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Analytical Solutions of Upper Convected Maxwell Fluid with Exponential Dependence of Viscosity under the Influence of Pressure

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Abstract: Some unsteady motions of incompressible upper-convected Maxwell (UCM) fluids with exponential dependence of viscosity on the pressure are analytically studied. The fluid motion between two infinite horizontal parallel plates is generated by the lower plate, which applies time-dependent shear stresses to the fluid. Exact expressions, in terms of standard Bessel functions, are established both for the dimensionless velocity fields and the corresponding non-trivial shear stresses using the Laplace transform technique and suitable changes of the unknown function and the spatial variable in the transform domain. They represent the first exact solutions for unsteady motions of non-Newtonian fluids with pressure-dependent viscosity. The similar solutions corresponding to the flow of the same fluids due to an exponential shear stress on the boundary as well as the solutions of ordinary UCM fluids performing the same motions are obtained as limiting cases of present results. Furthermore, known solutions for unsteady motions of the incompressible Newtonian fluids with/without pressure-dependent viscosity induced by oscillatory or constant shear stresses on the boundary are also obtained as limiting cases. Finally, the influence of physical parameters on the fluid motion is graphically illustrated and discussed. It is found that fluids with pressure-dependent viscosity flow are slower when compared to ordinary fluids.

Keywords: analytical solutions; fluid mechanics; UCM fluids; pressure-dependent viscosity; shear stresses on the boundary; starting solutions

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1. Introduction

The concept of fluid with pressure-dependent viscosity is due to Stokes [1] who, in his celebrated paper on the fluid response, remarked that the liquid viscosity can depend on the pressure. Over the time, the experimental research has certified this dependence (see for instance the authoritative book of Bridgman [2] for the pertinent literature prior to 1931 and the papers of Griest et al. [3], Johnson and Cameron [4], Johnson and Tevaarwerk [5], and more recently, Bair et al. [6,7] and Prusa et al. [8]). In the case of incompressible Newtonian fluids, whose constitutive equation is

$$\boldsymbol{T} = -p\boldsymbol{I} + \eta(p)\boldsymbol{A}, \text{ with } tr\boldsymbol{A} = 0, \tag{1}$$

the hydrostatic pressure p due to the incompressibility constraint is the Lagrange multiplier, i.e., $p = (tr\boldsymbol{T})/3$ Fusi [9]. In the above relations, $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{I}$ the unit tensor, $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{L}^T$ is the first Rivlin-Ericksen tensor, in which $\boldsymbol{L}$ is the gradient of the velocity vector, and $\eta(p)$ is the fluid viscosity. From the previous constitutive relation, it results that the frictional forces exerted by adjacent layers of the fluid depend on the normal force that acts between the layers Fusi [9].

If the viscosity variation with the pressure p is small enough, $\eta(p)$ can be approximated by a constant, and the above equations correspond to the ordinary incompressible Newtonian fluids. Stokes was careful to delineate flows in channels and pipes at moderate pressures in which the viscosity can be considered as being constant. Barus [10,11] suggested a linear or exponential dependence

$$\eta(p) = \mu[\alpha(p - p_0) + 1] \text{ or } \eta(p) = \mu e^{\alpha(p-p_0)}, \tag{2}$$

of viscosity on the pressure for low until medium, respectively high, pressure differences. Here, μ is the fluid viscosity at the reference pressure p_0, and the positive constant α is the dimensional pressure–viscosity coefficient. Dowson and Higginson [12] as well as Rajagopal [13] noticed that the change in density is of order $3 - 5\%$ at changes of viscosity of the order $10^8\%$. Consequently, it seems reasonable to study these liquids as incompressible fluids with pressure-dependent viscosity. Moreover, at high pressures, the energy dissipation can be large enough, and it may be necessary to study the motions of such fluids in a thermo mechanical framework, following Hron et al. [14]. However, similar to the work by Karra et al. [15], we are going to study the motion problems of the incompressible UCM fluids with exponential dependence of viscosity on the pressure in an isothermal process.

It is a known fact that for pressures of the order 1000 atm or higher, the viscosity increases more than an order of magnitude, as showed by Renardy [16]. These situations appear at polymer processing operations (Lord [17] and Denn [18]), fluid film lubrication (Szeri [19]), microfluidics (Cui et al. [20]), crude oil and fuel oil pumping (Martinez-Boza et al. [21]), geophysics (Stemmer et al. [22]), food processing, pharmaceutical tablet manufacturing. For different experimental methods or techniques for measuring the variation of viscosity with the pressure, as well as the measurement of the dimensional pressure–viscosity coefficient, one can refer to studies reported by Goubert et al. [23] and Park et al. [24]. It was found that the dimensional pressure–viscosity coefficient α varies within the values 10–50 GPa^{-1} for the polymer melts (Carreras et al. [25] and Sorrentino and Pantani [26]), 10–70 GPa^{-1} for lubricants (Kottke et al. [27]) and 10–20 GPa^{-1} for mineral oils (Venner and Lubrecht [28]). These values are also valid for small or medium pressure differences $p - p_0$.

In addition, the gravity effects are significant in many flows of fluids with practical applications. Due to the gravity, the pressure inside a fluid changes with the depth, and its effects are more pronounced for fluid motions in which the pressure varies along the direction in which the gravity acts. The first explicit exact solutions for the flow of the incompressible Newtonian fluids with pressure-dependent viscosity between infinite horizontal parallel plates were presented by Rajagopal [29]. He also established steady solutions for the motion of the same fluids over an inclined plane due to gravity [30]. Analytical expressions for the steady-state or starting solutions corresponding to the modified Stokes' problems for different classes of the fluids with pressure-dependent viscosity have been established by Prusa [31], Fetecau and Bridges [32], Vieru et al. [33], and Fetecau and Vieru [34]. However, the most general solutions for these problems are those obtained by Rajagopal et al. [35] in terms of the eigenfunctions of a suitable boundary value problem. Exact solutions for the steady flow of Newtonian fluids with a linear dependence of viscosity on the pressure in a rectangular duct were obtained by Akyildiz and Siginer [36] and Housiadas and Georgiou [37].

However, exact solutions for motions of viscoelastic fluids with pressure-dependent viscosity are very few in the literature, although they can be used as tests to verify different numerical methods for the study of complex motion problems. An interesting study of the unsteady flow of incompressible UCM fluids with viscosity and relaxation time depending on the pressure, but without exact solutions, has been developed by Karra et al. [15]. Subsequently, Housiadas [38] established approximate solutions for steady flows of these fluids when the shear viscosity and the relaxation time exponentially depend on the pressure. The case when the fluid viscosity linearly depends on the pressure was analytically studied by Housiadas in [39]. Other steady solutions for pressure-driven unidirectional flows of incompressible UCM fluids with linear dependence of viscosity on the pressure through a straight channel and a circular tube were also established by Housiadas [40] using a slightly different model.

It is worth pointing out that the problem of the existence and uniqueness of solutions corresponding to motions of fluids with pressure-dependent viscosity was also approached and studied. For instance, Renardy [41] has proved the existence and uniqueness of the solutions of Navier–Stokes equations in a three-dimensional bounded domain, while Malek et al. [42] has studied the same problem for the flow of fluids whose viscosity depends both on pressure and the symmetric part of the velocity gradient. Relatively recently, using the theory of Lebesque and Sobolev spaces with variable exponents, Malek et al. [43] proved the existence of solutions of Stokes' problems for a generalization of the power-law fluids with an exponential dependence of viscosity on the pressure.

In this work, exact expressions are established for the velocity and shear stress fields corresponding to unsteady motions of incompressible UCM fluids with exponential dependence of viscosity on the pressure between two infinite horizontal parallel plates. Actually, they are the first exact solutions corresponding to unsteady motions of the non-Newtonian fluids with pressure-dependent viscosity. The fluid motion is due to the lower plate that applies time-dependent shear stresses to the fluid. The analytic solutions are obtained using the Laplace transform technique without any restriction on the physical parameters. Similar solutions corresponding to motions of the same fluids induced by exponential shear stresses on the boundary as well as the solutions for ordinary incompressible UCM fluids performing the same motions are obtained as limiting cases of general results. The influence of the physical parameters on the fluid motion is graphically underlined and discussed.

2. Presentation of the Problem

The constitutive equations of incompressible UCM fluids with pressure-dependent viscosity are given by the relations (see Karra et al. [15])

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S},\ \boldsymbol{S} + \lambda\frac{\delta \boldsymbol{S}}{\delta t} = \eta(p)\boldsymbol{A}, \tag{3}$$

where $\boldsymbol{S}$ is the extra-stress tensor, λ is the relaxation time, $\eta(p)$ is the pressure-dependent viscosity, and the upper-convected derivative $\delta/\delta t$ is defined by

$$\frac{\delta \boldsymbol{S}}{\delta t} = \frac{d\boldsymbol{S}}{dt} - \boldsymbol{L}\boldsymbol{S} - \boldsymbol{S}\boldsymbol{L}^{\mathrm{T}}. \tag{4}$$

In the above relations, the tensor $\boldsymbol{L}$ is the gradient of the velocity vector v, and the Laplace multiplier p in Equation (3) is not the same as the mean normal stress, as shown by Karra et al. [15].

In the following, we shall study some unsteady motions of incompressible UCM fluids whose pressure-dependent viscosity $\eta(p)$ is of the exponential form $(2)_2$. If $\alpha \to 0$ in this relation, then $\eta(p) \to \mu$, and Equation (3) reduces to the constitutive equations of the ordinary incompressible UCM fluids. In addition, $\eta(p) \to \infty$ if $p \to \infty$, which is a property that has been experimentally proved.

Let us now consider an incompressible UCM fluid with exponential dependence of viscosity on the pressure at rest between two infinite horizontal parallel plates at the distance d apart. When $t = 0^+$, the lower plate begins to apply a time-dependent shear stress of the form

$$S\frac{\cos(\omega t) + \lambda\omega\sin(\omega t)}{(\lambda\omega)^2 + 1} - \frac{S}{(\lambda\omega)^2 + 1}\exp\left(-\frac{t}{\lambda}\right),\ \text{or} \tag{5}$$

$$S\frac{\sin(\omega t)-\lambda\omega\cos(\omega t)}{(\lambda\omega)^2+1}+\frac{\lambda\omega S}{(\lambda\omega)^2+1}\exp\left(-\frac{t}{\lambda}\right), \tag{6}$$

to the fluid. When $\lambda \to 0$, Equation (3) reduces to the constitutive equations of incompressible Newtonian fluids with pressure-dependent viscosity, and the Expressions (5) and (6) take the simple forms

$$S\cos(\omega t),\ \text{respectively}\ S\sin(\omega t). \tag{7}$$

In this limit case, when the lower plate applies oscillatory shear stresses of the form (7) to the fluid, S and ω represent the amplitude and the frequency of oscillations respectively.

Due to shear, the fluid begins to move, and following Kara et al. [15], we are looking for a solution of the form

$$\boldsymbol{v}=\boldsymbol{v}(y,t)=u(y,t)\mathbf{e}_x,\ p=p(y), \tag{8}$$

where $\mathbf{e}_x$ is the unit vector lengthways in the x direction of a convenient Cartesian coordinate system x, y, and z. We also assume that the extra-stress tensor $\boldsymbol{S}$, as well as the velocity field $\boldsymbol{v}$, is a function of y and t only.

Introducing Equation $(8)_1$ in $(3)_2$ and bearing in mind the above assumption, we find that

$$S_{xx}+\lambda\frac{\partial S_{xx}}{\partial t}-2\lambda S_{xy}\frac{\partial u}{\partial y}=0,\ S_{yy}+\lambda\frac{\partial S_{yy}}{\partial t}=0,\ S_{zz}+\lambda\frac{\partial S_{zz}}{\partial t}=0, \tag{9}$$

$$S_{xy}+\lambda\frac{\partial S_{xy}}{\partial t}-\lambda S_{yy}\frac{\partial u}{\partial y}=\eta(p)\frac{\partial u}{\partial y},\ S_{yz}+\lambda\frac{\partial S_{yz}}{\partial t}=0,\ S_{xz}+\lambda\frac{\partial S_{xz}}{\partial t}-\lambda S_{yz}\frac{\partial u}{\partial y}=0. \tag{10}$$

Since the fluid is at rest when $t=0$, it results that

$$\boldsymbol{v}(y,0)=\mathbf{0},\ \mathbf{S}(y,0)=\mathbf{0}\ \text{for}\ 0\le y\le d. \tag{11}$$

Based on the initial condition $(11)_{2,}$ it clearly results from Equations $(9)_2$, $(9)_3$, and $(10)_2$ that $S_{yy}=S_{yz}=S_{zz}=0$. Using $S_{yz}=0$ in Equation $(10)_{3,}$ it also results that $S_{xz}=0$.

Consequently, Equations (9) and (10) reduce to

$$\sigma_x(y,t)+\lambda\frac{\partial\sigma_x(y,t)}{\partial t}=2\lambda\tau(y,t)\frac{\partial u(y,t)}{\partial y},\ \tau(y,t)+\lambda\frac{\partial\tau(y,t)}{\partial t}=\eta(p)\frac{\partial u(y,t)}{\partial y}, \tag{12}$$

where $\sigma_x(y,t)=S_{xx}(y,t)$ and $\tau(y,t)=S_{xy}(y,t)$ are the normal tension and the shear stress respectively, which are different from zero. The incompressibility condition is identically satisfied while the balance of the linear momentum reduces to the next relevant partial or ordinary differential equations, as shown by Fetecau et al. [44].

$$\frac{\partial\tau(y,t)}{\partial y}=\rho\frac{\partial u(y,t)}{\partial t},\ \frac{\mathrm{d}p(y)}{\mathrm{d}y}+\rho g=0, \tag{13}$$

where ρ is the fluid density while g is the gravitational acceleration. Integrating Equation $(13)_2$ between 0 and d, it results that

$$p=p(y)=\rho g(d-y)+p_0\ \text{with}\ p_0=p(d). \tag{14}$$

Eliminating $\tau(y,t)$ between Equations $(12)_2$ and $(13)_1$ and using the expressions of $\eta(p)$ and $p(y)$ from Equations (2) and (14), one obtains the governing equation

$$\lambda\frac{\partial^2u(y,t)}{\partial t^2}+\frac{\partial u(y,t)}{\partial t}=\nu e^{\alpha\rho g(d-y)}\left[\frac{\partial^2u(y,t)}{\partial y^2}-\alpha\rho g\frac{\partial u(y,t)}{\partial y}\right]=0;\ \ 0<y<d,\ \ t>0, \tag{15}$$

for the velocity field $u(y,t)$. Here, $\nu=\mu/\rho$ is the kinematic viscosity of the fluid.

The appropriate initial and boundary conditions are

$$u(y,0) = 0, \ \left.\frac{\partial u(y,t)}{\partial t}\right|_{t=0} = 0 \text{ for } 0 \leq y \leq d, \tag{16}$$

$$\tau(0,t) + \lambda\frac{\partial \tau(0,t)}{\partial t} = \mu e^{\alpha\rho g d}\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = S\cos(\omega t), \ u(d,t) = 0 \text{ for } t > 0, \tag{17}$$

$$\tau(0,t) + \lambda\frac{\partial \tau(0,t)}{\partial t} = \mu e^{\alpha\rho g d}\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = S\sin(\omega t), \ u(d,t) = 0 \text{ for } t0. \tag{18}$$

Determining $\tau(0,t)$ from the ordinary differential Equations (17) and (18) with the initial condition $\tau(0,0) = 0$ clearly results that in these motions, the lower plate applies to the fluid shear stresses of the forms (5), respectively (6). Early enough, Renardy [45] considered the flow of a Maxwell fluid and showed that boundary conditions on stresses at the inflow boundary have to be imposed in order to formulate a well-posed boundary value problem. Renardy [46] also showed how well-posed boundary value problems can be formulated in such situations. A mixed initial-boundary value problem has been recently studied by Baranovskii [47] for the motion equations of Kelvin–Voight fluids. He proved the existence of a weak solution, which is unique and continuously depends on the surface forces and the initial data.

Once the fluid velocity $u(y,t)$ is known, the non-trivial shear stress $\tau(y,t)$ and the normal stress $\sigma_x(y,t)$ can be determined solving the next ordinary differential equations with initial conditions

$$\lambda\frac{\partial \tau(y,t)}{\partial t} + \tau(y,t) = \mu e^{\alpha\rho g(d-y)}\frac{\partial u(y,t)}{\partial y}; \ \tau(y,0) = 0 \text{ for } 0 < y < d, \tag{19}$$

$$\lambda\frac{\partial \sigma_x(y,t)}{\partial t} + \sigma_x(y,t) = 2\lambda\tau(y,t)\frac{\partial u(y,t)}{\partial y}; \ \sigma_x(y,0) = 0 \text{ if } 0 < y < d. \tag{20}$$

To obtain solutions that do not depend on the flow geometry, let us introduce the next non-dimensional variables, functions and constant

$$y^* = \frac{y}{d}, \quad t^* = \frac{\nu}{d^2}t, \quad u^* = \frac{\mu}{dS}u, \quad \tau^* = \frac{\tau}{S}, \quad \sigma_x^* = \frac{\sigma_x}{S}, \quad \omega^* = \frac{d^2}{\nu}\omega. \tag{21}$$

Using the dimensionless entities defined by Equation (21) and discarding the star notation, one obtains the next two initial and mixed boundary value problems

$$\mathrm{We}\frac{\partial^2 u(y,t)}{\partial t^2} + \frac{\partial u(y,t)}{\partial t} = e^{\beta(1-y)}\left[\frac{\partial^2 u(y,t)}{\partial y^2} - \beta\frac{\partial u(y,t)}{\partial y}\right] = 0; \ \ 0 < y < 1, \ \ t > 0, \tag{22}$$

$$u(y,0) = 0, \ \left.\frac{\partial u(y,t)}{\partial t}\right|_{t=0} = 0 \text{ for } 0 \leq y \leq 1, \tag{23}$$

$$\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = \frac{1}{e^{\beta}}\cos(\omega t), \ u(1,t) = 0 \text{ for } t > 0, \text{ or} \tag{24}$$

$$\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = \frac{1}{e^{\beta}}\sin(\omega t), \ u(1,t) = 0 \text{ for } t > 0, \tag{25}$$

where $\mathrm{We} = \lambda V/d$ (with $V = \nu/d$ a characteristic velocity) is the Weissenberg number and $\beta = \alpha\rho g d$ is the non-dimensional pressure-viscosity coefficient.

The corresponding non-dimensional shear stress $\tau(y,t)$ and the normal stress $\sigma_x(y,t)$ must satisfy the following differential equations

$$\mathrm{We}\frac{\partial \tau(y,t)}{\partial t} + \tau(y,t) = e^{\beta(1-y)}\frac{\partial u(y,t)}{\partial y}; \ 0 < y < 1, \ \ t > 0, \tag{26}$$

$$\mathrm{We}\frac{\partial\sigma_x(y,t)}{\partial t}+\sigma_x(y,t)=2\gamma\tau(y,t)\frac{\partial u(y,t)}{\partial y};\ 0<y<1,\ \ t>0, \tag{27}$$

with the initial conditions

$$\tau(y,0)=0,\ \text{respectively}\ \ \sigma_x(y,0)=0;\ 0\le y\le 1. \tag{28}$$

In Equation (27) the non-dimensional constant $\gamma=\lambda S/\mu$.

3. Exact Expressions for the Dimensionless Velocity and Shear Stress Fields

In this section, we present the exact solutions for the differential Equations (22) and (26) with the initial and boundary conditions (23)–(25), respectively, with the initial condition $(28)_1$. To avoid confusion, the starting solutions of the two distinct motion problems will be denoted by $u_c(y,t)$, $\tau_c(y,t)$ and $u_s(y,t)$, $\tau_s(y,t)$. In order to determine them, we shall use suitable changes of the spatial variable and the unknown function and the Laplace transform technique.

3.1. Calculation of the Velocity Field $u_c(y,t)$

Applying the Laplace transform to Equation (22) and bearing in mind the initial conditions (23), we find that the Laplace transform $\overline{u}(y,s)$ of $u(y,t)$ has to satisfy the differential equation

$$\frac{\partial^2\overline{u}(y,s)}{\partial y^2}-\beta\frac{\partial\overline{u}(y,s)}{\partial y}=s(s\mathrm{We}+1)\mathrm{e}^{\beta(y-1)}\overline{u}(y,s);\ \ 0<y<1, \tag{29}$$

where s is the transform parameter. Now, making the changes of independent variable and unknown function

$$z=-\frac{2i}{\beta}\sqrt{s(s\mathrm{We}+1)\mathrm{e}^{\beta(y-1)}},\ \overline{u}(y,s)=z\overline{v}(z,s), \tag{30}$$

where i is the imaginary unit, the governing Equation (29) reduces to the Bessel equation

$$z^2\frac{\partial^2\overline{v}(z,s)}{\partial z^2}+z\frac{\partial\overline{v}(z,s)}{\partial z}+(z^2-1)\overline{v}(z,s)=0, \tag{31}$$

whose general solution is

$$\overline{v}(z,s)=C_1(s)J_1(z)+C_2(s)Y_1(z), \tag{32}$$

where $J_1(\cdot)$ and $Y_1(\cdot)$ are standard Bessel functions of the first, respectively second kind of order one, and the functions $C_1(s)$, $C_2(s)$ will be determined from the boundary conditions.

Coming back to the initial variables and unknown function, we find that

$$\overline{u}(y,s)=\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\Big\{C_1(s)J_1\Big[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\Big]+C_2(s)Y_1\Big[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\Big]\Big\}, \tag{33}$$

where $\zeta(s)=-\frac{2i}{\beta}\sqrt{s(s\mathrm{We}+1)}$. The Laplace transforms of the boundary conditions (24) are given by the equations

$$\left.\frac{\partial\overline{u}(y,s)}{\partial y}\right|_{y=0}=\frac{s\mathrm{e}^{-\beta}}{s^2+\omega^2},\ \overline{u}(1,s)=0. \tag{34}$$

Direct computations show that the derivative of the function $\overline{u}(y,s)$ with respect to y is given by the following equality

$$\frac{\partial\overline{u}(y,s)}{\partial y}=-\frac{2}{\beta}s(s\mathrm{We}+1)\mathrm{e}^{\beta(y-1)}\Big\{C_1(s)J_0\Big[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\Big]+C_2(s)Y_0\Big[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\Big]\Big\}. \tag{35}$$

The functions $C_1(s)$ and $C_2(s)$ can be immediately determined introducing $\overline{u}(y,s)$ and its derivative with respect to y from Equations (33) and (35) in (34) and solving the obtained algebraic system. The obtained expression for $\overline{u}(y,s)$ is

$$\overline{u}(y,s) = \frac{i\sqrt{s\mathrm{e}^{\beta(y-1)}}}{(s^2+\omega^2)\sqrt{s\mathrm{We}+1}} \frac{J_1[\zeta(s)]Y_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right] - Y_1[\zeta(s)]J_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right]}{J_1[\zeta(s)]Y_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right] - Y_1[\zeta(s)]J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]}. \quad (36)$$

In order to determine the inverse Laplace transform of $\overline{u}(y,s)$ we use the residue theorem. The solitary poles of this function, which are non-removable, are

$$s_{01} = -i\omega,\ s_{02} = i\omega,\ \ s_{1k} = a_0 + a_k,\ s_{2k} = a_0 - a_k, \quad (37)$$

where $a_0 = -1/(2\mathrm{We}),\ a_k = \sqrt{1-\mathrm{We}\beta^2 q_k^2}/(2\mathrm{We})$ and $q_k\ (k = 1,2,3,\ldots)$ are the roots of the transcendental equation

$$B_{01}\left(\mathrm{e}^{-\beta}, q\right) = J_1(q)Y_0\left(q\sqrt{\mathrm{e}^{-\beta}}\right) - Y_1(q)J_0\left(q\sqrt{\mathrm{e}^{-\beta}}\right) = 0. \quad (38)$$

Lengthy but straightforward computations show that the sum of the residues of the function $\overline{u}(y,s)$ in $i\omega$ and $-i\omega$ is given by the relation

$$\begin{aligned}&\mathrm{Rez}\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=i\omega} + \mathrm{Rez}\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=-i\omega}\\ &= \frac{\sqrt{\mathrm{e}^{\beta(y-1)}}}{\sqrt{\delta}}\mathrm{Re}\left\{\frac{J_1(\kappa)Y_1\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)-Y_1(\kappa)J_1\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)}{J_1(\kappa)Y_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)-Y_1(\kappa)J_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)}\exp\left(\omega t + \frac{\theta}{2}\right)i\right\},\end{aligned} \quad (39)$$

in which

$$\delta = \omega\sqrt{1+(\omega\mathrm{We})^2},\ \theta = \mathrm{arctg}\left(\frac{1}{\omega\mathrm{We}}\right) \in (0,\pi/2]\ \text{ and }\ \kappa = \frac{2\sqrt{\delta}}{\beta}\exp\left(-i\frac{\theta}{2}\right) \quad (40)$$

and Re denotes the real part of what follows.

As regards the sum of the residues of $\overline{u}(y,s)$ in $s = s_{1k}$ and $s = s_{2k}$, as a result, using the equalities (A1)–(A5) from the Appendix A, it is given by the relation

$$\begin{aligned}&\mathrm{Rez}\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=s_{1k}} + \mathrm{Rez}\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=s_{2k}} = 2\pi\beta\sqrt{\mathrm{e}^{\beta(y-1)}}\mathrm{e}^{a_0 t}\\ &\times\sum_{k=1}^{\infty}\frac{q_k J_0\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)J_1(q_k)}{J_0^2\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)-J_1^2(q_k)}[b_k\mathrm{ch}(a_k t) + c_k\mathrm{sh}(a_k t)]B_{11}\left(\mathrm{e}^{\beta(y-1)},\ q_k\right),\end{aligned} \quad (41)$$

where

$$B_{11}\left(\mathrm{e}^{\beta(y-1)}, q\right) = Y_1(q)J_1\left(q\sqrt{\mathrm{e}^{\beta(y-1)}}\right) - J_1(q)Y_1\left(q\sqrt{\mathrm{e}^{\beta(y-1)}}\right), \quad (42)$$

$$b_k = \frac{\beta^2 q_k^2 - 4\mathrm{We}\omega^2}{\left(\beta^2 q_k^2 - 4\mathrm{We}\omega^2\right)^2 + 16\omega^2},\ c_k = \frac{\beta^2 q_k^2 + 4\mathrm{We}\omega^2}{\left[\left(\beta^2 q_k^2 - 4\mathrm{We}\omega^2\right)^2 + 16\omega^2\right]\sqrt{1-\mathrm{We}\beta^2 q_k^2}}. \quad (43)$$

In conclusion, the dimensionless velocity $u_c(y,t)$ corresponding to the unsteady motion of the incompressible upper-convected Maxwell (UCM) fluids with exponential dependence of viscosity on the pressure induced by the lower plate that applies a shear stress $\tau(0,t)$ of the form (5) to the fluid can be presented as a sum of steady-state (permanent or long time) and transient solutions, namely

$$u_c(y,t) = u_{cp}(y,t) + u_{ct}(y,t), \quad (44)$$

where

$$u_{cp}(y,t) = \frac{\sqrt{e^{\beta(y-1)}}}{\sqrt{\delta}} \mathrm{Re}\left\{ \frac{J_1(\kappa)Y_1\left(\kappa\sqrt{e^{\beta(y-1)}}\right) - Y_1(\kappa)J_1\left(\kappa\sqrt{e^{\beta(y-1)}}\right)}{J_1(\kappa)Y_0\left(\kappa\sqrt{e^{-\beta}}\right) - Y_1(\kappa)J_0\left(\kappa\sqrt{e^{-\beta}}\right)} \exp\left(\omega t + \frac{\theta}{2}\right)i \right\}, \quad (45)$$

$$u_{ct}(y,t) = 2\pi\beta\sqrt{e^{\beta(y-1)}}e^{a_0 t} \times \sum_{k=1}^{\infty} \frac{q_k J_0(q_k\sqrt{e^{-\beta}})J_1(q_k)}{J_0^2(q_k\sqrt{e^{-\beta}}) - J_1^2(q_k)}[b_k \mathrm{ch}(a_k t) + c_k \mathrm{sh}(a_k t)]B_{11}\left(e^{\beta(y-1)}, q_k\right). \quad (46)$$

At the beginning of the movement the fluid flows according to the starting solution $u_c(y,t)$. After some time, when the transients disappear, the fluid motion is characterized by the steady-state solution $u_{cp}(y,t)$, which is independent of the initial conditions but satisfies the boundary conditions (24) and the governing Equation (22).

3.2. Calculation of the Velocity Field $u_s(y,t)$

In order to determine the starting solution $u_s(y,t)$ corresponding to the second initial and boundary value problem, we follow the same line as before using the boundary conditions

$$\left.\frac{\partial \overline{u}(y,s)}{\partial y}\right|_{y=0} = \frac{\omega e^{-\beta}}{s^2+\omega^2},\ \overline{u}(1,s) = 0, \quad (47)$$

for the Laplace transform $\overline{u}(y,s)$ of $u(y,t)$. The solution of the ordinary differential Equation (29) with the boundary conditions (47) is given by the relation

$$\overline{u}(y,s) = \frac{i\omega\sqrt{e^{\beta(y-1)}}}{(s^2+\omega^2)\sqrt{s(s\mathrm{We}+1)}} \frac{J_1[\zeta(s)]Y_1\left(\zeta(s)\sqrt{e^{\beta(y-1)}}\right) - Y_1[\zeta(s)]J_1\left(\zeta(s)\sqrt{e^{\beta(y-1)}}\right)}{J_1[\zeta(s)]Y_0\left(\zeta(s)\sqrt{e^{-\beta}}\right) - Y_1[\zeta(s)]J_0\left(\zeta(s)\sqrt{e^{-\beta}}\right)}, \quad (48)$$

and its non-removable poles are identical to those from Equation (37).

Following the same way as in the previous case and using Equation (A6) from the Appendix A, it can be proved that the starting solution $u_s(y,t)$ corresponding to this motion can be also written as

$$u_s(y,t) = u_{sp}(y,t) + u_{st}(y,t), \quad (49)$$

where its steady-state and transient components $u_{sp}(y,t)$ and $u_{st}(y,t)$ respectively are given by the relations

$$u_{sp}(y,t) = \frac{\sqrt{e^{\beta(y-1)}}}{\sqrt{\delta}} \mathrm{Im}\left\{ \frac{J_1(\kappa)Y_1\left(\kappa\sqrt{e^{\beta(y-1)}}\right) - Y_1(\kappa)J_1\left(\kappa\sqrt{e^{\beta(y-1)}}\right)}{J_1(\kappa)Y_0\left(\kappa\sqrt{e^{-\beta}}\right) - Y_1(\kappa)J_0\left(\kappa\sqrt{e^{-\beta}}\right)} \exp\left(\omega t + \frac{\theta}{2}\right)i \right\}, \quad (50)$$

$$u_{st}(y,t) = 4\pi\beta\omega\sqrt{e^{\beta(y-1)}}e^{a_0 t} \times \sum_{k=1}^{\infty} \frac{q_k J_0\left(q_k\sqrt{e^{-\beta}}\right)J_1(q_k)}{J_0^2\left(q_k\sqrt{e^{-\beta}}\right) - J_1^2(q_k)}[d_k \mathrm{ch}(a_k t) + e_k \mathrm{sh}(a_k t)]B_{11}\left(e^{\beta(y-1)}, q_k\right), \quad (51)$$

where $d_k = \frac{-2}{\left(\beta^2 q_k^2 - 4\mathrm{We}\omega^2\right)^2 + 16\omega^2}$, $e_k = \frac{(\beta^2 q_k^2 - 4\mathrm{We}\omega^2)\mathrm{We} - 2}{\left[\left(\beta^2 q_k^2 - 4\mathrm{We}\omega^2\right)^2 + 16\omega^2\right]\sqrt{1 - \mathrm{We}\beta^2 q_k^2}}$.

3.3. Calculation of the Shear Stress $\tau_c(y,t)$

Applying the Laplace transform to Equation (26) and bearing in mind the corresponding initial condition $(28)_1$, we find that

$$\overline{\tau}(y,s) = \frac{e^{\beta(1-y)}}{s\mathrm{We}+1}\frac{\partial \overline{u}(y,s)}{\partial y}, \quad (52)$$

where $\overline{\tau}(y,s)$ is the Laplace transform of $\tau(y,t)$ and $\overline{u}(y,s)$ is given by Equation (36). Calculating the derivative of $\overline{u}(y,s)$ with respect to y and using again Equation $(2)_2$ and its correspondent for the function $Y_1(\cdot)$, the result is

$$\overline{\tau}(y,s)=\frac{s}{(s^2+\omega^2)(s\mathrm{We}+1)}\frac{J_1[\zeta(s)]Y_0\left(\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right)-Y_1[\zeta(s)]J_0\left(\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right)}{J_1[\zeta(s)]Y_0\left(\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right)-Y_1[\zeta(s)]J_0\left(\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right)}. \tag{53}$$

The poles of the function $\overline{\tau}(y,s)$ are the same as those given in the relation (37) and $s_0=-1/\mathrm{We}$. The residues of $\overline{\tau}(y,s)$ in $i\omega$ and $-i\omega$ are determined following the same procedure as that given in the Section 3.1 and the steady-state component $\tau_{cp}(y,t)$ of $\tau_c(y,t)$ is given by the relation

$$\tau_{cp}(y,t)=\frac{1}{\sqrt{1+(\omega\mathrm{We})^2}}\mathrm{Re}\left\{\frac{J_1(\kappa)Y_0\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)-Y_1(\kappa)J_0\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)}{Y_1(\kappa)J_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)-J_1(\kappa)Y_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)}\mathrm{e}^{i\,(\omega t+\theta+\frac{\pi}{2})}\right\}. \tag{54}$$

Direct computations show that $u_{cp}(y,t)$ and $\tau_{cp}(y,t)$ given by Equations (45) and (54), respectively, satisfy the differential Equation (26).

In order to determine the residue of $\overline{\tau}(y,s)$ in $s_0=-1/\mathrm{We}$, namely

$$\mathrm{Rez}\{\overline{\tau}(y,s)\mathrm{e}^{st}\}_{s=-1/\mathrm{We}}=-\frac{1}{1+(\omega\mathrm{We})^2}\exp\left(-\frac{t}{\mathrm{We}}\right), \tag{55}$$

we used the asymptotic approximations from Equation (A7) from the Appendix A and the fact that $Y_0(az)/Y_1(z)\to 0$ and $J_0(az)/J_0(z)\to 1$ if $z\to 0$.

The sum of residues of $\overline{\tau}(y,s)$ in s_{1k} and s_{2k} is immediately obtained using Equations (A4) and (A8) from the Appendix A, and the transient component $\tau_{ct}(y,t)$ of $\tau_c(y,t)$ is given by

$$\begin{aligned}\tau_{ct}(y,t)&=-\tfrac{1}{1+(\omega\mathrm{We})^2}\exp\left(-\tfrac{t}{\mathrm{We}}\right)\\&+2\pi\mathrm{e}^{a_0t}\sum_{k=1}^{\infty}\frac{J_0\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)J_1(q_k)}{J_0^2\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)-J_1^2(q_k)}[f_k\mathrm{ch}(a_kt)+g_k\mathrm{sh}(a_kt)]B_{01}\left(\mathrm{e}^{\beta(y-1)},q_k\right),\end{aligned} \tag{56}$$

where $f_k=\frac{-8\omega^2}{(\beta^2q_k^2-4\mathrm{We}\omega^2)^2+16\omega^2}$, $g_k=\frac{\beta^2q_k^2(\beta^2q_k^2-4\mathrm{We}\omega^2)+8\omega^2}{[(\beta^2q_k^2-4\mathrm{We}\omega^2)^2+16\omega^2]\sqrt{1-\mathrm{We}\beta^2q_k^2}}$.

3.4. Calculation of the Shear Stress $\tau_s(y,t)$

Introducing the derivative of $\overline{u}(y,s)$ with respect to y from Equation (48) in Equation (52), it results that the Laplace transform of the corresponding shear stress, namely

$$\overline{\tau}(y,s)=\frac{\omega}{(s^2+\omega^2)(s\mathrm{We}+1)}\frac{J_1[\zeta(s)]Y_0\left(\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right)-Y_1[\zeta(s)]J_0\left(\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right)}{J_1[\zeta(s)]Y_0\left(\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right)-Y_1[\zeta(s)]J_0\left(\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right)}, \tag{57}$$

has the same poles as $\overline{\tau}(y,s)$ from the previous section.

The residue of $\overline{\tau}(y,s)$ in $s_0=-1/\mathrm{We}$ is given by

$$\mathrm{Rez}\{\overline{\tau}(y,s)\mathrm{e}^{st}\}_{s=-1/\mathrm{We}}=-\frac{\omega\mathrm{We}}{1+(\omega\mathrm{We})^2}\exp\left(-\frac{t}{\mathrm{We}}\right), \tag{58}$$

and the steady-state and transient components of $\tau_s(y,t)$ are given by the equalities

$$\tau_{sp}(y,t)=\frac{1}{\sqrt{1+(\omega\mathrm{We})^2}}\mathrm{Im}\left\{\frac{J_1(\kappa)Y_0\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)-Y_1(\kappa)J_0\left(\kappa\sqrt{\mathrm{e}^{\beta(y-1)}}\right)}{Y_1(\kappa)J_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)-J_1(\kappa)Y_0\left(\kappa\sqrt{\mathrm{e}^{-\beta}}\right)}\mathrm{e}^{i\,(\omega t+\theta+\frac{\pi}{2})}\right\}, \tag{59}$$

$$\begin{aligned}\tau_{st}(y,t)&=-\tfrac{\omega\mathrm{We}}{1+(\omega\mathrm{We})^2}\exp\left(-\tfrac{t}{\mathrm{We}}\right)\\&-4\pi\omega\mathrm{e}^{a_0t}\sum_{k=1}^{\infty}\frac{J_0\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)J_1(q_k)}{J_0^2\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)-J_1^2(q_k)}[b_k\mathrm{ch}(a_kt)+c_k\mathrm{sh}(a_kt)]B_{01}\left(\mathrm{e}^{\beta(y-1)},q_k\right).\end{aligned} \tag{60}$$

4. Limiting Cases

In order to obtain similar solutions corresponding to the flow of same fluids induced by the lower plate that applies an exponential shear stress to the fluid, as well as recover some known results from the existing literature for the Newtonian fluids with/without pressure-dependent viscosity, we consider some limiting cases.

4.1. Case $\omega \to 0$ (Lower Plate Applies an Exponential Shear Stress to the Fluid)

In order to provide the starting velocity field $u_{Se}(y,t)$ and the adequate shear stress $\tau_{Se}(y,t)$ corresponding to the motion of incompressible UCM fluids with exponential dependence of viscosity on the pressure induced by the lower plate that applies an exponential shear stress $S(1-\exp(-t/\lambda))$ to the fluid (see the limit of the expression (5) when $\omega \to 0$), we need the limits of Equations (45), (46), (54), and (56) when $\omega \to 0$. It is easy to show that the solutions of this new problem will be given by the relations

$$u_{Se}(y,t) = u_{Sep}(y) + u_{Set}(y,t),\ \tau_{Se}(y,t) = \tau_{Sep}(y) + \tau_{Set}(y,t), \tag{61}$$

where the transient components

$$\begin{gathered} u_{Set}(y,t) = \lim_{\omega\to 0} u_{ct}(y,t) = \tfrac{2\pi}{\beta}\sqrt{e^{\beta(y-1)}}\, e^{a_0 t} \\ \times \sum_{k=1}^{\infty} \frac{J_0(q_k\sqrt{e^{-\beta}})J_1(q_k)}{q_k[J_0^2(q_k\sqrt{e^{-\beta}})-J_1^2(q_k)]}\left[\mathrm{ch}(a_k t) + \frac{\mathrm{sh}(a_k t)}{\sqrt{1-\mathrm{We}\beta^2 q_k^2}}\right] B_{11}\left(e^{\beta(y-1)},\, q_k\right), \end{gathered} \tag{62}$$

$$\begin{gathered} \tau_{Set}(y,t) = \lim_{\omega\to 0} \tau_{ct}(y,t) = -\ \exp\left(-\tfrac{t}{\mathrm{We}}\right) \\ +2\pi e^{a_0 t}\sum_{k=1}^{\infty} \frac{J_0(q_k\sqrt{e^{-\beta}})J_1(q_k)}{J_0^2(q_k\sqrt{e^{-\beta}})-J_1^2(q_k)}\left[\frac{\mathrm{sh}(a_k t)}{\sqrt{1-\mathrm{We}\beta^2 q_k^2}}\right] B_{01}\left(e^{\beta(y-1)},\, q_k\right), \end{gathered} \tag{63}$$

are immediately obtained from Equations (46) and (56), making $\omega = 0$.

In order to get the steady components $u_{Sep}(y)$ and $\tau_{Sep}(y)$ of the velocity and shear stress fields corresponding to this motion, we first use the asymptotic approximations from Equation (A7) in Equations (45) and (54) to show that for small enough values of ω

$$u_{cp}(y,t) \approx \frac{e^{\beta(y-1)}-1}{\sqrt{\delta}}\mathrm{Re}\left\{\frac{e^{i\,[\omega t+(\theta/2)]}}{\frac{2}{\kappa}+\frac{\kappa}{\pi^2}\left[\ln\left(\kappa\sqrt{e^{-\beta}}/2\right)+\gamma\right]}\right\}, \tag{64}$$

$$\tau_{cp}(y,t) \approx -\frac{1}{\sqrt{1+(\omega\mathrm{We})^2}}\mathrm{Re}\left\{\frac{\frac{2\pi}{\kappa}+\frac{\kappa}{\pi}\left[\ln\left(\kappa\sqrt{e^{\beta(y-1)}}/2\right)+\gamma\right]}{\frac{2\pi}{\kappa}+\frac{\kappa}{\pi}\left[\ln\left(\kappa\sqrt{e^{-\beta}}/2\right)+\gamma\right]}e^{i\,[\omega t+\theta+(\pi/2)]}\right\}, \tag{65}$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. Substituting κ from Equation (40) in (64) and (65) and taking the limit of the obtained results for $\omega \to 0$, we find that

$$u_{Sep}(y) = \lim_{\omega\to 0} u_{cp}(y,t) = \frac{e^{\beta(y-1)}-1}{\beta},\ \tau_{Sep}(y) = \lim_{\omega\to 0}\tau_{cp}(y,t) = \tau_{Sep} = \text{const.} = 1, \tag{66}$$

which are just the steady velocity and shear stress fields $u_{NSp}(y)$ and τ_{NSp} corresponding to the motion of incompressible Newtonian fluids with exponential dependence of viscosity on the pressure induced by the lower plate that applies a constant shear stress S to the fluid (see for instance Fetecau et al. [44], Equation $(58)_1$). This is not a surprising case, because the ordinary differential equations governing the steady motions of the incompressible UCM or Newtonian fluids with pressure-dependent viscosity are identical. What is very important is the fact that the non-dimensional steady shear stress corresponding to this motion is constant on the entire flow domain, although the adequate velocity field depends of the spatial variable. This constant is just the dimensionless shear stress applied to the fluid by the lower plate.

It is also important to observe that, as expected, the transient components of the starting solutions $u_{NS}(y,t)$ and $\tau_{NS}(y,t)$ corresponding to the motion of incompressible Newtonian fluids with exponential dependence of viscosity on the pressure induced by the lower plate that applies a constant shear stress S to the fluid, namely (see Fetecau et al. [44], Equation (57) for the velocity field only)

$$u_{NSt}(y,t) = \frac{2\pi\sqrt{\mathrm{e}^{\beta(y-1)}}}{\beta}\sum_{k=1}^{\infty}\left[\frac{J_0\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)J_1(q_k)}{q_k\left[J_0^2\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)-J_1^2(q_k)\right]}B_{11}\left(\mathrm{e}^{\beta(y-1)},q_k\right)\exp\left(-\frac{\beta^2q_k^2}{4}t\right)\right], \quad (67)$$

$$\tau_{NSt}(y,t) = \pi\sum_{k=1}^{\infty}\left[\frac{J_0\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)J_1(q_k)}{J_0^2\left(q_k\sqrt{\mathrm{e}^{-\beta}}\right)-J_1^2(q_k)}B_{01}\left(\mathrm{e}^{\beta(y-1)},q_k\right)\exp\left(-\frac{\beta^2q_k^2}{4}t\right)\right], \quad (68)$$

are immediately obtained, making $\mathrm{We}\to 0$ in Equations (62) and (63).

4.2. Case $\beta \to 0$ (Flows of Ordinary Incompressible UCM Fluids)

Based on the asymptotic approximations given by Equation (A9) from the Appendix A, we can show that for small enough values of the non-dimensional pressure–viscosity coefficient β, the steady-state solutions $u_{cp}(y,t)$ and $u_{sp}(y,t)$ can be approximated by the following relations

$$u_{cp}(y,t) \approx \frac{\sqrt[4]{\mathrm{e}^{\beta y}}}{\sqrt{\delta\mathrm{e}^{\beta}}}\mathrm{Re}\left\{\frac{\sin\{\kappa(1-\exp[\beta(y-1)/2])\}}{\cos\{\kappa[1-\exp(-\beta/2)]\}}\mathrm{e}^{i(\omega t+\theta/2)}\right\}, \quad (69)$$

$$u_{sp}(y,t) \approx \frac{\sqrt[4]{\mathrm{e}^{\beta y}}}{\sqrt{\delta\mathrm{e}^{\beta}}}\mathrm{Im}\left\{\frac{\sin\{\kappa(1-\exp[\beta(y-1)/2])\}}{\cos\{\kappa[1-\exp(-\beta/2)]\}}\mathrm{e}^{i(\omega t+\theta/2)}\right\}. \quad (70)$$

Introducing the Maclaurin series expansions of the exponential functions $\exp[\beta(y-1)/2]$ and $\exp(-\beta/2)$ in Equations (69) and (70) and making $\beta\to 0$, we get the steady-state solutions

$$u_{Ocp}(y,t) = \frac{1}{\sqrt{\delta}}\mathrm{Re}\left\{\frac{\sin\left[(y-1)\sqrt{\delta}\exp\left(-i\frac{\theta}{2}\right)\right]}{\cos\left[\sqrt{\delta}\exp\left(-i\frac{\theta}{2}\right)\right]}\mathrm{e}^{i(\omega t+\theta/2)}\right\}. \quad (71)$$

$$u_{Osp}(y,t) = \frac{1}{\sqrt{\delta}}\mathrm{Im}\left\{\frac{\sin\left[(y-1)\sqrt{\delta}\exp\left(-i\frac{\theta}{2}\right)\right]}{\cos\left[\sqrt{\delta}\exp\left(-i\frac{\theta}{2}\right)\right]}\mathrm{e}^{i(\omega t+\theta/2)}\right\}, \quad (72)$$

corresponding to ordinary incompressible UCM fluids performing the same motions. The steady-state solutions given by Equations (71) and (72), as well as those given by Equations (45) and (50), can be used to determine the required time to reach the permanent state for the two motions of the fluids in discussion. This is the time after which the fluid flows according to the steady-state solutions. More exactly, in the last case, it is the time after which the diagrams of the starting solutions $u_{Oc}(y,t)$ or $u_{Os}(y,t)$ corresponding to the ordinary UCM fluids performing the initial motions are almost identical to those of $u_{Ocp}(y,t)$, respectively $u_{Osp}(y,t)$. To the best of our knowledge, the steady solutions given by Equations (71) and (72) are not known in the literature.

4.3. Case We $\to 0$ (Flows of Incompressible Newtonian Fluids with Exponential Dependence of Viscosity on the Pressure)

Finally, making $\mathrm{We}\to 0$ in Equations (45), (46), (50), and (51) and bearing in mind the fact that

$$\begin{aligned}&\delta\to\omega,\ \ \theta\to\tfrac{\pi}{2},\ \ \kappa\to\tfrac{2}{\beta}\sqrt{-i\omega},\ \ \mathrm{e}^{a_0t}\mathrm{ch}(a_kt)\to\tfrac{1}{2}\exp\left(-\tfrac{\beta^2q_k^2}{4}t\right),\\&\mathrm{e}^{a_0t}\mathrm{sh}(a_kt)\to\tfrac{1}{2}\exp\left(-\tfrac{\beta^2q_k^2}{4}t\right),\end{aligned} \quad (73)$$

we recover the similar solutions (see Danish et al. [44], Equations (35), (36), (43), and (44))

$$u_{Ncp}(y,t)=\frac{\sqrt{e^{\beta(y-1)}}}{\sqrt{\omega}}\mathrm{Re}\left\{\frac{J_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)Y_1\left(\frac{2}{\beta}\sqrt{-i\omega e^{\beta(y-1)}}\right)-Y_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)J_1\left(\frac{2}{\beta}\sqrt{-i\omega e^{\beta(y-1)}}\right)}{J_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)Y_0\left(\frac{2}{\beta}\sqrt{-i\omega e^{-\beta}}\right)-Y_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)J_0\left(\frac{2}{\beta}\sqrt{-i\omega e^{-\beta}}\right)}e^{i(\omega t+\pi/4)}\right\}, \tag{74}$$

$$u_{Nct}(y,t)=2\pi\sqrt{e^{\beta(y-1)}}\sum_{k=1}^{\infty}\left[\frac{(\beta q_k)^3B_{11}\left(e^{\beta(y-1)},q_k\right)}{(\beta q_k)^4+16\omega^2}\frac{J_0(q_k\sqrt{e^{-\beta}})J_1(q_k)}{J_0^2(q_k\sqrt{e^{-\beta}})-J_1^2(q_k)}\exp\left(-\frac{\beta^2q_k^2}{4}t\right)\right], \tag{75}$$

$$u_{Nsp}(y,t)=\frac{\sqrt{e^{\beta(y-1)}}}{\sqrt{\omega}}\mathrm{Im}\left\{\frac{J_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)Y_1\left(\frac{2}{\beta}\sqrt{-i\omega e^{\beta(y-1)}}\right)-Y_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)J_1\left(\frac{2}{\beta}\sqrt{-i\omega e^{\beta(y-1)}}\right)}{J_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)Y_0\left(\frac{2}{\beta}\sqrt{-i\omega e^{-\beta}}\right)-Y_1\left(\frac{2}{\beta}\sqrt{-i\omega}\right)J_0\left(\frac{2}{\beta}\sqrt{-i\omega e^{-\beta}}\right)}e^{i(\omega t+\pi/4)}\right\}, \tag{76}$$

$$u_{Nst}(y,t)=-8\pi\beta\omega\sqrt{e^{\beta(y-1)}}\sum_{k=1}^{\infty}\frac{q_kB_{11}\left(e^{\beta(y-1)},q_k\right)}{(\beta q_k)^4+16\omega^2}\frac{J_0(q_k\sqrt{e^{-\beta}})J_1(q_k)}{J_0^2(q_k\sqrt{e^{-\beta}})-J_1^2(q_k)}\exp\left(-\frac{\beta^2q_k^2}{4}t\right), \tag{77}$$

corresponding to motions of incompressible Newtonian fluids with exponential dependence of viscosity on the pressure generated by the lower plate that applies oscillatory shear stresses $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid.

It is worth pointing out the fact that for small enough values of the dimensionless pressure–viscosity coefficient β, the steady-state solutions $u_{Ncp}(y,t)$ and $u_{Nsp}(y,t)$ given by Equation (74), respectively (76) can be approximated by the relations (see Equation (9)

$$u_{Ncp}(y,t)\approx-\frac{\sqrt[4]{e^{\beta y}}}{\sqrt{\omega e^{\beta}}}\mathrm{Re}\left\{\frac{\sin\left[\frac{2}{\beta}\sqrt{-i\omega}\left(1-e^{\beta(y-1)/2}\right)\right]}{\cos\left[\frac{2}{\beta}\sqrt{-i\omega}\left(1-e^{-\beta/2}\right)\right]}e^{i\left(\omega t+\frac{\pi}{4}\right)}\right\}, \tag{78}$$

$$u_{Nsp}(y,t)\approx-\frac{\sqrt[4]{e^{\beta y}}}{\sqrt{\omega e^{\beta}}}\mathrm{Im}\left\{\frac{\sin\left[\frac{2}{\beta}\sqrt{-i\omega}\left(1-e^{\beta(y-1)/2}\right)\right]}{\cos\left[\frac{2}{\beta}\sqrt{-i\omega}\left(1-e^{-\beta/2}\right)\right]}e^{i\left(\omega t+\frac{\pi}{4}\right)}\right\}. \tag{79}$$

Now, again using the Maclaurin series expansions of the two exponential functions $\exp[\beta(y-1)/2]$ and $\exp(-\beta/2)$ in Equations (78) and (79), making $\beta\to 0$ and making in mind the identities (10), we recover the known solutions of Fetecau et al. [44], as shown in Equation (45)

$$u_{ONcp}(y,t)=\mathrm{Re}\left\{\frac{\mathrm{sh}[(y-1)\sqrt{i\omega}]}{\mathrm{ch}(\sqrt{i\omega})}\frac{e^{i\omega t}}{\sqrt{i\omega}}\right\},\quad u_{ONsp}(y,t)=\mathrm{Im}\left\{\frac{\mathrm{sh}[(y-1)\sqrt{i\omega}}{\mathrm{ch}(\sqrt{i\omega})}\frac{e^{i\omega t}}{\sqrt{i\omega}}\right\}, \tag{80}$$

corresponding to ordinary incompressible Newtonian fluids performing the same motions. Of course, making $\mathrm{We}\to 0$ in Equations (71) and (72) and using again the identities (10), we attain the same solutions given by Equation (80).

5. Some Numerical Results and Discussion

Exact solutions for unsteady motions of the non-Newtonian fluids with pressure-dependent viscosity seem to be absent in the existing literature. Here, the constitutive Equation (1) of the incompressible UCM fluids are used to describe the viscoelastic response of such a fluid (typically a polymer melt), which is moving in a horizontal rectangular channel. The fluid movement, induced by the lower plate that applies time-dependent shear stresses to the fluid, is characterized by mixed boundary value problems in which the velocity gradient is given on the bottom plate. Exact expressions are established for the dimensionless velocity fields and the adequate non-trivial shear stresses when the gravity effects are taken into consideration. These expressions, presented in simple forms in the terms of some standard Bessel functions, have been easily particularized to give similar solutions corresponding to motions of the same fluids due to the lower plate that applies an exponential $S(1-\exp(-t/\lambda))$ shear stress to the fluid. Similar solutions for the ordinary UCM fluids performing the same motions, as well as the solutions corresponding to motions of the incompressible Newtonian fluids with/without pressure-dependent viscosity induced by oscillatory $S\cos(\omega t)$, $S\sin(\omega t)$ or constant S shear stresses on the boundary, have been also obtained as limiting cases of general results.

It is worth pointing out that the general solutions corresponding to unsteady motions of the incompressible UCM fluids with exponential dependence of viscosity on the pressure have been established using the Laplace transform with respect to time t and appropriate changes of the unknown function $u(y,t)$ and of the spatial variable y. The inverse Laplace transforms of $\overline{u}(y,s)$ and $\overline{\tau}(y,s)$ have been obtained using the residue theorem from the complex analysis. The poles of these functions s_{1k} and s_{2k}, which are the roots of the transcendental Equation (38), have been determined using the subroutine $root(f(x),x,a,b)$ of the software Mathcad. The fact that different known solutions from the existing literature have also been obtained as limiting cases of present results could be validation of their correctness.

It is also important to note that the expressions for the dimensionless velocity and shear stress fields that have been here obtained depend on two essential parameters, namely the dimensionless pressure-viscosity coefficient β and the Weissenberg number We, which represents the ratio of elastic to viscous forces Poole [48]. Consequently, to bring to light some physical insight of obtained results, the influence of these parameters on the fluid velocity and the frictional force per unit area exerted by the fluid on the stationary plate has been graphically illustrated in Figures 1–6 for cosine oscillations of the velocity gradient on the bottom wall, more precisely, in the case when $\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = e^{-\beta}\cos(\omega t)$. All graphical representations are prepared for $\omega = \pi/2$ and different values of β, We, and the time t.

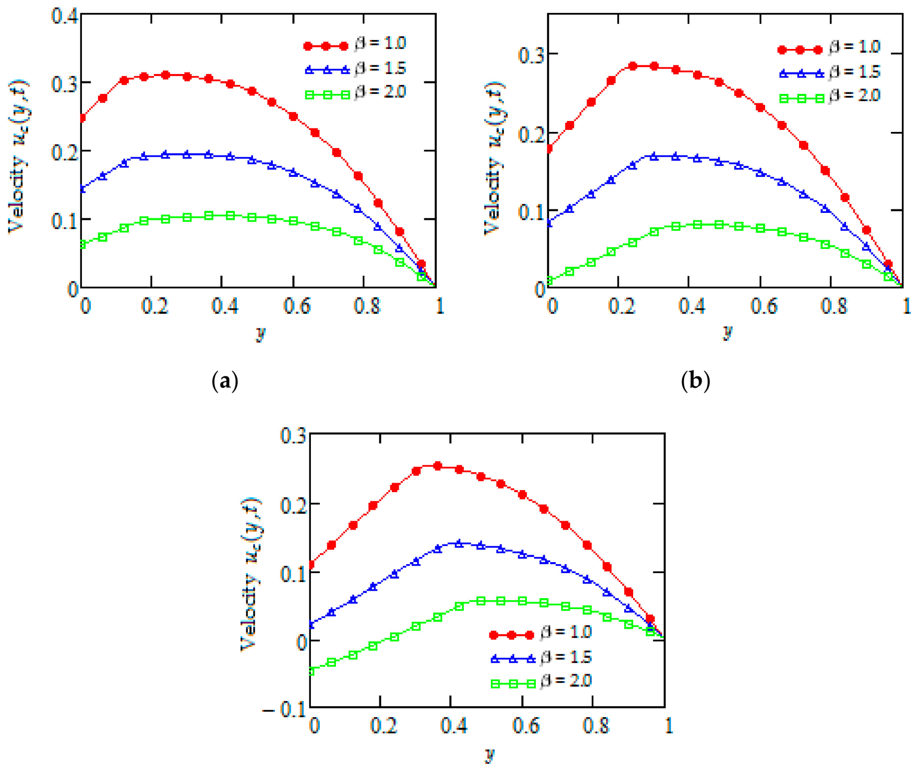

Figure 1. The profiles of velocity $u_c(y,t)$ for We = 0.5 and different values of the pressure–viscosity parameter β at (**a**) t = 0.05, (**b**) t = 0.10, (**c**) t = 0.15.

The profiles of the dimensionless velocity field $u_c(y,t)$ given by Equation (44) are presented in Figures 1 and 2 at different values of time t, for three values of the pressure–viscosity coefficient β and two values of the Weissenberg number We. It is clearly seen from these figures that the fluid velocity is a decreasing function with respect to β. This behavior is due to the fact that the fluid viscosity increases for increasing values of β, and the fluid velocity diminishes. On the other hand, the fluid velocity increases for increasing values of the Weissenberg number We, because at the same elastic properties of the fluid, an increase of this number means a decrease of viscous forces, which implies an increase of the fluid velocity. In all cases, the boundary conditions are clearly satisfied.

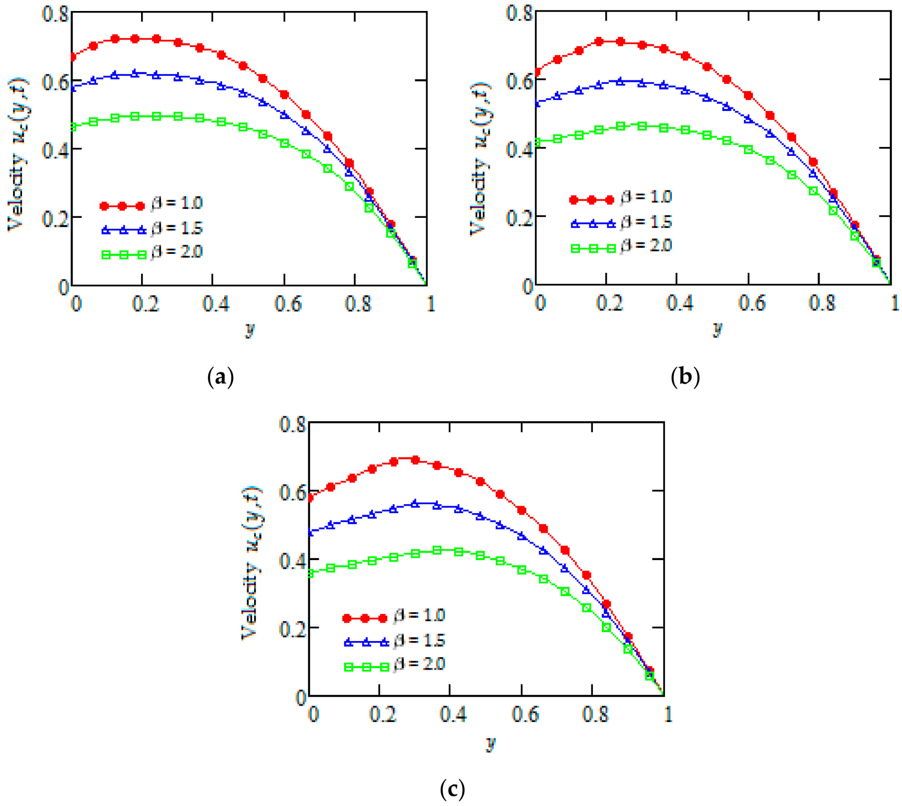

Figure 2. The profiles of velocity $u_c(y,t)$ for We = 0.75 and different values of the pressure–viscosity parameter β at (**a**) t = 0.05, (**b**) t = 0.10, (**c**) t = 0.15.

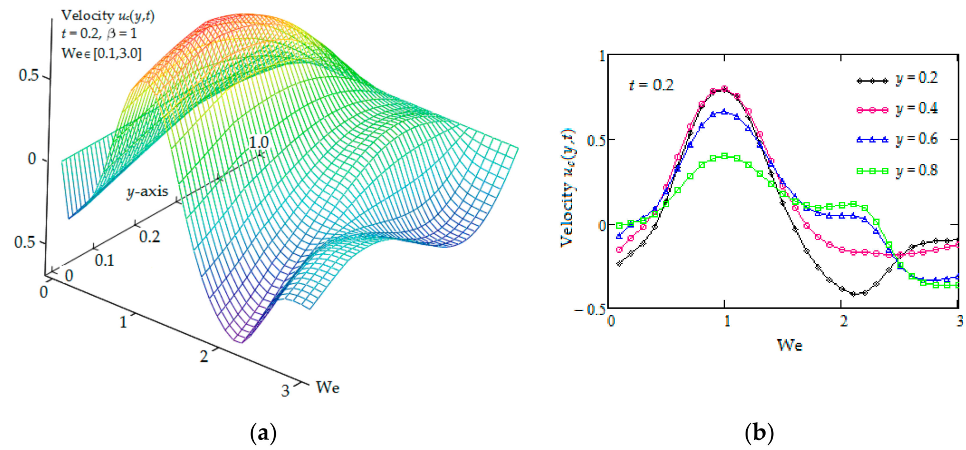

Figure 3. Spatial variation of the velocity $u_c(y,t)$ in the channel (**a**), and its profiles from four transversal sections (**b**), when the Weissenberg parameter belongs to the interval [0.1, 3].

The spatial variation of the velocity $u_c(y,t)$ in the channel and its profiles from four transversal sections with the planes $y = 0.2,\ 0.4,\ 0.6$ and 0.8, are presented in Figure 3a,b respectively at the moment $t = 0.2$ when the Weissenberg number We varies in the interval $[0.1,\ 3]$.

It can be observed that both from the spatial representation and the transversal sections that the fluid velocity attains maximum values for We $= 1$ when the elastic and viscous forces have the same intensities. For all values of the Weissenberg number We, as expected, the fluid velocity decreases to zero near the upper wall. The time evolution of its transient component $u_{ct}(y,t)$ is presented in Figure 4a,b in the plane section $y = 0.4$, for two values of the Weissenberg number We and three values of the pressure–viscosity coefficient β. As expected, the transient component $u_{ct}(y,t)$ in absolute value tends to zero for increasing values of the time t. In all cases, the transient velocity can be neglected after the moment $t = 6$. This is the approximate time to reach the steady-state (permanent state). After this moment, the fluid motion can be well enough described by the permanent component $u_{cp}(y,t)$.

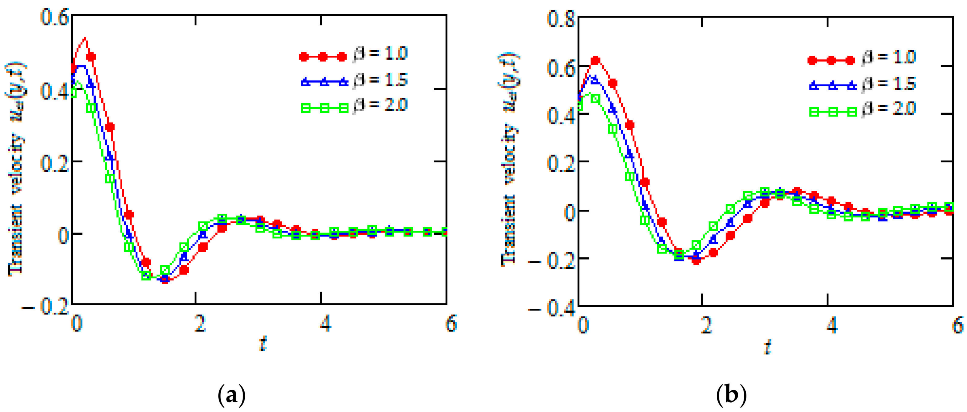

Figure 4. Profiles of the transient velocity $u_{ct}(y,t)$ versus t, for three different values of pressure–viscosity parameter β at (**a**) y = 0.4, We = 0.5, and (**b**) y = 0.4, We = 0.75.

The variation of the frictional force per unit area $\tau_c(1,t)$ exerted by the fluid on the upper plate is presented in Figure 5 when the time t and the Weissenberg number We vary in the intervals $[0,\ 4]$ and $[0.1,\ 3]$, respectively.

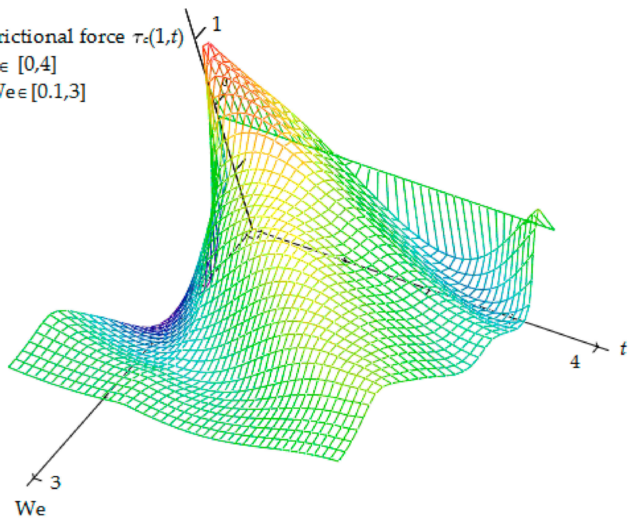

Figure 5. Spatial profile of the frictional force $\tau_c(1,t)$ on the upper plate for $t \in [0,4]$ and We $\in [0.1,3]$.

The profiles of $\tau_c(1,t)$ versus t for different values of Weissenberg number and versus We for some values of the time t are presented in Figure 6a,b respectively. Their oscillatory specific feature is clearly observed from Figure 6a and the oscillations' amplitude decreases for increasing values of We. From Figure 6b, it can be seen that the frictional force per unit area in absolute value tends to zero for large values of We. This behavior is because at the same elastic properties of the fluid, the intensity of the viscous forces diminishes for increasing values of the Weissenberg number We.

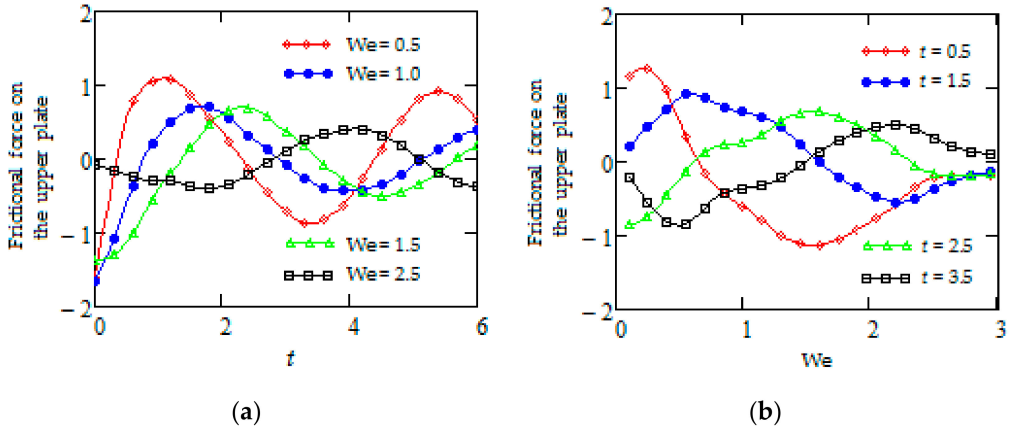

Figure 6. The profiles of the frictional force $\tau_c(1,t)$ on the upper plate (**a**) versus time t, and (**b**) versus Weissenberg number We.

6. Conclusions

Exact solutions are established for some unsteady motions of incompressible UCM fluids with exponential dependence of viscosity on the pressure between two infinite horizontal parallel plates. Contrary to what is usually assumed in the existing literature, the non-slip condition is used on a part of the boundary and the shear stress is specified in rest. The corresponding mixed initial-boundary value problems are solved in a simple way using the Laplace transform and suitable changes of the spatial variable and the unknown function. All solutions that have been here obtained are presented as sums of the steady-state and transient components, and they are important for the experimentalists who want to eliminate the transients from their experiments.

The main results that have been here obtained are as follows:

- Analytical solutions are for the first time obtained for mixed boundary value problems associated with unsteady motions of UCM fluids with pressure-dependent viscosity.

- Exact expressions are established both for the dimensionless velocity fields and the adequate non-trivial shear stresses using the Laplace transform and the residue theorem.

- Similar solutions corresponding to motions of the same fluids due to an exponential shear stress on the boundary are obtained as limiting cases of the general results.

- The solutions corresponding to ordinary UCM fluids performing the same motions as well as some known results for Newtonian fluids are also obtained as limiting cases.

- Obtained solutions can be used as tests to verify numerical schemes that are developed for more complex problems or to find the required time to reach the steady state.

- Fluid velocity is a decreasing function with respect to the pressure–viscosity coefficient β and increases for increasing values of the Weissenberg number We.

- The transient component $u_{ct}(y,t)$, in absolute value, tends to zero for increasing values of time t. The required dimensionless time to reach the steady state is about 6 (six).

- Steady shear stress for the motion due to an exponential shear on the boundary is constant on entire flow domain, although the velocity depends on the spatial variable y.

- The frictional force $\tau_c(1,t)$ per unit area exerted by the fluid on the upper plate, in absolute value, tends to zero for increasing values of the Weissenberg number We.

- The present results could be extended to unsteady motions of the incompressible UCM fluids through porous media Fusi [49], Ullah et al. [50], and Fetecau et al. [51].

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Abbreviations

$\boldsymbol{T}$	Cauchy stress tensor	$\boldsymbol{S}$	Extra stress tensor
$\boldsymbol{A}$	First Rivlin–Ericksen tensor	$\boldsymbol{v}$	Velocity vector
$\boldsymbol{L}$	The velocity gradient	u	Fluid velocity
p	Pressure	x, y, z	Cartesian coordinates
τ	Non-trivial shear stress	σ_x	Non-trivial normal stress
t	Time	g	Gravitational acceleration
λ	Relaxation time	ρ	Fluid density
μ	Fluid viscosity at the reference pressure	ν	Kinematic viscosity
α	Pressure–viscosity coefficient	S	Constant shear-stress
d	Distance between plates	We	Weissenberg number

Appendix A

Bearing in mind the expression of $\overline{u}(y,s)$ from Equation (36), it results that

$$\begin{aligned}
&\mathrm{Re}z\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=s_{1k}} + \mathrm{Re}z\{\overline{u}(y,s)\mathrm{e}^{st}\}_{s=s_{2k}} \\
&= \lim_{s\to s_{1k}}\left[\frac{i\sqrt{s\mathrm{e}^{\beta(y-1)}}\,\mathrm{e}^{st}}{(s^2+\omega^2)\sqrt{s\mathrm{We}+1}}\right]\frac{J_1[\zeta(s)]Y_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right]-Y_1[\zeta(s)]J_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right]}{\frac{d}{ds}\left\{J_1[\zeta(s)]Y_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]-Y_1[\zeta(s)]J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]\right\}} \\
&+ \lim_{s\to s_{2k}}\left[\frac{i\sqrt{s\mathrm{e}^{\beta(y-1)}}\,\mathrm{e}^{st}}{(s^2+\omega^2)\sqrt{s\mathrm{We}+1}}\right]\frac{J_1[\zeta(s)]Y_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right]-Y_1[\zeta(s)]J_1\left[\zeta(s)\sqrt{\mathrm{e}^{\beta(y-1)}}\right]}{\frac{d}{ds}\left\{J_1[\zeta(s)]Y_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]-Y_1[\zeta(s)]J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]\right\}}.
\end{aligned} \tag{A1}$$

Calculating the derivative of the denominator with respect to the variable s and using the identities

$$J'_0(z) = -J_1(z),\ J'_1(z) = J_0(z) - J_1(z)/z,\ J_0(z)Y_1(z) - J_1(z)Y_0(z) = -\frac{2}{\pi z}, \tag{A2}$$

as well as the fact that the poles q_k are roots of the transcendental Equation (38) and

$$1 + 2s_{1k}\mathrm{We} = 2a_k\mathrm{We},\ 1 + 2s_{2k}\mathrm{We} = -2a_k\mathrm{We}, \tag{A3}$$

one obtains after lengthy but straightforward computations that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{ds}}\left\{J_1[\zeta(s)]Y_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]-Y_1[\zeta(s)]J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]\right\} \\
&= \frac{2\zeta'(s)}{\pi\zeta(s)}\frac{J_0^2\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]-J_1^2[\zeta(s)]}{J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]J_1[\zeta(s)]} = \frac{2s\mathrm{We}+1}{\pi s(s\mathrm{We}+1)}\frac{J_0^2\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]-J_1^2[\zeta(s)]}{J_0\left[\zeta(s)\sqrt{\mathrm{e}^{-\beta}}\right]J_1[\zeta(s)]}.
\end{aligned} \tag{A4}$$

Now, using the expressions of s_{1k} and s_{2k} from Equation (37), one can show, after lengthy but straightforward computations, that

$$\frac{s_{1k}e^{s_{1k}t}}{(s_{1k}^2+\omega^2)(2s_{1k}\mathrm{We}+1)}+\frac{s_{2k}e^{s_{2k}t}}{(s_{2k}^2+\omega^2)(2s_{2k}\mathrm{We}+1)} \\ =-4e^{a_0t}[b_k\mathrm{ch}(a_kt)+c_k\mathrm{sh}(a_kt)]\,;\ \ i\sqrt{s_{1k}(s_{1k}\mathrm{We}+1)}=i\sqrt{s_{2k}(s_{2k}\mathrm{We}+1)}=-\frac{\beta q_k}{2}. \tag{A5}$$

$$\frac{e^{s_{1k}t}}{(s_{1k}^2+\omega^2)(2s_{1k}\mathrm{We}+1)}+\frac{e^{s_{2k}t}}{(s_{2k}^2+\omega^2)(2s_{2k}\mathrm{We}+1)}=d_k\mathrm{ch}(a_kt)+e_k\mathrm{sh}(a_kt)\,, \tag{A6}$$

$$J_0(z)\approx 1,\ J_1(z)\approx\frac{z}{2},\ Y_0(z)\approx\frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right)+\gamma\right],\ Y_1(z)\approx-\frac{2}{\pi z}\ \text{ for } z<<1, \tag{A7}$$

$$\frac{s_{1k}^2e^{s_{1k}t}}{(s_{1k}^2+\omega^2)(2s_{1k}\mathrm{We}+1)}+\frac{s_{2k}^2e^{s_{2k}t}}{(s_{2k}^2+\omega^2)(2s_{2k}\mathrm{We}+1)}=2e^{a_0t}[f_k\mathrm{ch}(a_kt)+g_k\mathrm{sh}(a_kt)]\,, \tag{A8}$$

$$J_\nu(z)\approx\sqrt{\frac{2}{\pi z}}\cos\left[z-\frac{(2\nu+1)\pi}{4}\right],\ Y_\nu(z)\approx\sqrt{\frac{2}{\pi z}}\sin\left[z-\frac{(2\nu+1)\pi}{4}\right]\ \text{ for } z>>1, \tag{A9}$$

$$\sin(iz)=i\mathrm{sh}(z)\,,\ \cos(iz)=\mathrm{ch}(z)\,. \tag{A10}$$

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