



Article Diffusion-Advection Equations on a Comb: Resetting and Random Search

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Abstract: This review addresses issues of various drift–diffusion and inhomogeneous advection problems with and without resetting on comblike structures. Both a Brownian diffusion search with drift and an inhomogeneous advection search on the comb structures are analyzed. The analytical results are verified by numerical simulations in terms of coupled Langevin equations for the comb structure. The subordination approach is one of the main technical methods used here, and we demonstrated how it can be effective in the study of various random search problems with and without resetting.

Keywords: diffusion–advection equation; stochastic resetting; comb structure; random search; first arrival time density; efficiency

1. Introduction

In the standard theory of Brownian motion, the probability density function (PDF) $P_0(x, t)$ for finding a particle at position x at time t has a Gaussian form,

$$P_0(x,t) = \frac{1}{\sqrt{4\pi\mathcal{D}t}}e^{-\frac{(x-x_0)^2}{4\mathcal{D}t}},$$

for the initial conditions given by the Dirac delta function $P_0(x, t = 0) = \delta(x - x_0)$ and for the natural (vanishing) boundary conditions at infinity, $P_0(\pm \infty, t) = 0$ and $\frac{\partial}{\partial x}P(x,t)|_{x=\pm\infty} = 0$. Brownian motion is characterized by linear growth of the mean squared displacement (MSD), $\langle x^2(t) \rangle = 2Dt$, which means normal diffusion. However, in a large variety of transport phenomena in randomly inhomogeneous media, one observes deviation from this linear growth in time such that the MSD has a power-law dependence on time, $\langle x^2(t) \rangle \sim t^{\alpha}$, which is a signature of anomalous diffusion (see, for example, [1]). When $0 < \alpha < 1$, it corresponds to subdiffusion, while $\alpha > 1$ corresponds to superdiffusion.

One of the well-known examples of anomalous diffusion is Brownian motion on a comb, which is governed by the Fokker–Planck equation [2,3]

$$\partial_t P(x, y, t) = \delta(y) \mathcal{D}_x \, \partial_x^2 P(x, y, t) + \mathcal{D}_y \, \partial_y^2 P(x, y, t), \tag{1}$$

with the initial condition $P(x, y, t = 0) = \delta(x - x_0)\delta(y)$, and the boundary conditions for P(x, y, t) and $\frac{\partial}{\partial a}P(x, y, t)$, $q = \{x, y\}$, are set to zero at infinity, $x = \pm \infty$, $y = \pm \infty$. Here,



Citation: Sandev, T.; Domazetoski, V.; Iomin, A.; Kocarev, L. Diffusion–Advection Equations on a Comb: Resetting and Random Search. *Mathematics* **2021**, *9*, 221. https://doi.org/10.3390/math9030221

Academic Editor: Andrea Scapellato Received: 25 December 2020 Accepted: 20 January 2021 Published: 22 January 2021

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Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/). $D_x \delta(y)$ and D_y are diffusion coefficients along the *x* and *y* directions, respectively. The δ -function in the Fokker–Planck operator

$$L_{\rm FP} = \delta(y)\mathcal{D}_x\,\partial_x^2 + \mathcal{D}_y\,\partial_y^2 \tag{2}$$

means that diffusion along the *x* direction (the so-called backbone) is allowed only at y = 0. Along the *y* direction (the so-called fingers), the particle performs normal diffusion. The MSD along the backbone has a power-law dependence on time $\langle x^2(t) \rangle = \frac{D_x}{\sqrt{D_y}} \frac{t^{1/2}}{\Gamma(3/2)}$, i.e.,

subdiffusion is observed in the system [2,3], while normal diffusion takes place along the fingers. Different generalizations of the comb geometry have been considered. For example, various diffusion processes have been considered in a comb with a finite finger length [4-6], diffusion on cylindrical [7,8] and circular combs [9-11], more complex branched structures [12], random comb models [13], and a comb with ramified teeth [14], as well as the problem of first encounters for two workers [15]. Diffusion processes in fractal mesh and grid structures have been considered as well: In this case, anomalous diffusion of a particle is affected by the fractal structure of the infinite numbers of backbones and fingers [16]. It has been shown that these models are useful for description of anomalous transport through porous solid pellets with various porous geometries [17]. Comb models are also applicable for describing diffusion in percolation clusters [2,18,19], anomalous transport of inert compounds in spiny dendrites [20–22], modeling electron transport in disordered nanostructured semiconductors [23,24], dispersive transport of charge carriers in two-layer polymers [25], percolative phonon-assisted hopping in twodimensional disordered systems [26,27], and anomalous diffusion of fluorescence recovery after photobleaching in a random-comb model [13]. Another interesting realization is that turbulent diffusion in a comb appears to be due to multiplicative noise [28,29].

Nowadays, one of the most explored problems in stochastic processes is the problem of stochastic resetting, meaning that a particle is reset to the initial (or any other) position from time to time. The one-dimensional Brownian motion with Markovian resetting with a constant resetting rate r was introduced by Evans and Majumdar [30]. It was shown that the solution for the PDF approaches a non-equilibrium steady state and, in the long-time limit, its MSD is saturated, $\langle x^2(t) \rangle \sim 1/r$ (also see the review paper [31] for more details). Moreover, Brownian motion in a two-dimensional comb in the presence of stochastic (Markovian) resetting can be solved analytically [32–34]. The marginal PDFs along both the backbone and fingers approach non-equilibrium steady states, and the MSDs are saturated according to the resetting rate: $\langle x^2(t) \rangle \sim 1/\sqrt{r}$ and $\langle y^2(t) \rangle \sim 1/r$ [32–34]. These models have been extended to diffusion processes with non-static resetting [34]. Stochastic resetting is a natural mechanism in various search processes, such as foraging [35], population dynamics [36], Michaelis–Menten enzymatic reactions [37], and human behavior of finding resources [38], to mention but a few. Resetting may also affect the first-passage properties and completion of the process. In particular, in the case of the one-dimensional Brownian search, the mean arrival time at the absorbing boundary becomes finite in the presence of resetting [30], while it is infinite in the absence of resetting [39]. The resetting dynamics of a Brownian particle under external potentials have been analyzed in detail, as well [40–45]. This issue can also be employed to understand resetting in molecular reaction systems.

Another important topic in stochastic processes is the random search problem. Many studies on random searches in foraging theory with incomplete information have employed a Brownian search as a default strategy [46], while others have proposed Lévy flights as an efficient strategy for searching for sufficiently sparse targets [47], stating that the Lévy process is one of the most natural and optimal search strategies [48–52]. Various search strategies have been introduced and proposed, including different combinations of search processes [53–57].

The corresponding Fokker–Planck equation of a Brownian random search process for the non-normalized density function f(x, t) with a δ -sink of strength \mathcal{P}_{fa} reads [53–55]

$$\partial_t f(x,t) = \mathcal{D}\partial_x^2 f(x,t) - \mathcal{P}_{fa}(t)\delta(x-X), \tag{3}$$

where \mathcal{D} is a diffusion coefficient. One assumes here that the initial position is given at $x = x_0$ by $f(x, t = 0) = \delta(x - x_0)$. The δ -sink means that the random searcher positioned at the beginning at $x = x_0$ will be removed at the first arrival at x = X, i.e., f(x = X, t) = 0. Therefore, $\mathcal{P}_{fa}(t)$ represents the first arrival time distribution (FATD) [53–55], which is obtained from Equation (3):

$$\mathcal{P}_{fa}(t) = -\frac{d}{dt} \int_{-\infty}^{\infty} f(x,t) \, dx = -\frac{d}{dt} \mathcal{S}(t), \tag{4}$$

which is a negative time derivative of the survival probability $S(t) = \int_{-\infty}^{\infty} f(x, t) dx$. The FATD for the Brownian search is described by the Lévy–Smirnov density:

$$\mathcal{P}_{fa}(t) = \frac{|X - x_0|}{\sqrt{4\pi D t^3}} \times e^{-\frac{(X - x_0)^2}{4D t}}$$
(5)

with the long-time asymptotics, $\mathcal{P}_{fa}(t) \sim |X - x_0|t^{-3/2}$. Other important characteristics of searching are the search reliability and the efficiency. The search reliability is considered as the cumulative arrival probability [54]:

$$\mathcal{P} = \int_0^\infty \mathcal{P}_{fa}(t) \, dt = \mathcal{P}_{fa}(s=0),\tag{6}$$

which, for the Brownian search, is $\mathcal{P} = 1$ (the searcher will find the target with the probability of one), while the search efficiency [54],

$$\mathcal{E} = \left\langle \frac{1}{t} \right\rangle = \int_0^\infty \mathcal{P}_{fa}(s) \, ds,\tag{7}$$

represents the averaged inverse search time. Here, $\mathcal{P}_{fa}(s) = \mathcal{L}[\mathcal{P}_{fa}(t)](s) = \int_0^\infty \wp_{fa}(t)e^{-st} dt$ is the Laplace image of $\mathcal{P}_{fa}(t)$. For a one-dimensional Brownian search, it is given by [54]:

$$\mathcal{E} = \frac{2\mathcal{D}}{(X - x_0)^2}.$$
(8)

In a similar way, a Brownian random search on a comblike structure has also been considered. In this case, the initial position of the searcher is located at the backbone at $(x, y) = (x_0, 0)$, and the target is also located at the backbone at (x, y) = (X, 0), with a δ -sink $\delta(x - X)\delta(y)$ of the strength $\wp_{fa}(t)$ [58]. The FATD is given in terms of the Fox *H*-function with the long-time asymptotics reducing to the power law, $\wp_{fa}(t) \sim |X - x_0|t^{-5/4}$, while the search reliability equals one, and the efficiency becomes [58]:

$$\mathcal{E} = \frac{24 \left(\frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}\right)^2}{(X - x_0)^4}.$$
(9)

The inhomogeneous advection on the comb, where the motion along the backbone is interrupted by Brownian motion in the fingers, can be described by the Fokker–Planck operator

$$L_{\rm FP} = -v\,\delta(y)\,\partial_x|x| + \mathcal{D}_y\,\partial_y^2.\tag{10}$$

It results in turbulent diffusion, which is characterized by the log-normal distribution and exponential growth of the MSD in time [29]. This behavior is analogous to one-dimensional

geometric Brownian motion [29], which is used in the Black–Scholes model for option pricing [59,60]. The FATD is the Lévy–Smirnov distribution, and the process is suitable for searching for long-distance targets. Turbulent diffusion occurs due to a multiplicative noise, in contrast to the additive noise in Brownian diffusion with a drift. The impact of the resetting mechanism on turbulent diffusion is one of the main issues in this paper.

This paper is organized as follows. In Section 2, the one-dimensional Brownian motion in the presence of a drift and resetting to the initial position of the particle is considered. As the main characteristics of the process, the PDF and the MSD are obtained. The analytical treatments of the corresponding Brownian search problem with the drift in the cases of a single target and two targets are presented. The main features of a subordination approach as the analytical tool used throughout the analysis—are described as well. The analytical results obtained for the FATD, the search reliability, and the efficiency are verified by numerical simulations. Brownian motion with drift in the presence of stochastic resetting on a two-dimensional comb is investigated in Section 3. Both analytical and numerical results are presented. The problem of a Brownian search with a drift on a two-dimensional comb in the cases of a single target and two targets is analyzed in detail. Section 4 is devoted to inhomogeneous advection with stochastic resetting on the comb. It is shown that three different scenarios for the MSD, depending on the resetting parameter, can be observed. These are: (i) exponential growth of the MSD in time, (ii) linear growth of the MSD in time, and (iii) saturation of the MSD. The results for the FATD, the search reliability, and the efficiency are presented as well. A generalization of the inhomogeneous advection search is also considered. A summary of the analysis is provided in Section 5.

2. One-Dimensional Brownian Motion with Drift

In this section, we consider resetting and search problems in the framework of a one-dimensional diffusion process with a drift. We show that this "simple" addition of a drift term in the corresponding equations leads to new physical effects, which are also based on well-known results of the one-dimensional diffusion–advection equation.

Therefore, to set the stage for the clear presentation of the analysis, we first offer a short overview of the results related to the one-dimensional diffusion–advection equation without stochastic resetting, which will be used later to find the corresponding results for more general problems. The corresponding Fokker–Planck equation with a constant velocity V reads [1]

$$\partial_t P_0(x,t) = \left[\mathcal{D} \, \partial_x^2 - V \, \partial_x \right] P_0(x,t). \tag{11}$$

The initial condition $P_0(x, t = 0) = \delta(x - x_0)$ and vanishing boundary conditions at infinity, $P_0(\pm \infty, t) = 0$ and $\frac{\partial}{\partial x} P_0(\pm \infty, t) = 0$, are imposed. In Laplace space, it reads

$$sP_0(x,s) - \delta(x-x_0) = \left[\mathcal{D}\,\partial_x^2 - V\,\partial_x\right]P_0(x,s). \tag{12}$$

The solution of Equation (11) is the Galilei-shifted Gaussian (see, for example, Ref. [1]),

$$P_0(x,t) = \frac{1}{\sqrt{4\pi\mathcal{D}t}} e^{-\frac{(x-x_0-Vt)^2}{4\mathcal{D}t}},$$
(13)

and in the following, we shall need its Laplace image, which reads

$$P_0(x,s) = \frac{1}{2\mathcal{D}} \frac{1}{\sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}} e^{\frac{V}{2\mathcal{D}}(x-x_0) - \sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}|x-x_0|}}.$$
 (14)

From here, we find that the PDF is normalized, since

$$\langle x^{0}(t) \rangle_{0} = \int_{-\infty}^{\infty} P_{0}(x,t) \, dx = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_{0}-Vt)^{2}}{4Dt}} \, dx = \frac{2}{\sqrt{4\pi Dt}} \int_{0}^{\infty} e^{-\frac{y^{2}}{4Dt}} \, dy = 1, \tag{15}$$

and, respectively,

$$\langle x^0(s) \rangle_0 = \int_{-\infty}^{\infty} P_0(x,s) \, dx = \frac{1}{s}.$$
 (16)

The MSD is given by

$$\langle x^2(s) \rangle_0 = \int_{-\infty}^{\infty} x^2 P_0(x,s) \, dx = \frac{x_0^2}{s} + \frac{2(\mathcal{D} + x_0 V)}{s^2} + \frac{2V^2}{s^3},\tag{17}$$

$$\langle x^2(t) \rangle_0 = \int_{-\infty}^{\infty} x^2 P_0(x,t) \, dx = 2\mathcal{D}t + (x_0 + Vt)^2,$$
 (18)

which means that the short-time diffusive behavior ($\langle x^2(t) \rangle_0 \sim t$) turns into ballistic motion in the long-time limit $\langle x^2(t) \rangle_0 \sim t^2$.

2.1. One-Dimensional Diffusion-Advection Equation with Stochastic Resetting

Now, we consider the diffusion-advection equation in the presence of stochastic resetting. The Fokker–Planck equation reads

$$\partial_t P_r(x,t) = \left[\mathcal{D}_x \,\partial_x^2 - V \,\partial_x \right] P_r(x,t) - r P_r(x,t) + r \delta(x-x_0) \tag{19}$$

with the same initial condition $P_r(x, t = 0) = \delta(x - x_0)$. Here, *r* is the rate of resetting to the initial position x_0 . The last two terms of the equation represent the loss of the probability from position *x* due to the reset to the initial position and the gain of the probability at x_0 due to resetting from all other positions, respectively. This equation means that between any two consecutive resetting events, the particle undergoes diffusion with a constant drift.

From the Laplace transform of Equation (19), one finds

$$sP_r(x,s) - \delta(x-x_0) = \frac{s}{s+r} \Big[\mathcal{D} \,\partial_x^2 - V \,\partial_x \Big] P_r(x,s). \tag{20}$$

Then, the inverse Laplace transform yields Equation (19) in the equivalent form

$$\partial_t P_r(x,t) = \frac{d}{dt} \int_0^t \eta(t-t') \Big[\mathcal{D} \,\partial_x^2 - V \,\partial_x \Big] P_r(x,t') \,dt', \tag{21}$$

where $\eta(t) = e^{-rt}$ and $\eta(s) = \frac{1}{s+r}$.

This equation can be solved by using a subordination approach [1,59,61–63]. Equation (21) in Laplace space reads

$$sP_r(x,s) - \delta(x-x_0) = s\eta(s) \Big[\mathcal{D}\,\partial_x^2 - V\,\partial_x \Big] P_r(x,s).$$
⁽²²⁾

Let us present the solution of Equation (21) in the subordination form with the integral

$$P_r(x,t) = \int_0^\infty P_0(x,u)h(u,t)\,du,$$
(23)

where $P_0(x, t)$ is the solution in Equation (13). Here, h(u, t) is the so-called subordination function. The latter is the PDF, which subordinates the process governed by Equation (21)

to the process governed by Equation (13). By the Laplace transform of Equation (23), and by using the subordination function

$$h(u,s) = \frac{1}{s\eta(s)} e^{-u/\eta(s)},$$
(24)

we find

$$P_r(x,s) = \int_0^\infty P_0(x,u)h(u,s)\,du = \frac{1}{s\eta(s)}\int_0^\infty P_0(x,u)e^{-u/\eta(s)}\,du = \frac{1}{s\eta(s)}P_0(x,1/\eta(s)).$$
(25)

Performing the variable change $s \rightarrow 1/\eta(s)$ in Equation (12), we have

$$\frac{1}{\eta(s)}P_0(x,1/\eta(s)) - \delta(x-x_0) = \left[\mathcal{D}\,\partial_x^2 - V\,\partial_x\right]P_0(x,1/\eta(s)).\tag{26}$$

Therefore, from Equations (25) and (26), we obtain Equation (22). Eventually, from Equations (14) and (25), we obtain the PDF in the presence of resetting,

$$P_r(x,s) = \frac{s+r}{s} \frac{1}{2\mathcal{D}} \frac{1}{\sqrt{\frac{s+r}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}} e^{\frac{V}{2\mathcal{D}}(x-x_0) - \sqrt{\frac{s+r}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}|x-x_0|}}.$$
 (27)

From here, one finds

$$P_r(x,t) = e^{-rt} P_0(x,t) + r \int_0^t e^{-rt'} P_0(x,t') dt',$$
(28)

where $P_0(x, t)$ is defined by Equation (13), which is the solution of the corresponding Fokker–Planck equation without resetting. We also note that the case without the drift (V = 0) yields the known result for free diffusion with stochastic resetting [31].

From Equations (27) and (16), we find that the PDF $P_r(x, t)$ is normalized ($\langle x^0(t) \rangle_r = 1$), since

$$\langle x^{0}(s) \rangle_{r} = \int_{-\infty}^{\infty} P_{r}(x,s) \, dx = \frac{1}{s\eta(s)} \int_{-\infty}^{\infty} P_{0}(x,1/\eta(s)) \, dx = \frac{1}{s\eta(s)} \langle x^{0}(1/\eta(s)) \rangle_{0} = \frac{\eta(s)}{s\eta(s)} = \frac{1}{s}.$$
 (29)

From Equation (29) for the MSD, we find

$$\langle x^{2}(t) \rangle_{r} = \int_{-\infty}^{\infty} x^{2} P_{r}(x,s) \, dx = \frac{1}{s\eta(s)} \int_{-\infty}^{\infty} x^{2} P_{0}(x,1/\eta(s)) \, dx = \frac{1}{s\eta(s)} \langle x^{2}(1/\eta(s)) \rangle_{0}$$

$$= \frac{x_{0}^{2} \eta(s) + 2(\mathcal{D} + x_{0}V) \, \eta^{2}(s) + 2V^{2} \, \eta^{3}(s)}{s\eta(s)} = \frac{x_{0}^{2}}{s} + \frac{2(\mathcal{D} + x_{0}V)\eta(s)}{s} + \frac{2V^{2} \, \eta^{2}(s)}{s},$$
(30)

which results in

$$\langle x^{2}(t) \rangle_{r} = \mathcal{L}^{-1} \left[\frac{x_{0}^{2}}{s} + \frac{2(\mathcal{D} + x_{0}V)}{s(s+r)} + \frac{2V^{2}}{s(s+r)^{2}} \right] = x_{0}^{2} + \frac{2(\mathcal{D} + x_{0}V)(1 - e^{-rt})}{r} + \frac{2V^{2}(1 - e^{-rt} - rte^{-rt})}{r^{2}}.$$
 (31)

Then, the long-time limit yields saturation of the MSD,

$$\langle x^2(t) \rangle_r \sim x_0^2 + \frac{2(\mathcal{D} + x_0 V)}{r} + \frac{2V^2}{r^2},$$

while the short-time limit corresponds to the result without resetting, Equation (18). In the absence of the drift, the MSD reads $\langle x^2(t) \rangle_r = x_0^2 + \frac{2D}{r}(1 - e^{-rt})$ [31].

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Langevin Equation

We compare the analytical results against the ones obtained from direct numerical simulation of the dynamics by considering a Langevin equation in the presence of stochastic resetting to the initial position [44],

$$x(t + \Delta t) = \begin{cases} x(0), & \text{with probability } r\Delta t, \\ x(t) + V\Delta t + \sqrt{2D\Delta t}\zeta(t), & \text{with probability } (1 - r\Delta t), \end{cases}$$
(32)

where $\zeta(t)$ is a zero-mean Gaussian noise and *V*, D, and *r* are parameters that are used equivalently in the analytical case. Regarding the temporal evolution of the variance, ensembles of 10⁴ particle positions were simulated considering a time step of $\Delta t = 0.01$ across a time span of 10³ in order to observe convergence of the processes.

A graphical representation of the PDF is given in Figure 1 (left panel). The numerical results for the MSD, represented by dots, triangles, and squares in Figure 1 (right panel), show excellent agreement with the analytical results, represented by lines. A typical trajectory of a particle is shown in Figure 2.



Figure 1. Left panel: Probability density function (PDF) (27) at t = 1; right panel: Mean squared displacement (MSD) (30) for $\mathcal{D} = 1$, $x_0 = 0$, V = -1 and r = 0 (blue solid line), r = 1 (red dashed line), and r = 2 (black dot-dashed line).



Figure 2. A typical trajectory of particles in the presence of stochastic resetting to the initial position $x_0 = 0$, for r = 1, D = 1, and V = -1. The resetting events are represented by black dots. Dashed regions are introduced for these resetting events to be more visible.

2.2. One-Dimensional Brownian Search with Drift

The random Brownian search with drift in one dimension is described by the Fokker– Planck equation [54]

$$\partial_t f(x,t) = \left[\mathcal{D} \,\partial_x^2 - V \,\partial_x \right] f(x,t) - \mathcal{P}_{fa}(t)\delta(x-X), \tag{33}$$

where f(x, t) is the non-normalized density function, and the last term in the equation is a δ -sink of strength $\mathcal{P}_{fa}(t)$, which is considered as the FATD. From the condition f(x = X, s) = 0, one finds the FATD [39,54]

$$\mathcal{P}_{fa}(s) = e^{\frac{V}{2D}(X-x_0) - \sqrt{\frac{s}{D} + \frac{V^2}{4D^2}}|X-x_0|} \quad \to \quad \mathcal{P}_{fa}(t) = \frac{|X-x_0|}{\sqrt{4\pi Dt^3}} e^{-\frac{(X-x_0-Vt)^2}{4Dt}}, \quad (34)$$

which is the exponentially truncated Lévy–Smirnov probability density. The search reliability is [54,55]

$$\mathcal{P} = \mathcal{P}_{fa}(s=0) = e^{\frac{V(X-x_0)}{2D} - \frac{|V(X-x_0)|}{2D}} = \begin{cases} 1, & \text{for } V(X-x_0) > 0, \\ e^{-\frac{V(x_0-X)}{D}}, & \text{for } V(X-x_0) < 0, \end{cases}$$
(35)

while the search efficiency has the form [54,55]

$$\mathcal{E} = \frac{2\mathcal{D} + |V(X - x_0)|}{(X - x_0)^2} \times \begin{cases} 1, & \text{for } V(X - x_0) > 0, \\ e^{-\frac{V(x_0 - X)}{\mathcal{D}}}, & \text{for } V(X - x_0) < 0. \end{cases}$$
(36)

For V = 0, one recovers the known result for the random Brownian search (8). The FATD and the search efficiency are depicted in Figure 3. In Figure 4, we present the survival probability obtained by numerical simulations (dots, triangles, and squares) in the framework of the Langevin equation approach and the numerical inverse Laplace transform in MATHEMATICA [64].



Figure 3. Left panel: First arrival time distribution (FATD) (34) for $\mathcal{D} = 1$, V = 0, and $X - x_0 = 2$ (blue solid line), V = 1 and $X - x_0 = -2$ (red dashed line), and V = 1 and $X - x_0 = 2$ (black dot-dashed line). Right panel: Efficiency (36) for $\mathcal{D} = 1$ and V = 0 (blue solid line), v = 10 and $X > x_0$ (red dashed line), and v = 10 and $X < x_0$ (black dot-dashed line).



Figure 4. The survival probability for the one-dimensional search with drift for D = 1, V = 0, and $X - x_0 = 2$ (blue solid line), V = 1 and $X - x_0 = -2$ (red dashed line), and V = 1 and $X - x_0 = 2$ (black dot-dashed line). The numerical results are represented by dots, triangles, and squares.

Next, we extend our analysis to the case of the Brownian search with drift in the presence of two sinks located at $x = X_1$ and $x = X_2$, where $X_1 < X_2$. The problem is governed by the Fokker–Planck equation

$$\partial_t f(x,t) = \left[\mathcal{D} \,\partial_x^2 - V \,\partial_x \right] f(x,t) - \mathcal{P}_{\mathsf{fa},1}(t)\delta(x-X_1) - \mathcal{P}_{\mathsf{fa},2}(t)\delta(x-X_2), \tag{37}$$

where $f(x = X_1, t) = f(x = X_2, t) = 0$ and $\mathcal{P}_{fa,1}(t) + \mathcal{P}_{fa,2}(t) = \mathcal{P}_{fa}(t)$ is the FATD. Without loss of generality, we consider V > 0. By the Fourier–Laplace transformation, we obtain

$$f(k,s) = \frac{e^{ikx_0} - \mathcal{P}_{fa,1}(s) e^{ikX_1} - \mathcal{P}_{fa,2}(s) e^{ikX_2}}{s + \mathcal{D} k^2 + i \, V \, k}.$$
(38)

By the inverse Fourier transform, we find

$$f(x,s) = \frac{1}{2\mathcal{D}} \frac{1}{\sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}} \left[e^{\frac{V}{2\mathcal{D}}(x-x_0)} e^{-\sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}|x-x_0|} - \mathcal{P}_{\text{fa},1}(s) e^{\frac{V}{2\mathcal{D}}(x-X_1)} e^{-\sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}|x-X_1|} - \mathcal{P}_{\text{fa},2}(s) e^{\frac{V}{2\mathcal{D}}(x-X_2)} e^{-\sqrt{\frac{s}{\mathcal{D}} + \frac{V^2}{4\mathcal{D}^2}}|x-X_2|} \right].$$
(39)

Three cases of the initial position of the searcher are considered. These are: (i) $x_0 < X_1 < X_2$, (ii) $X_1 < x_0 < X_2$, and (iii) $X_1 < X_2 < x_0$. Following the same approach as is used in the case of one sink, we find the FATD in the Laplace space as follows: (i) $x_0 < X_1 < X_2$:

$$\mathcal{P}_{fa}(s) = e^{-\left[\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}} - \frac{V}{2D}\right](X_1 - x_0)},\tag{40}$$

(ii)
$$X_1 < x_0 < X_2$$

$$\mathcal{P}_{fa}(s) = \frac{e^{-\left(\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}} + \frac{V}{2D}\right)(x_0 - X_1)} \left[1 - e^{-\left(\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}} - \frac{V}{2D}\right)(X_2 - X_1)}\right]}{1 - e^{-2\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}}(X_2 - X_1)}} + \frac{e^{-\left(\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}} - \frac{V}{2D}\right)(X_2 - x_0)} \left[1 - e^{-\left(\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}} + \frac{V}{2D}\right)(X_2 - X_1)}\right]}{1 - e^{-2\sqrt{\frac{s}{D} + \frac{V^2}{4D^2}}(X_2 - X_1)}}, \quad (41)$$

(iii) $X_1 < X_2 < x_0$

$$\mathcal{P}_{fa}(s) = e^{-\left[\sqrt{\frac{s}{D} + \frac{V^2}{4D^2} + \frac{V}{2D}}\right](x_0 - X_2)}.$$
(42)

Then, the search reliability reads

$$\mathcal{P} = \mathcal{P}_{fa}(s=0) = \begin{cases} 1 & \text{for } x_0 < X_1 < X_2, \\ 1 & \text{for } X_1 < x_0 < X_2, \\ e^{-\frac{V}{\mathcal{D}}(x_0 - X_2)} & \text{for } X_1 < X_2 < x_0. \end{cases}$$
(43)

The search efficiency for the case $X_1 < x_0 < X_2$ can be analyzed numerically, while the other two cases can be calculated exactly and correspond to the single-target problem.

3. Brownian Motion with Drift on a Comb: Stochastic Resetting and Random Search Problem

3.1. Diffusion-Advection Equation on a Comb with Stochastic Resetting

The problem of the diffusion–advection equation on a comblike structure was introduced by Arkhincheev and Baskin in Ref. [2], and was also studied in Refs. [65,66] in terms of random walks with anisotropy that appear due to the presence of an external electrical field. We extend the model with stochastic resetting, which results in the Fokker–Planck equation

$$\partial_t P_r(x,y,t) = \delta(y) \left[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \right] P_r(x,y,t) + \mathcal{D}_y \,\partial_y^2 P_r(x,y,t) - r P_r(x,y,t) + r \delta(x-x_0) \delta(y) \tag{44}$$

with the initial condition $P_r(x, y, t = 0) = \delta(x - x_0)\delta(y)$, where v is a constant velocity. For the Laplace image, one looks for the solution in the form

$$P_r(x, y, s) = g(x, s) e^{-\sqrt{\frac{s+r}{D_y}}|y|},$$
(45)

which yields the following marginal PDF:

$$p_{1,r}(x,s) = \int_{-\infty}^{\infty} P_r(x,y,s) \, dy = 2\sqrt{\frac{\mathcal{D}_y}{s+r}g(x,s)}.$$
(46)

Performing the Laplace transform of Equation (44) and taking into account Equation (46), one obtains

$$sp_{1,r}(x,s) - \delta(x-x_0) = \frac{s(s+r)^{-1/2}}{2\sqrt{\mathcal{D}_y}} \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] p_{1,r}(x,s). \tag{47}$$

By the inverse Laplace transform, we find the equation

$$\partial_t p_{1,r}(x,t) = \frac{1}{2\sqrt{\mathcal{D}_y}} \frac{d}{dt} \int_0^t \eta_r(t-t') \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] p_{1,r}(x,t') \,dt', \tag{48}$$

where $\eta_r(t) = \mathcal{L}^{-1}\left[(s+r)^{-1/2}\right] = e^{-rt} \frac{t^{-1/2}}{\Gamma(1/2)}$ (for more details on tempered operators, see Ref. [67]). Disregarding resetting, and taking into account that $\eta_{r=0}(t) = t^{-1/2}/\Gamma(1/2)$, one obtains

$$\partial_t p_{1,0}(x,t) = \frac{1}{2\sqrt{\mathcal{D}_y}} \partial_t^{1/2} \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] p_{1,0}(x,t), \tag{49}$$

where $\partial_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-t')^{-\alpha} f(t') dt'$ is the Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ [68].

In the framework of the subordination approach, the Laplace image of the solution to Equation (49) is presented as follows:

$$p_{1,r}(x,s) = \frac{1}{s\eta_r(s)} P_0(x, 1/\eta_r(s)) = \frac{(s+r)^{1/2}}{s} P_0(x, (s+r)^{1/2}),$$
(50)

where $P_0(x, u)$ is the PDF (13). Then, the PDF (50) reads

$$p_{1,r}(x,s) = \frac{e^{\frac{v}{2D_x}(x-x_0)}}{2\frac{D_x}{2\sqrt{D_y}}} \frac{1}{s\eta_r(s)} \frac{e^{-\sqrt{\frac{2\sqrt{D_y}}{D_x}}\frac{1}{\eta_r(s)} + \frac{v^2}{4D_x^2}|x-x_0|}}{\left(\frac{2\sqrt{D_y}}{D_x}\frac{1}{\eta_r(s)} + \frac{v^2}{4D_x^2}\right)^{1/2}},$$
(51)

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which is

$$p_{1,r}(x,s) = \frac{e^{\frac{v}{2D_x}(x-x_0)}}{2\frac{D_x}{2\sqrt{D_y}}} \frac{(s+r)^{1/2}}{s} \frac{e^{-\sqrt{\frac{2\sqrt{D_y}}{D_x}(s+r)^{1/2} + \frac{v^2}{4D_x^2}|x-x_0|}}}{\left(\frac{2\sqrt{D_y}}{D_x}(s+r)^{1/2} + \frac{v^2}{4D_x^2}\right)^{1/2}}.$$
(52)

The case without resetting (r = 0) yields the PDF

$$p_{1,0}(x,s) = \frac{e^{\frac{v}{2D_x}(x-x_0)}}{2\frac{D_x}{2\sqrt{D_y}}} s^{-1/2} \frac{e^{-\sqrt{\frac{2\sqrt{D_y}}{D_x}s^{1/2} + \frac{v^2}{4D_x^2}|x-x_0|}}}{\left(\frac{2\sqrt{D_y}}{D_x}s^{1/2} + \frac{v^2}{4D_x^2}\right)^{1/2}}.$$
(53)

Eventually, the solution to Equation (48) reads

$$p_{1,r}(x,t) = e^{-rt} p_{1,0}(x,t) + r \int_0^t e^{-rt'} p_{1,0}(x,t') \, dt'.$$
(54)

From Equations (50) and (16), one can easily check that the PDF is normalized, since

$$\langle x^{0}(s) \rangle_{r} = \frac{(s+r)^{1/2}}{s} \int_{-\infty}^{\infty} P_{0}(x, (s+r)^{1/2}) \, dx = \frac{(s+r)^{1/2}}{s} \langle x^{0}((s+r)^{1/2}) \rangle_{0} = \frac{1}{s}, \quad (55)$$

that is, $\langle x^0(t) \rangle_r = 1$. For the MSD in the case of resetting, we find

$$\langle x^2(t) \rangle_r = x_0^2 + \left(\frac{\mathcal{D}_x}{\sqrt{\mathcal{D}_y}} + \frac{x_0 v}{\sqrt{\mathcal{D}_y}}\right) \frac{\operatorname{erf}(\sqrt{rt})}{\sqrt{r}} + \frac{v^2}{2\mathcal{D}_y} \frac{1 - e^{-rt}}{r},\tag{56}$$

which, in the absence of resetting, reduces to

$$\langle x^{2}(t) \rangle_{0} = x_{0}^{2} + 2(\mathcal{D} + x_{0}V)\frac{t^{1/2}}{\Gamma(3/2)} + 2V^{2}t = x_{0}^{2} + \left(\frac{\mathcal{D}_{x}}{\sqrt{\mathcal{D}_{y}}} + \frac{x_{0}v}{\sqrt{\mathcal{D}_{y}}}\right)\frac{t^{1/2}}{\Gamma(3/2)} + \frac{v^{2}}{2\mathcal{D}_{y}}t.$$
 (57)

Therefore, in the long-time limit, saturation of the MSD is due to the resetting,

$$\langle x^2(t) \rangle \sim x_0^2 + \frac{\mathcal{D}_x}{\sqrt{\mathcal{D}_y}} \frac{1}{\sqrt{r}} + \frac{v^2}{2\mathcal{D}_y} \frac{1}{r},$$

while in the short-time limit (when $\operatorname{erf}(x) \sim 2z/\sqrt{\pi}$ and $e^{-z} \sim 1-z$ for $z \ll 1$), we recover the result obtained for the resetting-free case in Equation (57).

Langevin Equation

To analyze the diffusion dynamics numerically, we use a system of Langevin equations in the presence of drift and stochastic resetting to the initial position [33,44]:

$$x(t + \Delta t) = \begin{cases} x(0), & \text{with probability } r\Delta t, \\ x(t) + A(y) [v\Delta t + \sqrt{2D_1\Delta t} \zeta_1(t)], & \text{with probability } (1 - r\Delta t), \end{cases}$$
(58)
$$y(t + \Delta t) = \begin{cases} y(0), & \text{with probability } r\Delta t, \\ y(t) + \sqrt{2D_2\Delta t} \zeta_2(t), & \text{with probability } (1 - r\Delta t), \end{cases}$$
(59)

where A(y) is a function that mimics the Dirac δ -function. To simulate A(y), diffusion across the *x* directions is permitted in a narrow strip with a width of 2ϵ along the *y* axis such that the value of ϵ is of the same order of magnitude as the diffusion coefficients [69]. A graphical representation of the PDF is depicted in Figure 5 (left panel), while the MSD is presented in Figure 5 (right panel). A good agreement between the analytical and numerical results for the MSDs is obtained. The individual trajectory along the backbone in the presence of stochastic resetting is shown in Figure 6. The case without resetting is shown in Figures 7 and 8.



Figure 5. Left panel: PDF (51) at t = 1; right panel: MSD (56) for $D_x = 1$, $D_y = 1$, $x_0 = 0$, v = -2, and r = 0 (blue solid line), r = 0.5 (red dashed line), and r = 2 (black dot-dashed line).



Figure 6. A typical trajectory of particles in the presence of stochastic resetting to the initial position $(x_0, y_0) = (0, 0)$ for r = 0.5, $\mathcal{D}_x = 1$, $\mathcal{D}_y = 1$, and v = -2. The resetting events are represented by black dots. Dashed regions are introduced for these resetting events to be more visible.



Figure 7. Left panel: PDF (53) at t = 1; right panel: MSD (57) for $D_x = 1$, $D_y = 1$, $x_0 = 0$, and v = 0 (blue solid line), v = -2 (red dashed line), and v = -5 (black dot-dashed line).



Figure 8. A typical trajectory of particles in the absence of stochastic resetting for $D_x = 1$, $D_y = 1$, and v = -2.

3.2. Brownian Search with Drift on a Comb

In this section, we consider a Brownian random search process on a comblike structure in the presence of an external drift along the backbone (constant external force). The Fokker–Planck equation then reads

$$\partial_t f(x, y, t) = \delta(y) \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] f(x, y, t) + \mathcal{D}_y \,\partial_y^2 f(x, y, t) - \wp_{fa}(t) \delta(x - X) \delta(y) \tag{60}$$

with the initial condition $f(x, y, t = 0) = \delta(x - x_0)\delta(y)$. The δ -sink means that the searcher is annihilated by reaching the point (x, y) = (X, 0). According to the definition in Equation (4), the FATD results from double integration

$$\wp_{fa}(t) = -\frac{d}{dt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, t) \, dx \, dy = -\frac{d}{dt} \mathcal{S}(t), \tag{61}$$

where $S(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, t) dx dy$ is the survival probability. By the Laplace transform of Equation (61), we find

$$sf(x,y,t) - \delta(x-x_0)\delta(y) = \delta(y) \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] f(x,y,s) + \mathcal{D}_y \,\partial_y^2 f(x,y,s) - \wp_{\mathrm{fa}}(s)\delta(x-X)\delta(y). \tag{62}$$

Again, using the substitution

$$f(x,y,s) = g(x,s) e^{-\sqrt{\frac{s}{\mathcal{D}_y}}|y|} \quad \to \quad f_1(x,s) = \int_{-\infty}^{\infty} f(x,y,s) \, dy = 2\sqrt{\frac{\mathcal{D}_y}{s}} g(x,s),$$

we find

$$sf_1(x,s) - \delta(x-x_0) = \frac{1}{2\sqrt{\mathcal{D}_y}} s \times s^{-1/2} \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] f_1(x,s) - \wp_{\mathsf{fa}}(s) \delta(x-X). \tag{63}$$

By the Fourier transform with respect to *x*, we find

$$sf_1(k,s) - e^{\imath k x_0} = \frac{1}{2\sqrt{\mathcal{D}_y}} s \times s^{-1/2} \Big[-\mathcal{D}_x \, k^2 - \imath \, v \, k \Big] f_1(k,s) - \wp_{\mathrm{fa}}(s) e^{\imath k X}, \tag{64}$$

and thus,

$$f_1(k,s) = \frac{s^{-1/2}}{s^{1/2} + \frac{D_x}{2\sqrt{D_y}}k^2 + \iota \frac{v}{2\sqrt{D_y}}k} \Big[e^{\iota kx_0} - \wp_{fa}(s) e^{\iota kX} \Big],$$
(65)

which is the solution of the equation

$$\frac{\partial}{\partial t}f_1(x,t) = \frac{1}{2\sqrt{\mathcal{D}_y}}\frac{\partial^{1/2}}{\partial t^{1/2}} \Big[\mathcal{D}_x\,\partial_x^2 - v\,\partial_x\Big]f_1(x,t) - \wp_{\rm fa}(t)\delta(x-X). \tag{66}$$

The inverse Fourier transform of Equation (65) with respect to k gives

$$f_{1}(x,s) = \frac{1}{2\sqrt{\mathcal{D}}} \frac{s^{-1/2}}{\left(s^{1/2} + \frac{V^{2}}{4\mathcal{D}}\right)^{1/2}} \left[e^{\frac{V}{2\mathcal{D}}(x-x_{0}) - \sqrt{\frac{s^{1/2}}{\mathcal{D}} + \frac{V^{2}}{4\mathcal{D}^{2}}|x-x_{0}|}} - \wp_{fa}(s) e^{\frac{V}{2\mathcal{D}}(x-X) - \sqrt{\frac{s^{1/2}}{\mathcal{D}} + \frac{V^{2}}{4\mathcal{D}^{2}}|x-X|}} \right], \tag{67}$$

where $\mathcal{D} = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}$ and $V = \frac{v}{2\sqrt{\mathcal{D}_y}}$. From the condition

$$f(x = X, y = 0, t) = g(x = X, t) = \frac{s^{1/2}}{2\sqrt{D_y}}f_1(x = X, t) = 0,$$

we conclude that it corresponds to $f_1(x = X, t) = 0$, i.e., $f_1(x = X, s) = 0$. Thus, for the FATD in Laplace space, we have

$$\wp_{fa}(s) = e^{\frac{v}{2D_x}(X-x_0) - \sqrt{\frac{2\sqrt{D_y}}{D_x}}s^{1/2} + \frac{v^2}{4D_x^2}}|X-x_0|.$$
(68)

We also note that the FATD (68) can be directly obtained from the FATD for the onedimensional search with drift (see Equation (34)). Then, we have $\wp_{fa}(s) = \mathcal{P}_{fa}(s^{1/2})$, where we use $\mathcal{D} = \frac{\mathcal{D}_x}{2\sqrt{\mathcal{D}_y}}$ and $V = \frac{v}{2\sqrt{\mathcal{D}_y}}$. The search reliability becomes

$$\mathcal{P} = \wp_{fa}(s=0) = e^{\frac{v(X-x_0)}{2\mathcal{D}_X} - \frac{|v(X-x_0)|}{2\mathcal{D}_X}} = \begin{cases} 1, & \text{for } v(X-x_0) > 0, \\ e^{-\frac{v(x_0-X)}{\mathcal{D}_X}}, & \text{for } v(X-x_0) < 0, \end{cases}$$
(69)

while the efficiency is given by

$$\mathcal{E} = \frac{4\left(\frac{\mathcal{D}_{x}}{2\sqrt{\mathcal{D}_{y}}}\right)}{(X-x_{0})^{4}} e^{\frac{v(X-x_{0})}{2\mathcal{D}_{x}} - \frac{|v(X-x_{0})|}{2\mathcal{D}_{x}}} \left[6\frac{\mathcal{D}_{x}}{2\sqrt{\mathcal{D}_{y}}} \left(1 + \frac{|v(X-x_{0})|}{2\mathcal{D}_{x}} \right) + \frac{v^{2}(X-x_{0})^{2}}{2\sqrt{\mathcal{D}_{y}}} \right]$$

$$= \frac{4\left(\frac{\mathcal{D}_{x}}{2\sqrt{\mathcal{D}_{y}}}\right)}{(X-x_{0})^{4}} \times \begin{cases} 6\frac{\mathcal{D}_{x}}{2\sqrt{\mathcal{D}_{y}}} \left(1 + \frac{v(X-x_{0})}{2\mathcal{D}_{x}} \right) + \frac{v^{2}(X-x_{0})^{2}}{2\sqrt{\mathcal{D}_{y}}}, & \text{for } v(X-x_{0}) > 0, \\ e^{\frac{v(X-x_{0})}{\mathcal{D}_{x}}} \left[6\frac{\mathcal{D}_{x}}{2\sqrt{\mathcal{D}_{y}}} \left(1 - \frac{v(X-x_{0})}{2\mathcal{D}_{x}} \right) + \frac{v^{2}(X-x_{0})^{2}}{2\sqrt{\mathcal{D}_{y}}} \right], & \text{for } v(X-x_{0}) < 0. \end{cases}$$

$$(70)$$

For v = 0, we recover the known result (9). The FATD and the efficiency for the random search with drift on the comb are depicted in Figure 9. In Figure 10, we give the survival probability obtained by numerical simulations of the Langevin equation and the numerical inverse Laplace transform obtained by MATHEMATICA [64].

One can also consider the Brownian search with drift on a comb in the presence of two sinks located on the backbone at $x = X_1$ and $x = X_2$, where $X_1 < X_2$. The corresponding Fokker–Planck equation is

$$\partial_t f(x, y, t) = \delta(y) \Big[\mathcal{D}_x \,\partial_x^2 - v \,\partial_x \Big] f(x, y, t) + \mathcal{D}_y \,\partial_y^2 f(x, y, t) \\ - \big[\wp_{\text{fa},1}(t) \delta(x - X_1) + \wp_{\text{fa},2}(t) \delta(x - X_2) \big] \delta(y)$$
(71)

with the initial condition $f(x, y, t = 0) = \delta(x - x_0)\delta(y)$. We consider v > 0, while the case for v < 0 can be treated in a similar way. Following the same approach suggested for the

one-sink problem, we consider the marginal PDF $f_1(x, t) = \int_{-\infty}^{\infty} f(x, y, t)$. Then, its Laplace image reads

$$f_1(k,s) = \frac{s^{-1/2}}{s^{1/2} + \frac{D_x}{2\sqrt{D_y}}k^2 + \iota \frac{v}{2\sqrt{D_y}}k} \Big[e^{\iota kx_0} - \wp_{fa,1}(s)e^{\iota kX_1} - \wp_{fa,2}(s)e^{\iota kX_2} \Big].$$
(72)

Due to the presence of two sinks, the absorbing condition is $f(x = X_1, y = 0, t) = f(x = X_2, y = 0, t) = 0$, and correspondingly, $f_1(x = X_1, t) = f_1(x = X_2, t) = 0$. We will also consider three cases of the initial positions of the searcher: (i) $x_0 < X_1 < X_2$, (ii) $X_1 < x_0 < X_2$, and (iii) $X_1 < X_2 < x_0$. For the FATD $\wp_{fa}(t) = \wp_{fa,1}(t) + \wp_{fa,2}(t)$, we obtain: (i) $x_0 < X_1 < X_2$:

$$\wp_{fa}(s) = e^{-\left(\sqrt{\frac{2\sqrt{D_x}}{D_x}s^{1/2} + \frac{v^2}{4D_x^2}} - \frac{v}{2D_x}\right)(X_1 - x_0)},$$
(73)

(iii) $X_1 < X_2 < x_0$

$$p_{\rm fa}(s) = e^{-\left(\sqrt{\frac{2\sqrt{D_x}}{D_x}s^{1/2} + \frac{v^2}{4D_x^2}} + \frac{v}{2D_x}\right)(x_0 - X_2)}.$$
(75)

Therefore, the search reliability is

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$$\mathcal{P} = \begin{cases} 1 & \text{for } x_0 < X_1 < X_2, \\ 1 & \text{for } X_1 < x_0 < X_2, \\ e^{-\frac{v}{\mathcal{D}_x}(x_0 - X_2)} & \text{for } X_1 < X_2 < x_0. \end{cases}$$
(76)

The search efficiency for the case $X_1 < x_0 < X_2$ can be analyzed numerically, while the other two cases can be calculated exactly, and the results are the same as for the single-target problem on the comb.



Figure 9. Left panel: FATD (68) for $D_x = 1$, $D_y = 1$, v = 0, and $X - x_0 = 2$ (blue solid line), v = 1 and $X - x_0 = -2$ (red dashed line), and v = 1 and $X - x_0 = 2$ (black dot-dashed line). Right panel: Efficiency (70) for $D_x = 1$, $D_y = 1$, and v = 0 (blue solid line), v = 10 and $X > x_0$ (red dashed line), and v = 10 and $X < x_0$ (black dot-dashed line).



Figure 10. The survival probability for the random search with the drift on the comb for $D_x = 1$, $D_y = 1$, v = 0, and $X - x_0 = 2$ (blue solid line), v = 1 and $X - x_0 = -2$ (red dashed line), and v = 1 and $X - x_0 = 2$ (black dot-dashed line). The numerical results are represented by dots, triangles, and squares.

4. Inhomogeneous Advection on a Comb

4.1. Inhomogeneous Advection with Stochastic Resetting

Inhomogeneous advection on a comb in the presence of stochastic resetting is described by the following equation:

$$\partial_t P(x, y, t) = -v \,\delta(y) \,\partial_x \{x \, P(x, y, t)\} + \mathcal{D} \,\partial_y^2 P(x, y, t) - r P(x, y, t) + r \delta(x - x_0) \delta(y) \tag{77}$$

with the initial condition $P(x, y, t = 0) = \delta(x - x_0)\delta(y)$. The case without resetting was considered in Ref. [29]. By the Laplace transform, we have

$$sP(x,y,s) - \delta(x-x_0)\delta(y) = -v\frac{s}{s+r}\,\delta(y)\,\partial_x\{x\,P(x,y,s)\} + \mathcal{D}\,\frac{s}{s+r}\,\partial_y^2P(x,y,s).$$
 (78)

The solution can be represented in the form $P(x, y, s) = g(x, s) e^{-\sqrt{\frac{s+r}{D}}|y|}$. Then, the marginal PDF along the backbone is $p_1(x, s) = \int_{-\infty}^{\infty} P(x, y, s) dy = 2\sqrt{\frac{D}{s+r}} g(x, s)$, and the equation for the marginal PDF reads

$$sp_1(x,s) - \delta(x-x_0) = -\frac{v}{2\sqrt{D}} \frac{s}{\sqrt{s+r}} \partial_x \{x \, p_1(x,s)\}.$$
 (79)

The inverse Laplace transform yields

$$\partial_t p_1(x,t) = -\frac{v}{2\sqrt{\mathcal{D}}} \frac{d}{dt} \int_0^t \eta(t-t') \partial_x \{x \, p_1(x,t')\} dt',\tag{80}$$

where $\eta(t) = \mathcal{L}^{-1}\left[(s+r)^{-1/2}\right] = e^{-rt} \frac{t^{-1/2}}{\Gamma(1/2)}$. In order to find the solution of this equation, we use the subordination approach. Let us consider the standard inhomogeneous advection equation

$$\partial_t P_0(x,t) = -V \,\partial_x \{ x \, P_0(x,t) \},\tag{81}$$

which, in Laplace space, reads

$$sP_0(x,s) - \delta(x - x_0) = -V \,\partial_x \{ x \, P_0(x,s) \}.$$
(82)

The solution of the equation for $x > x_0$ in Laplace space is [29]

$$P_0(x,s) = \frac{\theta(x-x_0)}{Vx} e^{-\frac{s}{V}\log\frac{x}{x_0}}.$$
(83)

From the subordination approach, we obtain

$$p_{1}(x,s) = \frac{1}{s\eta(s)} P_{0}(x, 1/\eta(s)) = \frac{(s+r)^{1/2}}{s} P_{0}(x, (s+r)^{1/2})$$

$$= \frac{2\sqrt{D}}{v} \frac{\theta(x-x_{0})}{x} \frac{(s+r)^{1/2}}{s} e^{-\frac{2\sqrt{D}}{v}(s+r)^{1/2}\log\frac{x}{x_{0}}}$$

$$= \frac{s+r}{s} \frac{2\sqrt{D}}{v} \frac{\theta(x-x_{0})}{x} (s+r)^{-1/2} e^{-\frac{2\sqrt{D}}{v}(s+r)^{1/2}\log\frac{x}{x_{0}}},$$
(84)

where we used $V \rightarrow \frac{v}{2\sqrt{D}}$. The inverse Laplace transform yields the solution

$$p_1(x,t) = e^{-rt} P_1(x,t) + r \int_0^t e^{-rt'} P_1(x,t') \, dt', \tag{85}$$

where

$$P_1(x,t) = \frac{2\sqrt{\mathcal{D}}}{v} \frac{\theta(x-x_0)}{x} \mathcal{L}^{-1} \left[s^{-1/2} e^{-\frac{2\sqrt{\mathcal{D}}}{v} s^{1/2} \log \frac{x}{x_0}} \right] = \frac{2\theta(x-x_0)}{x\sqrt{4\pi \left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}} \times \exp\left(-\frac{\log^2 \frac{x}{x_0}}{4\left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2 t}\right)$$
(86)

is the solution of Equation (77) without resetting [29].

From Equation (84), the Laplace image of the MSD reads

$$\langle x^2(s) \rangle = x_0^2 \frac{s+r}{s} \frac{(s+r)^{-1/2}}{(s+r)^{1/2} - \frac{v}{\sqrt{D}}},$$
(87)

and the inverse Laplace transform yields

$$\langle x^{2}(t) \rangle = x_{0}^{2} \bigg[e^{-rt} E_{1/2} \bigg(\frac{v}{\sqrt{\mathcal{D}}} t^{1/2} \bigg) + r \int_{0}^{t} e^{-rt'} E_{1/2} \bigg(\frac{v}{\sqrt{\mathcal{D}}} t'^{1/2} \bigg) dt' \bigg].$$
(88)

Here, $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ is the one-parameter Mittag–Leffler function, and its Laplace image is $\mathcal{L}[E_{\alpha}(at^{\alpha})] = \frac{s^{\alpha-1}}{s^{\alpha}-a}$ [68]. For the large argument, the Mittag–Leffler function reads $E_{1/2}\left(\frac{v}{\sqrt{\mathcal{D}}}t^{1/2}\right) \sim e^{\frac{v^2}{\mathcal{D}}t}$ (see Ref. [68]). Therefore, this yields the long-time limit of the MSD, which has three different regimes: (i) exponential growth in time for $r < v^2/\mathcal{D}$, (ii) linear growth with time for $r = v^2/\mathcal{D}$, and (iii) saturation for $r > v^2/\mathcal{D}$.

4.2. Langevin Equation

In the case of inhomogeneous advection on a comb, completely different results are obtained due to the multiplicative noise. The microscopic approach to the process without resetting is described by the Langevin equation:

$$\dot{x}(t) = vA(y)x(t), \tag{89}$$

$$\dot{y}(t) = \sqrt{2\mathcal{D}\,\zeta(t)},\tag{90}$$

where $\zeta(t)$ is a Gaussian noise. Therefore, y(t) is a random Brownian motion, so A(y) is a function of the random variable. Without any restriction of generality, we may use $A(y) = \delta(y)$, which corresponds to the inhomogeneous advection motion on the comb.

To validate the analytical results, an ensemble of 10^6 particles was simulated based on the system of Langevin Equations (89) and (90), which, in the presence of resetting, are

$$x(t + \Delta t) = \begin{cases} x(0), & \text{with probability } r\Delta t, \\ x(t) + v A(y)x(t) \Delta t, & \text{with probability } (1 - r\Delta t), \end{cases}$$
(91)

$$y(t + \Delta t) = \begin{cases} y(0), & \text{with probability } r\Delta t, \\ y(t) + \sqrt{2D\Delta t} \zeta(t), & \text{with probability } (1 - r\Delta t), \end{cases}$$
(92)

where the time step is $\Delta t = 0.01$. It should be admitted that for the multiplicative noise, one cannot use the numerical approximation of the Dirac δ -function considered in Section 3. For numerical purposes, one can use another approximation of the Dirac δ -function (see Ref. [4]). Consequently, we use a zero-mean Gaussian function $A(y) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-y^2/[2\sigma^2])$ in the limit $\sigma \to 0$.

A graphical representation of the PDF is shown in Figure 11 (left panel), while the MSD obtained analytically and by numerical simulations is shown in Figure 11 (right panel). A typical trajectory along the backbone in the presence of stochastic resetting is shown in Figure 12.



Figure 11. Left panel: PDF (85) at t = 2; right panel: MSD (88) for $\mathcal{D} = 1$, $x_0 = 1$, v = 1, and r = 0.7 (blue solid line), r = 1 (red dashed line), and r = 2 (black dot-dashed line).



Figure 12. A typical trajectory of particles in the presence of stochastic resetting to the initial position $(x_0, y_0) = (1, 0)$ for r = 1, D = 1, and v = 1. The resetting events are represented by black dots. Dashed regions are introduced for these resetting events to be more visible.

4.3. Inhomogeneous Advection Search on a Comb

For the completeness of the analysis, we consider an inhomogeneous advection search on the comb, or the so-called turbulent diffusion search. The corresponding Fokker–Planck equation reads [29]

$$\partial_t F(x, y, t) = -v \,\delta(y) \,\partial_x \{ x F(x, y, t) \} + \mathcal{D} \,\partial_y^2 F(x, y, t) - \wp_{\mathsf{fa}}(t) \,\delta(x - X) \,\delta(y). \tag{93}$$

Following the standard procedure described in Section 3.2, we find the FATD in Laplace space as follows:

$$\wp_{fa}(s) = e^{-\frac{2\sqrt{D}}{v}\sqrt{s}\log\frac{X}{x_0}}.$$
(94)

The inverse Laplace transform of Equation (94) for $X > x_0$ gives the Lévy–Smirnov distribution [29]:

$$\wp_{fa}(t) = \frac{\log \frac{X}{x_0}}{\sqrt{4\pi \left(\frac{v}{2\sqrt{D}}\right)^2 t^3}} \times \exp\left(-\frac{\log^2 \frac{X}{x_0}}{4\left(\frac{v}{2\sqrt{D}}\right)^2 t}\right). \tag{95}$$

The search reliability equals one ($\mathcal{P} = 1$), while the efficiency is

$$\mathcal{E} = \int_0^\infty e^{-\frac{2\sqrt{D}}{v}\sqrt{s}\log\frac{X}{x_0}} \, ds = \frac{2\left(\frac{v}{2\sqrt{D}}\right)^2}{\log^2\frac{X}{x_0}}.$$
(96)

Therefore, the turbulent diffusion search is more efficient than the Brownian search for long-distance targets, but the searcher should have a prior knowledge of the direction of the target, which is not the case for the Brownian search. The FATD and the efficiency are shown in Figure 13. The survival probability is shown in Figure 14.



Figure 13. Left panel: FATD (95) for $\mathcal{D} = 1$, v = 1, and $X/x_0 = 2.5$ (blue solid line), $X/x_0 = 5$ (red dashed line), and $X/x_0 = 7.5$ (black dot-dashed line). Right panel: Efficiency (96) for $\mathcal{D} = 1$, V = 1 (blue solid line), v = 5 (red dashed line), and v = 10 (black dot-dashed line).



Figure 14. The survival probability for the inhomogeneous advection search on a comb for $\mathcal{D} = 1$, v = 1, and $X/x_0 = 2$ (blue solid line), $X/x_0 = 5$ (red dashed line), and $X/x_0 = 10$ (black dot-dashed line). The numerical results are represented by dots, triangles, and squares.

Remark 1. As observed above in Equations (95) and (96), the corresponding inhomogeneous advection search problem cannot be considered for the initial position of the searcher at the origin $(x_0 = 0)$. To consider a random search with the initial condition at $x_0 = 0$, we add some constant advection $v_0 = v \epsilon$, where $\epsilon > 0$. In this case, $x_0 = 0$ is no longer a singular point for the space derivative. The search equation reads

$$\partial_t F(x, y, t) = -v \,\delta(y) \,\partial_x \{(x + \epsilon) F(x, y, t)\} + \mathcal{D} \,\partial_y^2 F(x, y, t) - \wp_{\mathsf{fa}}(t) \,\delta(x - X) \,\delta(y). \tag{97}$$

Following the same procedure for the FATD described above, we obtain

$$\wp_{fa}(s) = e^{-\frac{2\sqrt{D}}{v}\sqrt{s}\log\frac{X+\epsilon}{x_0+\epsilon}}.$$
(98)

For $x_0 = 0$, it reads $\wp_{fa}(s) = e^{-\frac{2\sqrt{D}}{v}\sqrt{s}\log\frac{X+\epsilon}{\epsilon}}$. The reliability becomes $\mathcal{P} = 1$, and the efficiency becomes

$$\mathcal{E} = \frac{2\left(\frac{v}{2\sqrt{D}}\right)^2}{\log^2 \frac{X+\epsilon}{x_0+\epsilon}},\tag{99}$$

which is also valid for $x_0 = 0$. However, the searcher moves only in the advection direction.

Remark 2. For completeness of the analysis, we also consider a search model with inhomogeneous advection of the form $\partial_x |x|^{\lambda}$, $0 < \lambda < 1$ [28]. Note that for $\lambda = 1$, it corresponds to the inhomogeneous advection search problem considered above. The search Equation (93) now reads

$$\partial_t F(x,y,t) = -v\,\delta(y) \Big\{ \operatorname{sgn}(x)\,\lambda\,|x|^{\lambda-1} + |x|^\lambda\,\partial_x \Big\} F(x,y,t) + \mathcal{D}\,\partial_y^2 F(x,y,t) - \wp_{\mathsf{fa}}(t)\,\delta(x-X)\,\delta(y), \tag{100}$$

where

$$sgn(x) = \partial_x |x| = 2\theta(x) - 1 = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

Following the same procedure for the FATD at $X > x_0 > 0$ *, we obtain*

$$\wp_{\text{fa}}(s) = e^{-\frac{2\sqrt{D}}{v}\sqrt{s}\frac{X^{1-\lambda} - x_0^{1-\lambda}}{1-\lambda}}.$$
(101)

The search reliability is $\mathcal{P} = \wp_{fa}(s = 0) = 1$ *. The inverse Laplace transform of Equation (101) yields*

$$\wp_{fa}(t) = \frac{X^{1-\lambda} - x_0^{1-\lambda}}{\sqrt{4 \pi (1-\lambda)^2 \left(\frac{v}{2\sqrt{D}}\right)^2 t^3}} \times \exp\left(-\frac{\left(X^{1-\lambda} - x_0^{1-\lambda}\right)^2}{4(1-\lambda)^2 \left(\frac{v}{2\sqrt{D}}\right)^2 t}\right).$$
 (102)

Therefore, the efficiency has the form

$$\mathcal{E} = \frac{2 \left(1 - \lambda\right)^2 \left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2}{\left(X^{1-\lambda} - x_0^{1-\lambda}\right)^2}.$$
(103)

Remark 3. We note that in the limit $\lambda \rightarrow 1$, the FATD (101) and the efficiency (103) become

$$\lim_{\lambda \to 1} \wp_{\mathrm{fa}}(s) = e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}\lim_{\lambda \to 1}\frac{X^{1-\lambda} - x_0^{1-\lambda}}{1-\lambda}} = e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}\log\frac{X}{x_0}},\tag{104}$$

1 1 1 1

$$\lim_{\lambda \to 1} \mathcal{E} = \lim_{\lambda \to 1} \frac{2 (1-\lambda)^2 \left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2}{\left(X^{1-\lambda} - x_0^{1-\lambda}\right)^2} = \lim_{\lambda \to 1} \frac{2 (1-\lambda) \left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2}{\left(X^{1-\lambda} - x_0^{1-\lambda}\right) (\log X - \log x_0)} = \frac{2 \left(\frac{v}{2\sqrt{\mathcal{D}}}\right)^2}{\log^2 \frac{X}{x_0}}, \quad (105)$$

where we use L'Hôpital's rule, and thus, we recover the corresponding results ((94) and (96)) for the inhomogeneous advection search.

Remark 4. For $\lambda = 0$, in Equation (101), for the FATD, we obtain $\wp_{fa}(s) = e^{-\frac{2\sqrt{D}}{v}\sqrt{s}(X-x_0)}$. This case corresponds to homogeneous advection on the comb, and thus, the FATD can be obtained from Equation (68) for $\mathcal{D}_x \to 0$. Namely, we have

$$\lim_{\mathcal{D}_{x}\to 0} \wp_{fa}(s) = \lim_{\mathcal{D}_{x}\to 0} e^{\frac{v}{2\mathcal{D}_{x}}(X-x_{0}) - \sqrt{\frac{2\sqrt{\mathcal{D}_{y}}}{\mathcal{D}_{x}}s^{1/2} + \frac{v^{2}}{4\mathcal{D}_{x}^{2}}}|X-x_{0}|} = \lim_{\mathcal{D}_{x}\to 0} e^{\frac{v}{2\mathcal{D}_{x}}(X-x_{0}) - \frac{v}{2\mathcal{D}_{x}}\sqrt{1 + \frac{8\sqrt{\mathcal{D}_{y}}\mathcal{D}_{x}}{v^{2}}s^{1/2}}|X-x_{0}|} = \lim_{\mathcal{D}_{x}\to 0} e^{\frac{v}{2\mathcal{D}_{x}}(X-x_{0}) - \frac{v}{2\mathcal{D}_{x}}\left(1 + \frac{4\sqrt{\mathcal{D}_{y}}\mathcal{D}_{x}}{v^{2}}s^{1/2}\right)|X-x_{0}|} = e^{-\frac{2\sqrt{\mathcal{D}}}{v}\sqrt{s}(X-x_{0})} \quad for \quad X > x_{0}.$$
(106)

The efficiency reads

$$\mathcal{E} = \frac{2\left(\frac{v}{2\sqrt{D}}\right)^2}{\left(X - x_0\right)^2},\tag{107}$$

which has the same behavior as the efficiency for the one-dimensional Brownian search (see Equation (8)).

5. Summary

We present an overview of various drift–diffusion and inhomogeneous advection problems with and without resetting on comblike structures. Both the Brownian diffusion search with drift and the inhomogeneous advection search on the comb structures were analyzed. The analytical results were verified by numerical simulations in terms of coupled Langevin equations for the comb structures. The subordination approach was one of the main technical methods used here, and we demonstrated how it can be effective in the study of various random search problems with and without resetting.

In conclusion, consideration of the comb model (or comb geometry) is an important issue for investigation of the interplay between diffusion, drift, and geometry. For example, further modification of the comb model, like a fractal tartan [16,70–72], can be an interesting task for understanding of the impact of fractal geometry on the fractional transport, as well as for the experimental implementation and the technological design and development of sparse sensor arrays [73,74].

Author Contributions: Conceptualization, T.S., A.I. and L.K.; Formal analysis, V. D.; Investigation, T.S., V.D. and A.I.; Methodology, T.S., V.D. and A.I.; Supervision, L.K.; Visualization, T.S. and V.D.; Writing—original draft, T.S., V.D. and A.I.; Writing—review and editing, A.I. and L.K. All authors contributed to each part of this work equally. All authors have read and agreed to the published version of the manuscript.

Funding: T.S. was supported by the Alexander von Humboldt Foundation.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PDF probability density function

MSD mean squared displacement

FATD first arrival time distribution

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