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New Generalizations and Results in Shift-Invariant Subspaces of Mixed-Norm Lebesgue Spaces $L_{\vec{p}}(\mathbb{R}^d)$

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Abstract: In this paper, we establish new generalizations and results in shift-invariant subspaces of mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$. We obtain a mixed-norm Hölder inequality, a mixed-norm Minkowski inequality, a mixed-norm convolution inequality, a convolution-Hölder type inequality and a stability theorem to mixed-norm case in the setting of shift-invariant subspace of $L_{\vec{p}}(\mathbb{R}^d)$. Our new results unify and refine the existing results in the literature.

Keywords: mixed-norm; shift-invariant space; stability theorem; convolution type inequality; Hölder type inequality; Minkowski type inequality

MSC: 35Q30; 41A15; 42C15; 42C40

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1. Introduction

A mixed-norm Lebesgue space is a natural generalization of the classical Lebesgue space $L_p(\mathbb{R}^d)$, in which independent variables that may have different meanings are considered. We first recall the notion of mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ with $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$, which was originally introduced by Benedek and Panzone [1] in 1961.

Definition 1 (see [1,2]). *Let $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$. The mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ (in Benedek and Panzone's sense) is defined to be the set of all measurable functions f such that their norms $\|f\|_{L_{\vec{p}}(\mathbb{R}^d)}$ (abbreviated as $\|f\|_{L_{\vec{p}}}$ or $\|f\|_{\vec{p}}$) defined by*

$$\begin{aligned}\|f\|_{L_{\vec{p}}(\mathbb{R}^d)} &= \left\{ \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_d)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right]^{\frac{p_3}{p_2}} dx_3 \ldots dx_d \right\}^{\frac{1}{p_d}} \\ &= \left\| \cdots \left\| \|f(x_1, x_2, \ldots, x_d)\|_{L_{p_1}(x_1)} \right\|_{L_{p_2}(x_2)} \cdots \right\|_{L_{p_d}(x_d)} < \infty.\end{aligned}$$

Here, $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$ means that $1 \le p_i \le \infty$ for $i = 1, 2, \ldots, d$, and is usually abbreviated by $\vec{p} \in [1, \infty]^d$. When $p_i = \infty$ for $i = 1, \ldots, d$, the L_{p_i}-norm is replaced by L_∞-norm.

In fact, the function spaces with mixed norms have practical significance and have rapidly been developed and applied in many fields of mathematics. For instance, in PDEs as an example, functions defined by spacial and time variables may belong to certain mixed-norm spaces; inhomogeneous Besov spaces with mixed Lebesgue norms were studied recently (see, e.g., [3–5]); sampling theory based on mixed-norm theories was studied in [6]. Moreover, Triebel–Lizorkin spaces or Hardy spaces [7,8] were also studied under mixed-norm theories. Kowalski [9] studied the sparse methods in signal regression with mixed norms in 2009, where the author used the Besov and Triebel–Lizorkin spaces under mixed-norm characterization. In 2012, Kolyada [10] developed further properties of the mixed-norm Fournier–Gagliardo spaces and Lorentz-type spaces with iterative rearrangements, which gave a sharp constant in a Sobolev embedding problem. For more progress about various mixed-norm Lebesgue spaces and their applications, we refer the reader to [3–5,7,8,11,12] and references therein. A natural question is whether a mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ has important results and properties, such as Hölder inequality, stability property, etc., for shift-invariant subspaces of $L_p(\mathbb{R}^d)$ which were established in [13]. It is known that $L_{\vec{p}}(\mathbb{R}^d)$ is a type of Banach space (see [1]), so it inherits many excellent properties of traditional $L_p(\mathbb{R}^d)$ space. However, the order of integration in the definition of $L_{\vec{p}}(\mathbb{R}^d)$ is not commutative. Therefore, research on some issues in mixed-norm Lebesgue spaces also brings various challenges. For example, the characterization study on the revelent mixed-norm space [7,14] is troublesome because of the noncommutative order of the integral. In this paper, we will consider these properties for shift-invariant subspaces of mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$. To some extent, our research in this paper promotes the existing conclusions of shift-invariant subspaces, such as [6,13] and can be used in the study of characterization of a mixed norm space in the future.

The paper is organized as follows. In Section 2, we establish a mixed-norm Hölder inequality, a mixed-norm Minkowski inequality and a mixed-norm convolution inequality. A convolution-Hölder type inequality and a stability theorem to mixed-norm case in the setting of shift-invariant subspace of mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ are given in Section 3. Our new results unify and refine the relevant existing results in the literature [6,13,15,16].

2. New Mixed-Norm Inequalities and Generalizations

Let us begin with some basic definitions and notation that will be needed in this paper. Let $\mathbb{Z}$ be the integer set, and define $\mathbb{Z}^d = \{k = (k_1, k_2, \ldots, k_d) : k_i \in \mathbb{Z}, i = 1, 2, \ldots, d\}$. For $c = (c_k)_{k\in\mathbb{Z}}$, we define the discrete space $l_p(\mathbb{Z})$ with its finite norm

$$\|c\|_{l_p} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}} |c_k|^p\right)^{\frac{1}{p}}, & 0 < p < \infty; \\ \sup_{k\in\mathbb{Z}} |c_k|, & p = \infty. \end{cases}$$

The mixed-norm of $l_{\vec{p}}(\mathbb{Z}^d)$ is defined by

$$\|c\|_{l_{\vec{p}}} = \left\| \left(\ldots \|c(k_1, k_2, \ldots, k_d)\|_{l_{p_1(k_1)}} \cdots \right) \right\|_{l_{p_d(k_d)}}.$$

The Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$ for every $f \in L_1(\mathbb{R}^d)$. Other types of Fourier transform are the classical extension of this form.

Let $f(x)$ be measurable for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then $f(x) \in \mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$, if it satisfies

$$\begin{aligned} &\|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} \\ &:= \left\| \textstyle\sum_{k_d\in\mathbb{Z}} \left(\cdots \| \sum_{k_1\in\mathbb{Z}} |f(x_1 + k_1, \cdots, x_d + k_d)| \|_{L_{p_1(x_1)}([0,1])} \cdots \right) \right\|_{L_{p_d(x_d)}([0,1])} < \infty. \end{aligned}$$

Notice that $\|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)}$ is usually abbreviated by

$$\|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} = \left\| \left[\ldots \left\| \|f(x_1, x_2, \ldots, x_d)\|_{\mathfrak{L}_{p_1}(x_1)(\mathbb{R})} \right\|_{\mathfrak{L}_{p_2}(x_2)(\mathbb{R})} \cdots \right] \right\|_{\mathfrak{L}_{p_d}(x_d)(\mathbb{R})}.$$

For example, when $\vec{p} \in [1, \infty)^d$, we have

$$\|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} = \left(\int_{[0,1]} \left\{ \sum_{k_d\in\mathbb{Z}} \cdots \left[\int_{[0,1]} \left(\sum_{k_1\in\mathbb{Z}} |f(x_1 + k_1, \ldots, x_d + k_d)| \right)^{p_1} dx_1 \right]^{\frac{1}{p_1}} \cdots \right\}^{p_d} dx_d \right)^{\frac{1}{p_d}},$$

and for $\vec{p} = \{\infty, \ldots, \infty\} = \vec{\infty}$, we have

$$\|f\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} = \operatorname{esssup}_{x_d \in [0,1]} \left(\sum_{k_d \in \mathbb{Z}} \left\{ \cdots \left[\operatorname{esssup}_{x_1 \in [0,1]} \left(\sum_{k_1 \in \mathbb{Z}} |f(x_1 + k_1, \ldots, x_d + k_d)| \right) \right] \cdots \right\} \right).$$

Here, the difference between $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$ and $L_{\vec{p}}(\mathbb{R}^d)$ will be explained. To this end, we assume $d = 1$ and $p \in [1, \infty)$, then

$$\begin{aligned} \|f\|^p_{L_p(\mathbb{R})} &= \int_{[0,1]} \sum_{k \in \mathbb{Z}} |f(x+k)|^p dx \\ &\leq \int_{[0,1]} [\sum_{k \in \mathbb{Z}} |f(x+k)|]^p dx \\ &= \|f\|^p_{\mathfrak{L}_p(\mathbb{R})}. \end{aligned}$$

So it follows that $\|f\|_{L_p(\mathbb{R})} \leq \|f\|_{\mathfrak{L}_p(\mathbb{R})}$. However, $\|f\|_{L_p(\mathbb{R})} \leq \|f\|_{\mathfrak{L}_p(\mathbb{R})}$ cannot be reversed, that is, $\|f\|_{\mathfrak{L}_p(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R})}$ or $\|f\|_{\mathfrak{L}_p(\mathbb{R})} \leq C\|f\|_{L_p(\mathbb{R})}$ (C is a positive constant) cannot be correct. Then, we only have $\mathfrak{L}_p(\mathbb{R}) \subset L_p(\mathbb{R})$ and $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d) \subset L_{\vec{p}}(\mathbb{R}^d)$. In this paper, $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$ is a fundamental space, which will be used in the estimation of the upper bound for all kinds of inequalities, such as stability property inequality, convolution-type inequality. It can replace the classical L_p space at any needed time to obtain some similar conclusions in $L_{\vec{p}}$ space. Meanwhile, after simple calculations, we can obtain the following inclusion relations

$$\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d) \subset \mathfrak{L}_{\vec{p}}(\mathbb{R}^d) \subset L_1(\mathbb{R}^d).$$

In order to establish our important generalizations and results in the next section, we should give some extensions of known results in this section. First of all, we recall a fundamental lemma ([17], Theorem 6.18) as follows.

Lemma 1. *Let $1 \leq p \leq \infty$, $c \in l_p(\mathbb{Z})$ and $h \in l_1(\mathbb{Z})$. Then $\|\sum_k c(k)h(l-k)\|_{l_p} \leq \|c\|_{l_p}\|h\|_{l_1}$.*

With the help of Lemma 1, we establish the following new inequality.

Lemma 2. *Assume $1 \leq p \leq \infty$, $c \in l_p(\mathbb{Z})$ and $\varphi \in \mathfrak{L}_p(\mathbb{R})$. Then*

$$\left\| \sum_k c_k \varphi(x-k) \right\|_{L_p(\mathbb{R})} \leq \|c\|_{l_p} \|\varphi\|_{\mathfrak{L}_p(\mathbb{R})}.$$

Proof. Let $1 \leq p < \infty$. Then, we get

$$\begin{aligned} \left\| \sum_k c_k \varphi(x-k) \right\|^p_{L_p(\mathbb{R})} &= \int_{\mathbb{R}} \left[|\sum_k c_k \varphi(x-k)| \right]^p dx \\ &\leq \int_{\mathbb{R}} \left[\sum_k |c_k||\varphi(x-k)| \right]^p dx \\ &= \int_{[0,1]} \sum_{l \in \mathbb{Z}} \left[\sum_k |c_k||\varphi(x-l-k)| \right]^p dx \\ &\leq \int_{[0,1]} \|(c_k)_k\|^p_{l_p} \left[\sum_k |\varphi(x-k)| \right]^p dx \\ &= \|(c_k)_k\|^p_{l_p} \int_{[0,1]} \left[\sum_k |\varphi(x-k)| \right]^p dx, \end{aligned}$$

where the last inequality holds by using Lemma 1. Finally, one can verify the case $p = \infty$ in a very similar way as above. The proof is completed. □

By applying Lemma 2, we can give an extension to the mixed-norm case.

Theorem 1. *Assume $\vec{p} \in [1,\infty]^d$, $c \in l_{\vec{p}}(\mathbb{Z}^d)$ and $\varphi \in \mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$. Then*

$$\left\| \sum_k c_k \varphi(x-k) \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_{\vec{p}}(\mathbb{R}^d)} \|\varphi\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)}.$$

Proof. Without loss of generality, we assume $d = 2$, $\vec{p} = (p_1, p_2)$ and $\vec{p} \in [1,\infty)^2$. Then, we obtain

$$\begin{aligned}
&\left\| \sum_k c_{k_1,k_2} \varphi(x_1 - k_1, x_2 - k_2) \right\|_{L_{(p_1,p_2)}(\mathbb{R}^2)} \\
&= \left\| \left\| \textstyle\sum_{k_1} \sum_{k_2} c_{k_1,k_2} \varphi(x_1 - k_1, x_2 - k_2) \right\|_{L_{p_1}(x_1)(\mathbb{R})} \right\|_{L_{p_2}(x_2)(\mathbb{R})} \\
&\leq \left(\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \left[\textstyle\sum_{k_2} \sum_{k_1} |c_{k_1,k_2}| |\varphi(x_1 - k_1, x_2 - k_2)| \right]^{p_1} dx_1 \right\}^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}} \\
&\leq \left(\int_{\mathbb{R}} \left\{ \textstyle\sum_{k_2} \left(\int_{\mathbb{R}} \left[\sum_{k_1} |c_{k_1,k_2}| |\varphi(x_1 - k_1, x_2 - k_2)| \right]^{p_1} dx_1 \right)^{\frac{1}{p_1}} \right\}^{p_2} dx_2 \right)^{\frac{1}{p_2}} \\
&= \left(\int_{\mathbb{R}} \left\{ \textstyle\sum_{k_2} \left(\sum_{l_1} \int_{[0,1]} \left[\sum_{k_1} |c_{k_1,k_2}| |\varphi(x_1 - k_1 - l_1, x_2 - k_2)| \right]^{p_1} dx_1 \right)^{\frac{1}{p_1}} \right\}^{p_2} dx_2 \right)^{\frac{1}{p_2}},
\end{aligned}$$

where the second inequality holds by using the triangle inequality on $L_{p_1}(x_1)$. Now, with the help of Lemma 2, we have

$$\begin{aligned}
&\left\| \sum_k c_{k_1,k_2} \varphi(x_1 - k_1, x_2 - k_2) \right\|_{L_{(p_1,p_2)}(\mathbb{R}^2)} \\
&\leq \left(\int_{\mathbb{R}} \left\{ \sum_{k_2} \|(c_{k_1,k_2})\|_{l_{p_1}(k_1)} \|\varphi(x_1, x_2 - k_2)\|_{\mathfrak{L}_{p_1}(x_1)} \right\}^{p_2} dx_2 \right)^{\frac{1}{p_2}} \\
&\leq \left\| \|(c_{k_1,k_2})\|_{l_{p_1}(k_1)} \right\|_{l_{p_2}(k_2)} \left\| \|\varphi(x_1, x_2)\|_{\mathfrak{L}_{p_1}(x_1)} \right\|_{\mathfrak{L}_{p_2}(x_2)} \\
&= \|c\|_{l_{\vec{p}}(\mathbb{R}^2)} \|\varphi\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^2)}.
\end{aligned}$$

A similar argument could be made for $p_1 = \infty$ or $p_2 = \infty$. The proof is completed. □

In order to prove the property of Theorem 6 in Scetion 3, we need the classical Hölder inequality. In fact, Theorem 6 is very similar to the combination of Hölder inequality and convolution inequality (see below). Now, we give these two classical inequalities here.

Lemma 3 (Hölder inequality, [17])**.** *Let $f(x) \in L_p(\mathbb{R})$, $g(x) \in L_q(\mathbb{R})$ for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f(x)g(x)\|_{L_1(\mathbb{R})} \leq \|f(x)\|_{L_p(\mathbb{R})} \|g(x)\|_{L_q(\mathbb{R})}.$$

Lemma 4 (Convolution inequality, [11])**.** *Let $f(x) \in L_p(\mathbb{R}^d)$, $g(x) \in L_1(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. Then*

$$\|(f * g)(x)\|_{L_p(\mathbb{R}^d)} \leq \|f(x)\|_{L_p(\mathbb{R}^d)} \|g(x)\|_{L_1(\mathbb{R}^d)}.$$

The convolution formula is defined by $(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$ with $f(x)$ and $g(x)$ are meaningful.

Moreover, in order to make a comparison in format with Theorem 6, we generalize the classical Hölder inequality as in Lemma 3 and extend the classical convolution inequality as in Lemma 4 to the mixed-norm cases. We give a simple proof of the mixed-norm Hölder inequality as follows.

Theorem 2 (Mixed-norm Hölder inequality). *Let $f(x) \in L_{\vec{p}}(\mathbb{R}^d)$, $g(x) \in L_{\vec{q}}(\mathbb{R}^d)$ for $\vec{p}, \vec{q} \in [1,\infty]^d$ with $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, 2, \ldots, d$. Then*

$$\|f(x)g(x)\|_{L_1(\mathbb{R}^d)} \le \|f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \|g(x)\|_{L_{\vec{q}}(\mathbb{R}^d)}.$$

Proof. Without loss of generality, we assume $d = 2$ and then $\vec{p} = (p_1, p_2)$. So

$$\begin{aligned}
\|fg\|_{L_1(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x_1, x_2) g(x_1, x_2)| dx_1 dx_2 \\
&\le \int_{\mathbb{R}} \|f(x_1, x_2)\|_{L_{p_1}(x_1)} \|g(x_1, x_2)\|_{L_{p_1}(x_1)} dx_2 \\
&\le \Big\| \|f(x_1, x_2)\|_{L_{p_1}(x_1)} \Big\|_{L_{p_2}(x_2)} \Big\| \|g(x_1, x_2)\|_{L_{p_1}(x_1)} \Big\|_{L_{p_2}(x_2)},
\end{aligned}$$

where the inequalities can be obtained by using Hölder inequality as in Lemma 3. The proof is completed. □

Next, we will generalize the classical convolution inequality as in Lemma 4 to the mixed-norm case. To prove this mixed-norm convolution inequality, we need to generalize the classical Minkowski inequality to the mixed-norm case in advance.

Lemma 5 (Minkowski inequality [12]). *Let $f(x, y)$ be a Borel function on $\mathbb{R} \times \mathbb{R}$ and $1 \le p \le \infty$. Then*

$$\left\| \int_{\mathbb{R}} f(x, y) dx \right\|_{L_p(\mathbb{R})} \le \int_{\mathbb{R}} \|f(x, \cdot)\|_{L_p(\mathbb{R})} dx.$$

By applying Lemma 5, we extend the traditional Minkowski inequality to the mixed-norm case as follows.

Theorem 3 (Mixed-norm Minkowski inequality). *Let $f(x, y)$ be a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ and $\vec{p} \in [1, \infty]^d$. Then*

$$\left\| \int_{\mathbb{R}^d} f(x, y) dx \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} \|f(x, \cdot)\|_{L_{\vec{p}}(\mathbb{R}^d)} dx.$$

Proof. Without loss of generality, we only need to show the case $d = 2$. Thus $\vec{p} = (p_1, p_2)$ and

$$\left\| \int_{\mathbb{R}^d} f(x, y) dx \right\|_{L_{\vec{p}}(\mathbb{R}^2)} = \left\| \int_{\mathbb{R}^2} f(x, y) dx \right\|_{L_{(p_1, p_2)}} = \left\| \left(\left\| \int_{\mathbb{R}^2} f(x, y) dx \right\|_{L_{p_1}(y_1)} \right) \right\|_{L_{p_2}(y_2)}.$$

By Lemma 5,

$$\begin{aligned}
\left\| \left(\left\| \int_{\mathbb{R}^2} f(x, y) dx \right\|_{L_{p_1}(y_1)} \right) \right\|_{L_{p_2}(y_2)} &\le \left\| \left(\int_{\mathbb{R}^2} \|f(x, y_1, y_2)\|_{L_{p_1}(y_1)} dx \right) \right\|_{L_{p_2}(y_2)} \\
&\le \int_{\mathbb{R}^2} \left\| \left(\|f(x, y_1, y_2)\|_{L_{p_1}(y_1)} \right) \right\|_{L_{p_2}(y_2)} dx.
\end{aligned}$$

This completes the proof. □

By applying Theorem 3, we establish the following new generalized convolution inequality in mixed-norm Lebesgue spaces.

Theorem 4 (Mixed-norm convolution inequality)**.** *Let $f(x) \in L_{\vec{p}}(\mathbb{R}^d)$, $g(x) \in L_1(\mathbb{R}^d)$ for $\vec{p} \in [1,\infty]^d$. Then*

$$\|(f*g)(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \|f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\|g(x)\|_{L_1(\mathbb{R}^d)}.$$

Proof. Without loss of generality, we assume $d = 2$. So $\vec{p} = (p_1, p_2)$ and

$$\begin{aligned}\|f*g\|_{L_{(p_1,p_2)}} &= \left\|\int_{\mathbb{R}}\int_{\mathbb{R}} f(x_1-y_1, x_2-y_2)g(y_1,y_2)dy_1dy_2\right\|_{L_{(p_1,p_2)}} \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \|f(x_1-y_1,x_2-y_2)\|_{L_{(p_1,p_2)}(x_1,x_2)}|g(y_1,y_2)|dy_1dy_2 \\ &= \|f\|_{L_{(p_1,p_2)}}\|g\|_1.\end{aligned}$$

The inequality comes from Theorem 3 and the conclusion is proved. □

3. New Convolution-Type Inequality and Stability Theorem in Shift-Invariant Subspaces of $L_{\vec{p}}(\mathbb{R}^d)$

Based on the careful preparation in Section 2, some important new inequalities and results are established in this section, which are almost based on the setting of $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$. We expect that these inequalities will contribute to the characterization and application of mixed-norm Besov spaces and Triebel–Lizorkin spaces, and so forth. We first give a generalization of Theorem 1 as follows.

Theorem 5. *Assume $c \in l_1(\mathbb{Z}^d)$ and $\varphi(x) \in \mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)$. Then*

$$\left\|\sum_k c_k\varphi(x-k)\right\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} \le \|c\|_{l_1}\|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}.$$

Proof. Here, we also assume $d = 2$. Then, one has

$$\begin{aligned}&\left\|\sum_k c_k\varphi(x-k)\right\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^2)} \\ &= \operatorname{esssup}_{x_2\in[0,1]}\sum_{l_2\in\mathbb{Z}}\operatorname{esssup}_{x_1\in[0,1]}\sum_{l_1\in\mathbb{Z}}\left|\sum_{(k_1,k_2)} c_{(k_1,k_2)}\varphi(x_1-k_1-l_1, x_2-k_2-l_2)\right| \\ &\le \operatorname{esssup}_{x_2\in[0,1]}\sum_{l_2\in\mathbb{Z}}\operatorname{esssup}_{x_1\in[0,1]}\sum_{l_1\in\mathbb{Z}}\sum_{(k_1,k_2)}|c_{(k_1,k_2)}||\varphi(x_1-k_1-l_1, x_2-k_2-l_2)| \\ &= \operatorname{esssup}_{x_2\in[0,1]}\sum_{l_2\in\mathbb{Z}}\operatorname{esssup}_{x_1\in[0,1]}\sum_{(k_1,k_2)}|c_{(k_1,k_2)}|\sum_{l_1\in\mathbb{Z}}|\varphi(x_1-k_1-l_1, x_2-k_2-l_2)|.\end{aligned}$$

So, it is easy to see that

$$\begin{aligned}&\left\|\sum_k c_k\varphi(x-k)\right\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^2)} \\ &\le \operatorname{esssup}_{x_2\in[0,1]}\sum_{l_2\in\mathbb{Z}}\sum_{(k_1,k_2)}|c_{(k_1,k_2)}|\operatorname{esssup}_{x_1\in[0,1]}\sum_{l_1\in\mathbb{Z}}|\varphi(x_1-k_1-l_1, x_2-k_2-l_2)| \\ &\le \sum_{(k_1,k_2)}|c_{(k_1,k_2)}|\operatorname{esssup}_{x_2\in[0,1]}\sum_{l_2\in\mathbb{Z}}\operatorname{esssup}_{x_1\in[0,1]}\sum_{l_1\in\mathbb{Z}}|\varphi(x_1-k_1-l_1, x_2-k_2-l_2)| \\ &= \|c\|_{l_1}\|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^2)}.\end{aligned}$$

The proof is completed. □

It is worth noting here that

$$\|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} \leq \|f\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)},$$

so we have

$$\left\|\sum_k c_k \varphi(x-k)\right\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} \leq \|c\|_{l_1} \|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}.$$

Next, we will establish the following new mixed-norm convolution-type inequality based on $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$.

Theorem 6. *Let $f(x) \in L_{\vec{p}}(\mathbb{R}^d)$ and $g(x) \in L_{\vec{q}}(\mathbb{R}^d)$ for $\vec{p}, \vec{q} \in [1,\infty]^d$. Then*

$$\|f * g\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} \leq \|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)} \|g\|_{\mathfrak{L}_{\vec{q}}(\mathbb{R}^d)}.$$

Proof. Without loss of generality, we assume $d = 2$. First, we note that

$$\begin{aligned} & f * g(x_1 + k_1, x_2 + k_2) \\ = & \int_{\mathbb{R}} \int_{\mathbb{R}} f(y_1, y_2) g(x_1 + k_1 - y_1, x_2 + k2 - y_2) dy_1 dy_2. \end{aligned}$$

So, we obtain

$$\begin{aligned} & \operatorname{esssup}_{x_1 \in [0,1]} \sum_{k_1 \in \mathbb{Z}} |f * g(x_1 + k_1, x_2 + k_2)| \\ \leq & \operatorname{esssup}_{x_1 \in [0,1]} \textstyle\sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \int_{[0,1]} \sum_{l_1} |f(y_1 - l_1, y_2)| |g(x_1 - l_1 + k_1 - y_1, x_2 + k_2 - y_2)| dy_1 dy_2 \\ = & \operatorname{esssup}_{x_1 \in [0,1]} \textstyle\int_{\mathbb{R}} \int_{[0,1]} \sum_{l_1} |f(y_1 - l_1, y_2)| \sum_{k_1 \in \mathbb{Z}} |g(x_1 - l_1 + k_1 - y_1, x_2 + k_2 - y_2)| dy_1 dy_2 \\ = & \operatorname{esssup}_{x_1 \in [0,1]} \textstyle\int_{\mathbb{R}} \int_{[0,1]} \left(\sum_{l_1} |f(y_1 - l_1, y_2)|\right) \left(\sum_{k_1 \in \mathbb{Z}} |g(x_1 + k_1 - y_1, x_2 + k_2 - y_2)|\right) dy_1 dy_2, \end{aligned}$$

which leads to

$$\begin{aligned} & \operatorname{esssup}_{x_1 \in [0,1]} \sum_{k_1 \in \mathbb{Z}} |f * g(x_1 + k_1, x_2 + k_2)| \\ \leq & \operatorname{esssup}_{x_1 \in [0,1]} \textstyle\int_{\mathbb{R}} \Big(\left\| \left(\sum_{l_1} |f(y_1 - l_1, y_2)|\right) \right\|_{L_{p_1}(y_1)[0,1]} \\ & \qquad \left\| \left(\sum_{k_1 \in \mathbb{Z}} |g(x_1 + k_1 - y_1, x_2 + k2 - y_2)|\right) \right\|_{L_{q_1}(y_1)[0,1]} \Big) dy_2 \\ \leq & \textstyle\int_{\mathbb{R}} \|f(y_1, y_2)\|_{\mathfrak{L}_{p_1}(y_1)} \|g(y_1, x_2 + k_2 - y_2)\|_{\mathfrak{L}_{q_1}(y_1)} dy_2, \end{aligned}$$

where the first inequality holds by using the Hölder inequality as in Lemma 3. Similarly, we have

$$\begin{aligned} & \operatorname{esssup}_{x_2 \in [0,1]} \sum_{k_2 \in \mathbb{Z}} \operatorname{esssup}_{x_1 \in [0,1]} \sum_{k_1 \in \mathbb{Z}} |f * g(x_1 + k_1, x_2 + k_2)| \\ \leq & \operatorname{esssup}_{x_2 \in [0,1]} \textstyle\sum_{k_2 \in \mathbb{Z}} \int_{\mathbb{R}} \|f(y_1, y_2)\|_{\mathfrak{L}_{p_1}(y_1)} \|g(y_1, x_2 + k_2 - y_2)\|_{\mathfrak{L}_{q_1}(y_1)} dy_2 \\ = & \operatorname{esssup}_{x_2 \in [0,1]} \textstyle\int_{\mathbb{R}} \|f(y_1, y_2)\|_{\mathfrak{L}_{p_1}(y_1)} \sum_{k_2 \in \mathbb{Z}} \|g(y_1, x_2 + k_2 - y_2)\|_{\mathfrak{L}_{q_1}(y_1)} dy_2 \\ = & \operatorname{esssup}_{x_2 \in [0,1]} \textstyle\int_{[0,1]} \sum_{l_2 \in \mathbb{Z}} \Big(\|f(y_1, y_2 - l_2)\|_{\mathfrak{L}_{p_1}(y_1)} \sum_{k_2 \in \mathbb{Z}} \|g(y_1, x_2 + k_2 - y_2 - l_2)\|_{\mathfrak{L}_{q_1}(y_1)} \Big) dy_2, \end{aligned}$$

which deduces that

$$
\begin{aligned}
&\operatorname{esssup}_{x_2\in[0,1]}\sum_{k_2\in\mathbb{Z}}\operatorname{esssup}_{x_1\in[0,1]}\sum_{k_1\in\mathbb{Z}}|f*g(x_1+k_1,x_2+k_2)| \\
&\le \operatorname{esssup}_{x_2\in[0,1]}\int_{[0,1]}\Big(\textstyle\sum_{l_2\in\mathbb{Z}}\|f(y_1,y_2-l_2)\|_{\mathfrak{L}_{p_1}(y_1)}\Big)\Big(\textstyle\sum_{k_2\in\mathbb{Z}}\|g(y_1,x_2+k_2-y_2)\|_{\mathfrak{L}_{q_1}(y_1)}\Big)dy_2 \\
&\le \operatorname{esssup}_{x_2\in[0,1]}\Big\|\Big(\textstyle\sum_{l_2\in\mathbb{Z}}\|f(y_1,y_2-l_2)\|_{\mathfrak{L}_{p_1}(y_1)}\Big)\Big\|_{L_{p_2}(y_2)([0,1])} \\
&\qquad\qquad \Big\|\Big(\textstyle\sum_{k_2\in\mathbb{Z}}\|g(y_1,x_2+k_2-y_2)\|_{\mathfrak{L}_{q_1}(y_1)}\Big)\Big\|_{L_{q_2}(y_2)([0,1])} \\
&\le \Big\|\|f(y_1,y_2)\|_{\mathfrak{L}_{p_1}(y_1)}\Big\|_{\mathfrak{L}_{p_2}(y_2)([0,1])}\Big\|\|g(y_1,y_2)\|_{\mathfrak{L}_{q_1}(y_1)}\Big\|_{\mathfrak{L}_{p_2}(y_2)([0,1])} \\
&= \|f\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^2)}\|g\|_{\mathfrak{L}_{\vec{q}}(\mathbb{R}^2)}.
\end{aligned}
$$

The proof is completed. □

Remark 1. *It is very obvious that Theorem 6 is a unified format of the mixed-norm Hölder inequality (i.e., Theorem 2) and the mixed-norm convolution inequality (i.e., Theorem 4).*

We will end this section with an important stability theorem in mixed-norm Lebesgue spaces. Note that this stability conclusion is based on a setting of the shift-invariant subspace of $L_{\vec{p}}(\mathbb{R}^d)$ space. Of course, the upper and lower bounds of the stability theorem are given in the setting of $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$. The shift-invariant subspace $\mathrm{Span}(\varphi)$ (generated by φ) is defined by

$$
\mathrm{Span}(\varphi)=\left\{f=\sum_k c_k\varphi(x-k) : c\in l_{\vec{p}}(\mathbb{Z}^d),\varphi\in\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)\right\}.
$$

The following known result is crucial for proving our new stability theorem.

Theorem 7 (see [13])**.** *Let $\varphi\in\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)$ and C_1 and C_2 be positive constants. Then*

$$
C_1\|c\|_{l_2}\le\left\|\sum_k c_k\varphi(x-k)\right\|_{L_2(\mathbb{R}^d)}\le C_2\|c\|_{l_2}
$$

holds if and only if one of the following conditions holds:

(1) There exists a function $h\in \mathrm{Span}(\varphi)$ such that

$$
\langle\varphi(x-l),h(x)\rangle=\delta_{0,l},\ l\in\mathbb{Z}^d,
$$

and $\delta_{0,l}$ is defined by $\delta=1$ for $l=0$, $\delta=0$ for $l\neq0$;

(2) $\sum_l|\hat{\varphi}(\xi+2\pi l)|^2>0$ for every $\xi\in\mathbb{R}^d$.

By applying Theorems 1 and 2, we prove the following stability theorem.

Theorem 8. *Let $\vec{p}\in[1,\infty]^d$, $\sum_l|\hat{\varphi}(\xi+2\pi l)|^2>0$ for every $\xi\in\mathbb{R}^d$, $c\in l_{\vec{p}}(\mathbb{Z}^d)$ and $\varphi\in\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)$. Then*

$$
C_1\|c\|_{l_{\vec{p}}}\le\left\|\sum_k c_k\varphi(x-k)\right\|_{L_{\vec{p}}(\mathbb{R}^d)}\le C_2\|c\|_{l_{\vec{p}}}
$$

holds for some positive constants C_1 and C_2.

Proof. By Theorem 1, one has

$$\left\|\sum_k c_k\varphi(x-k)\right\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \|\varphi\|_{\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)}\|c\|_{l_{\vec{p}}} \le \|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}\|c\|_{l_{\vec{p}}}.$$

That means $C_2 = \|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}$. So the upper bound is arrived.

Next, we will find its lower bound. Assume $g = \sum_k c_k\varphi(x-k)$, by Theorem 7, there exists a function $h \in \mathrm{Span}(\varphi)$ such that

$$\langle \varphi(x-l), h(x)\rangle = \delta_{0,l},\ l \in \mathbb{Z}^d.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^d} g(x)\overline{h(x-k)}dx &= \int_{\mathbb{R}^d}\left(\sum_m c_m\varphi(x-m)\right)\overline{h(x-k)}dx\\ &= \sum_m c_m\int_{\mathbb{R}^d}\varphi(x-m+k)\overline{h(x)}dx\\ &= c_k.\end{aligned}$$

Let $\{(\tilde{c}_k)_k\} \in l_{\vec{q}}$ for $\frac{1}{p_i}+\frac{1}{q_i}=1$ with $i=1,2,\ldots,d$. So we have

$$\begin{aligned}|\langle c,\tilde{c}\rangle| &= \left|\sum_k c_k\overline{\tilde{c}_k}\right|\\ &= \left|\sum_k \overline{\tilde{c}_k}\int_{\mathbb{R}^d} g(x)\overline{h(x-k)}dx\right|\\ &= \left|\int_{\mathbb{R}^d} g(x)\sum_k \overline{\tilde{c}_k h(x-k)}dx\right|.\end{aligned}$$

Taking into account Theorems 1 and 2, we get

$$\begin{aligned}|\langle c,\tilde{c}\rangle| &\le \|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\left\|\sum_k \overline{\tilde{c}_k h(x-k)}\right\|_{L_{\vec{q}}(\mathbb{R}^d)}\\ &\le \|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\|\tilde{c}\|_{l_{\vec{q}}}\|h(x)\|_{\mathfrak{L}_{\vec{q}}(\mathbb{R}^d)},\end{aligned}$$

which deduces

$$\|c\|_{l_{\vec{p}}} \le \|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\|h(x)\|_{\mathfrak{L}_{\vec{q}}(\mathbb{R}^d)} \le \|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\|h(x)\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}.$$

Then,

$$\frac{1}{\|h(x)\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}}\|c\|_{l_{\vec{p}}} \le \|g(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

We now take $C_1 = \frac{1}{\|h(x)\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)}}$ and the lower bound is arrived. □

Remark 2. *As an example, we can take*

$$\varphi(x_1,x_2,\ldots,x_d) = \phi_1(x_1)\otimes\phi_2(x_2)\otimes\cdots\otimes\phi_d(x_d).$$

Here, $\otimes$ means the traditional tensor product. The functions ϕ_i $(i=1,2,\ldots,d)$ can be taken as the orthonormal or biothogonal scaling functions with compact support in wavelet analysis theories. In fact, we can easily prove these scaling functions naturally satisfy the inequalities $\|\varphi\|_{\mathfrak{L}_{\vec{\infty}}(\mathbb{R}^d)} < \infty$ and $\sum_l |\hat{\varphi}(\xi+2\pi l)|^2 > 0$ for every $\xi \in \mathbb{R}^d$.

4. Conclusions

The main goal of the current study is to give some generalizations of inequalities under mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$ with the help of $\mathfrak{L}_{\vec{p}}(\mathbb{R}^d)$. We particularly establish the convolution-Hölder-type inequality and the stability theorem of mixed-norm case. It is obvious that the convolution-Hölder-type inequality unifies the mixed-norm Hölder inequality and the mixed-norm convolution inequality in format. This is an interesting result. In addition, a generalization of stability result under mixed-norm unifies and refines the existing results. We hope that it can be used in the characterization of relevant mixed norm spaces in the future.

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