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An Improved Nordhaus–Gaddum-Type Theorem for 2-Rainbow Independent Domination Number

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Abstract: For a graph G, its k-rainbow independent domination number, written as $\gamma_{\mathrm{ri}k}(G)$, is defined as the cardinality of a minimum set consisting of k vertex-disjoint independent sets $V_1, V_2, \ldots, V_k$ such that every vertex in $V_0 = V(G) \setminus (\cup_{i=1}^{k} V_i)$ has a neighbor in V_i for all $i \in \{1, 2, \ldots, k\}$. This domination invariant was proposed by Kraner Šumenjak, Rall and Tepeh (in Applied Mathematics and Computation 333(15), 2018: 353–361), which aims to compute the independent domination number of $G \Box K_k$ (the generalized prism) via studying the problem of integer labeling on G. They proved a Nordhaus–Gaddum-type theorem: $5 \leq \gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 3$ for any n-order graph G with $n \geq 3$, in which $\overline{G}$ denotes the complement of G. This work improves their result and shows that if $G \ncong C_5$, then $5 \leq \gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 2$.

Keywords: k-rainbow independent domination; Nordhaus–Gaddum; bounds

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1. Introduction

Throughout the paper, only simple graphs are considered. We refer to [1] for undefined notations. For a graph G, the *edge set* and *vertex set* of G are denoted by $E(G)$ and $V(G)$, respectively. For any $v_1, v_2 \in V(G)$, they are *adjacent* in G if v_1 and v_2 are the endpoints of an identical edge of G. A vertex $w \in V(G)$ is *adjacent* to a set $W \subseteq V(G)$ in G if W contains a vertex w' s.t. $ww' \in E(G)$. $N_G(w) = \{v | vw \in E(G)\}$ is called the *open neighborhood* of w and $N_G[w] = N_G(w) \cup \{w\}$ is the *closed neighborhood* of w. $d_G(w) = |N_G(w)|$ denotes the *degree* of w in G and $\Delta(G) = \max\{d_G(w) | w \in V(G)\}$. A vertex that has degree ℓ and at least ℓ is called an ℓ-vertex and ℓ^+-vertex, respectively. For any $W \subseteq V(G)$, let $N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W$ and $N_G[W] = N_G(W) \cup W$. We say that W *dominates* a set W' if $W' \subseteq N_G[W]$. Moreover, we use the notation $G - W$ to denote the subgraph of G by deleting vertices in W and their incident edges in G, and $G[W] = G - (V(G) \setminus W)$ the subgraph of G induced by W. The ℓ-order complete graph and the ℓ-length cycle are denoted by K_ℓ and C_ℓ, respectively. As usual, for any two natural numbers p, q with $p < q$, $[p, q]$ represents $\{p, p+1, \ldots, q\}$.

Given a graph G and a subset $W \subseteq V(G)$, we call W a *dominating set* (abbreviated as DS) of G if W dominates $V(G)$. An *independent set* (abbreviated as IS) of a graph is a set of vertices, no two of which are adjacent in the graph. If a DS W of G is an IS, then W is called an *independent dominating set* (IDS for short) of G. The *independent domination number* of G, denoted by $i(G)$, is the cardinality of a minimum IDS of G. Domination and independent domination in graphs have always attracted extensive attention [2,3] and many variants of domination [4] have been introduced increasingly, for the applications in diverse fields, such as electrical networks, computational biology, and land surveying. Recent studies on these variations include (total) roman domination [5,6], strong roman domination [7], semitotal domination [8,9], relating domination [10], just to name a few.

Let $G \Box H$ be the Cartesian product of G and H. In order to reduce the problem of determining $i(G \Box K_k)$ into the problem of integer labeling on G, Kraner Šumenjak et al. [11] proposed a new variation of domination, called *k-rainbow independent dominating function* of a graph G (kRiDF for short), which is a function f from $V(G)$ to $[0, k]$, s.t., for each

$i \in [1,k]$, V_i is an IS and every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = i$. Alternatively, a kRiDF f of G may be viewed as an ordered partition $(V_0, V_1, \ldots, V_k)$ such that for each $i \in [1,k]$, V_i is an IS and $N_G(x) \cap V_i \neq \emptyset$ for every $x \in V_0$, where V_j, $j \in [0,k]$, denotes the set of vertices assigned value j under f. The *weight* $w(f)$ of a kRiDF f is defined as the number of nonzero vertices, i.e., $w(f) = |V(G)| - |V_0|$. The *k-rainbow independent domination number* of G, denoted by $\gamma_{\mathrm{rik}}(G)$, is the minimum weight of a kRiDF of G. From the definition, we have $\gamma_{ri1}(G) = i(G)$. A $\gamma_{rik}(G)$*-function* represents a kRiDF of G which has weight $\gamma_{rik}(G)$.

Let G be a graph and H a subgraph of G. Suppose that g is a kRiDF of H. We say that a kRiDF f of G is *extended* from g if $f(v) = g(v)$ for every $v \in V(H)$. To prove that a graph G has a kRiDF, we will first find a k'RiDF g of a subgraph G' of G, $k' \leq k$, and then extend g to a kRiDF f of G. By using this approach, we describe the structure characterization of graphs G with $\gamma_{\mathrm{ri2}}(G) = |V(G)| - 1$ (Section 2), and then obtain an improved Nordhaus–Gaddum-type theorem with regard to γ_{ri2} (Section 3).

2. Structure Characterization of Graphs G s.t., $\gamma_{\mathrm{ri2}}(G) = |V(G)| - 1$

To get the improved Nordhaus–Gaddum-type theorem in terms of γ_{ri2}, we have to characterize the graphs G s.t., $\gamma_{\mathrm{ri2}}(G) = |V(G)| - 1$. For this, we need the following special graphs.

A star S_n, $n \geq 1$, is a complete bipartite graph $G[X,Y]$ with $|X|=1$ and $|Y| = n$, where the vertex in X is called the *center* of S_n and the vertices in Y are *leaves* of S_n. Let S_n^+ be the graph obtained from S_n by adding a single edge connecting an arbitrary pair of leaves of S_n [11]. A *double star* [12] is defined as the union of two vertex-disjoint stars with an edge connecting their centers. Specifically, for two integers n, m such that $n \geq m \geq 0$ the *double star*, denoted by $S(n,m)$, is the graph with vertex set $\{u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_m\}$ and edge set $\{u_0v_0, u_0u_i, v_0v_j | i \in [1,n], j \in [1,m]\}$, where u_0v_0 is called the *bridge* of $S(n,m)$ and the subgraphs induced by $\{u_i | i \in [0,n]\}$ and $\{v_j | j \in [0,m]\}$ are called the *n-star at* u_0 and *m-star at* v_0, respectively. Observe that $S(n,m)$ is defined on the premise of $n \geq m$. For mathematical convenience, we denote a double star $S(n,m)$ as a vertex-sequence $v_m v_{m-1} \ldots v_0 u_0 u_1 \ldots u_n$.

We start with a known result, which characterizes graphs G with $\gamma_{\mathrm{ri2}}(G) = n$. For a fixed graph G, its *complement* is written as $\overline{G}$.

Lemma 1 ([11])**.** *Let G be a graph of order n. Then, $\gamma_{\mathrm{ri2}}(G) = n$ iff G only contains components isomorphic to K_1 or K_2. And, if $\gamma_{\mathrm{ri2}}(G) = n$, then $\gamma_{\mathrm{ri2}}(\overline{G}) = 2$.*

The following conclusion is simple but will be used throughout this paper.

Lemma 2. *Let H be a subgraph of a fixed graph G and $g = (V_0, V_1, \ldots, V_k)$ be a $\gamma_{\mathrm{rik}}(H)$-function. Then g can be extended to a kRiDF of G with weight at most $|V(G)| - |V_0|$.*

Proof. Let $V(G) \setminus V(H) = \{x_1, \ldots, x_\ell\}$. We will deal with these vertices in the order of $x_1, \ldots, x_\ell$ by the following rule: for each x_i, $i \in [1,\ell]$, let $j \in [1,k]$ be the smallest one such that x_i is not adjacent to V_j in G. If such j does not exist, we update V_0 by $V_0 \cup \{x_i\}$; otherwise we update V_j by $V_j \cup \{x_i\}$. After the last one, i.e., x_ℓ is handled, we obtain a kRiDF of G. Obviously, the weight of the resulting kRiDF of G is not more than $|V(G)| - |V_0|$. □

The following theorem clarifies the structure of connected graphs G with $\gamma_{\mathrm{ri2}}(G) = |V(G)| - 1$.

Theorem 1. *Let G be a connected graph with order $n \geq 3$. Then, $\gamma_{\mathrm{ri2}}(G) = n - 1$ iff G is isomorphic to one among S_{n-1}, S_{n-1}^+, $S(n-3,1)$ ($n \geq 4$) and C_5.*

Proof. Let $f = (V_0, V_1, V_2)$ be an arbitrary $\gamma_{\mathrm{ri2}}(G)$-function. Observe that V_0 does not contain any 1-vertex; one can readily derive that $\gamma_{\mathrm{ri2}}(G) = n - 1$ when G is isomorphic to one of S_{n-1}, S^+_{n-1}, $S(n-3,1)$ and C_5. Conversely, suppose that $\gamma_{\rm ri2}(G) = n-1$, that is, $|V_0| = 1$. By Lemma 2, G contains no subgraph H that has a 2RiDF of weight at most $|V(H)|-2$. Since $\gamma_{\rm ri2}(C_4) = 2 = |V(C_4)|-2$ and each C_k for $k \geq 6$ contains a subgraph isomorphic to a 6-order path P_6 with $\gamma_{\rm ri2}(P_6) = 4 = |V(P_6)|-2$, G does not contain any subgraph isomorphic to C_4 or C_k for $k \geq 6$. This also shows that every two vertices of G share at most one neighbor in G.

Observation 1. *If G contains a 3^+-vertex x, then every 2^+-vertex of G belongs to $N_G(x)$.* Suppose to the contrary that G contains a 2^+-vertex y such that $y \notin N_G(x)$. Let $\{x_1, x_2, x_3\} \subseteq N_G(x)$ and $\{y_1, y_2\} \subseteq N_G(y)$. Observe that $|\{x_1, x_2, x_3\} \cap \{y_1, y_2\}| \leq 1$ and $|N_G(y_i) \cap \{x_1, x_2, x_3\}| \leq 1$ for $i \in [1,2]$; we WLOG assume that $y_2 \notin \{x_1, x_2, x_3\}$, $y_2x_2 \notin E(G)$ and $y_2x_3 \notin E(G)$. Let f be: $f(x) = f(y) = 0, f(x_2) = 1, f(x_3) = 2$. Notice that either $y_1 = x_j$ or $y_1x_j \notin E(G)$ for some $j \in [2,3]$; we further let $f(y_1) = f(x_j)$ and $f(y_2) = [1,2] \setminus \{f(y_1)\}$. Clearly, f is a 2RiDF of $G[\{x, x_2, x_3, y, y_1, y_2\}]$ of weight $|\{x, x_2, x_3, y, y_1, y_2\}|-2$, a contradiction.

Observation 2. *G contains at most one 3^+-vertex*. Suppose that G has two distinct 3^+-vertices, say x and y. By Observation 1, $xy \in E(G)$. Let $\{y, x_1, x_2\} \subseteq N_G(x)$ and $\{x, y_1, y_2\} \subseteq N_G(y)$. Since G contains no subgraph isomorphic to C_4, $|\{x_1, x_2\} \cap \{y_1, y_2\}| \leq 1$ and there are no edges between $\{x_1, x_2\}$ and $\{y_1, y_2\}$. Assume that $x_2 \notin \{y_1, y_2\}$ and $y_2 \notin \{x_1, x_2\}$. Then, the function f: $\{x, x_1, x_2, y, y_1, y_2\} \to \{0,1,2\}$ such that $f(x) = f(y) = 0$, $f(x_2) = f(y_2) = 2$ and $f(x_1) = f(y_1) = 1$, is a 2RiDF of $G[\{x, y, x_1, x_2, y_1, y_2\}]$ of weight $|\{x, y, x_1, x_2, y_1, y_2\}|-2$, a contradiction.

Observation 3. *If G contains a 3^+-vertex x, $N_G(x)$ has not more than two 2-vertices; in particular, when $N_G(x)$ contains two 2-vertices, in G these two 2-vertices are adjacent*. If not, suppose that $N_G(x)$ contains three 2-vertices, say x_1, x_2, x_3. We WLOG assume that $x_3 \notin N_G(\{x_1, x_2\})$ and let $N_G(x_3) = \{x, y_3\}$. Let $N_G(x_1) = \{x, y_1\}$ (possibly $y_1 = x_2$, but $y_1 \neq y_3$). By Observation 1, $d_G(y_3) = 1$, i.e., $y_1y_3 \notin E(G)$. Let f be: $f(x) = 1, f(x_1) = f(x_3) = 0, f(y_1) = f(y_3) = 2$. Obviously, f is a 2RiDF of $G[\{x, x_1, y_1, x_3, y_3\}]$ of weight $|\{x, x_1, y_1, x_3, y_3\}|-2$, a contradiction. Now, suppose that $N_G(x)$ contains two 2-vertices, say x_1, x_2. If $x_1x_2 \notin E(G)$, let $N_G(x_i) = \{x, y_i\}$, $i \in [1,2]$. Clearly, $y_1 \neq y_2$ and $y_1y_2 \notin E(G)$. Let f be: $f(x) = 1, f(x_1) = f(x_2) = 0, f(y_1) = f(y_2) = 2$. Then, f is a 2RiDF of $G[\{x, x_1, y_1, x_2, y_2\}]$ of weight $|\{x, x_1, x_2, y_1, y_2\}|-2$, a contradiction.

By the above three observations and the assumption that G is connected, we see that if G contains a 3^+-vertex x, then $V(G) \setminus \{x\}$ contains either only 1-vertices ($G \cong S_{n-1}$), or one 2-vertex and $n-2$ 1-vertices ($G \cong S(n-3,1)$), or two adjacent 2-vertices and $n-3$ 1-vertices ($G \cong S^+_{n-1}$); if $\Delta(G) = 2$, then G is isomorphic to one of $S^+_2, S_2, S(1,1)$ and C_5. □

The theorem below follows from Theorem 1, Lemma 1, and $\gamma_{\rm ri2}(G) = \sum_{i=1}^{k} \gamma_{\rm ri2}(G_i)$, where $G_1, \ldots, G_k$ are the components of G.

Theorem 2. *Given a graph G with order $n \geq 3$, $\gamma_{\rm ri2}(G) = n-1$ iff G has one component G_1 isomorphic to one among S_{n_1-1} ($n_1 \geq 3$), $S^+_{n_1-1}$ ($n_1 \geq 3$), $S(n_1-3,1)$ ($n_1 \geq 4$) and C_5, and other components are isomorphic to K_1 or K_2, where $n_1 = |V(G_1)|$.*

3. An Improved Nordhaus–Gaddum Type Theorem for $\gamma_{\rm ri2}(G)$

This section is devoted to achieve an improved Nordhaus–Gaddum type theorem by showing that $\gamma_{\rm ri2}(G) + \gamma_{\rm ri2}(\overline{G}) \leq n+2$ for every graph $G \ncong C_5$ of order $n \geq 2$, which improves a result obtained by Kraner Šumenjak et al., et al [11]. We first present some fundamental lemmas.

Lemma 3. *For an n-order graph G with $n \geq 3$, if G is S_{n-1}, S^+_{n-1} or $S(n-3,1)$, then $\gamma_{\rm ri2}(\overline{G}) \leq 3$.*

Proof. If $G \cong S_{n-1}$ or $G \cong S^+_{n-1}$, let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ where v_0 is the center and $v_1v_2 \in E(G)$ when $G \cong S^+_{n-1}$. Define a function f such that $f(v_1) = 1, f(v_0) = f(v_2) = 2$

and $f(v) = 0$ for every $v \in V(\overline{G}) \setminus \{v_0, v_1, v_2\}$. Since every vertex in $V(\overline{G}) \setminus \{v_0, v_1, v_2\}$ is a neighbor of v_1 and also v_2 in $\overline{G}$, it follows that f is a 2RiDF of $\overline{G}$ of weight 3.

If $G \cong S(n-3, 1)$, then $n \geq 4$. Let $V(G) = \{v_1, v_0, u_0, u_1, \ldots, u_{n-3}\}$, where v_0u_0 is the bridge of G and $E(G) = \{v_0v_1, v_0u_0, u_0u_i | i \in [1, n-3]\}$. If $n = 4$, then both G and $\overline{G}$ are isomorphic to P_4, the path of length 3, and the conclusion holds. If $n \geq 5$, then the function f from $V(\overline{G})$ to $[0, 2]$ such that $f(u_2) = 2, f(u_1) = f(u_0) = 1$, and $f(v) = 0$ for every $v \in V(\overline{G}) \setminus \{u_0, u_1, u_2\}$ is a 2RiDF of $\overline{G}$ with weight 3. □

Lemma 4. *For a graph n-order G, if $G \ncong C_5$ and $\gamma_{\text{ri2}}(G) = 4$, then $\gamma_{\text{ri2}}(\overline{G}) \leq n-2$.*

Proof. Clearly, $n \geq 4$. When $n = 4$, $\gamma_{\text{ri2}}(G) = 4$ implies that $\gamma_{\text{ri2}}(\overline{G}) = 2 = n-2$ by Lemma 1. Therefore, we assume that $n \geq 5$. Suppose that $\gamma_{\text{ri2}}(\overline{G}) \geq n-1$. If $\gamma_{\text{ri2}}(\overline{G}) = n$, by Lemma 1 we have $\gamma_{\text{ri2}}(G) = 2$, a contradiction. Therefore, $\gamma_{\text{ri2}}(\overline{G}) = n-1$. By Theorem 2 $\overline{G}$ has one component isomorphic to $S_{n_1}, S^+_{n_1}, S(n_2, 1)$ or C_5 where $n_1 \geq 2, n_2 \geq 1$, and all of the other components of $\overline{G}$ are isomorphic to K_1 or K_2.

If $\overline{G}$ contains two vertices u and v s.t. $N_{\overline{G}}(\{u, v\}) = \emptyset$, then in G both u and v are adjacent to every vertex in $V(G) \setminus \{u, v\}$. We can obtain a 2RiDF of G by assigning 1 to u, 2 to v, and 0 to the remained vertices of G. This indicates that $\gamma_{\text{ri2}}(G) \leq 2$ and a contradiction. Therefore, $\overline{G}$ contains no K_2 components and contains at most one K_1 component, implying that $\overline{G}$ contains at most two components. If $\overline{G}$ contains only one component, it follows that $\overline{G}$ is S_{n-1}, S^+_{n-1} or $S(n-3, 1)$ (since $G \ncong C_5$). By Lemma 3 $\gamma_{\text{ri2}}(G) \leq 3$ and a contradiction. Therefore, $\overline{G}$ has two components, denoted by G_1 and G_2, where $G_1 \cong K_1$ and G_2 is isomorphic to $S_{n-2}, S^+_{n-2}, S(n-4, 1)$ or C_5. Let $V(G_1) = \{u\}$ and define a function f as follows: let $f(u) = 1$; $f(v_0) = f(v') = 2$ when $G_2 \cong S_{n-2}$ or $G_2 \cong S^+_{n-2}$ (where v_0 is the center of G_2 and v' is a 1-vertex of G_2 by the assumption of $n \geq 5$), $f(v_0) = f(u_0) = 2$ when $G_2 \cong S(n-4, 1)$ (where v_0u_0 is the bridge of G_2), or $f(u_1) = f(u_2) = 2$ when $G_2 \cong C_5$ (where $C_5 = u_1u_2u_3u_4u_5u_1$); and all of the other remained vertices are assigned value 0. Clearly, all vertices with value 0 are adjacent to u and a vertex with value 2. Hence, f is a 2RiDF of G, which has weight 3, a contradiction. □

Lemma 5. *Suppose that G is an n-order graph satisfying that $\gamma_{\text{ri2}}(G) \geq 4$ and $\gamma_{\text{ri2}}(G)$+$\gamma_{\text{ri2}}(\overline{G})$ = $n+3$. Let $f = (V_0, V_1, V_2)$ be an arbitrary $\gamma_{\text{ri2}}(G)$-function. We have*

(1) If $|V_0| \geq 2$, then for any $u, v \in V_0$, there does not exist u_1, u_2, v_1, v_2 such that $\{u_1, u_2\} \in N_{\overline{G}}(u)$, $\{v_1, v_2\} \in N_{\overline{G}}(v)$ and $u_iv_i \notin E(\overline{G})$ for $i \in [1, 2]$, where $u_1 \neq u_2, v_1 \neq v_2$ but possibly $u_i = v_i$;

(2) If u, v are two arbitrary different vertices of V_0, then $|N_{\overline{G}}(\{u, v\})| \geq 3$;

(3) $|V_i| \geq 2$ for $i \in [0, 2]$.

Proof. For (1), if the conclusion is false, then let g be: $g(u) = g(v) = 0$ and $g(u_i) = g(v_i) = i$, $i \in [1, 2]$. Then, g is a 2RiDF of $\overline{G}[\{u, v, u_1, v_1, u_2, v_2\}]$ with weight $|\{u, v, u_1, v_1, u_2, v_2\}| - 2$. Since V_1 and V_2 are cliques in $\overline{G}$, V_i contains at most two vertices not assigned 0 under every 2RiDF of $\overline{G}$ for $i \in [1, 2]$. Hence, we can extend g to a 2RiDF of $\overline{G}$ with weight at most $|V_0| - 2 + 4 = |V_0| + 2$, according to Lemma 2. This shows that $\gamma_{\text{ri2}}(\overline{G}) \leq |V_0| + 2$ and $\gamma_{\text{ri2}}(G)$+$\gamma_{\text{ri2}}(\overline{G}) \leq |V_1| + |V_2| + |V_0| + 2 = n+2$, a contradiction.

For (2), if $|N_{\overline{G}}(\{u, v\})| \leq 2$, let f be: $f(v) = 2, f(u) = 1$, and $f(x) = 0$ for $x \in V(G) \setminus N_{\overline{G}}[\{u, v\}]$. It is clear that f is a 2RiDF of $G[V(G) \setminus N_{\overline{G}}(\{u, v\})]$ with weight 2. According to Lemma 2, we can extend f to a 2RiDF of G with weight at most 4 (since $|N_{\overline{G}}(\{u, v\})| \leq 2$). Thus, $\gamma_{\text{ri2}}(G) = 4$ and by Lemma 4 $\gamma_{\text{ri2}}(\overline{G}) \leq n-2$, a contradiction.

For (3), if $|V_0| = 1$, then $\gamma_{\text{ri2}}(G) = n-1$. By an analogous argument as that in Lemma 4, we can derive that $\gamma_{\text{ri2}}(G) + \gamma_{\text{ri2}}(\overline{G}) \leq n+2$, a contradiction. In the following, we prove that $|V_1| \geq 2$ (the proof of $|V_2| \geq 2$ is similar to that of $|V_2| \geq 2$). Suppose that $|V_1| = 1$ and let $V_1 = \{u\}$. Then, every vertex of V_0 is adjacent to u in G, i.e., u is not adjacent to V_0 in $\overline{G}$. By Lemma 4 we assume that $|V_1| + |V_2| \geq 5$. If V_0 contains a vertex v with two neighbors v_1, v_2 in $\overline{G}$, then $u \notin \{v_1, v_2\}$. Let g be: $g(v) = 0, g(v_1) = 1, g(v_2) = 2$. Since V_2 is a clique in $\overline{G}$, we can extend g to a 2RiDF of $\overline{G}$ with weight at most $|V_0| - 1 + 3 = |V_0| + 2$, according to Lemma 2. This shows that $\gamma_{\mathrm{ri2}}(\overline{G}) \leq |V_0| + 2$ and hence $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 2$, a contradiction. Therefore, every vertex in V_0 has degree at most 1 in $\overline{G}$, which implies that $|N_{\overline{G}}(\{x,y\})| \leq 2$ for any two vertices $x \in V_0, y \in V_0$ (observe that $|V_0| \geq 2$). This contradicts (2). □

Lemma 6. *Let G be an n-order graph, $n \geq 4$. For any $u \in V(G)$, if $H = G - u$, the resulting graph by deleting u and its incident edges from G, is connected and $\gamma_{\mathrm{ri2}}(H) = |V(H)| - 1$, then G has a 2RiDF f satisfying $f(u) = 1$ and $f(v) = 0$ for some $v \in V(H)$.*

Proof. Clearly, $|V(H)| \geq 3$. If u has no neighbor in $V(H)$, then let f be: $f(v) = g(v)$ for every $v \in V(H)$, and $f(u) = 1$, where g is a $\gamma_{\mathrm{ri2}}(H)$-function of H. Since $\gamma_{\mathrm{ri2}}(H) = |V(H)| - 1$, there exists $v \in V(H)$ satisfying $f(v) = g(v) = 0$. If u has a neighbor $u_1 \in V(H)$, there exists a $u_2 \in V(H)$ s.t. $u_1u_2 \in E(H)$ since H is connected. Let f be: $f(u_1) = 0, f(u) = 1, f(u_2) = 2$. Then, we can extend f to a desired 2RiDF of G according to Lemma 2. □

Now, we turn to the proof of the main result.

Theorem 3. *Suppose that G is an n-order graph, $n \geq 2$. If $G \ncong C_5$, then $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 2$.*

Proof. We are sufficient to handle the situation $n \geq 5$ since cases of $n \leq 4$ are trivial. Let $f_0 = (V_0, V_1, V_2)$ be a $\gamma_{\mathrm{ri2}}(G)$-function such that $\overline{G}[V_0]$ contains the maximum number of components isomorphic to K_2. Suppose to the contrary that $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) > n + 2$. Then, $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) = n + 3$ since $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 3$ [11], that is,

$$\gamma_{\mathrm{ri2}}(\overline{G}) = |V_0| + 3 \tag{1}$$

Formula (1) indicates that every 2RiDF of $\overline{G}$ has weight at least $|V_0| + 3$. We will complete our proof by constructing a 2RiDF of $\overline{G}$ of weight at most $|V_0| + 2$ or a 2RiDF of G of weight less than $|V_1| + |V_2|$.

If $|V_1 \cup V_2| = 2$, then $\gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq 2 + n$, a contradiction; if $|V_1 \cup V_2| = 3$, then $\gamma_{\mathrm{ri2}}(\overline{G}) = n$ and by Lemma 1 $\gamma_{\mathrm{ri2}}(G) = 2$, also a contradiction. Therefore, by Lemma 4,

$$|V_1| + |V_2| \geq 5 \tag{2}$$

Then, by Lemma 5 (3) we have $|V_i| \geq 2$ for $i \in [0,2]$. In addition, because, by definition, $\overline{G}[V_i]$ is a clique, $i \in [1,2]$, it follows that for every 2RiDF $g_0 = (V_0', V_1', V_2')$ of $\overline{G}$,

$$|(V_1' \cup V_2') \cap V_i| \leq 2, i \in [1,2] \tag{3}$$

Therefore, by Lemma 2 we can extend every $\gamma_{\mathrm{ri2}}(\overline{G}[V_0])$-function to a 2RiDF of $\overline{G}$ with weight at most $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) + 4$, i.e., $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) \geq |V_0| - 1$ by Formula (1).

Claim 1. *Denote by ℓ the number of vertices in $V_1 \cup V_2$, which have degree $|V_1| + |V_2| - 1$ in $\overline{G}[V_1 \cup V_2]$. Then, $\ell \leq 1 - \ell'$ where $\ell' = |V_0| - \gamma_{\mathrm{ri2}}(\overline{G}[V_0]) \leq 1$.* If not, either ℓ is at least 2 or both ℓ and ℓ' are equal to 1. Suppose that $\ell \geq 2$ and take two vertices v_1,v_2 $\in (V_1 \cup V_2)$ such that they are adjacent to all vertices of $(V_1 \cup V_2) \setminus \{u, v\}$ in $\overline{G}$. Let g' be: $g'(v_1) = 1, g'(v_2) = 2, g'(x) = 0$ for $x \in V_1 \cup V_2 \setminus \{v_1, v_2\}$. Clearly, g' is a 2RiDF of $\overline{G}[V_1 \cup V_2]$ and by Lemma 2 we can extend g' to a 2RiDF of $\overline{G}$, which has weight at most $|V_0| + 2$, a contradiction. Now, suppose that $\ell = \ell' = 1$. Then, $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) = |V_0| - 1$, which indicates that $\overline{G}[V_0]$ contains a component H' s.t. $\gamma_{\mathrm{ri2}}(H') = |V(H')| - 1$. Since $\ell = 1$, there is a vertex v, say $v \in V_1$, which is adjacent to every vertex of V_2 in $\overline{G}$. By Lemma 6 $\overline{G}[V(H') \cup \{v\}]$ has a 2RiDF g' s.t. $g'(x) = 0$ for some $x \in V(H')$ and $g'(v) = 1$. Observe that in $\overline{G}$ v is adjacent to all vertices of $(V_1 \cup V_2) \setminus \{v\}$; by the rule of Lemma 2 we can extend g' to a 2RiDF g of $\overline{G}$ under which there is at most one vertex in $V_1 \setminus \{v\}$ (and V_2)

not assigned value 0. Thus, $w(g) \leq |V_0| - 1 + 3 = |V_0| + 2$, a contradiction. This completes the proof of Claim 1.

Now, we WLOG assume $|V_1| \geq |V_2|$. Then, $|V_1| \geq 3$ by Formula (2).

Claim 2. $\overline{G}[V_0]$ *does not contain any isolated vertex* v *s.t.* $N_{\overline{G}}(v) \cap V_1 = \emptyset$. Otherwise, define f' as: for $x \in V_2$ $f'(x) = 2$, and $f'(v) = 1$. By Claim 1, in $\overline{G}$, V_1 has not more than one vertex adjacent to every vertex in V_2; say v' if such a vertex exists. We further let $f'(y) = 0$ for $y \in V_1 \cup (V_0 \setminus \{v\})$ (or for $y \in (V_1 \setminus \{v'\}) \cup (V_0 \setminus \{v\})$ if v' exists). Since in G every vertex in $V_1 \cup V_0$ (except for v') is adjacent to v and also V_2, f is a 2RiDF of G of weight at most $|V_2| + 2$, a contradiction. This completes the proof of Claim 2.

We proceed by distinguishing two cases: $\gamma_{\rm ri2}(\overline{G}[V_0]) = |V_0| - 1$ and $\gamma_{\rm ri2}(\overline{G}[V_0]) = |V_0|$.

Case 1. $\gamma_{\rm ri2}(\overline{G}[V_0]) = |V_0| - 1$. In this case, by Claim 1 each vertex of V_i owns a neighbor belonging to V_j in G where $\{i, j\}$=[1,2]; by Theorem 2, $\overline{G}[V_0]$ has one component H isomorphic to one of $S_{|V(H)|-1}$ ($|V(H)| \geq 3$), $S^+_{|V(H)|-1}$ ($|V(H)| \geq 3$), $S(|V(H)| - 3, 1)$ ($|V(H)| \geq 4$) and C_5, and other components of $\overline{G}[V_0]$ are isomorphic to K_1 or K_2. Let $u_0 \in V(H)$ be a vertex with $d_H(u_0) = \Delta(H)$. Clearly, $d_H(u_0) \geq 2$. Let $u_1 \in N_H(u_0)$ and $u_2 \in N_H(u_0)$ be two vertices such that every vertex in $V(H) \setminus \{u_0, u_1, u_2\}$ has degree in H not exceeding $\min\{d_H(u_1), d_H(u_2)\}$. By the structure of H, for $i \in [1,2]$, we have that $d_H(u_i) \leq 2$ and if u_i has a neighbor $u'_i (\notin \{u_0, u_1, u_2\})$ in H, then $u_0u'_i \notin E(H)$. Moreover, by Lemma 5 (1), $(N_{\overline{G}}(u_1) \cap N_{\overline{G}}(u_2)) \setminus \{u_0\} = \emptyset$, which implies that each vertex of $V_1 \cup V_2$ is adjacent to u_1 or u_2 in G.

Claim 3. $|V_0 \setminus V(H)| \leq 1$. Otherwise, let $\{v_1, v_2\} \subseteq (V_0 \setminus V(H))$. Then, $d_{\overline{G}[V_0]}(v_1) \leq 1$ and $d_{\overline{G}[V_0]}(v_2) \leq 1$. Suppose that $d_{\overline{G}[V_0]}(v_1) = 1$ (the case of $d_{\overline{G}[V_0]}(v_2) = 1$ can be similarly discussed). Let $v_1v'_1 \in E(\overline{G}[V_0])$ and clearly $d_{\overline{G}[V_0]}(v'_1) = 1$. By Lemma 5 (2), a vertex $v_0 \in (V_1 \cup V_2)$ is adjacent to $\{v_1, v'_1\}$ in $\overline{G}$. We WLOG assume that $v_1v_0 \in E(\overline{G})$. According to Lemma 6, $\overline{G}[V(H) \cup \{v_0\}]$ admits a 2RiDF g' satisfying $g'(v_0) = 1$ and $g'(x) = 0$ for some $x \in V(H)$. Further, let $g'(v_1) = 0$ and $g'(v'_1) = 2$. So g' is a 2RiDF of $\overline{G}[V(H) \cup \{v_0, v_1, v'_1\}]$, and by Lemma 2 and Formula (3) we can extend g' to a 2RiDF of $\overline{G}$ with weight at most $|V_0| - 2 + 4 = |V_0| + 2$ (since $g'(v_1) = g'(x) = 0$), a contradiction. We therefore assume that $d_{\overline{G}[V_0]}(v_1) = d_{\overline{G}[V_0]}(v_2) = 0$. By Lemma 5 (2) we have $|N_{\overline{G}}(\{v_1, v_2\}) \cap (V_1 \cup V_2)| \geq 3$. WLOG, suppose that in $\overline{G}$, v_1 has two neighbors belonging to $V_1 \cup V_2$, say v_{11} and v_{12}. By Lemma 5 (1), u_i is not adjacent to both v_{11} and v_{12}, and v_{1j} is not adjacent to both u_1 and u_2 in $\overline{G}$, where $i \in [1,2]$ and $j \in [1,2]$. Thus, it follows that $u_1v_{11} \notin E(\overline{G})$ and $u_2v_{12} \notin E(\overline{G})$, or $u_1v_{12} \notin E(\overline{G})$ and $u_2v_{11} \notin E(\overline{G})$, which contradicts to Lemma 5 (1) again. This completes the proof of Claim 3.

By Claim 3, we see that $\overline{G}[V_0]$ contains no component isomorphic to K_2 and contains at most one K_1 component.

Claim 4. $\overline{G}[V_0]$ *contains a* K_1 *component*. If not, we have $\overline{G}[V_0] = H$.

Claim 4.1. $(N_{\overline{G}}(u_1) \cup N_{\overline{G}}(u_2)) \cap (V_1 \cup V_2) \neq \emptyset$.

Otherwise, for $i \in [1,2]$, u_i is adjacent to every vertex of $V_1 \cup V_2$ in G, and by Lemma 5 (2) $d_H(u_i) = 2$ and $u_1u_2 \notin E(\overline{G})$. Set $\{u'_i\} = N_H(u_i) \setminus \{u_0\}$, $i \in [1,2]$; then, $u_0u'_i \notin E(\overline{G})$. Let f be: $f(u_1) = f(u'_1) = 1$, $f(u_2) = f(u'_2) = 2$ and $f(x) = 0$ for any x in $V(G) \setminus \{u_1, u'_1, u_2, u'_2\}$. So, we get a 2RiDF f of G, which has weight 4, a contradiction. So, Claim 4.1 holds.

Claim 4.2. $|V_1| = 3$.

Observe that $|V_1| \geq 3$; it is enough by showing that G admits a 2RiDF f s.t. $w(f) \leq |V_2| + 3$. When $u_1u_2 \in E(\overline{G})$, let f be: $f(u_i) = 1$ for $i \in [0,2]$, $f(x) = 0$ for $x \in (V_1 \cup V_0) \setminus \{u_0, u_1, u_2\}$, and $f(y) = 2$ for $y \in V_2$. By Lemma 5 (1), in $\overline{G}$, $V_1 \cup V_0$ contains no vertex adjacent to u_1 and also u_2. Therefore, f is a 2RiDF of G of weight $|V_2| + 3$. Now, suppose that $u_1u_2 \notin E(\overline{G})$. By Lemma 5 (1), V_1 contains at most one vertex adjacent to both u_0 and u_1 in $\overline{G}$; say u if such a vertex exists. Let f be: $f(u_0) = f(u_1) = 1$ (or $f(u) = f(u_0) = f(u_1) = 1$ if u exists), $f(x) = 0$ for $x \in (V_1 \cup (V_0 \setminus \{u_0, u_1\}))$ (or $x \in (V_1 \cup V_0) \setminus \{u_0, u_1, u\}$) and $f(y) = 2$ for $y \in V_2$. Notice that by Claim 1 every vertex of $V_0 \cup V_1$ is adjacent to V_2 in G, and by the structure of H and the selection of u_1 and

u_2, every vertex of $(V_0 \cup V_1) \setminus \{u, u_0, u_1\}$ is adjacent to $\{u_0, u_1\}$ in G; f is a 2RiDF of G of weight at most $|V_2| + 3$. This completes the proof of Claim 4.2.

By Claim 4.2, we have $2 \leq |V_2| \leq 3$. Let $V_1 = \{w_1, w_2, w_3\}$ in the following.

Claim 4.3. *In $\overline{G}$, for $\{i, j\} = [1, 2]$ every vertex in V_i has not more than one neighbor in V_j.*

If not, let $v \in V_2$ be adjacent to two vertices of V_1 in $\overline{G}$, say w_1, w_2. By Lemma 5 (1) u_1 or u_2 is not adjacent to v in $\overline{G}$, say $u_1 v \notin E(\overline{G})$. If $u_2 w_3 \notin E(\overline{G})$, define g' as: $g'(u_i) = i$ for every $i \in [0, 2]$, $g'(w_1) = g'(w_2) = 0, g'(w_3) = 2$, $g'(v) = 1$. If $u_2 w_3 \in E(\overline{G})$, then $u_1 w_3 \notin E(\overline{G})$ and let g' be: $g'(u_1) = g'(w_3) = 1, g'(w_1) = g'(w_2) = 0, g'(v) = 2$; further, let $g'(u_2) = 0$ when $u_2 v \in E(\overline{G})$, or let $g'(u_2) = 2$ and $g'(u_0) = 0$ when $u_2 v \notin E(\overline{G})$. According to Lemma 2, in either case the g' defined above can be extended to a 2RiDF g of $\overline{G}$ under which $g(w_1) = g(w_2) = 0$ and $g(u_0) = 0$ or $g(u_2) = 0$. Therefore, by Formula (3) $w(g) \leq |V_0| - 1 + 3 = |V_0| + 2$, a contradiction. With a similar discussion, there is also a contradiction if we assume V_1 contains a vertex that has two neighbors in V_2 in $\overline{G}$. This completes the proof of Claim 4.3.

Now, we consider $|V_2|$. Suppose that $|V_2| = 3$ and let $V_2 = \{w_4, w_5, w_6\}$. According to Claim 4.1, we WLOG assume that $u_1 w_1 \in E(\overline{G})$. This indicates that $u_2 w_1 \notin E(\overline{G})$ by Lemma 5 (1). If u_2 has a neighbor in V_2, say $u_2 w_4 \in E(\overline{G})$, then according to Lemma 5 (1), $u_1 w_4 \notin E(\overline{G})$, $w_1 w_4 \in E(\overline{G})$, and u_1 (resp. u_2) is not adjacent to $\{w_2, w_3\}$ (resp. $\{w_5, w_6\}$) in $\overline{G}$ (otherwise w_4 or w_1 has two neighbors in V_1 or V_2 in $\overline{G}$, respectively. This contradicts to Claim 4.3). Let f be: $f(u_1) = f(w_1) = 1, f(u_2) = f(w_4) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{u_1, u_2, w_1, w_4\}$. Observe that w_1 (resp. w_4) is not adjacent to $\{w_5, w_6\}$ (resp. $\{w_2, w_3\}$) in $\overline{G}$ and by Lemma 5 (1) $V_0 \setminus \{u_0, u_1, u_2\}$ contains no vertex adjacent to both u_i and w_i for some $i \in [1, 2]$. Hence, f is a 2RiDF of $G[V(G) \setminus \{u_0\}]$ of weight 4 and we are able to extend f to a 2RiDF of G with weight at most $5 < |V_1| + |V_2|$ according to Lemma 2, a contradiction. Therefore, we may assume that $N_{\overline{G}}(u_2) \cap V_2 = \emptyset$. In this case, when $N_{\overline{G}}(u_2) \cap V_1 = \emptyset$, let f be: $f(u_2) = 2$, $f(u_0) = f(u_1) = 1$. By Lemma 5 (1) $V_1 \cup V_2$ has not more than one vertex w' adjacent to both u_0 and u_1 in $\overline{G}$ and $V_0 \setminus \{u_0\}$ has not more than one vertex u' adjacent to u_2 in $\overline{G}$; for $x \in V(G) \setminus \{u_0, u_1, u_2, u', w'\}$ we further let $f(x) = 0$. Then, f is a 2RiDF of $G[V(G) \setminus \{u', w'\}]$ of weight 3 and according to Lemma 2 we can extend f to a 2RiDF of G of weight at most $5 < |V_1| + |V_2|$, a contradiction. We therefore suppose that u_2 has a neighbor in V_1 in $\overline{G}$, say $u_2 w_2 \in E(\overline{G})$. With the same argument as $N_{\overline{G}}(u_2) \cap V_2 = \emptyset$, we can show that $N_{\overline{G}}(u_1) \cap V_2 = \emptyset$ as well.

Then, if $w_3 u_1 \notin E(\overline{G})$ and $w_3 u_2 \notin E(\overline{G})$, the function f: $f(u_1) = f(w_1) = 1, f(u_2) = f(w_4) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{u_1, u_2, w_1, w_4, u_0\}$, is a 2RiDF of $G[V(G) \setminus \{u_0\}]$ with weight 4, and according to Lemma 2, we are able to extend f to a 2RiDF of G with weight at most $5 < |V_1| + |V_2|$, a contradiction. Therefore, we suppose that $w_3 u_1 \in E(\overline{G})$ by the symmetry. By Lemma 5 (1), it has that $w_3 u_2 \notin E(\overline{G})$, and $u_0 w_1 \notin E(\overline{G})$ or $u_0 w_3 \notin E(\overline{G})$, say $u_0 w_1 \notin E(\overline{G})$ by the symmetry. Let f be: $f(u_0) = f(u_1) = 1, f(u_2) = f(w_2) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{u_1, u_2, u_0, w_2, w_3\}$. Since in G every vertex in $V(G) \setminus \{u_1, u_2, u_0, w_2, w_3\}$ has a neighbor in $\{u_0, u_1\}$ and also $\{u_2, w_2\}$, f is a 2RiDF of $G[V(G) \setminus \{w_3\}]$ of weight 4 and according to Lemma 2 we can extend f to a 2RiDF of G of weight at most $5 < |V_1| + |V_2|$, and a contradiction.

A similar line of thought leads to a contradiction if we assume that $|V_2| = 2$, and so Claim 4 holds.

By Claim 4, we see that $\overline{G}[V_0]$ contains one component isomorphic to K_1. Let s be the vertex of the K_1 component. We first show that $|N_{\overline{G}}(s) \cap (V_1 \cup V_2)| \leq 1$. If not, in $\overline{G}$ we assume that s has two neighbors in $V_1 \cup V_2$, say s_1, s_2. By Lemma 5 (1) for $i, j \in [1, 2]$, s_i (resp. u_j) can not be adjacent to u_1 and u_2 (resp. s_1 and s_2) simultaneously in $\overline{G}$. This implies that either $s_i u_i \notin E(\overline{G})$, $i \in [1, 2]$, or $s_1 u_2 \notin E(\overline{G})$ and $s_2 u_1 \notin E(\overline{G})$, which violates Lemma 5 (1) as well. Thus, by Claim 2 $|N_{\overline{G}}(s) \cap (V_1 \cup V_2)| = 1$ and the vertex s' adjacent to s in $\overline{G}$ belongs to V_1. Let f be: $f(x) = 1$ for $x \in V_1$, $f(s) = 2$, $f(y) = 0$ for $y \in V_2 \cup V(H))$. Observe that by Claim 1 all vertices in V_2 are adjacent to V_1 in G. Hence, every vertex in $V_2 \cup V(H)$ is adjacent to s and also V_1 in G. Therefore, f is a 2RiDF of G with weight $|V_1| + 1 < |V_1| + |V_2|$ (since $|V_2| \geq 2$), a contradiction.

The foregoing discussion shows that there exists a contradiction if we assume that $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) = |V_0| - 1$. In what remains, we handle the case when $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) = |V_0|$.

Case 2. $\gamma_{\mathrm{ri2}}(\overline{G}[V_0]) = |V_0|$. Then by Lemma 1 every component of $\overline{G}[V_0]$ is isomorphic to K_1 or K_2. Recall that $|V_i| \geq 2$ for $i \in [0,2]$. Take two vertices u, v in V_0 s.t. $uv \in E(\overline{G})$ if $\overline{G}[V_0]$ contains a K_2 component and u, v are isolated vertices in $\overline{G}[V_0]$ otherwise. By Lemma 5 (1), we have

$$|(N_{\overline{G}}(u) \cap N_{\overline{G}}(v)) \cap (V_1 \cup V_2)| \leq 1 \tag{4}$$

We deal with two subcases in terms of the adjacency property of u and v.

Case 2.1. $uv \in E(\overline{G})$. Then in $\overline{G}$, $V_0 \setminus \{u, v\}$ contains no vertex adjacent to $\{u, v\}$.

Claim 5. *In $\overline{G}[V_1 \cup V_2]$, $V_1 \cup V_2$ contains only vertices with degree at most $|V_1| + |V_2| - 2$.* Suppose that V_1 contains a vertex w such that $ww' \in E(\overline{G})$ for every $w' \in V_2$. If $uw \in E(\overline{G})$ (or $vw \in E(\overline{G})$), define a 2RiDF g' of $\overline{G}[\{u, v, w\}]$ as: $g'(u) = 0$ (or $g'(v) = 0$), $g'(w) = 1$ and $g'(v) = 2$ ($g'(u) = 2$). According to Lemma 2 we can extend g' to a 2RiDF of $\overline{G}$, under which $(V_1 \cup V_2) \setminus \{w\}$ contains at most two vertices not assigned 0. Thus, $w(g) \leq |V_0| - 1 + 3 = |V_0| + 2$, a contradiction. We therefore assume that $uw \notin E(\overline{G})$ and $vw \notin E(\overline{G})$. By Lemma 5 (2), there are at least three vertices in $(V_1 \cup V_2)$ that are adjacent to u or v. We WLOG assume that $V_1 \cup V_2$ contains a vertex u' s.t. $u'u \in E(\overline{G})$. Construct a 2RiDF g' of $\overline{G}[\{u, v, u', w\}]$ as follows: $g'(u) = 0, g'(u') = 2$, and $g'(v) = g'(w) = 1$. Then, by Lemma 2 g' can be extended to a 2RiDF g of $\overline{G}$, under which $(V_1 \cup V_2) \setminus \{w, u'\}$ contains at most one vertex not assigned value 0. Therefore, $w(g) \leq |V_0| - 1 + 3 = |V_0| + 2$, a contradiction. Similarly, we can also obtain a contradiction if we assume that V_2 contains a vertex adjacent to every vertex of V_1. So, Claim 5 holds.

By Claim 5, for $\{i, j\}$=[1,2], each vertex of V_i is adjacent to a vertex of V_j in G. If $V_1 \cap (N_{\overline{G}}(u) \cap N_{\overline{G}}(v)) = \emptyset$, then in G all vertices of V_1 are adjacent to $\{u, v\}$. Let f be: $f(x) = 2$ for $x \in V_2$, $f(y) = 0$ for $y \in V_1 \cup (V_0 \setminus \{u, v\})$, and $f(u) = f(v) = 1$. Obviously, f is a 2RiDF of G s.t. $w(f) = |V_2| + 2 < |V_1| + |V_2|$, a contradiction. We therefore assume that V_1 contains a vertex s s.t. $su \in E(\overline{G})$ and $sv \in E(\overline{G})$. Then, in $\overline{G}$, by Lemma 5 (1) no vertex in $V_2 \cup (V_1 \setminus \{s\})$ is adjacent to u and v simultaneously. Analogously, the function f: $f(v) = f(u) = 1, f(x) = 2$ for $x \in V_1$, and $f(y) = 0$ for $y \in V_2 \cup (V_0 \setminus \{u, v\})$ (and $f(s) = f(v) = f(u) = 1, f(x) = 2$ for $x \in V_2$, and $f(y) = 0$ for $y \in (V_1 \setminus \{s\}) \cup (V_0 \setminus \{u, v\})$) is a 2RiDF of G with weight $|V_1| + 2$ (and $|V_2| + 3$). This implies that $|V_1| = 3$ and $|V_2| = 2$. Let $V_1 = \{s, s_1, s_2\}$ and $V_2 = \{s_3, s_4\}$. Then, in $\overline{G}$, neither u nor v is a neighbor of s_1 and s_2 simultaneously; otherwise, we, by the symmetry, suppose that $us_1 \in E(\overline{G})$ and $us_2 \in E(\overline{G})$. Let g' be: $g'(v) = g'(s_1) = g'(s_2) = 0, g'(u) = 1$, and $g'(s) = 2$. Obviously, g' is a 2RiDF of $\overline{G}[\{u, v, s, s_1, s_2\}]$ with weight 2. According to Lemma 2, we can extend g' to a 2RiDF of $\overline{G}$ with weight at most $|V_0| - 1 + |V_2| + 1 = |V_0| + 2$, a contradiction. In addition, in $\overline{G}$, s_i, $i \in [1,2]$, is not adjacent to u and v simultaneously according to Lemma 5 (1). Therefore, we may assume, by the symmetry, that $s_1 v \notin E(\overline{G})$ and $s_2 u \notin E(\overline{G})$.

If no edge between $\{u, v\}$ and V_2 in $\overline{G}$ exists, then by Lemmas 5 (2), $us_1 \in E(\overline{G})$ and $vs_2 \in E(\overline{G})$. Then, the function g' such that $g'(s) = g'(s_1) = g'(v) = 0, g'(s_2) = 2$, and $g'(u) = 1$ is a 2RiDF of $\overline{G}[\{u, v, s, s_1, s_2\}]$ with weight 2. According to Lemma 2, we can extend g' to a 2RiDF of $\overline{G}$ with weight at most $|V_2| + 1 + |V_0| - 1 = |V_0| + 2$, a contradiction. We therefore assume that $\overline{G}$ contains an edge connecting $\{u, v\}$ and V_2, say $vs_3 \in E(\overline{G})$ by the symmetry.

If $s_4 s \in E(\overline{G})$, define g' as: $g'(s_3) = 2, g'(s_4) = 0, g'(s) = 1, g'(v) = 0$. Then, g' is a 2RiDF of $\overline{G}[\{s, v, s_3, s_4\}]$ with weight 2. By Lemma 2 and Formula 3, we are able to extend g' to a 2RiDF of $\overline{G}$ of weight at most $|V_0| - 1 + 3 = |V_0| + 2$, a contradiction. Consequently, we have $s_4 s \notin E(\overline{G})$. Then, the function g' such that $g'(s_3) = 0, g'(s_4) = g'(s) = 2, g'(v) = 1$, $g'(u) = 0$ is a 2RiDF of $\overline{G}[\{s, u, v, s_3, s_4\}]$ with weight 3, and by Lemma 2 and Formula 3 we can extend g' to a 2RiDF of $\overline{G}$ with weight at most $|V_0| - 1 + 3 = |V_0| + 2$. This contradicts the assumption.

Case 2.2. $uv \notin E(\overline{G})$. Then, by the selection of u, v and f_0, $\overline{G}[V_0]$ contains only isolated vertices and G does not admit a $\gamma_{\mathrm{ri2}}(G)$-function for which the induced subgraph of $\overline{G}$ by vertices with value 0 contains K_2 components.

For every $x \in V_0$, let $U_i^x = N_{\overline{G}}(x) \cap V_i$ for $i \in [1,2]$. Let f' be: $f'(x) = 0$ for $x \in ((V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)) \cup (V_0 \setminus \{u,v\})$, $f'(v) = 2$, and $f'(u) = 1$. Apparently, f' is a 2RiDF of $G - (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v))$ with weight 2. According to Lemma 2, we can extend f' to a 2RiDF of G with weight at most $|(U_1^u \cup U_2^u \cup U_1^v \cup U_2^v))| + 2$. To ensure $|(U_1^u \cup U_2^u \cup U_1^v \cup U_2^v))| + 2 \geq |V_1| + |V_2|$, we have

$$|(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)| \leq 2 \tag{5}$$

Claim 6. *$|(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)| = 2$ and the two vertices in $(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)$ are adjacent in $\overline{G}$.* Define a 2RiDF g' of $\overline{G}[V_0]$ as: $g'(u) = g'(v) = 1$. Suppose that $|(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)| \leq 1$. Since V_1 and V_2 are cliques in $\overline{G}$ and every vertex in $U_1^u \cup U_2^u \cup U_1^v \cup U_2^v$ is adjacent to u or v in $\overline{G}$, by Lemma 2 we are able to extend g' to a 2RiDF g of $\overline{G}$ under which at most one vertex in V_i, $i \in [1,2]$, is not assigned value 0 (here if $(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)$ contains a vertex, say w, then let $g(w) = 2$). Clearly, $w(g) = w(g') + 2 \leq |V_0| + 2$, a contradiction. Moreover, if $(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)$ contains two nonadjacent vertices in $\overline{G}$, say w_1, w_2, then w_1 and w_2 are not in the same set V_i for some $i \in [1,2]$. Therefore, we can extend g' to a 2RiDF g of $\overline{G}$ via letting $g'(x) = 0$ when x is in $(V_1 \cup V_2) \setminus \{w_1, w_2\}$ and $g'(w_1) = g'(w_2) = 2$. However, $w(g) = w(g') + 2 \leq |V_0| + 2$, a contradiction. This completes the proof of Claim 6.

By Claim 6, $(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^v \cup U_2^v)$ contains two adjacent vertices in $\overline{G}$, say w_1, w_2. If there exists a $z \in (V_0 \setminus \{u,v\})$ s.t. $zw_1 \in E(\overline{G})$ (or $zw_2 \in E(\overline{G})$), then set g' as: $g'(z) = g'(u) = g'(v) = 1$, $g'(w_1) = 0$ (or $g'(w_2) = 0$), $g'(w_2) = 2$ (or $g'(w_1) = 2$). Since in $\overline{G}$ every vertex in $(V_1 \cup V_2) \setminus \{w_2\}$ has a neighbor in $\{z, u, v\}$ and every vertex in $V' \setminus \{w_2\}$ is a neighbor of w_2, where $w_2 \in V'$ for some $V' \in \{V_1, V_2\}$, we can extend g' to a 2RiDF g of $\overline{G}$ according to Lemma 2. Under g, every vertex in $V' \setminus \{w_2\}$ is assigned value 0 and at most one vertex in $\{V_1, V_2\} \setminus V'$ is not assigned value 0. Therefore, $w(g) \leq |V_0| + 2$, a contradiction. This demonstrates that in $\overline{G}$ no vertex in V_0 is adjacent to $\{w_1, w_2\}$. Furthermore, if there is a $z \in V_0 \setminus \{u,v\}$, then by Claim 6 we have $(V_1 \cup V_2) \setminus (U_1^u \cup U_2^u \cup U_1^z \cup U_2^z) = \{w_1, w_2\}$ and $(V_1 \cup V_2) \setminus (U_1^v \cup U_2^v \cup U_1^z \cup U_2^z) = \{w_1, w_2\}$, which implies that $N_{\overline{G}}(z) = U_1^u \cup U_2^u \cup U_1^v \cup U_2^v$. Set g' as: $g'(z) = 1, g'(u) = g'(v) = 2$ and $g'(x) = 0$ for $x \in U_1^u \cup U_2^u \cup U_1^v \cup U_2^v$. Then, g' is a 2RiDF of $\overline{G} - (\{w_1, w_2\} \cup (V_0 \setminus \{u, v, z\}))$ with weight 3, and we can extend g' to a 2RiDF of $\overline{G}$ with weight at most $(|V_0| + 2 - 3) + 3 = |V_0| + 2$ according to Lemma 2, a contradiction. So far, we have shown that $V_0 = \{u, v\}$, that is, $\gamma_{ri2}(G) = n - 2$.

Now, we define a 2RiDF f' of $G[\{u, v, w_1, w_2\}]$ as follows: $f'(w_1) = f'(w_2) = 0$, $f'(u) = 1$ and $f'(v) = 2$. According to Lemma 2, we can extend f' to a 2RiDF f of G with weight at most $n - 2$. To ensure $w(f) \geq \gamma_{ri2}(G) = n - 2$, f must be a $\gamma_{ri2}(G)$-function (since $w(f) = n - 2$). However, $\overline{G}[\{w_1 w_2\}]$ is isomorphic to K_2. This contradicts the selection of f_0. Eventually, the proof of Theorem 3 is finished. □

Based on the foregoing analysis, we observed that the upper bound $n + 2$ can be attained by graphs $S_r (r \geq 2)$, $S_r^+ (r \geq 2)$, and $S(r, 1)$ $(r \geq 1)$, while we did not find other graphs that possess this property. So, we propose a problem as follows.

Question 1. Is it enough to determine graphs G with $\gamma_{ri2}(G) + \gamma_{ri2}(\overline{G}) = |V(G)| + 2$ by $S_r (r \geq 2)$, $S_r^+ (r \geq 2)$, and $S(r, 1) (r \geq 1)$?

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