractional equations with delays were studied recently in [17,18], but there are unclear parts in the statement of the problem (the lower limit of the RL derivative is different than the initial time point) as well as in the initial condition (the RL fractional integral has no meaning, compared with [5] at the initial time). The Razumikhin method is applied to RL fractional differential equations in [11], but the initial condition is not connected with the RL fractional derivative.

In this paper, the initial value problem for nonlinear delay differential equations with the RL fractional derivative is studied. Based on the arguments in the books [5,6], we set up initial conditions expressed in terms of initial values of fractional derivatives of the unknown function. Any solution of the defined initial conditions with RL fractional derivatives is not continuous at zero (the initial point). We require a new definition for stability excluding a small interval around zero. We define stability in time and generalize Mittag-Leffler stability in time for RL fractional differential equations. The stability properties of the zero solution are studied by Lyapunov functions. An appropriate modification of the Razumikhin method is suggested. Two types of derivatives of Lyapunov functions are applied: the RL fractional derivative when the argument of the Lyapunov function is a solution of the studied problem and the Dini fractional derivative among the studied problem.

The main contribution in the paper could be summarized as follows:

- the initial conditions connected with the RL fractional derivative are set up in an appropriate way;
- new types of stability connected with the type of initial conditions are defined;
- the RL fractional modification of the Razumikhin method is presented;
- new sufficient conditions for the defined stability are obtained;
- two types of fractional derivatives of the Lyapunov functions are used.

## 2. Preliminary Notes

In this paper, we will use the following definitions that are well known in the literature ([5,19]):

- Riemann–Liouville fractional integral of order $q > 0$

$$ {}_0I_t^q m(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{m(s)}{(t-s)^{1-q}} ds, \quad t \geq 0, $$

where $\Gamma(.)$ is the gamma function.
Note that the notation ${}_0D_t^{-\alpha} m(t) = {}_0I_t^{\alpha} m(t)$ is sometimes used.

- Riemann–Liouville fractional derivative of order $q \in (0,1)$

$$ {}_0^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t (t-s)^{-q} m(s) ds, \ t \geq 0 $$

- Grünwald–Letnikov fractional derivative of order $q \in (0,1)$

$$ {}_0^{GL}D_t^q m(t) = \lim_{h \to 0+} \frac{1}{h^q} \sum_{j=0}^{\frac{[t]}{h}} (-1)^j \, {}_qC_j \, m(t-jh), \quad t \geq 0, $$

where

$$ {}_qC_j = \frac{q(q-1)(q-2)\dots(q-j-1)}{j!} = \frac{\Gamma(q+1)}{j!\Gamma(q-j+1)}. \tag{1} $$

**Remark 1.** *If $m \in C([0,T],\mathbb{R})$, then (see, Theorem 2.25 [19]),*

$$ {}_0^{GL}D_t^q m(t) = {}_0^{RL}D_t^q m(t), \quad t \in (0,T]. $$

The fractional derivatives for scalar functions could be easily generalized to the vector case, by taking fractional derivatives with the same fractional order for all components.

Define the class

$$ C_{1-q}([0,T],\mathbb{R}) = \{m : [0,T] \to \mathbb{R} : \ t^{1-q} m(t) \in C([0,T],\mathbb{R})\}, \ \text{with } T \leq \infty. $$

We will provide some well-known results from the literature (see, for example, [5,19]):

**Proposition 1.** *For $q \in (0,1), \beta > 0$:*

$$ {}_0^{RL}D_t^q \, t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-q)} t^{\beta-q-1}. $$

**Proposition 2.** *For $q \in (0,1)$:*

$$ {}_0^{RL}D_t^{-q} \, t^{-q} = \Gamma(1-q), $$

$$ {}_0^{RL}D_t^q 1 = \frac{1}{\Gamma(1-q)} t^{-q}, $$

$$ {}_0^{RL}D_t^q \, t^{q-1} = 0. $$

**Proposition 3** (Property 4 [20])**.** *If the inequalities ${}_0^{RL}D_t^q x(t) \geq {}_0^{RL}D_t^q y(t), \ t \in (0,b]$ and ${}_0^{RL}D_t^{q-1} x(t)|_{t=0} \geq {}_0^{RL}D_t^{q-1} y(t)|_{t=0}$ hold, then $x(t) \geq y(t), \ t \in (0,b]$*

**Remark 2.** *As it is mentioned in [20] (see Example 1 [20]), a function might not be differentiable at one point in the classical sense, but it is RL differentiable. The positive RL fractional derivative ${}_0^{RL}D_t^q m(t) > 0$ of order $q \in (0,1)$ only means that the RL fractional integral ${}_0^{RL}I_t^{1-q} m(t)$*

*is monotonously increasing with respect to $t$ and it does not imply that the function $m(t)$ is monotonously increasing.*

*So, we cannot regard RL and Caputo derivatives as the generalization of the ordinary derivative in a rigorous mathematical way.*

**Proposition 4** (Lemma 2.3 [21]). *Let $m \in C_{1-q}([0,T],\mathbb{R})$. Suppose that for any $t_1 \in (0,T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $0 \le t < t_1$. Then, it follows that ${}_0^{RL}D_t^q m(t)|_{t=t_1} \ge 0$.*

**Proposition 5.** *The initial value problem*

$$
\begin{aligned}
&{}_0^{RL}D_t^q x(t) = Ax(t) + h(t) \;\; \text{for } t > 0, \\
&{}_0^{RL}I_t^{1-q} x(t)|_{t=0} = x_0,
\end{aligned}
$$

*has a unique solution*

$$
x(t) = x_0 t^{q-1} E_{q,q}(At^q) + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) h(s) ds,
$$

*where $E_{q,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(q(k+1))}$ is the two-parameter Mittag-Leffler function.*

Now, based on Proposition 5, we will illustrate the importance of the initial condition when the RL fractional derivative is used in the equation. For simplicity, we will consider the case of equations without any delays.

**Remark 3.** *Consider the scalar linear RL fractional equation ($q \in (0,1)$)*

$$
{}_0^{RL}D_t^q x(t) = x(t) \;\; \text{for } t > 0,
$$

*It is well known that ${}_0^{RL}D_t^q t^{q-1} E_{q,q}(t^q) = t^{q-1}E_{q,q}(t^q)$, i.e., the solution of the above RL fractional differential equation, is $x(t) = ct^{q-1}E_{q,q}(t^q)$ where $c$ is a real constant.*

*Now, consider the initial condition $x(0) = k$ where $k$ is a real constant. However, $t^{q-1}E_{q,q}(t^q)|_{t=0+} = \infty \ne k$. This illustrates that the initial condition of the type $x(0) = k$ is not applicable for RL fractional equations (see, for example, [11]).*

*Now, consider the initial condition $t^{1-q}x(t)|_{t=0+} = k$ where $k$ is a real constant. Then, $t^{1-q}x(t)|_{t=0+} = t^{1-q}ct^{q-1}E_{q,q}(t^q)|_{t=0+} = cE_{q,q}(0) = \frac{c}{\Gamma(q)}$, i.e., the initial condition $t^{1-q}x(t)|_{t=0+} = k$ has a meaning for the RL fractional derivative with $k = \frac{c}{\Gamma(q)}$.*

The practical definition of the initial condition of fractional differential equations with RL derivatives is based on the following result:

**Lemma 1** ([2]). *Let $q \in (0,1)$ and $b > 0$, $m : [0,b] \to \mathbb{R}$ be a Lebesgue measurable function.*

*(a) If there exists a.e. a limit $\lim_{t\to 0+}[t^{1-q}m(t)] = c \in \mathbb{R}$, then there also exists a limit*

$$
{}_0I_t^{1-q} m(t)|_{t=0} = {}_0D_t^{q-1} m(t)|_{t=0} := \lim_{t\to 0+} \frac{1}{\Gamma(1-q)} \int_0^t \frac{m(s)}{(t-s)^q} ds = c\Gamma(q).
$$

*(b) If there exists a.e. a limit $\lim_{t\to 0+} {}_0I_t^{1-q} m(t) = c \in \mathbb{R}$, and if there exists the limit $\lim_{t\to 0+}[t^{1-q}m(t)]$, then*

$$
\lim_{t\to 0+}[t^{1-q}m(t)] = \frac{c}{\Gamma(q)}.
$$

## 3. Statement of the Problem

Consider the following nonlinear Riemann–Liouville fractional delay differential equation (RLFrDDE) of fractional order $q \in (0,1)$:

$$ {}_0^{RL}D_t^q x(t) = f(t, x_t) \ \text{ for } t > 0, \tag{2} $$

with initial conditions

$$ \begin{aligned} x(t) &= \phi(t), \text{ for } t \in [-\tau, 0], \\ \lim_{t \to 0+} [t^{1-q} x(t)] &= \frac{\phi(0)}{\Gamma(q)}, \end{aligned} \tag{3} $$

where $x \in \mathbb{R}^n$, $x_t(\Theta) = x(t+\Theta)$, $\Theta \in [-\tau, 0]$, $\phi : [-\tau, 0] \to \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

**Remark 4.** *According to Lemma 1, the second equation in the initial conditions (3) could be replaced by the equality* ${}_0I_t^{1-q} x(t)|_{t=0} = \phi(0)$.

We denote the solution of the initial value problem (IVP) for RLFrDDE (2) and (3) by $x(t) = x(t;\phi)$ for $t \geq -\tau$. In this paper, we will assume that the function $f$ is such that for any continuous initial function $\phi$ the IVP for RLFrDDE (2) and (3) has a solution. Note that some existence and uniqueness results to RL fractional differential equations with delay were obtained in [22].

For any $\phi \in C([-\tau, 0], \mathbb{R}^n)$, we denote $||\phi||_0 = \max_{t \in [-\tau, 0]} ||\phi(t)||$ where $||.||$ is a norm in $\mathbb{R}^n$.

We will introduce the following conditions

**Hypothesis 1** (H1). *The function* $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *is such that for any initial function* $\phi \in C([-\tau, 0], \mathbb{R}^n))$, *the corresponding IVP for RLFrDDE (2) and (3) has a solution* $x(t;\phi) \in C_{1-q}([0,\infty), \mathbb{R}^n)$;

**Hypothesis 2** (H2). $f(t, 0) = 0$ *for all* $t \in \mathbb{R}_+$.

**Remark 5.** *If* $\phi(t) \equiv 0$ *and condition (H2) is satisfied, then, because of the equality* ${}_0^{RL}D_t^q 0 = 0$, *the IVP for RLFrDDE (2) and (3) has the zero solution.*

We will give the basic definitions for stability:

**Definition 1.** *The zero solution of RLFrDDE (2) and (3) (with the zero initial function) is said to be*

- *stable in time if for any number* $\varepsilon > 0$ *there exist numbers* $\delta > 0$ *and* $T_\varepsilon > 0$ *depending on* $\varepsilon$ *such that for any initial functions* $\phi \in C_0 : \ ||\phi||_0 < \delta$, *the corresponding solution* $x(t;\phi)$ *of IVP (2) and (3) satisfies* $||x(t;\phi)|| < \varepsilon$ *for* $t \geq T_\varepsilon$;
- *asymptotically stable if it is stable in time and additionally* $x(t;\phi) \to 0$ *as* $t \to +\infty$;
- *generalized Mittag-Leffler stable in time if there exist positive numbers* $\lambda, b$ *and* $\gamma \in (0,1)$ *and a locally Lipschitz function* $h \in C([0,\infty), [0,\infty)) : \ h(0) = 0$ *such that for any* $\varepsilon > 0$, *there exists* $T_\varepsilon > 0$ *the solution of IVP (2) and (3) satisfies*

$$ ||x(t;\phi)|| \leq \varepsilon \{h(||\phi||_0) t^{-\gamma} E_{q,1-\gamma}(-\lambda t^q)\}^b, \quad t \geq T_\varepsilon. $$

As an example to discuss stability in time, we consider a scalar RL fractional equation without any delay, whose exact solution is known.

**Example 1.** *According to Proposition 5, the scalar initial value problem*

$$ {}_0^{RL}D_t^q x(t) = -ax(t) \ \text{ for } t > 0, \quad {}_0^{RL}I_t^{1-q} x(t)|_{t=0} = x_0 $$

*has a unique solution*

$$x(t) = x_0 t^{q-1} E_{q,q}(-at^q).$$

*Therefore, the zero solution is generalized Mittag-Leffler stable with* $\gamma = 1 - q$, $m(u) = u$ *and* $\lambda = a$.

*The zero solution is stable in time because for any* $\varepsilon > 0$ *there exist* $\delta, T_\varepsilon > 0$ *such that* $\frac{\delta}{T_\varepsilon^{1-q}} E_{q,q}(-aT_\varepsilon^q) = \varepsilon$. *At the same time, the zero solution is not stable (in the regular sense) because* $\lim_{t\to 0} t^{q-1}E_{q,q}(-at^q) = \infty$ *and* $|t^{q-1}E_{q,q}(-at^q)| < \varepsilon$ *cannot be satisfied for values of t close to 0.*

*For example, if* $a = 1, q = 0.5, \varepsilon = 0.1, \delta = 0.2 > \varepsilon$ *then* $T_\varepsilon \approx 0.257459$, *if* $a = 1, q = 0.5, \varepsilon = 0.1, \delta = 0.01 < \varepsilon$ *then* $T_\varepsilon \approx 0.00265746$.

Throughout the paper, we shall use the class

$$\mathcal{K} = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : \ w(s) \text{ is strictly increasing and } w(0) = 0\}$$

and

$$\bar{B}(\rho) = \{x \in \mathbb{R}^n : \ ||x|| \le \rho\}.$$

## 4. Stability of Nonlinear RL Fractional Differential Equations

### *4.1. Lyapunov Functions and Their Derivatives*

One approach to study stability properties of nonlinear RL fractional differential equations is based on the application of Lyapunov functions and an appropriate modification of the Razumikhin method. The first step is to define a Lyapunov function. The second step is to define its derivative among the studied equation.

We will use the following class of functions called Lyapunov functions:

**Definition 2** ([8])**.** *Let* $J = [-\tau, T)$, $T \le \infty$, *be a given interval, and* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ *be a given set. We will say that the function* $V(t,x) : J \times \Delta \to \mathbb{R}_+$ *belongs to the class* $\Lambda(J,\Delta)$ *if* $V(t,x)$ *is continuous on* $J/\{0\} \times \Delta$, *and it is locally Lipschitzian with respect to its second argument.*

In our study, we will use the Razumikhin condition for the Lyapunov function $V \in \Lambda(J,\Delta)$ and any $\psi \in C([-\tau,0],\mathbb{R}^n)$:

$$V(t+\Theta, \psi(\Theta)) \le V(t, \psi(0)), \ \Theta \in [-\tau, 0].$$

We will give a brief overview of the derivatives of Lyapunov functions among solutions of fractional differential equations in the literature. There are three main types of derivatives of Lyapunov functions from the class $\Lambda(J,\Delta)$ used in the literature to study stability properties of solutions of fractional differential Equation (2):

- <u>RL fractional derivative</u>—Let $x(t) : [0,T) \to \Delta$ be a solution of the IVP for the RL-FrDDE (2) and (3). Then, we consider

$$ {}_0^{RL}D^q V(t,x(t)) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} V(s,x(s)) ds, \quad t \in [0,T). \tag{4}$$

- <u>Dini fractional derivative</u>—Let $\phi : [-\tau, 0] \to \Delta$. Then, consider (see [8])

$$\begin{aligned} &D^+_{(2)} V(t,\phi(0),\phi) \\ &= \limsup_{h\to 0} \frac{1}{h^q} \Big[ V(t,\phi(0)) - \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1} {}_qC_r V(t-rh, \phi(0) - h^q f(t,\phi_0)) \Big], \end{aligned} \tag{5}$$

where ${}_qC_r$ is defined by (1) and $\phi_0(\Theta) = \phi(\Theta)$, $\Theta \in [-\tau, 0]$.

The Dini fractional derivative is applicable for continuous Lyapunov functions.

We will provide an example concerning some Lyapunov functions and their Dini fractional derivatives. To simplify the calculations and to emphasize on the derivative, we will consider the scalar case, i.e., $n = 1$.

**Example 2.** *Let $V(t,x) = m(t)\,x^2$ where $m \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\phi \in C([-\tau,0],\mathbb{R})$. Apply (5) and obtain*

$$
\begin{aligned}
&D^+_{(2)}V(t,\phi(0),\phi)\\
&= \limsup_{h\to 0}\frac{1}{h^q}\Big[m(t)\,(\phi(0))^2 - \sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r m(t-rh)(\phi(0)-h^q f(t,\phi_0))^2\Big]\\
&= \limsup_{h\to 0}\frac{1}{h^q}\Big[[m(t)\,(\phi(0))^2 - m(t)\,(\phi(0))^2 - h^q f(t,\phi_0))^2\\
&\quad -(\phi(0))^2 - h^q f(t,\phi_0))^2(-1)^{0+1}\,{}_qC_0 m(t-0h)\\
&\quad -(\phi(0)-h^q f(t,\phi))^2\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r m(t-rh)\Big]\\
&= \limsup_{h\to 0}\frac{1}{h^q}\Big[m(t)h^q f(t,\phi_0)(2\phi(0)-h^q f(t,\phi_0))\\
&\quad +(\phi(0))^2 - h^q f(t,\phi_0))^2\sum_{r=0}^{[\frac{t}{h}]}(-1)^r\,{}_qC_r m(t-rh)\Big]\\
&= 2\phi(0)\,m(t)f(t,\phi) + (\phi(0))^2\,{}^{RL}_{0}D^q\Big(m(t)\Big).
\end{aligned}
\tag{6}
$$

*Special case 1. Let $V(t,x) = t^{1-q}\,x^2$ for $t > 0,\ x \in \mathbb{R}$. According to Proposition 1 with $\beta = 2-q$, we get ${}^{RL}_{0}D^q_t\,t^{1-q} = \frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{1-2q}$ and from (6), we get*

$$
D^+_{(2)}V(t,\phi(0),\phi) = t^{1-q}\Big(2\phi(0)f(t,\phi) + (\phi(0))^2\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big). \tag{7}
$$

*Special case 2. Let $V(t,x) = x^2$ for $x \in \mathbb{R}$. From Proposition 2, with $m(t) \equiv 1$, we obtain*

$$
D^+_{(2)}V(t,\phi(0),\phi) = 2\phi(0)f(t,\phi) + \frac{(\phi(0))^2}{t^q\Gamma(1-q)}.
$$

*Special case 3. Let $V(t,x) = t^{q-1}\,x^2$ for $t > 0,\ x \in \mathbb{R}$. According to Proposition 2, we get from (6)*

$$
D^+_{(2)}V(t,\phi(0),\phi) = 2\,t^{q-1}\phi(0)f(t,\phi). \tag{8}
$$

*Special case 4. Let $V(t,x) = t^{1-q}m(t)x^2$ where $A \le t^{1-q}m(t) \le B$ for all $t \ge 0$. For example, if $m(t) = \frac{1}{t^{1-q}} + \frac{1}{1+t^{1-q}}$, then $t^{1-q}m(t) = 1 + \frac{t^{1-q}}{1+t^{1-q}} \in (1,2)$, ${}^{RL}_{0}D^q_t\Big(1+\frac{t^{1-q}}{1+t^{1-q}}\Big)$ exists and*

$$
D^+_{(2)}V(t,\phi(0),\phi) = 2\phi(0)\Big(1+\frac{t^{1-q}}{1+t^{1-q}}\Big)f(t,\phi) + (\phi(0))^2\Big(\frac{2}{t^q\Gamma(1-q)} - {}^{RL}_{0}D^q_t\frac{1}{1+t^{1-q}}\Big).
$$

**Remark 6.** *Note that $V(t,x(t)) = t^{1-q}m(t)\sum_{i-1}^n x_i^2(t) \in C_{1-q}([0,\infty),\mathbb{R}_+)$, $m \in C([0,\infty),\mathbb{R}_+)$ for any solution $x(t) = (x_1(t),x_2(t),\dots,x_n(t))$ of (2) and (3) because*

$$
t^{1-q}V(t,x(t))|_{t=0} = m(t)\sum_{i=1}^n(t^{1-q}\,x_i(t))^2|_{t=0} = m(0)\frac{\sum_{i=1}^n\phi_i^2(0)}{\Gamma^2(q)} = \frac{m(0)}{\Gamma^2(q)}||\phi||_0^2.
$$

*If $m(t) \equiv 1$, then the function $V(t, x(t)) = t^{1-q} \sum_{i-1}^{n} x_i^2(t) \in C_{1-q}([0, \infty), \mathbb{R}_+)$ for any solution $x(t)$ of (2) and (3).*

*The quadratic Lyapunov function $V(t, x(t)) = \sum_{i=1}^{n} x_i(t)^2$ is not from the set $C_{1-q}([0, \infty), \mathbb{R}_+)$ for any solution $x(t)$ of (2) and (3), because*

$$t^{1-q} V(t, x(t))|_{t=0} = t^{1-q} \sum_{i=1}^{n} x_i(t)^2|_{t=0} = t^{q-1} \sum_{i=1}^{n} (t^{1-q} x_i(t))^2|_{t=0} = \infty.$$

*The functions $V(t, x(t)) = t^{q-1} \sum_{i=1}^{n} x_i^2(t)$ is not from the set $C_{1-q}([0, \infty), \mathbb{R}_+)$ for any solution $x(t)$ of (2) and (3).*

We will study stability properties of the zero solution of RLFrDDE (2) by an application of both defined types of fractional derivatives of Lyapunov functions.

### *4.2. Stability by the RL Fractional Derivative of Lyapunov Functions*

We will obtain some sufficient conditions for stability with applications of the RL fractional derivative of Lyapunov functions.

**Theorem 1.** *Let conditions (H1) and (H2) be satisfied, and there exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*

*(i) for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that*

$$\varepsilon a(||x||) \leq V(t, x) \text{ for } t > T_\varepsilon,\ x \in \mathbb{R}^n, \tag{9}$$

*where $a \in \mathcal{K}$;*

*(ii) there exists an increasing function $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any function $y \in C_{1-q}([0, \infty), \mathbb{R}^n) :\ t^{1-q} y(t)|_{t=0+} = y_0$ the inequality*

$$t^{1-q} V(t, y(t))|_{t=0+} = \lim_{t \to 0+} t^{1-q} V(t, y(t)) \leq g(||y_0||)$$

*holds;*

*(iii) for any point $t > 0$ such that $(t + \Theta)^{1-q} V(t + \Theta, x(t + \Theta)) < t^{1-q} V(t, x(t))$ for $\Theta \in (-\min\{t, \tau\}, 0)$, the RL fractional derivative ${}_0^{RL}D_t^q V(t, x(t))$ exists and the inequality*

$${}_0^{RL}D_t^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} V(s, x(s)) ds < 0, \tag{10}$$

*holds where $x(t)$ is a solution of the IVP for RLFrDDE (2) and (3).*

*Then, the zero solution of (2) and (3) with the zero initial function is stable in time.*

**Proof.** Let $\epsilon > 0$ be an arbitrary number. According to condition (i), there exists $\tilde{T}_\varepsilon > 0$ such that inequality (9) holds for $t > \tilde{T}_\varepsilon,\ \ x \in \mathbb{R}^n$.

Let $\delta > 0$ be such that for $||\phi||_0 < \delta$, the inequality $g(\frac{||\phi||_0}{\Gamma(q)}) < \varepsilon$ holds.

Consider the solution $x(t) = x(t; \phi)$ of the IVP for RLFrDDE (2) and (3) with the initial function $\phi :\ ||\phi||_0 < \delta$.

From condition 2(ii) with $y(t) \equiv x(t)$ and $y_0 = \frac{\phi(0)}{\Gamma(q)}$, we get

$$t^{1-q} V(t, x(t))|_{t=0+} = \lim_{t \to 0+} t^{1-q} V(t, x(t)) \leq g(\frac{||\phi(0)||}{\Gamma(q)}) \leq g(\frac{||\phi||_0}{\Gamma(q)}) < \varepsilon. \tag{11}$$

Therefore, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that $V(t, x(t)) < \varepsilon t^{q-1}$ for $t < \delta_1$.

Consider the function $H(t) = \varepsilon t^{q-1} \in C_{1-q}([0,\infty), \mathbb{R}_+)$ and $\lim_{t\to\infty} H(t) = 0$. Therefore, there exists $\hat{T}_\varepsilon > \delta_1$ such that

$$H(t) < \varepsilon a(\varepsilon) \ \text{ for } t > \hat{T}_\varepsilon. \tag{12}$$

Define the function $m(t) = V(t, x(t))$ for $t \geq 0$. From (11), it follows that $m \in C_{1-q}([0,\infty), \mathbb{R}_+)$.

We will prove that

$$m(t) < H(t), \quad t > 0. \tag{13}$$

Note that the inequality (13) holds for $t \in (0, \delta_1)$. Assume inequality (13) is not true for all $t > 0$. Therefore, there exists a point $\xi \geq \delta_1 > 0$ such that

$$m(\xi) = H(\xi), \ \text{ and } \ m(t) < H(t), \ \ t \in (0, \xi). \tag{14}$$

Therefore, $m(t) - H(t) \in C_{1-q}([0,\xi], \mathbb{R})$. According to Proposition 4 with $t_1 = \xi$, the inequality ${}_0^{RL}D_t^q\Big(m(t) - H(t)\Big)|_{t=\xi} \geq 0$ holds. From Proposition 2, we get ${}_0^{RL}D_t^q\, t^{q-1} = 0$ and therefore,

$${}_0^{RL}D_t^q m(t)|_{t=\xi} = {}_0^{RL}D_t^q\Big(m(t) - H(t)\Big)|_{t=\xi} \geq 0. \tag{15}$$

*Case 1.* Let $\xi > \tau$. Then, $\min\{\xi, \tau\} = \tau$. From (14), it follows that $\xi^{q-1}m(\xi) = \varepsilon > t^{q-1}m(t),\ t \in (0,\xi)$, or $(\xi+\Theta)^{q-1}m(\xi+\Theta) = (\xi+\Theta)^{q-1}V(\xi+\Theta, x(\xi+\Theta)) < \xi^{q-1}m(\xi) = \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\tau, 0)$. According to condition 2(iii)

$${}_0^{RL}D_t^q V(t, x(t))|_{t=\xi} < 0. \tag{16}$$

The inequality (16) contradicts (15).

*Case 2.* Let $\xi \leq \tau$. Then, $\min\{\xi, \tau\} = \xi$. From (14), it follows that $\xi^{q-1}m(\xi) = \varepsilon > t^{q-1}m(t),\ t \in (0,\xi)$, or $(\xi_\Theta)^{q-1}m(\xi+\Theta) = (\xi_\Theta)^{q-1}V(\xi+\Theta, x(\xi+\Theta)) < \xi^{q-1}m(\xi) = \xi^{q-1}V(\xi, x(\xi))$ for $\Theta \in (-\xi, 0)$ and the proof is similar to the one of Case 1.

From inequality (13) and condition (i), it follows that

$$\varepsilon a(||x(t)||) \leq V(t, x(t)) = m(t) < H(t) < \varepsilon a(\varepsilon) \ \text{ for } t > T_\varepsilon, \tag{17}$$

where $T_\varepsilon = \max\{\hat{T}_\varepsilon, \tilde{T}_\varepsilon\}$.

This proves the stability in time of the zero solution. □

**Remark 7.** *The main condition in Theorem 1 is condition (iii), which is connected with any solution of the IVP (2) and (3).*

**Remark 8.** *According to Remark 6, the Lyapunov function $V(t,y) = t^{1-q}m(t)\sum_{i=1}^n y_i^2$ with $m \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies condition (ii) with $g(s) = m(0)s^2$. Condition (i) is satisfied with $a(u) = u^2$ if $m(t) \geq C = const > 0$ for $t \geq 0$.*

**Remark 9.** *Condition (i) is different than the corresponding condition for Lyapunov functions for differential equations with ordinary derivatives as well as with Caputo fractional differential equations. This is because of the type of initial condition (see Remark 3).*

**Theorem 2.** *Let conditions (H1) and (H2) be satisfied and there exists a function $V \in \Lambda([-\tau,\infty), \mathbb{R}^n)$ such that conditions (i) and (ii) of Theorem 1 hold with $a(s) = ks^p$, $k, p > 0$, locally Lipschitz function $g \in C([0,\infty), [0,\infty)) :\ g(0) = 0$ and*

*(iii*) for any point $t > 0$ such that $V(t+\Theta, x(t+\Theta)) < V(t, x(t))$ for $\Theta \in (-\min\{t,\tau\}, 0)$, the RL fractional derivative ${}_0^{RL}D_t^q V(t, x(t))$ exists and the inequality*

$${}_0^{RL}D_t^q V(t, x(t)) < -cV(t, x(t)), \tag{18}$$

*holds where $c > 0$, $x(t)$ is a solution of the IVP for RLFrDDE (2) and (3).*

*Then, the zero solution of (2) and (3) with the zero initial function is generalized Mittag-Leffler stable.*

**Proof.** Consider any solution $x(t) = x(t;\phi)$ of the IVP for RLFrDDE (2) and (3) with the initial function $\phi \in C([-\tau,0],\mathbb{R}^n)$.

From condition (ii) of Theorem 1, similar to inequality (11), we get

$$t^{1-q}V(t,x(t))|_{t=0+} \le g\Big(\frac{||\phi||_0}{\Gamma(q)}\Big). \tag{19}$$

Consider the function $H(t) = \Gamma(q)g(\frac{||\phi||_0}{\Gamma(q)})t^{q-1}E_{q,q}(-ct^q) \in C_{1-q}([0,\infty),(0,\infty))$.

Define the function $m(t) = V(t,x(t))$ for $t \ge 0$. From (19), it follows that $m \in C_{1-q}([0,\infty),\mathbb{R}_+)$.

Let $\varepsilon > 0$ be an arbitrary number. We will prove that

$$m(t) < H(t) + \varepsilon t^{q-1}, \quad t > 0. \tag{20}$$

We have $t^{1-q}V(t,x(t))|_{t=0+} \le g\Big(\frac{||\phi||_0}{\Gamma(q)}\Big) < g\Big(\frac{||\phi||_0}{\Gamma(q)}\Big) + \varepsilon = t^{1-q}H(t)|_{t=0+} + \varepsilon$. Therefore, there exists $\delta_2 > 0$ such that for $t \in (0,\delta_2)$, the inequality $V(t,x(t)) \le H(t) + \varepsilon t^{q-1}$ holds.

Assume inequality (20) is not true for all $t > 0$. Therefore, there exists a point $\xi \ge \delta_2 > 0$ such that

$$m(\xi) = H(\xi) + \varepsilon t^{q-1}, \ \text{and} \ m(t) < H(t) + \varepsilon t^{q-1}, \ t \in (0,\xi). \tag{21}$$

Therefore, $m(t) - H(t) - \varepsilon t^{q-1} \in C_{1-q}([0,\xi],\mathbb{R})$. According to ${}_0^{RL}D_t^q\, t^{q-1} = 0$ and Proposition 4 with $t_1 = \xi$, the inequality ${}_0^{RL}D_t^q\Big(m(t) - H(t)\Big)|_{t=\xi} \ge 0$ holds. From ${}_0^{RL}D_t^q\Big(t^{q-1}E_{q,q}(-ct^q)\Big) = -ct^{q-1}E_{q,q}(-ct^q)$, we have,

$$\begin{aligned} &{}_0^{RL}D_t^q m(t)|_{t=\xi} + cm(\xi) = {}_0^{RL}D_t^q m(t)|_{t=\xi} + cH(\xi) \\ &= {}_0^{RL}D_t^q m(t)|_{t=\xi} + cg(\frac{||\phi||_0}{\Gamma(q)})\xi^{q-1}E_{q,q}(-c\xi^q) = {}_0^{RL}D_t^q\Big(m(t) - H(t)\Big)|_{t=\xi} \ge 0. \end{aligned} \tag{22}$$

*Case 1.* Let $\xi > \tau$. Then, $\min\{\xi,\tau\} = \tau$. Therefore, $m(\xi+\Theta) = V(\xi+\Theta, x(\xi+\Theta)) < m(\xi) = V(\xi,x(\xi))$ for $\Theta \in (-\tau,0)$. Let $\psi(\Theta) = x(\xi-\Theta) \in C([-\tau,0],\mathbb{R}^n)$, $\Theta \in [-\tau,0]$. Then, $V(\xi+\Theta,\psi(\Theta)) < V(\xi,\psi(0))$ for $\Theta \in [-\tau,0)$ and, according to condition (iii*),

$${}_0^{RL}D^q V(\xi,x(\xi)) < -cV(t,x(t)). \tag{23}$$

The inequality (23) contradicts (22).

*Case 2.* Let $\xi \le \tau$. Then, $\min\{\xi,\tau\} = \xi$ and $V(\xi+\Theta,x(\xi+\Theta)) < V(\xi,x(\xi))$ for $\Theta \in (-\xi,0)$ and the proof is similar to the one of Case 1.

Since $\varepsilon > 0$ is an arbitrary number from inequality (20), it follows that

$$m(t) \le H(t), \quad t > 0. \tag{24}$$

Now, let $\varepsilon > 0$. According to condition (i), there exists $T_\varepsilon > 0$ such that $(\frac{1}{\varepsilon})^p k||x(t)||^p \le V(t,x(t))$ for $t \ge T_\varepsilon$. Then, from inequality (24), it follows that

$$(\frac{1}{\varepsilon})^p k||x(t)||^p \le V(t,x(t)) = m(t) < \Gamma(q)g(\frac{||\phi||_0}{\Gamma(q)})t^{q-1}E_{q,q}(-ct^q) \ \text{for}\ t \ge T_\varepsilon. \tag{25}$$

This proves the generalized Mittag-Leffler stability in time of the zero solution with $b = \frac{1}{p}, \gamma = 1 - q,\ \lambda = c$ and $h(s) = \frac{g(\frac{s}{\Gamma(q)})}{k}$ (see Definition 1). $\square$

**Corollary 1.** *If all the conditions of Theorem 2 are satisfied, then the zero solution of (2) and (3) is asymptotically stable.*

*4.3. Stability by the Dini Fractional Derivative of Lyapunov Functions*

We will study stability by the application of the defined above Dini fractional derivative of Lyapunov functions among the studied delay fractional differential equations.

Initially, we will prove a comparison result for Lyapunov functions.

**Lemma 2.** *Assume:*

1. *The function $x(t) = x(t;\phi) \in C_{1-q}([0,T],\Delta)$ is a solution of the IVP for RLFrDDE (2) and (3) with $\phi \in C([-\tau,0],\Delta)$ where $0 < T \leq \infty$.*
2. *The function $V \in \Lambda([-\tau,T],\Delta)$, $\Delta \subset \mathbb{R}^n$, is such that :*
   - *(i) There exists an increasing function $g \in C(\mathbb{R}_+,\mathbb{R}_+)$ such that the inequality $t^{1-q}V(t,x(t))|_{t=0+} = \lim_{t\to 0+} t^{1-q}V(t,x(t)) \leq g(\frac{||\phi(0)||}{\Gamma(q)}) \leq g(||\phi||_0)$ holds;*
   - *(ii) for any point $t > 0$ such that $V(t+\Theta,x(t+\Theta)) < V(t,x(t))$ for $\Theta \in (-\min\{t,\tau\},0)$, the inequality*
   $$D^+_{(2)}V(t,\psi(0),\psi) < 0 \tag{26}$$
   *holds where $D^+_{(2)}V(t,\psi(0),\psi)$ is the Dini fractional derivative defined by (5) and $\psi(\Theta) = x(t+\Theta)$, $\Theta \in [-\tau,0]$.*

   *Then, $V(t,x(t;\phi)) \leq t^{q-1}g(||\phi||_0)$ for $t \in [0,T]$.*

**Remark 10.** *Let us, for simplicity, again consider the RL fractional differential equation without any delay ${}_0^{RL}D_t^q x(t) = -x(t)$ for $t > 0$, ${}_0^{RL}I_t^{1-q}x(t)|_{t=0+} = c$ with a solution $x(t) = ct^{q-1}E_{q,q}(-t^q)$ (see Example 1). If we consider the quadratic Lyapunov function $V(x) = x^2$, then $D^+_{(2)}V(t,\psi(0),\psi) \leq 0$ and $(ct^{q-1}E_{q,q}(-t^q))^2 \leq c^2$ is not satisfied. However, if $V(x) = t^{1-q}x^2$ then $D^+_{(2)}V(t,\psi(0),\psi) \leq 0$ and $t^{1-q}(ct^{q-1}E_{q,q}(-t^q))^2 \leq c^2t^{q-1}$ is satisfied. This example again illustrates the changes in the applied Lyapunov functions and their conditions in the application of RL fractional derivatives comparatively with the application in Caputo fractional derivatives.*

**Remark 11.** *Let $V(t,x) = t^{1-q}m(t)\sum_{i=1}^n x_i^2$, where $m \in C([0,T])$, $T \leq \infty$, and $x(t)$ be a solution of (2) and (3). Then, the following*

$$t^{1-q}V(t,x(t))|_{t=0+} = \lim_{t\to 0+} m(t)t^{2-2q}\sum_{i=1}^n x_i^2(t) = m(0)\sum_{i=1}^n \frac{\phi_i(0)^2}{\Gamma^2(q)}$$

*holds, i.e., $t^{1-q}V(t,x(t))|_{t=0+} \leq \frac{m(0)}{\Gamma^2(q)}(||\phi||_0)^2 < \infty$ and condition 2 (i) of Lemma 2 is satisfied with $g(s) = \frac{m(0)}{\Gamma^2(q)}s^2$.*

**Proof.** Define the function $m(t) = V(t,x(t))$ for $t \in (0,T]$. According to condition 2 (i), $m \in C_{1-q}([0,T],\mathbb{R})$.

Let $H(t) = m(t) - t^{q-1}(B+\varepsilon)$ where $B = g(||\phi||_0) \geq 0$ and $\varepsilon > 0$ be an arbitrary number. We will prove

$$H(t) < 0,\ \ t \in [0,T] \tag{27}$$

For $t = 0$ from condition 2(i), we get $t^{1-q}H(t)|_{t=0} = t^{1-q}V(t,x(t))|_{t=0} - (B+\varepsilon) \leq g(||\phi||_0) - B - \varepsilon < 0$, i.e., the inequality (27) is true.

Assume (27) is not true. Therefore, there exist $t^* \in (0, T]$ such that

$$H(t) < 0,\ t \in [0, t^*), \qquad H(t^*) = 0. \tag{28}$$

From Proposition 4, we have the inequality ${}_0^{RL}D_t^q H(t)|_{t=t^*} \geq 0$. Then, applying Proposition 2 and ${}_0^{RL}D_t^q\, t^{q-1} = 0$, we obtain

$$ {}_0^{RL}D_t^q H(t)|_{t=t^*} = {}_0^{GL}D_t^q m(t)|_{t=t^*} = {}_0^{RL}D_t^q m(t)|_{t=t^*} \geq 0. \tag{29}$$

For any $t \in (0, t^*]$ and $h > 0$, we let

$$S(x(t), h) = \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1}\, {}_qC_r x(t - rh).$$

From Remark 1 and Equation (2), it follows that the function $x(t)$ satisfies for $t \in [t_0, t^*]$, the equalities ${}_0^{RL}D_t^q x(t) = {}_0^{GL}D_t^q x(t) = \limsup_{h\to 0+} \frac{1}{h^q}\Big[x(t) - S(x(t), h)\Big] = f(t, x_t)$ and

$$\limsup_{h\to 0+} \frac{1}{h^q}\Big[x(t) - S(x(t), h)\Big] = f(t, x_t).$$

Therefore,

$$S(x(t), h) = x(t) - h^q f(t, x_t) - \Lambda(h^q)$$

or

$$x(t) - h^q f(t, x_t) = S(x(t), h) + \Lambda(h^q) \tag{30}$$

with $\frac{||\Lambda(h^q)||}{h^q} \to 0$ as $h \to 0$. Then, for any $t \in [0, t^*]$ we obtain

$$\begin{aligned} m(t) &- \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1}\, {}_qC_r m(t - rh) \\ &= \Big\{ V(t, x(t)) - \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1}\, {}_qC_r \Big[ V(t - rh, x(t) - h^q f(t, x_t) \Big] \Big\} \\ &\quad + \sum_{r=1}^{[\frac{t}{h}]} (-1)^{r+1}\, {}_qC_r \Big\{ \Big[ V(t - rh, S(x(t), h) + \Lambda(h^q)) \Big] \\ &\quad - \Big[ V(t - rh, x(t - rh)) \Big] \Big\}. \end{aligned} \tag{31}$$

Since $V$ is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$,, we obtain

$$
\begin{aligned}
&\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big\{V(t-rh,S(x(t),h)+\Lambda(h^q))-V(t-rh,x(t-rh))\Big\}\\
&\le L\Big|\Big|\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big(S(x(t),h)+\Lambda(h^q)-(x(t-rh))\Big)\Big|\Big|\\
&\le L\Big|\Big|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}_qC_r\sum_{j=1}^{[\frac{t}{h}]}(-1)^{j+1}\,{}_qC_jx(t-jh)\\
&\quad-\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_rx(t-rh)\Big|\Big|+L||\Lambda(h^q)||\,\Big|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}_qC_r\Big|\\
&=L\Big|\Big|\Big(\sum_{r=0}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big)\Big(\sum_{j=1}^{[\frac{t}{h}]}(-1)^{j+1}\,{}_qC_jx(t-jh)\Big)\Big|\Big|\\
&\quad+L||\Lambda(h^q)||\,\Big|\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big|.
\end{aligned}
\tag{32}
$$

Substitute (32) in (31), divide both sides by $h^q$, take the limit as $h \to 0^+$, use $\sum_{r=0}^{\infty}\,{}_qC_rz^r = (1+z)^q$ if $|z| \le 1$ and we obtain for any $t \in (0, t^*]$ the inequality

$$
\begin{aligned}
&{}_0^{GL}D_t^q m(t)\le\lim_{h\to 0+}\frac{1}{h^q}\Big\{V(t,x(t))\\
&\quad-\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big[V(t-rh,x(t)-h^qf(t,x^*(t))\Big]\Big\}\\
&\quad+L\lim_{h\to 0+}\frac{||\Lambda(h^q)||}{h^q}\lim_{h\to 0+}\Big|\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big|\\
&\quad+L\lim_{h\to 0^+}\sup\Big|\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_r\Big|\,\Big|\Big|\frac{1}{h^q}\sum_{j=1}^{[\frac{t}{h}]}\,{}_qC_jx(t-jh)\Big|\Big|\\
&\le\lim_{h\to 0+}\frac{1}{h^q}\Big\{V(t,x(t))-\sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\,{}_qC_rV(t-rh,x(t)-h^qf(t,x^*(t)))\Big\}.
\end{aligned}
\tag{33}
$$

Let $t = t^*$. Define the function $\psi(\Theta) = x(t^* + \Theta)$, $\Theta \in [-\tau, 0]$. From the choice of the point $t^*$, it follows that $V(t^* + \Theta, x(t^* + \Theta)) \le V(t^*, x(t^*))$, $\Theta \in [-\tau, 0]$ and from inequalities (26) and (33) for $t = t^*$, we get

$$
\begin{aligned}
&{}_0^{GL}D_t^q m(t)|_{t=t^*}\\
&\le\lim_{h\to 0+}\frac{1}{h}\Big\{V(t^*,\psi(0))-\sum_{r=1}^{[\frac{t^*}{h}]}(-1)^{r+1}\,{}_qC_rV(t^*-rh,\psi(0)-h^qf(t^*,\psi))\Big\}\\
&=D_{(2)}^{+}V(t^*,\psi(0),\psi)<0.
\end{aligned}
\tag{34}
$$

Now (34) contradicts (29). Therefore, inequality (27) holds for an arbitrary $\varepsilon > 0$. Thus, the claim in our Lemma is true. □

**Theorem 3.** *Let conditions (H1) and (H2) be satisfied and there exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that $V(t, 0) = 0,\ t \ge 0$ and*

(i) *for any* $\varepsilon > 0$*, there exists* $T_\varepsilon > 0$ *such that*

$$\varepsilon a(||x||) \le V(t,x) \text{ for } t > T_\varepsilon,\ x \in \mathbb{R}^n, \tag{35}$$

*where* $a \in \mathcal{K}$*;*

(ii) *there exists an increasing function* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *such that for any function* $y \in C_{1-q}([0,\infty), \mathbb{R}^n) :\ t^{1-q}y(t)|_{t=0+} = y_0$ *the inequality*

$$t^{1-q}V(t,y(t))|_{t=0+} = \lim_{t\to 0+} t^{1-q}V(t,y(t)) \le g(||y_0||)$$

*holds;*

(iii) *for any function* $\psi \in C([-\tau,0], \mathbb{R}^n$ *such that if for a point t we have* $V(t+\Theta, \psi(\Theta)) < V(t,\psi(0))$ *for* $\Theta \in [-\tau, 0)$*, then the inequality*

$$D^+_{(2)}V(t,\psi(0),\psi) < 0 \tag{36}$$

*holds.*

*Then, the zero solution of (2) with the zero initial function is stable.*

**Proof.** Let $\epsilon > 0$ be a positive number. According to condition (i), there exists $\tilde{T}_\varepsilon > 0$ such that inequality (9) holds for $t > \tilde{T}_\varepsilon,\ \ x \in \mathbb{R}^n$.

There exists a positive number $\delta = \delta(\varepsilon)$ such that $g(u) < \sqrt{a(\varepsilon)}$ for $u \in \mathbb{R}_+ :\ u < \delta$. Choose the function $\phi \in C([-\tau,0], \mathbb{R}^n)$ such that $||\phi||_0 < \delta$. Consider the solution $x(t) = x(t;\phi)$ of the IVP for RLFrDDE (2) and (3) with initial function $\phi$ and define the function $m(t) = V(t,x(t))$ for $t \ge 0$. From condition 2(ii), it follows that $m \in C_{1-q}([0,\infty), \mathbb{R}_+)$.

Since $\lim_{t\to\infty} t^{q-1} = 0$, there exists $\hat{T}_\varepsilon > 0$ such that $t^{q-1} < \varepsilon\sqrt{a(\varepsilon)}$ for $t > \hat{T}_\varepsilon$. Therefore,

$$t^{q-1}g(||\phi||_0) < \varepsilon a(\varepsilon) \text{ for } t > \hat{T}_\varepsilon. \tag{37}$$

According to Lemma 2 with $T = \infty$, $\frac{\phi(0)}{\Gamma(q)} = y_0$, and $\psi(\Theta) \equiv x(t+\Theta)$, $\Theta \in [-\tau, 0]$,, we obtain the following inequality

$$V(t,x(t)) \le t^{q-1}g(||\phi||_0), \quad t \ge 0. \tag{38}$$

From inequalities (37), (38) and condition (i), it follows that $\varepsilon a(||x(t)||) \le V(t,x(t)) \le t^{q-1}g(||\phi||_0) < \varepsilon a(\varepsilon)$ for $t > T_\varepsilon$ with $T_\varepsilon = \max\{\hat{T}_\varepsilon, \tilde{T}_\varepsilon\}$.

This proves the stability in time of the zero solution. □

**Example 3.** *Consider the scalar RL fractional differential equation*

$$^{RL}_{0}D^q_t x(t) = f(t, x(t), x(t-1)) \text{ for } t > 0, \tag{39}$$

*with initial conditions*

$$\begin{aligned} x(t) &= \phi(t), \text{ for } t \in [-1,0], \\ \lim_{t\to 0+} [t^{1-q}x(t)] &= \frac{\phi(0)}{\Gamma(q)}, \end{aligned} \tag{40}$$

*where* $x \in \mathbb{R}$, $\phi \in C([-1,0], \mathbb{R})$ *and* $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is defined by*

$$f(t,x,y) = \begin{cases} -1.6x\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{1-2q} & \text{if } 0 < t \le 1,\ x,y \in \mathbb{R} \\ -1.6x\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{1-2q} + t^{-q}(t-1)^{1-q}y\frac{\Gamma(2-q)}{\Gamma(2-2q)} & \text{if } t > 1,\ x,y \in \mathbb{R}. \end{cases}$$

*Consider the Lyapunov function $V(t,y) = t^{1-q}y^2$. According to Remarks 8, this function satisfies condition (i) of Theorem 3 with $a(u) = u^2$, $u \in \mathbb{R}_+$ and $T_\varepsilon = \sqrt[1-q]{\varepsilon}$. It also satisfies condition (ii) with $g(u) = u^2$, $u \in \mathbb{R}_+$.*

*Now, let the function $\psi \in C([-1,0],\mathbb{R}^n)$ and the point $t > 0$ be such that $V(t+\Theta,\psi(\Theta)) < V(t,\psi(0))$ for $\Theta \in (-\min\{t,1\},0)$, i.e., $(t-1)^{1-q}\psi(-1)^2 < t^{1-q}\psi(0)^2$.*

*For $t > 1$, apply $2\phi(0)(t-1)^{1-q}\phi(-1) \le (t-1)^{1-q}\phi^2(0) + (t-1)^{1-q}\phi^2(-1) < 2t^{1-q}\phi^2(0)$ and Example 2 (Special case 1) and obtain*

$$
\begin{aligned}
D^+_{(2)}V(t,\phi(0),\phi) = {} & t^{1-q}\Big(2\phi(0)f(t,\phi(0),\phi(-1)) + (\phi(0))^2\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big) \\
= {} & t^{1-q}\Big(2\phi(0)(t-1)^{1-q}\phi(-1)\frac{\Gamma(2-q)t^{-q}}{t^{1-q}\Gamma(2-2q)} - 3.2\phi^2(0)\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q} + (\phi(0))^2\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big) \\
\le {} & t^{1-q}\Big(2t^{1-q}\phi^2(0)\frac{\Gamma(2-q)t^{-q}}{t^{1-q}\Gamma(2-2q)} - 3.2\phi^2(0)\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}) + \phi^2(0)\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big) \\
= {} & -0.2\phi^2(0)\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}) < 0.
\end{aligned}
\tag{41}
$$

*Let $t \le 1$. Then, we get*

$$
\begin{aligned}
D^+_{(2)}V(t,\phi(0),\phi) = {} & t^{1-q}\Big(2\phi(0)f(t,\phi(0),\phi(-1)) + (\phi(0))^2\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big) \\
= {} & t^{1-q}\Big(-3\phi^2(0)\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q} + (\phi(0))^2\frac{\Gamma(2-q)}{\Gamma(2-2q)}t^{-q}\Big) < 0.
\end{aligned}
\tag{42}
$$

*Therefore, all the conditions of Theorem 3 are satisfied, and thus the zero solution of (39) and (40) is stable in time.*

## 5. Conclusions

The nonlinear RL fractional differential equation is studied. The initial value problem is a subject that remains quite up-to-date (see, for example, the books [5,6]). Note the initial condition imposed to study fractional kinetic equations with RL fractional derivative. This point is critical in many physical situations, especially in astrophysical problems and the problem of anomalous subdiffusion ([23]). A good overview of the physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives was most clearly formulated by Diethelm [19] and it is detailed discussed in [7], where it is shown that initial conditions for RL fractional differential equations have physical meaning, and that the corresponding quantities can be obtained from measurements.

In this paper, some new definition for stability excluding a small interval around zero is defined and studied. These types of stability are called stability in time and generalize Mittag-Leffler stability in time for RL fractional differential equations. The definitions are deeply connected with the singularity at the initial time point. The stability properties of the zero solution are studied by Lyapunov functions. Two types of derivatives of Lyapunov functions: the RL fractional derivative when the argument of the Lyapunov function is a solution of the studied problem and the Dini fractional derivative among the studied problem.

**Author Contributions:** Conceptualization, R.A., S.H. and D.O.; methodology, R.A., S.H. and D.O.; validation, R.A., S.H. and D.O.; formal analysis, R.A., S.H. and D.O.; writing—original draft preparation, R.A., S.H. and D.O.; writing—review and editing, R.A., S.H. and D.O.; funding acquisition, S.H. All authors have read and agreed to the published version of the manuscript.

**Funding:** The research is supported by the Bulgarian National Science Fund under Project KP-06-N32/7.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

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