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Dynamic Analysis of a Stochastic Rumor Propagation Model with Regime Switching

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Abstract: We study the rumor propagation model with regime switching considering both colored and white noises. Firstly, by constructing suitable Lyapunov functions, the sufficient conditions for ergodic stationary distribution and extinction are obtained. Then we obtain the threshold \mathcal{R}^s which guarantees the extinction and the existence of the stationary distribution of the rumor. Finally, numerical simulations are performed to verify our model. The results indicated that there is a unique ergodic stationary distribution when $\mathcal{R}^s > 1$. The rumor becomes extinct exponentially with probability one when $\mathcal{R}^s < 1$.

Keywords: rumor propagation model; regime switching; ergodic stationary distribution; extinction; threshold



Citation: Jia, F.; Cao, C. Dynamic Analysis of a Stochastic Rumor Propagation Model with Regime Switching. *Mathematics* **2021**, *9*, 3277. <https://doi.org/10.3390/math9243277>

Academic Editor: Chao Huang

Received: 15 October 2021

Accepted: 13 December 2021

Published: 16 December 2021

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1. Introduction

Shakespeare once created a very vivid metaphor about rumor spreading, “Rumor is like a flute. Guesswork, suspicion, and speculation are the breath that makes it sound” [1]. Generally speaking, rumor can cause social panic, and have a certain impact on social stability [2]. People fabricate and spread rumors in order to self-slander, distract and cause panic [3–5]. In order to prevent the negative effects of rumors, it is necessary to analyze the internal mechanism of rumor propagation.

As a social contagion process, rumor propagation is very similar to the spread of epidemic diseases, but not identical. In order to distinguish it from the spread of epidemics, Daley-Kendall divided the D-K rumor spreading model [5] into three classes: stiflers, ignorants, and spreaders. Maki-Thompson [6] modified the D-K model to research rumor spreading from a Markov chain. Since then, many researchers have studied the dynamic behavior of rumors. For example, Moreno et al. [7] studied the propagation process for random scale-free networks. Wang et al. [8] simulated the spatio-temporal characteristics of information diffusion through a diffusive logistic equation. Doer et al. [9] analyzed how news spreads in social networks by simulating a simple information-spreading process in various network topologies. It is found that news spreads much more quickly in existing social-network topologies than other network topologies. Afassinou [10] studied the education rate affect on the rumor propagation mechanism. Ma et al. [11] found that the reinforcement factor can effectively suppress the spread of rumors. In addition, Zhu et al. [12] studied the spatial-temporal dynamics of rumor propagation. Tian and Ding [13] considered that when an ignorant was exposed to a rumor or counter-rumor, he or she would change into a latent with one of three different attitudes toward the rumor. Li et al. [14] discussed the local stability and global stability of rumor-free equilibrium by using the Lyapunov stability theory.

The spread of rumors often has randomness that cannot be ignored. Stochasticity is the adjustment of transmission rates due to changes in the interests and behavior of the user

group [15]. Mao et al. [16] studied the effects of the environmental noise in population systems and concluded that even a small noise can have an effect on public opinion. Dauhoo et al. [17] studied the spreading process of rumors in latent and migrating periods based on deterministic and stochastic rumor propagation models. Zhu et al. [18] found that noise accelerates the spread of rumors through an improved SIR model. Zhu et al. [19] investigated the influence of network topology by the rumor diffusion model with spatio-temporal diffusion. Jia et al. [20–22] studied the random factors effect on rumor propagation by introducing white noise and jumping noise into the model. Jain et al. [23] obtained the global and local asymptotic stability conditions of the deterministic and random models.

The rumor model is an important part of the ecosystem, which is inevitably affected by environmental noise. Moreover, there is usually another type of colored noise. It is easy to switch the population system from one state to another [24,25]. The colored noise can change randomly between two or more subsystems. For example, due to different social and cultural factors, rumors spread faster or slower. Traditional deterministic or random rumor models are usually unable to describe the changeable phenomenon. Therefore, it is important to consider additional factors in a random environment [26]. More precisely, the relationship between the rumor model and regime switching needs to be considered. To the best of our knowledge, the analysis methods with regime switching are often used in epidemic analysis, see [27,28], but are rarely used in the rumor propagation model.

The contributions of this article are as follows. First, a rumor propagation model with regime switching is constructed to point out the rumor propagation dynamics in a social network. Second, the stochasticity of the rumor propagation is taken into account, extending the previous approach of Zhao and Zhu [29]. Third, an emphasis is laid on the derivation of ergodic stationary distribution and extinction. Moreover, the threshold for ergodic stationary distribution and extinction are obtained.

The main novelty of our work is that we establish a rumor model with regime switching to describe the rumor propagation dynamics in social network. The existence of the ergodic stationary distribution of the solution is an important issue under white noise and colored noise. However, there are no research results on the influence of the correlation coefficient of regime switching and random disturbances on the dynamic of rumor models with variable population size. Motivated by Liu et al. [30,31] and Li et al. [32], we try to fill this gap.

The organization of this paper is as follows. Section 2 introduces the model formula. The conditions for the existence of ergodic stationary distributions are established in Section 3. The sufficient conditions for the extinction of rumors are established in Section 4. We evaluate the influence of the random noises by numerical simulations in Section 5 and conclude in Section 6.

2. Model Formulation

In the real world, social rumors usually include the following characteristics: fabricating false news, seeking economic benefits, aggression, retaliation, amongst others [33,34]. For example, The nuclear accident in Japan in 2011 is a notable example. It was rumored that that nuclear radiation would pollute sea salt and that salt can protect against radiation. The crazy rumors propagated in the coastline cities of China, prompting residents to buy and hoard sea salt. Even some merchants pushed up prices, leading to market failures [35].

When individuals infected by rumors communicate face to face with other individuals, the credibility of the rumors is increased. Similarly to the epidemic model, all users in social networks are divided into two class: the S -susceptible, those who do not know the rumor and the I -infected, those who know and transmit the rumor. Combined with the rumor propagation rules in social networks, Zhao and Zhu [29] studied the following rumor propagation model

$$\begin{cases} \frac{\partial S}{\partial t} = d \frac{\partial^2 S}{\partial x^2} - \mu S - \beta SI + \alpha I^2 + A, \\ \frac{\partial I}{\partial t} = d \frac{\partial^2 I}{\partial x^2} - \alpha I^2 - (\mu + \eta)I + \beta SI, \end{cases} \quad (1)$$

where $t > 0, x \in D = (0, L)$ with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu}(t, x) = \frac{\partial I}{\partial \nu}(t, x) = 0, \quad t \geq 0, \quad x \in \partial D, \tag{2}$$

in which ν denotes the unit outward normal on ∂D and

$$\begin{cases} S(0, x) = \rho_1(x), & x \in \bar{D}, \\ I(0, x) = \rho_2(x), & x \in \bar{D}, \end{cases}$$

where $S(t, x)$ and $I(t, x)$ represent, respectively, the densities of the rumor-susceptible users and the rumor-infected users with a distance of x at time t . $d \frac{\partial^2}{\partial x^2}$ is a diffusion term, being used to describe the impact of the mobility on the rate of change in the density of users with a distance of x at time t . L represents the upper limit of the distances between rumors and other social network users. D is the bounded region of the smooth boundary ∂D . The boundary condition in (2) implies that there are no rumors across the boundary of D . μ is the rate of social network users losing interest in early rumors. β is the rumor propagation ratio of rumor-susceptible users to rumor-infected users. α is the ratio that rumor-infected users convert to rumor-susceptible users. A is the rate at which the users continuously access the network. η is the forgetting rate of rumor-infected users who stop spreading rumors.

For convenience, we omit the spatial effect, that is, $d = 0$. As we know, similar to the epidemic model [30,36], environmental noise has an influence on the rumor model. For example, with the increasing popularity of the Internet, rumors can quickly spread all over the world with the help of the Internet, which is unimaginable in an era without the Internet. Inspired by [36], we assume that random disturbance is a typical type of white noise, which grows with $S(t)$ and $I(t)$. That is, we make the following changes:

$$-\mu \mapsto -\mu + \sigma_1 dB_1(t); \quad -(\mu + \eta) \mapsto -(\mu + \eta) + \sigma_2 dB_2(t).$$

Therefore, the system (1) becomes a stochastic one which can be written as follows:

$$\begin{cases} dS(t) = [-\mu S(t) - \beta S(t)I(t) + \alpha I^2(t) + A]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = [-\alpha I^2(t) - (\mu + \eta)I(t) + \beta S(t)I(t)]dt + \sigma_2 I(t)dB_2(t), \end{cases} \tag{3}$$

where $B_j(0) = 0, B_j(t)$ is a standard Brownian motion and they are independent from each other. The intensity of white noise is a scale parameter $\sigma_j > 0 (j = 1, 2)$.

However, in the ecosystem, colored noise will have an impact on the results of the rumor model, causing it to shift from one state to another. The transition between states usually has no trace [37]. Therefore, the system (3) with regime switching becomes

$$\begin{cases} dS(t) = [-\mu(r(t))S(t) - \beta(r(t))S(t)I(t) + \alpha(r(t))I^2(t) + A(r(t))]dt + \sigma_1(r(t))S(t)dB_1(t), \\ dI(t) = [-\alpha(r(t))I^2(t) - (\mu(r(t)) + \eta(r(t)))I(t) + \beta(r(t))S(t)I(t)]dt + \sigma_2(r(t))I(t)dB_2(t), \end{cases} \tag{4}$$

where $r(t)$ is a Markov chain with a finite state space $\mathbb{S} = \{1, \dots, N\}, 1 \leq N < \infty$.

Next, we suppose that there is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is right continuous, and \mathcal{F}_0 contains all \mathbb{P} -null sets. Furthermore, $B_j(t)$ is defined on the complete probability space, $j = 1, 2$. Denote $\mathbb{R}_+ = [0, \infty], \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}$. If $f(t)$ is a bounded function on \mathbb{R}_+ , define $f^u = \sup_{t \in \mathbb{R}_+} f(t)$ and $f^l = \inf_{t \in \mathbb{R}_+} f(t)$. If $f(t)$ is an integral function on \mathbb{R}_+ , define $\langle f \rangle_t = \frac{1}{t} \int_0^t f(s)ds, t > 0$. Set $\hat{g} = \min_{k \in \mathbb{S}} \{g(k)\}$ and $\check{g} = \max_{k \in \mathbb{S}} \{g(k)\}$ for any vector $g = (g(1), \dots, g(N))$. Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking a value on the finite state space $\mathbb{S} = \{1, 2, \dots, N\}$. The generator $\Gamma = (\gamma_{ij})_{N \times N}$ is given by

$$\mathbb{P}\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta t + o(\Delta t), & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. We suppose that the Markov chain is independent of the Brownian motion. We suppose further that the Markov chain $r(t)$ is irreducible, which means that the system can switch from one regime to the other regime. This means that $\pi = (\pi_1, \pi_2, \dots, \pi_N)$ is given by

$$\pi \Gamma = 0,$$

subject to

$$\sum_{h=1}^N \pi_h = 1 \text{ and } \pi_h > 0 \text{ for any } h \in \mathbb{S}.$$

Throughout this work, Brownian motion and the Markov chain are assumed to be mutually independent and defined on the same complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. For any $k \in \mathbb{S}$, $A(k), \mu(k), \beta(k), \alpha(k), \eta(k)$ and $\sigma_j(k), (j = 1, 2)$ are all positive constants.

3. Existence of Ergodic Stationary Distribution

In this section, we will discuss the sufficient conditions for the system (4) to satisfy a unique ergodic stationary distribution. To this end, according to Zhu and Yin [25], it is sufficient to prove that the system (4) is positive recurrent. First, we recall some results on the stationary distribution for stochastic differential equations under regime switching.

Theorem 1. *For any initial value $(S(0), I(0), r(0)) \in \mathbb{R}_+^2 \times \mathbb{S}$, there is a unique solution $(S(t), I(t), r(t))$ of system (4) on $t > 0$ and the solution will remain in $\mathbb{R}_+^2 \times \mathbb{S}$ with probability one, namely, for all $t > 0$, $(S(t), I(t), r(t)) \in \mathbb{R}_+^2 \times \mathbb{S}$ is almost surely (a.s).*

The proof process is similar to Theorem 2.1 [22], so it is omitted here.

Next, we give some results on the stationary distribution for stochastic differential equations under regime switching. For more details, we can refer the readers to [38]. Let $(x(t), r(t))$ be the diffusion process denoted by the following equation

$$dx(t) = \varphi(x(t), r(t))dt + \sigma(x(t), r(t))dB(t), \quad x(0) = z_0, \quad r(0) = r, \tag{5}$$

where $B(\cdot)$ and $r(\cdot)$ are the d -dimensional Brownian motion and the right continuous Markov chain, respectively. $\varphi(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$, $\sigma(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^{n \times d}$ and $\sigma(x, k)\sigma^T(x, k) = (d_{ij}(x, k))$. For each $k \in \mathbb{S}$, let $V(\cdot, k)$ be any twice continuously differentiable function; the operator L can be defined by

$$LV(x, k) = \sum_{i=1}^n \varphi_i(x, k) \frac{\partial V(x, k)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n d_{ij}(x, k) \frac{\partial^2 V(x, k)}{\partial x_i \partial x_j} + \sum_{l=1}^N \gamma_{kl} V(x, l).$$

According to the generalized Itô's formula [39], if $x(t) \in \mathbb{R}^n$, then

$$dV(x(t), r(t)) = LV(x(t), r(t))dt + V_x(x(t), r(t))\sigma(x(t), r(t))dB(t).$$

Lemma 1 ([25]). *If the following some conditions are satisfied:*

- (i) For any $i \neq j$, $\gamma_{ij} > 0$;
- (ii) For each $k \in \mathbb{S}$, $D(x, k) = (d_{ij}(x, k))$ is symmetric and satisfies

$$\theta |\xi|^2 \leq \langle D(x, k)\xi, \xi \rangle \leq \theta^{-1} |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n;$$

$\theta \in (0, 1]$ for all $x \in \mathbb{R}^n$;

- (iii) There exists a bounded open subset \mathcal{D} of \mathbb{R}^n with a regular (i.e., smooth) boundary satisfying that, for $k \in \mathbb{S}$, there is $V(\cdot, k) : \mathcal{D}^C \rightarrow \mathbb{R}$ such that $V(\cdot, k)$ is twice continuously differentiable and that for $v > 0$, $LV(x, k) \leq -v$ for any $(x, k) \in \mathcal{D}^C \times \mathbb{S}$, then $(x(t), r(t))$ of system (5)

is ergodic and positive recurrent. In other words, there exists a unique stationary distribution $\mu(\cdot, \cdot)$ such that for any Borel measurable function $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}$ satisfying

$$\sum_{k=1}^N \int_{\mathbb{R}^n} |f(x, k)| \mu(dx, k) < \infty.$$

Therefore,

$$\mathbb{P}\left\{\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(x(s), r(s)) ds = \sum_{k=1}^N \int_{\mathbb{R}^n} f(x, k) \mu(dx, k)\right\} = 1.$$

Through Theorem 1, we know that for $(S(0), I(0), r(0)) \in \mathbb{R}_+^2 \times \mathbb{S}$, there is a unique solution of (4). Now, let $z(t) = \ln S(t)$ and $h(t) = \ln I(t)$, then system (4) becomes

$$\begin{cases} dz(t) = [A(r(t))e^{-z(t)} - \beta(r(t))e^{h(t)} - c_1(r(t)) + \alpha(r(t))e^{-z(t)}e^{2h(t)}]dt + \sigma_1(r(t))dB_1(t), \\ dh(t) = [\beta(r(t))e^{z(t)} - c_2(r(t)) - \alpha(r(t))e^{h(t)}]dt + \sigma_2(r(t))dB_2(t), \end{cases} \tag{6}$$

where $c_1(i) = \mu(i) + \frac{\sigma_1^2(i)}{2}$, $c_2(i) = \mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}$. From the proof of Lemma 1 in [40], we know that the positive recurrence and ergodic properties of system (4) are equivalent to system (6). The following theorem is to verify that system (6) satisfies the three conditions of Lemma 1.

(A) Set $\mathcal{R}^s = \left(\frac{\check{\alpha}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_i(\mu(i) + \eta(i) + \frac{1}{2}\sigma_2^2(i))}\right) > 1$, which will be determined later (see (11)). If $\sigma_i = 0, (i = 1, 2)$, then \mathcal{R}^s will be the reproduction number [29]. The role of \mathcal{R}^s with $\sigma_i \neq 0, (i = 1, 2)$ is similar to that of the reproduction number of deterministic models. The derivation of reproduction number is given by [41].

Theorem 2. Let assumption (A) hold. Then for any $i \in \mathbb{S}$ and for any initial value $(S(0), I(0), r(0)) \in \mathbb{R}_+^2 \times \mathbb{S}$, the solution $(S(t), I(t))$ of system (4) is positive recurrent and admits an unique ergodic stationary distribution.

Proof of Theorem 2. We only need to validate conditions (i)–(iii) in Lemma 1. We assume $\gamma_{ij} > 0, i \neq j$ and then the condition (i) in Lemma 1 holds. By using the same method as those in [37], we obtain that condition (ii) holds. In detail, we consider the following bounded open subset:

$$D = (1/\epsilon, \epsilon) \times (1/\epsilon, \epsilon) \subset \mathbb{R}_+^2,$$

where ϵ is sufficiently large number. Then $\bar{D} \subset \mathbb{R}_+^2$. Note that $D(x, i) = \text{diag}(\sigma_1^2(i), \sigma_2^2(i))$, $i \in \mathbb{S}$, which is positive definite. Then

$$\theta_{\max}(D(x, i)) \geq \theta_{\min}(D(x, i)) > 0. \tag{7}$$

On the other hand, we have for all $\zeta \in D$

$$\theta_{\min}(D(x, i))|\zeta|^2 \leq \zeta^T D(x, i)\zeta \leq \theta_{\max}(D(x, i))|\zeta|^2, \tag{8}$$

where $\theta_{\min}(D(x, i))$ and $\theta_{\max}(D(x, i))$ are two continuous functions. Hence we have $\hat{\theta} = \min_{(x,i) \in \bar{D} \times \mathbb{S}} \theta_{\min}(D(x, i)) > 0$ and $\tilde{\theta} = \max_{(x,i) \in \bar{D} \times \mathbb{S}} \theta_{\max}(D(x, i)) > 0$ from (7). Moreover, (8) implies that

$$\theta|\zeta|^2 \leq \zeta^T D(x, i)\zeta \leq \theta^{-1}|\zeta|^2,$$

where $\theta = \min\{\hat{\theta}, \tilde{\theta}\}$. We have therefore verified condition (ii) in Lemma 1.

Now we verify the condition (iii) in Lemma 1. We can define a C^2 -function $\hat{V} : \mathbb{R}_+^2 \times \mathbb{S} \rightarrow \mathbb{R}$

$$\hat{V}(z, h, i) = \frac{1}{\lambda + 1}(e^z + e^h)^{\lambda+1} - q(h + \frac{\check{\beta}}{\hat{\mu}}(e^z + e^h) - \omega_i) - z.$$

Take $\lambda \in (0, 1)$ and $q > 0$ such that

$$\hat{\mu} - \frac{\lambda}{2}\check{\sigma}_1^2 > 0, \hat{\mu} + \hat{\eta} - \frac{\lambda}{2}\check{\sigma}_2^2 > \text{ and } \max_{z \in [0, \infty]} f(z) - q \sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})(\mathcal{R}^s - 1) \leq -2. \tag{9}$$

We can verify that there is a unique point (z_0, h_0, i) , which is the minimum $\hat{V}(z, h, i)$. Define a C^2 -function as follows

$$V(z, h, i) = \frac{1}{\lambda + 1}(e^z + e^h)^{\lambda+1} - q(h + \frac{\check{\beta}}{\hat{\mu}}(e^z + e^h) - \omega_i) - z - \hat{V}(z_0, h_0, i).$$

Denote $V_1 = \frac{1}{\lambda+1}(e^z + e^h)^{\lambda+1}$, $V_2 = h + \frac{\check{\beta}}{\hat{\mu}}(e^z + e^h) - \omega_i$ and $V_3 = -z - \hat{V}(z_0, h_0, i)$. Applying Itô's formula (see [39]), we obtain

$$\begin{aligned} LV_1 &= (e^z + e^h)^\lambda [A(i) - \mu(i)e^z - (\mu(i) + \eta(i))e^h] + \frac{\lambda}{2}(e^z + e^h)^{\lambda-1}(\sigma_1^2(i)e^{2z} + \sigma_2^2(i)e^{2h}) \\ &\leq -(\mu(i) - \frac{\lambda}{2}\sigma_1^2(i))e^{(1+\lambda)z} - (\mu(i) + \eta(i) - \frac{\lambda}{2}\sigma_2^2(i))e^{(1+\lambda)h} \\ &\quad + 2^\lambda A(i)e^{\lambda z} + 2^\lambda A(i)e^{\lambda h}, \\ LV_3 &= -\frac{A(i)}{e^z} + \beta(i)e^h + \mu(i) + \frac{\sigma_1^2(i)}{2} - \frac{\alpha(i)}{e^z}e^{2h} \\ &\leq -\frac{A(i)}{e^z} + \beta(i)e^h + \mu(i) + \frac{\sigma_1^2(i)}{2}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} LV_2 &= \beta(i)e^z - (\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}) - \alpha(i)e^h \\ &\quad + \frac{\check{\beta}}{\hat{\mu}}(A(i) - \mu(i)e^z - (\mu(i) + \eta(i))e^h) - \sum_{l \in \mathbb{S}} \gamma_{il}\omega_l \\ &\leq \frac{\check{\beta}\check{A}}{\hat{\mu}} - (\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}) - \sum_{l \in \mathbb{S}} \gamma_{il}\omega_l - \alpha(i)e^h - \frac{\check{\beta}}{\hat{\mu}}(\hat{\mu} + \hat{\eta})e^h. \end{aligned}$$

Since the generator matrix Γ is irreducible, for $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_N)$ with $\mathcal{R}_i = -(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})$, there exists $\omega = (\omega_1, \dots, \omega_N)^T$ satisfying the following Poisson system (see Lemma 2.3 in [42])

$$\Gamma\omega = (\sum_{i=1}^N \pi_i \mathcal{R}_i)\vec{\mathbf{1}} - \mathcal{R},$$

where $\vec{\mathbf{1}}$ denotes the column vector with all its entries equal to one. Then

$$-\sum_{l=1}^N \gamma_{il}\omega_l - (\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}) = -\sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}),$$

which shows that

$$\begin{aligned} LV_2 &\leq \frac{\check{\beta}\check{A}}{\hat{\mu}} - \sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2}) - \alpha(i)e^h - \frac{\check{\beta}}{\hat{\mu}}(\hat{\mu} + \hat{\eta})e^h \\ &\leq \sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})(\mathcal{R}^s - 1) - (\hat{\alpha} + \check{\beta} + \frac{\check{\beta}\hat{\eta}}{\hat{\mu}})e^h, \end{aligned} \tag{11}$$

where

$$\mathcal{R}^s = \left(\frac{\check{A}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_i(\mu(i) + \eta(i) + \frac{1}{2}\sigma_2^2(i))} \right).$$

Combining (10) and (11), we can derive that

$$\begin{aligned} LV &= LV_1 - qLV_2 + L V_3 \\ &\leq -(\hat{\mu} - \frac{\lambda}{2}\check{\sigma}_1^2)e^{(1+\lambda)z} + 2^\lambda \check{A}e^{\lambda z} - \frac{\hat{A}}{e^z} + \check{\mu} + \frac{\check{\sigma}_1^2}{2} \\ &\quad -(\hat{\mu} + \hat{\eta} - \frac{\lambda}{2}\check{\sigma}_2^2)e^{(1+\lambda)h} + 2^\lambda \check{A}e^{\lambda h} - q\left(\sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})\right)(\mathcal{R}^s - 1) \\ &\quad -(\hat{\alpha} + \frac{\check{\beta}\hat{\eta}}{\hat{\mu}})e^h \\ &= f(z) + g(h), \end{aligned}$$

where

$$f(z) = -(\hat{\mu} - \frac{\lambda}{2}\check{\sigma}_1^2)e^{(1+\lambda)z} + 2^\lambda \check{A}e^{\lambda z} - \frac{\hat{A}}{e^z} + \check{\mu} + \frac{\check{\sigma}_1^2}{2},$$

and

$$\begin{aligned} g(h) &= -(\hat{\mu} + \hat{\eta} - \frac{\lambda}{2}\check{\sigma}_2^2)e^{(1+\lambda)h} + 2^\lambda \check{A}e^{\lambda h} \\ &\quad -q\left(\sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})\right)(\mathcal{R}^s - 1) - (\hat{\alpha} + \frac{\check{\beta}\hat{\eta}}{\hat{\mu}})e^h. \end{aligned}$$

Case I: if $z \rightarrow \infty$, we have

$$f(z) + g(h) \leq f(z) + g^u \rightarrow -\infty.$$

If $h \rightarrow \infty$, we have

$$f(z) + g(h) \leq f^u + g(h) \rightarrow -\infty.$$

Case II: if $z \rightarrow -\infty$, we have

$$f(z) + g(h) \leq f(z) + g^u \rightarrow -\infty.$$

If $h \rightarrow -\infty$, we have

$$f(z) + g(h) \leq f^u + g(h) \rightarrow f^u - q\left(\sum_{i=1}^N \pi_i(\mu(i) + \eta(i) + \frac{\sigma_2^2(i)}{2})\right)(\mathcal{R}^s - 1) \leq -2.$$

Therefore, let $\kappa > 0$ be large enough and $U = [-\kappa, \kappa] \times [-\kappa, \kappa]$, we obtain

$$LV(z, h, k) \leq -1 \text{ for any } (z, h, k) \in U^C \times \mathbb{S}.$$

Therefore, the condition (iii) in Lemma 1 is satisfied. It can be concluded that the system (4) is positive recurrent and has a unique ergodic stationary distribution by Lemma 1. This proof is complete. \square

4. Extinction

When studying the dynamic behavior of rumor models, another concern is how to eradicate rumors in the long term. For example, the government will severely punish those

who spread rumors. Those who spread rumors fear that they will be punished when they spread rumors, thus reducing the spread of rumors. Based on this, we shall present a sufficient condition in the stochastic model (4).

Lemma 2 ([43]). *Let $M = \{M_t\}_t \geq 0$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty \text{ a.s.} \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0 \text{ a.s.},$$

and also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty \text{ a.s.} \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0 \text{ a.s.}$$

Lemma 3. *Strong ergodicity theorem [44]. Assume that (φ_t, Λ_t) and (ψ_t, Λ_t) are positive recurrent. Let f be a bounded measurable function on $\mathbb{R} \times \mathcal{S}$. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\varphi_s, \Lambda_s) ds = \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} f(x, i) \pi^\varphi(dx, i),$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\psi_s, \Lambda_s) ds = \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} f(y, i) \pi^\psi(dy, i) \text{ a.s.}$$

Moreover

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_1(\Lambda_s) \varphi_s ds \geq \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_1(i) x \pi^\varphi(dx, i),$$

and

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_2(\Lambda_s) \psi_s ds \geq \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i) y \pi^\psi(dy, i) \text{ a.s.}$$

Theorem 3. *For any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, let $(S(t), I(t))$ be the solution of system (4). If*

$$\mathcal{R}^s = \left(\frac{\check{A}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_k(\mu(k) + \eta(k) + \frac{1}{2}\sigma_2^2(k))} \right) < 1,$$

then rumor I tends to zero exponentially with probability one, i.e.,

$$\lim_{t \rightarrow +\infty} I(t) = 0 \text{ a.s.}$$

There are

$$\limsup_{t \rightarrow \infty} \langle S \rangle_t \leq \frac{\check{A}}{\hat{\mu}}.$$

Proof of Theorem 3. From model (4), there are

$$\begin{aligned} \frac{S(t) - S(0)}{t} &= \frac{1}{t} \int_0^t A(r(s))ds - \frac{1}{t} \int_0^t (\beta(r(s))S(s)I(s))ds - \frac{1}{t} \int_0^t (\mu(r(s))S(s))ds \\ &\quad + \frac{1}{t} \int_0^t (\alpha(r(s))I^2(s))ds + \frac{1}{t} \int_0^t \sigma_1(r(s))S(s)dB_1(s) \\ &\leq \check{A} - \frac{1}{t} \int_0^t (\beta(r(s))S(s)I(s))ds - \hat{\mu} \langle S \rangle_t \\ &\quad + \frac{1}{t} \int_0^t (\alpha(r(s))I^2(s))ds + \frac{1}{t} \int_0^t \sigma_1(r(s))S(s)dB_1(s), \end{aligned}$$

$$\begin{aligned} \frac{I(t) - I(0)}{t} &= \frac{1}{t} \int_0^t (\beta(r(s))S(s)I(s))ds - \frac{1}{t} \int_0^t (\mu(r(s)) + \eta(r(s))I(s))ds \\ &\quad - \frac{1}{t} \int_0^t (\alpha(r(s))I^2(s))ds + \frac{1}{t} \int_0^t \sigma_2(r(s))I(s)dB_2(s). \end{aligned}$$

Then

$$\begin{aligned} \frac{S(t) - S(0)}{t} + \frac{I(t) - I(0)}{t} &\leq \check{A} - \hat{\mu} \langle S \rangle_t - (\hat{\mu} + \hat{\eta}) \langle I \rangle_t \\ &\quad + \frac{\check{\sigma}_1}{t} \int_0^t S(s)dB_1(s) + \frac{\check{\sigma}_2}{t} \int_0^t I(s)dB_2(s). \end{aligned}$$

We can obtain

$$\langle S \rangle_t \leq \frac{\check{A}}{\hat{\mu}} - \frac{\hat{\mu} + \hat{\eta}}{\hat{\mu}} \langle I \rangle_t + H(t),$$

where

$$H(t) = \frac{1}{\hat{\mu}} \frac{\check{\sigma}_1}{t} \int_0^t S(s)dB_1(s) + \frac{1}{\hat{\mu}} \frac{\check{\sigma}_2}{t} \int_0^t I(s)dB_2(s) - \frac{1}{\hat{\mu}} \frac{S(t) - S(0)}{t} - \frac{1}{\hat{\mu}} \frac{I(t) - I(0)}{t}.$$

According to Lemma 1, we have

$$\lim_{t \rightarrow \infty} H(t) = 0 \text{ a.s.}$$

Then

$$\limsup_{t \rightarrow \infty} \langle S \rangle_t \leq \frac{\check{A}}{\hat{\mu}}. \tag{12}$$

According to Itô's formula applied to $\ln I$, we obtain

$$d \ln I(t) = [\beta(r(t))S(t) - (\mu(r(t)) + \eta(r(t)) + \frac{\sigma_2^2(r(t))}{2}) - \alpha(r(t))I(t)]dt + \sigma_2(r(t))dB_2(t). \tag{13}$$

Integrating both sides of (13), we have

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &= \frac{1}{t} \int_0^t \beta(r(s))S(s)ds - \frac{1}{t} \int_0^t (\mu(r(s)) + \eta(r(s)) + \frac{\sigma_2^2(r(s))}{2})ds \\ &\quad - \frac{1}{t} \int_0^t \alpha(r(s))I(s)ds + \frac{1}{t} \int_0^t \sigma_2(r(s))dB_2(s) \\ &\leq \check{\beta} \frac{1}{t} \int_0^t S(s)ds - \frac{1}{t} \int_0^t (\mu(r(s)) + \eta(r(s)) + \frac{\sigma_2^2(r(s))}{2})ds \\ &\quad - \hat{\alpha} \frac{1}{t} \int_0^t I(s)ds + \frac{1}{t} \int_0^t \sigma_2(r(s))dB_2(s). \end{aligned} \tag{14}$$

From the strong ergodicity theorem in Lemma 3, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(r(s)) ds = \sum_{k=1}^N \pi_k \mu(k).$$

Taking the upper limit in inequality (14), and according to the Lemmas 2 and 3, we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \sum_{k=1}^N \pi_k (\mu(k) + \eta(k) + \frac{\sigma_2^2(k)}{2})(\mathcal{R}^s - 1) \leq 0,$$

which implies that

$$\lim_{t \rightarrow \infty} I(t) = 0.$$

The proof is completed. □

Remark 1. It can be seen from Theorems 2 and 3 that the \mathcal{R}^s mainly determines the persistence or extinction of the rumor. If $\mathcal{R}^s > 1$, the system (4) has a unique ergodic stationary distribution. This shows that rumor-infected I persists, while if $\mathcal{R}^s < 1$, the rumor-infected I goes to extinction exponentially with probability one. Hence, parameter \mathcal{R}^s is a threshold of system (4).

5. Numerical Simulations

In this part, we verify the main results of this paper through Milstein’s Higher Order Method in [45]. System (4) becomes

$$\begin{cases} S_{j+1} = S_j + [A(k) - \beta(k)S_jI_j - \mu(k)S_j + \alpha(k)I_j^2]\Delta t + \sigma_1(k)S_j\sqrt{(\Delta t)}\xi_j + \frac{\sigma_1^2(k)}{2}S_j(\xi_j^2 - 1)\Delta t, \\ I_{j+1} = I_j + [\beta(k)S_jI_j - (\mu(k) + \eta(k))I_j - \alpha(k)I_j^2]\Delta t + \sigma_2(k)I_j\sqrt{(\Delta t)}\xi_j + \frac{\sigma_2^2(k)}{2}I_j(\xi_j^2 - 1)\Delta t, \end{cases}$$

where $\Delta t > 0$ and $\xi_j (j = 1, \dots, n)$ are independent Gaussian random variables with distribution $N(0, 1)$. For convenience, we assume that the $r(t)$ is on the state space $\mathbb{S} = [1, 2]$ with the generator

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

It is easy to know that $r(t)$ has a unique stationary distribution with $\pi = (\pi_1, \pi_2) = (\frac{1}{3}, \frac{2}{3})$. The movement of $r(t)$ in $\mathbb{S} = 1, 2$ is illustrated in Figure 1.

Without loss of generality, we choose parameters $(A(1), A(2)) = (0.6, 0.7), (\beta(1), \beta(2)) = (0.2, 0.3), (\mu(1), \mu(2)) = (0.2, 0.3), (\eta(1), \eta(2)) = (0.2, 0.4), (\alpha(1), \alpha(2)) = (0.2, 0.3)$. The values is $S(0) = 1.5, I(0) = 0.5$.

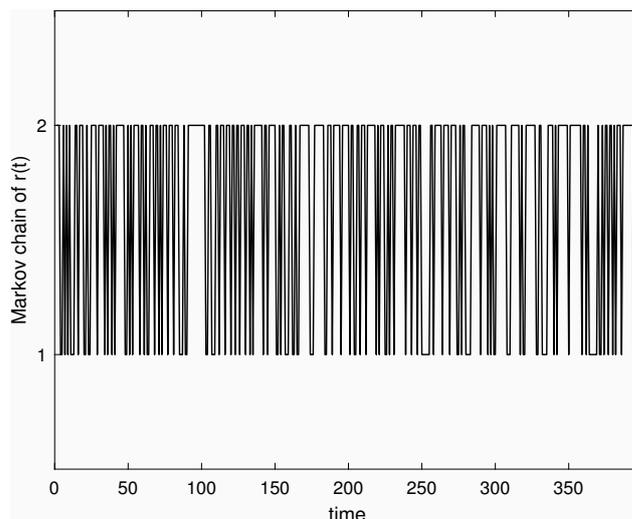


Figure 1. The movement of $r(t)$ taking values in $\mathbb{S} = 1, 2$.

Scenario 1. We simulate the solution $(S(t), I(t))$ of system (4) with middle noises ($\sigma_1(1) = 0.1, \sigma_1(2) = 0.12, \sigma_2(1) = 0.08, \sigma_2(2) = 0.1$, the top panel of Figure 2) and small noises ($\sigma_1(1) = 0.03, \sigma_1(2) = 0.05, \sigma_2(1) = 0.03, \sigma_2(2) = 0.05$, the bottom panel of Figure 2), respectively. By simple calculations, we obtain $\mathcal{R}^S = \left(\frac{\hat{A}\hat{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_k (\mu(k) + \eta(k) + \frac{1}{2} \sigma_2^2(k))} \right) > 1$. In view of Theorem 2, one can see that system (4) has a unique stationary distribution. It is consistent with the findings in the sample paths of $S(t)$ and $I(t)$ under both middle noises (Figure 2a) and small noises (Figure 2d). Moreover, one can see that when the noises become smaller, the fluctuation of system (4) becomes weaker. This phenomenon can also be shown more intuitively through the histograms and densities of the solution $(S(t), I(t))$ in Figure 2b,c (for middle noises) and Figure 2e,f (for small noises).

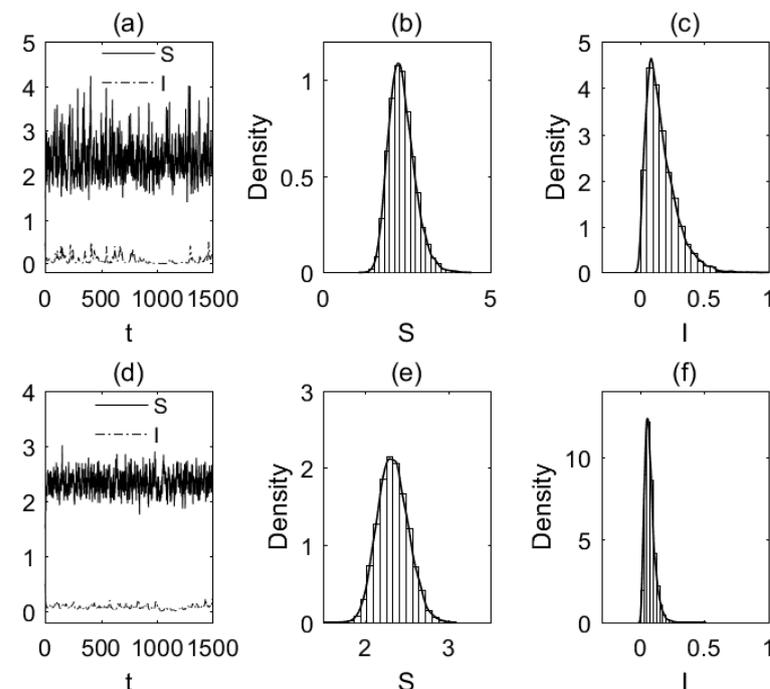


Figure 2. The simulated solution $(S(t), I(t))$ of system (4) with middle noises (top panel) and small noises (bottom panel). The left panel (a,d) is the sample paths of S (solid line) and I (dash-dot line), the middle panel (b,e) is the histograms and densities of S , and the right panel (c,f) is the histograms and densities of I .

Scenario 2. We now simulate the solution $(S(t), I(t))$ of system (4) with middle–large noises ($\sigma_1(1) = 0.2, \sigma_1(2) = 0.2, \sigma_2(1) = 1, \sigma_2(2) = 1$, the top panel of Figure 3) and large noises ($\sigma_1(1) = 0.3, \sigma_1(2) = 0.3, \sigma_2(1) = 1.2, \sigma_2(2) = 1.2$, the bottom panel of Figure 3), respectively. Direct calculation leads to $\mathcal{R}^s = \left(\frac{\check{A}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_k(\mu(k) + \eta(k) + \frac{1}{2}\sigma_2^2(k))}\right) < 1$. That is to say, the condition in Theorem 3 holds. By Theorem 3, one can obtain that the rumor I tends to zero exponentially with probability one. It can be seen from Figure 3a,d that $I(t)$ tends towards zero under middle–large and large noises. Due to the influence of random disturbance, $S(t)$ will fluctuate around the stable value. From the histograms and densities of the solution $(S(t), I(t))$ in Figure 3b,c (for middle–large noises) and Figure 3e,f (for large noises), the density function of $I(t)$ also tends towards zero, which means that the rumor will disappear.

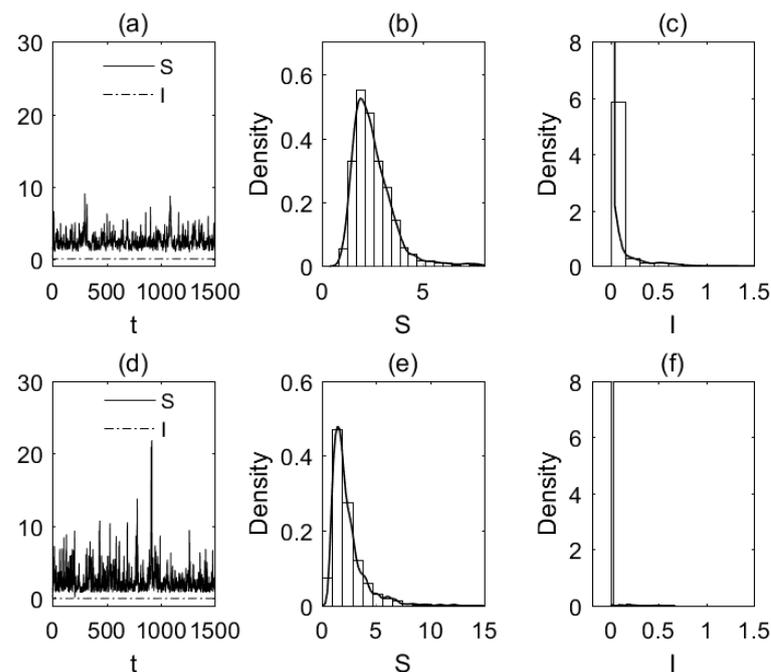


Figure 3. The simulated solution $(S(t), I(t))$ of system (4) with middle-large noises (top panel) and large noises (bottom panel). The left panel (a,d) is the sample path of S (solid line) and I (dash-dot line), the middle panel (b,e) is the histograms and densities of S , and the right panel (c,f) is the histograms and densities of I .

6. Conclusions

In this research article, the dynamical behavior of a regime-switching rumor model is considered. By designing a stochastic Lyapunov function, the sufficient conditions for ergodic stationary distribution and extinction are obtained. A stochastic reproduction number \mathcal{R}^s is obtained as a threshold to identify the stochastic persistence and extinction. If $\mathcal{R}^s > 1$, there is a unique ergodic stationary distribution of system (4). This means that the rumors are random and persistent. Whereas if $\mathcal{R}^s < 1$, the rumor becomes extinct exponentially with probability one. More precisely,

- If $\mathcal{R}^s = \left(\frac{\check{A}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_k(\mu(k) + \eta(k) + \frac{1}{2}\sigma_2^2(k))}\right) > 1$, the system (4) has a unique ergodic stationary distribution, which means that the rumor I is persistent in the mean a.s.
- If $\mathcal{R}^s = \left(\frac{\check{A}\check{\beta}}{\hat{\mu} \sum_{k=1}^N \pi_k(\mu(k) + \eta(k) + \frac{1}{2}\sigma_2^2(k))}\right) < 1$, the rumor becomes extinct exponentially with probability one.

One shortcoming of this article is that it just considers the influence of switching on rumor propagation. The influence of impulse disturbance on the rumor model can be further studied.

Author Contributions: Conceptualization, F.J. and C.C.; Methodology, F.J. and C.C.; Software, C.C.; Investigation, C.C.; Validation, F.J.; Writing—original draft preparation, F.J.; Writing—review and editing, F.J. and C.C. All authors have read and agreed to the published version of the manuscript.

Funding: The Natural Science Foundation of Jiangsu Province of China (Grant No. BK20191394) and the research was supported by the Major Project of the National Social Science Foundation (Grant No. 16ZDA047).

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the anonymous reviewers for their constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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