

Article

Novel Analytical Approach for the Space-Time Fractional (2+1)-Dimensional Breaking Soliton Equation via Mathematical Methods

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Abstract: The aim of this work is to build novel analytical wave solutions of the nonlinear space-time fractional (2+1)-dimensional breaking soliton equations, with regards to the modified Riemann–Liouville derivative, by employing mathematical schemes, namely, the improved simple equation and modified F-expansion methods. We used the fractional complex transformation of the concern fractional differential equation to convert it for the solvable integer order differential equation. After the successful implementation of the presented methods, a comprehensive class of novel and broad-ranging exact and solitary travelling wave solutions were discovered, in terms of trigonometric, rational and hyperbolic functions. Hence, the present methods are reliable and efficient for solving nonlinear fractional problems in mathematics physics.

Keywords: space-time fractional (2+1)-dimensional breaking soliton equation; modified mathematical method; fractional derivatives



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1. Introduction

The fractional order of nonlinear partial differential equations (NPDEs) arise in numerous grounds like the theoretical physics, dynamical system, fluid dynamics, elasticity, gas dynamics, plasma physics, solid state physics, MHD and many others. The construction of the exact solutions is one of the dominant leitmotifs in applied mathematics and theoretical physics. Moreover, the fractional order in calculus is one the developing fields of applied mathematics and physics and whose idea was first instigated with the possibility of fractional derivatives by Leibniz in 1695 [1]. Liouville laid the basic rules of this subject in (1832) and Riemann discovered the fractional derivative of a power function Riemann in (1847) [2]. The plethora of attention has been received to determine the incipient and closed form exact solutions of FPDEs by many researchers. The potential symbolic computer programming implements have been utilized to investigate opportune solutions to NFPDEs, namely, first integral method [3–5], modified simple equation schemes [6,7], direct algebraic methods and auxiliary equation techniques [8–10], fractional equation scheme [11–14], (G'/G) -expansion scheme [15–17], Lie-symmetry technique [18], Seadawy and Exp function methods [19–22], tanh-coth scheme [23], generalized Kudryashov scheme [24–29] and many more [30–38].

Let the time fractional breaking soliton model be stated as in [39]:

$$\begin{aligned} \frac{\partial^\alpha U}{\partial t^\alpha} + a \frac{\partial^{3\alpha} U}{\partial x^{2\alpha} y^\alpha} + 4aU \frac{\partial^\alpha V}{\partial x^\alpha} + 4a \frac{\partial^\alpha U}{\partial x^\alpha} V &= 0, \\ \frac{\partial^\alpha U}{\partial y^\alpha} - \frac{\partial^\alpha V}{\partial x^\alpha} &= 0, \quad 0 < \alpha \leq 1. \end{aligned} \tag{1}$$

Currently different schemes to find solutions of Equation (1) have been presented. Yildirim and Yasar discussed a wave solution of the (2+1)-dimensional breaking soliton model in [40]. In [41], the author derived multi-soliton solutions to Equation (1) by the variable coefficients method. Negative-order breaking soliton equations and breaking soliton equations of typical and higher orders have been assessed in [42]. Yildiz and Daghah [43], by using two different methods, investigated exact solutions of Equation (1). Wang in [44] and Chen and Ma in [45] suggested the analytical multi-soliton solutions and exact solutions to the (2+1)-dimensional breaking soliton equation respectively. New multi-soliton solutions and Symmetries solutions of Equation (1) have been established in [46,47]. Furthermore, a technologically advanced scheme in [48] utilized the three wave method to obtain some exact solutions for Equation (1). To the best of our understanding, Equation (1) has not been studied with our implemented mathematical methods, see details in [49–51].

The rest of manuscript is arranged as follows: in Section 2 the definition and rudimental properties of the modified Riemann-Liouville fractional order derivative are provided. In Section 3, we illustrate the sequence of the amended simple equation and modified F-expansion methods. In Section 4, we Implement these techniques to find incipient exact solitary wave solutions of Equation (1). Results, discussion and conclusion are mentioned in Sections 5 and 6, respectively.

2. Preliminaries and Basic Definitions

Definition 1. Let the Jumarie’s modified Riemann-Liouville derivative of order α with the continuous function $F:R \rightarrow R, \sim x \rightarrow F(x)$ be stated as in [52]:

$$\begin{aligned} D_x^\alpha &= \frac{1}{\Gamma(-\alpha)} \int_0^x (F(\eta) - F(0))(x - \eta)^{-\alpha-1} d\eta, \quad \alpha < 0, \\ &\frac{1}{\Gamma(-\alpha)} \int_0^x (F(\eta) - F(0))(x - \eta)^{-\alpha-1} d\eta, \quad 0 < \alpha < 1, \\ &((F^n(x))^{\alpha-n}), \quad n \leq \alpha \leq n + 1, \quad n \geq 1 \end{aligned} \tag{2}$$

Γ is defined as

$$\Gamma(\alpha) = \lim_{n \rightarrow \infty} \frac{n!n^\alpha}{(\alpha(\alpha + 1)(\alpha + 2)\dots(\alpha + n)} \tag{3}$$

$$\Gamma(x) = \int_0^\infty dx e^{-t} t^{x-1} \tag{4}$$

Definition 2. The Mittag-Leffler function via two parameters is explained in [53]:

$$e_{\alpha, \beta_1}^x = \sum_{i=0}^\infty \frac{x^i}{\Gamma(\alpha i + \beta_1)}, \quad \Re(\alpha) \geq 0, \sim \beta_1 \sim x \in \mathbb{C} \tag{5}$$

This function is used for utilization to scrutinize the fractional PDEs as the exponential function with integer order. For the fractional derivative, some postulates are as follows:

First Postulate:

$$D_x^\alpha x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha}, \quad r > 0 \tag{6}$$

where r is a real number.

Second Postulate:

$$D_x^\alpha (C_1 F(x)) = C_1 D_x^\alpha (F(x)), \quad C_1 = \text{constant}. \tag{7}$$

Third Postulate:

$$D_x^\alpha (AF(x) + BG(x)) = AD_x^\alpha (F(x)) + BD_x^\alpha (G(x)) \tag{8}$$

Fourth Postulate:

$$D_x^\alpha F(\eta) = \frac{dF}{d\eta} D_x^\alpha (\eta) \tag{9}$$

where $\eta = G(x)$.

3. Description of the Proposed Methods

Let the NFPDEs be

$$F(U, D_t^\alpha U, D_x^\alpha U, D_t^\alpha D_t^\alpha U, D_x^\alpha D_x^\alpha U, \dots) = 0. \tag{10}$$

Let

$$U = U(\xi), \quad \xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{\delta t^\alpha}{\Gamma(\alpha+1)} \tag{11}$$

Substitute Equation (11) into Equation (10),

$$G(U, U', U'', U''', \dots) = 0, \tag{12}$$

3.1. Improved Simple Equation Method

Let the solution of (12) be

$$U(\xi) = \sum_{i=-N}^N A_i \Psi^i(\xi) \tag{13}$$

Let Ψ satisfy

$$\Psi'(\xi) = c_0 + c_1 \Psi + c_2 \Psi^2 + c_3 \Psi^3 \tag{14}$$

Put Equation (13) with Equation (12) into Equation (14).

3.2. Modified F-Expansion Method

Let (12) have the following solution:

$$U = a_0 + \sum_{i=1}^N a_i F^i(\xi) + \sum_{i=1}^N b_i F^{-i}(\xi) \tag{15}$$

$$F' = A + BF + CF^2. \tag{16}$$

Put Equation (15) with Equation (16) in Equation (12), solve the obtained algebraic system of equations for investigating the solution of Equation (10).

4. Applications

The space-time fractional breaking soliton equation:

Put Equation (11) into Equation (1) (Figures 1–6), we have the following ODE form:

$$\begin{aligned} -\delta U' + ak^2\omega U''' + 4akUV' + 4akU'V &= 0, \\ \omega U' - kV' &= 0. \end{aligned} \tag{17}$$

Now we integrate the second equation, Equation (17), by taking the constant of integration as equal to zero,

$$\omega U = kV \tag{18}$$

Putting Equation (18) into the first equation of (17), yields

$$-\delta U + ak^2\omega U'' + 4aU^2\omega = 0, \tag{19}$$

4.1. Applications of Improved Simple Equation Method

Let Equation (19) have the following solution;

$$U = A_2\psi^2 + A_1\psi + \frac{A_{-2}}{\psi^2} + \frac{A_{-1}}{\psi} + A_0 \tag{20}$$

Put (20) with (14) in (19),

CASE 1: $c_3 = 0$,

Family-I

$$\begin{aligned} A_{-2} &= 0, \quad A_{-1} = 0, \quad A_2 = \frac{1}{2}(-3)c_2^2k^2, \quad A_1 = \frac{1}{2}(-3)c_1c_2k^2, \\ A_0 &= -\frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 + a(c_1^2 + 8c_0c_2)k^2}}{8a}, \quad \delta = -\omega\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4} \end{aligned} \tag{21}$$

Put (21) in (20),

$$\begin{aligned} U_1 &= -\frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 + a(c_1^2 + 8c_0c_2)k^2}}{8a} - \frac{3c_1c_2k^2\left(-\left(c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)\right)}{2(2c_2)} \\ &\quad - \frac{3}{2}c_2^2k^2\left(-\frac{c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)}{2c_2}\right)^2, \quad 4c_0c_2 > c_1^2. \end{aligned} \tag{22}$$

$$\begin{aligned} V_1 &= -\omega\frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 + a(c_1^2 + 8c_0c_2)k^2}}{8ka} - \frac{3\omega c_1c_2k^2\left(-\left(c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)\right)}{2(2kc_2)} \\ &\quad - \frac{3\omega}{2}c_2^2k\left(-\frac{c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)}{2c_2}\right)^2, \quad 4c_0c_2 > c_1^2. \end{aligned} \tag{23}$$

Family-II

$$A_{-2} = \frac{1}{2}(-3)c_0^2k^2, \quad A_{-1} = \frac{1}{2}(-3)c_0c_1k^2, \quad A_2 = 0, \quad A_1 = 0,$$

$$A_0 = \frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 - a(c_1^2 + 8c_0c_2)k^2}}{8a}, \quad \omega\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4} = \delta \quad (24)$$

put (24), in (20),

$$U_2 = \frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 - a(c_1^2 + 8c_0c_2)k^2}}{8a} - \frac{3c_0c_1(2c_2)k^2}{2\left(-\left(c_1 - \sqrt{4c_2c_0 - c_1^2} \tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)\right)}$$

$$\frac{1}{2}(-3)c_0^2k^2 \left(-\frac{2c_2}{c_1 - \sqrt{4c_2c_0 - c_1^2} \tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)} \right)^2, \quad 4c_0c_2 > c_1^2 \quad (25)$$

$$V_2 = \left(\frac{\sqrt{a^2(c_1^2 - 4c_0c_2)^2k^4 - a(c_1^2 + 8c_0c_2)k^2}}{8a} - \frac{3c_0c_1(2c_2)k^2}{2\left(-\left(c_1 - \sqrt{4c_2c_0 - c_1^2} \tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)\right)} \right)$$

$$\left(\frac{\omega}{k} \right) - \frac{1}{2\omega}(3)c_0^2k \left(-\frac{2c_2}{c_1 - \sqrt{4c_2c_0 - c_1^2} \tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)} \right)^2, \quad 4c_0c_2 > c_1^2 \quad (26)$$

CASE 2: $c_0 = 0, c_3 = 0,$

$$A_0 = 0, \quad A_{-2} = 0, \quad A_{-1} = 0, \quad A_2 = \frac{1}{2}(-3)c_2^2k^2, \quad A_1 = \frac{1}{2}(-3)c_1c_2k^2, \quad \delta = ac_1^2k^2\omega \quad (27)$$

Put (27) in (20),

$$U_3 = -\frac{1}{2}3c_2^2k^2 \left(\frac{c_1 \exp(c_1(\xi + \xi_0))}{1 - c_2 \exp(c_1(\xi + \xi_0))} \right)^2 - \frac{3c_1c_2k^2(c_1 \exp(c_1(\xi + \xi_0)))}{2(1 - c_2 \exp(c_1(\xi + \xi_0)))}, \quad c_1 > 0, \quad (28)$$

$$V_3 = \left(-\frac{1}{2}3c_2^2k^2 \left(\frac{c_1 \exp(c_1(\xi + \xi_0))}{1 - c_2 \exp(c_1(\xi + \xi_0))} \right)^2 - \frac{3c_1c_2k^2(c_1 \exp(c_1(\xi + \xi_0)))}{2(1 - c_2 \exp(c_1(\xi + \xi_0)))} \right) \frac{\omega}{k}, \quad c_1 > 0, \quad (29)$$

$$U_4 = -\frac{1}{2}3c_2^2k^2 \left(-\frac{c_1 \exp(c_1(\xi + \xi_0))}{c_2 \exp(c_1(\xi + \xi_0)) + 1} \right)^2 - \frac{3c_1c_2k^2(-c_1 \exp(c_1(\xi + \xi_0)))}{2(c_2 \exp(c_1(\xi + \xi_0)) + 1)}, \quad c_1 < 0. \quad (30)$$

$$V_4 = \left(-\frac{1}{2}3c_2^2k^2 \left(-\frac{c_1 \exp(c_1(\xi + \xi_0))}{c_2 \exp(c_1(\xi + \xi_0)) + 1} \right)^2 - \frac{3c_1c_2k^2(-c_1 \exp(c_1(\xi + \xi_0)))}{2(c_2 \exp(c_1(\xi + \xi_0)) + 1)} \right) \frac{\omega}{k}, \quad c_1 < 0. \quad (31)$$

CASE 3: $c_1 = 0, c_3 = 0,$

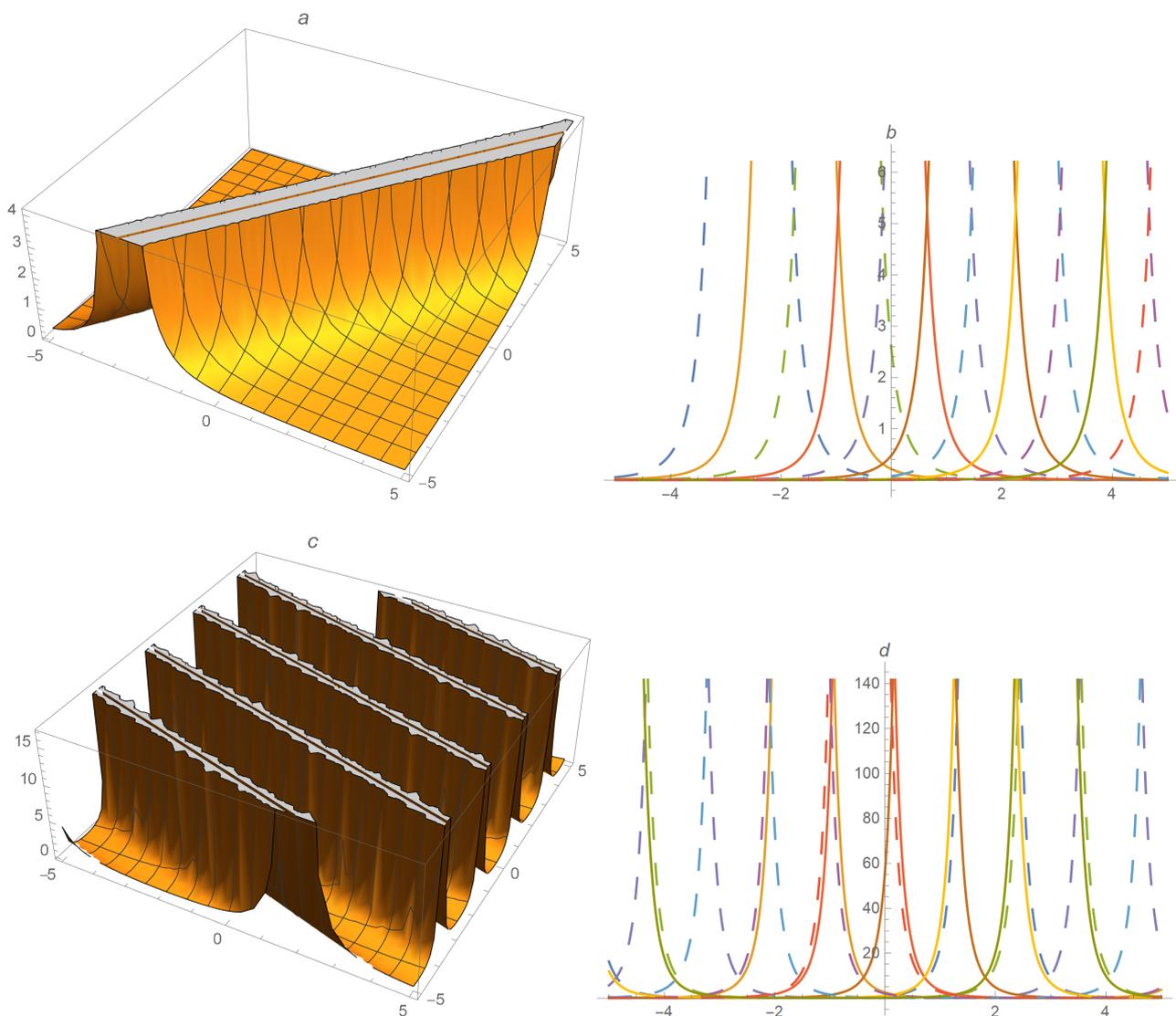


Figure 1. The profile of solutions U_2 (a,b) and V_2 (c,d) with $\alpha = 1, c_0 = 2, c_2 = 1, c_1 = 2, k = -0.2, a = 2.01, \omega = 0.5, \xi_0 = 10.5$ and $\alpha = 1, c_0 = 2, c_2 = 1, c_1 = 2, k = 0.4, a = 2.01, \omega = 1.05, \xi_0 = 0.5$ respectively.

Family-I

$$A_0 = -\frac{\delta}{8a\omega}, A_{-2} = 0, A_{-1} = 0, A_2 = -\frac{3c_2\delta}{8ac_0\omega}, A_1 = 0, k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{c_0}\sqrt{c_2}\sqrt{\omega}} \quad (32)$$

Put (32) in (20),

$$U_5 = -\frac{(3\delta c_2) \left(\frac{\sqrt{c_2 c_0} \tan(\sqrt{c_2 c_0}(\xi + \xi_0))}{c_2} \right)^2}{8ac_0\omega} - \frac{\delta}{8a\omega}, \quad c_0 c_2 > 0, \quad (33)$$

$$V_5 = \left(-\frac{(3\delta c_2) \left(\frac{\sqrt{c_2 c_0} \tan(\sqrt{c_2 c_0}(\xi + \xi_0))}{c_2} \right)^2}{8ac_0\omega} - \frac{\delta}{8a\omega} \right) \left(\frac{\omega}{k} \right), \quad c_0 c_2 > 0, \quad (34)$$

$$U_6 = -\frac{(3c_2\delta)\left(-\frac{\sqrt{-c_2c_0}\tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2}\right)^2}{8ac_0\omega} - \frac{\delta}{8a\omega}, \quad c_0c_2 < 0. \tag{35}$$

$$V_6 = \left(-\frac{(3c_2\delta)\left(-\frac{\sqrt{-c_2c_0}\tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2}\right)^2}{8ac_0\omega} - \frac{\delta}{8a\omega}\right)\left(\frac{\omega}{k}\right), \quad c_0c_2 < 0. \tag{36}$$

Family-II

$$A_0 = -\frac{\delta}{8a\omega}, \quad A_{-2} = -\frac{3c_0\delta}{8ac_2\omega}, \quad A_{-1} = A_2 = A_1 = 0, \quad k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{c_0}\sqrt{c_2}\sqrt{\omega}} \tag{37}$$

Put (37) in (20),

$$U_7 = -\frac{(3\delta c_0)\left(\frac{1}{\frac{\sqrt{c_2c_0}\tan(\sqrt{c_2c_0}(\xi+\xi_0))}{c_2}}\right)^2}{8ac_2\omega} - \frac{c}{8a\omega}, \quad c_0c_2 > 0, \tag{38}$$

$$V_7 = \left(-\frac{(3\delta c_0)\left(\frac{1}{\frac{\sqrt{c_2c_0}\tan(\sqrt{c_2c_0}(\xi+\xi_0))}{c_2}}\right)^2}{8ac_2\omega} - \frac{c}{8a\omega}\right)\left(\frac{\omega}{k}\right), \quad c_0c_2 > 0, \tag{39}$$

$$U_8 = -\frac{(3cc_0)\left(-\frac{1}{\frac{\sqrt{-c_2c_0}\tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2}}\right)^2}{8ac_2\omega} - \frac{c}{8a\omega}, \quad c_0c_2 < 0. \tag{40}$$

$$V_8 = \left(-\frac{(3cc_0)\left(-\frac{1}{\frac{\sqrt{-c_2c_0}\tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2}}\right)^2}{8ac_2\omega} - \frac{c}{8a\omega}\right)\left(\frac{\omega}{k}\right), \quad c_0c_2 < 0. \tag{41}$$

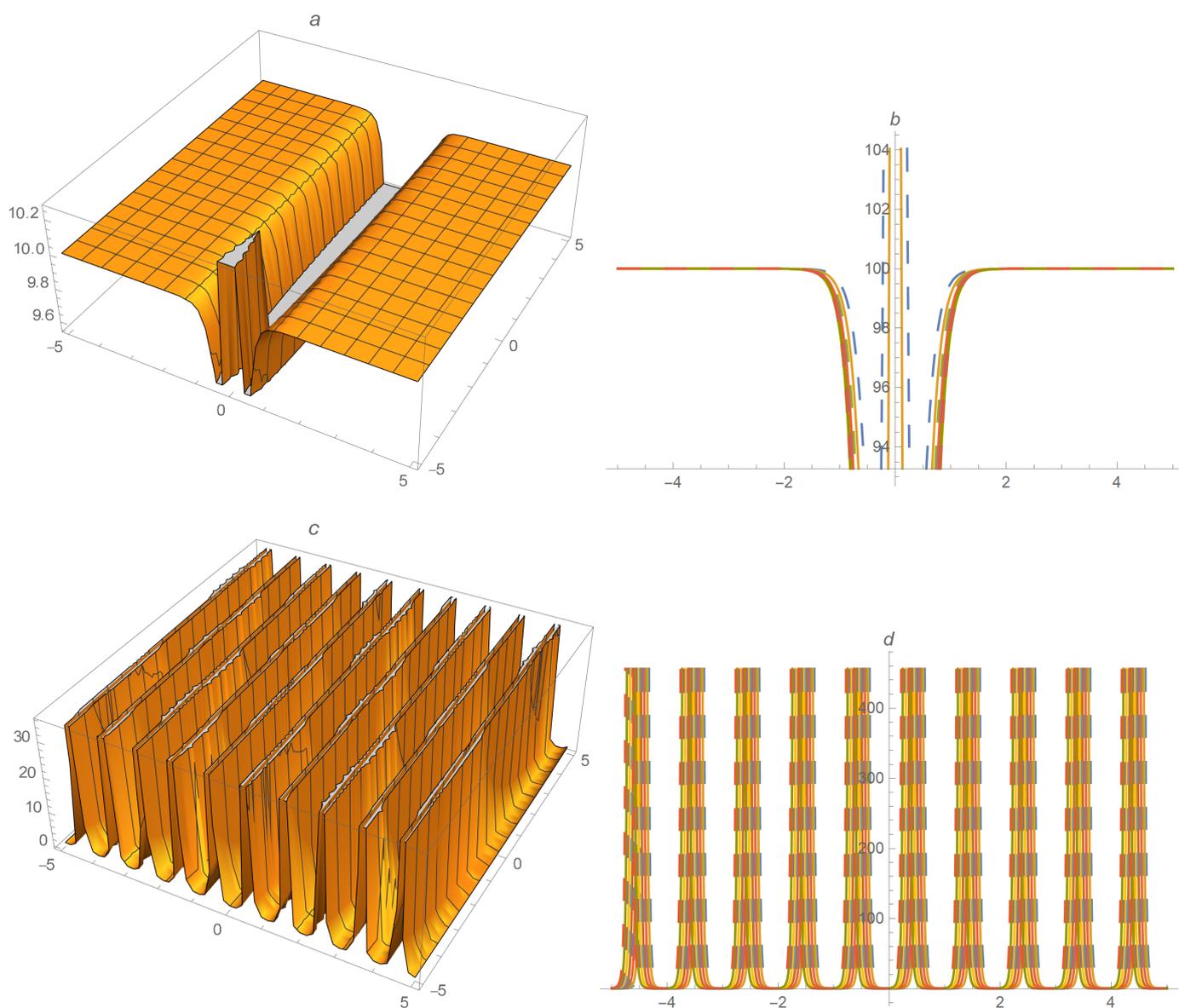


Figure 2. The profile of solutions U_5 (a,b) and V_5 (c,d) with $\alpha = 1, c_0 = 0.2, c_2 = 1, c_1 = 2, \delta = -0.2, a = 0.01, \omega = 0.5, \xi_0 = 0.5$ and $\alpha = 1, c_0 = 2, c_2 = 1, c_1 = 2, \delta = 0.2, a = 0.01, \omega = 0.05, \xi_0 = 0.5$ respectively.

Family-III

$$A_0 = \frac{\delta}{16a\omega}, A_{-2} = -\frac{3c_0\delta}{32ac_2\omega}, A_{-1} = 0, A_2 = -\frac{3c_2\delta}{32ac_0\omega}, A_1 = 0, k = \frac{\sqrt{\delta}}{4\sqrt{a}\sqrt{c_0}\sqrt{c_2}\sqrt{\omega}} \tag{42}$$

Put (42) in (20),

$$U_9 = -\frac{(3c_2\delta)\left(\frac{\sqrt{c_2c_0}\tan(\sqrt{c_2c_0}(\xi+\xi_0))}{c_2}\right)^2}{32ac_0\omega} - \frac{(3c_0\delta)\left(\frac{1}{\sqrt{c_2c_0}\tan(\sqrt{c_2c_0}(\xi+\xi_0))}\right)^2}{32ac_2\omega} + \frac{\delta}{16a\omega}, \quad c_0c_2 > 0, \tag{43}$$

$$V_9 = \left(-\frac{(3c_2\delta) \left(\frac{\sqrt{c_2c_0} \tan(\sqrt{c_2c_0}(\xi+\xi_0))}{c_2} \right)^2}{32ac_0\omega} - \frac{(3c_0\delta) \left(\frac{1}{\sqrt{c_2c_0} \tan(\sqrt{c_2c_0}(\xi+\xi_0))} \right)^2}{32ac_2\omega} + \frac{\delta}{16a\omega} \right) \left(\frac{\omega}{k} \right), \quad c_0c_2 > 0, \tag{44}$$

$$U_{10} = -\frac{(3c_2\delta) \left(-\frac{\sqrt{-c_2c_0} \tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2} \right)^2}{32ac_0\omega} - \frac{(3c_0\delta) \left(-\frac{1}{\sqrt{-c_2c_0} \tanh(\sqrt{-c_2c_0}(\xi+\xi_0))} \right)^2}{32ac_2\omega} + \frac{\delta}{16a\omega}, \quad c_0c_2 < 0. \tag{45}$$

$$V_{10} = \left(-\frac{(3c_2\delta) \left(-\frac{\sqrt{-c_2c_0} \tanh(\sqrt{-c_2c_0}(\xi+\xi_0))}{c_2} \right)^2}{32ac_0\omega} - \frac{(3c_0\delta) \left(-\frac{1}{\sqrt{-c_2c_0} \tanh(\sqrt{-c_2c_0}(\xi+\xi_0))} \right)^2}{32ac_2\omega} \right) \left(\frac{\omega}{k} \right) + \frac{\delta}{16a\omega} \left(\frac{\omega}{k} \right), \quad c_0c_2 < 0. \tag{46}$$

4.2. Applications of Modified F-Expansion Method

Let solution of Equation (19) is

$$U = a_2F^2 + a_1F + a_0 + \frac{b_2}{F^2} + \frac{b_1}{F} \tag{47}$$

Put (47) in (19) with (16),

For A = 0, B = 1, C = -1

$$a_0 = 0, \quad a_2 = -\frac{3\delta}{2a\omega}, \quad a_1 = \frac{3\delta}{2a\omega}, \quad b_1 = 0, \quad b_2 = 0, \quad k = \frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{48}$$

Put (48) in (47),

$$U_{11} = \frac{3\delta}{4a\omega} \left(\tanh\left(\frac{\xi}{2}\right) + 1 \right) - \frac{3\delta}{8a\omega} \left(\tanh\left(\frac{\xi}{2}\right) + 1 \right)^2 \tag{49}$$

$$V_{11} = \left(\frac{\omega}{k} \right) \left(\frac{3\delta}{4a\omega} \left(\tanh\left(\frac{\xi}{2}\right) + 1 \right) - \frac{3\delta}{8a\omega} \left(\tanh\left(\frac{\xi}{2}\right) + 1 \right)^2 \right) \tag{50}$$

For A = 0, B = -1, C = 1

$$a_0 = 0, \quad a_2 = -\frac{3\delta}{2a\omega}, \quad a_1 = \frac{3\delta}{2a\omega}, \quad b_1 = 0, \quad b_2 = 0, \quad k = \frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{51}$$

Substitute (51) into (47),

$$U_{12} = \frac{3\delta}{4a\omega} \left(1 - \coth\left(\frac{\xi}{2}\right) \right) - \frac{3\delta}{8a\omega} \left(1 - \coth\left(\frac{\xi}{2}\right) \right)^2 \tag{52}$$

$$V_{12} = \left(\frac{\omega}{k}\right) \left(\frac{3\delta}{4a\omega} \left(1 - \coth\left(\frac{\xi}{2}\right)\right) - \frac{3\delta}{8a\omega} \left(1 - \coth\left(\frac{\xi}{2}\right)\right)^2\right) \tag{53}$$

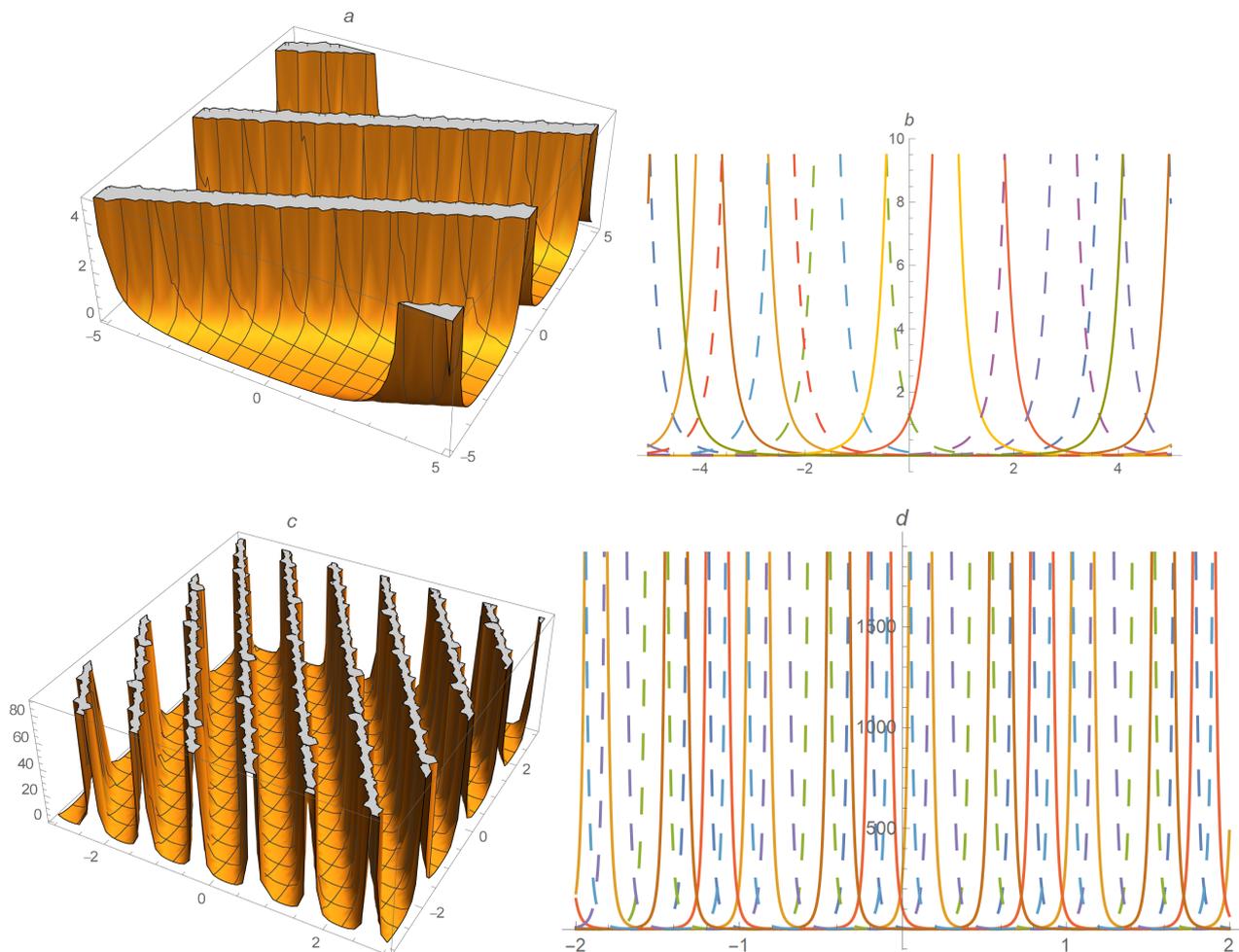


Figure 3. The profile of solutions U_9 (a,b) and V_9 (c,d) with $c_2 = 1, c_0 = 3.2, \alpha = 1, \delta = 0.2, a = -1, \omega = -0.5, \xi_0 = 0.5$ and $\alpha = 1, c_2 = -1, c_0 = -0.2, \delta = -2.2, \lambda = 0.11, \omega = -0.5, \xi_0 = -0.5$ respectively.

$A = 1/2, B = 0, C = -1/2$
 Family-I

$$a_0 = \frac{3\delta}{8a\omega}, a_2 = -\frac{3\delta}{8a\omega}, a_1 = 0, b_1 = 0, b_2 = 0, k = \frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{54}$$

Put (54) in (47),

$$U_{131} = \frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left((\coth(\xi) + \operatorname{csch}(\xi))^2 \right) \tag{55}$$

$$V_{131} = \left(\frac{\omega}{k}\right) \left(\frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left((\coth(\xi) + \operatorname{csch}(\xi))^2 \right)\right) \tag{56}$$

Family-II

$$a_0 = \frac{3\delta}{8a\omega}, a_2 = 0, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{8a\omega}, k = -\frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{57}$$

Put (57) in (47),

$$U_{132} = \frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} 1 \left(\frac{1}{(\coth(\xi) + \operatorname{csch}(\xi))} \right)^2 \tag{58}$$

$$V_{132} = \left(\frac{\omega}{k} \right) \left(\frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{(\coth(\xi) + \operatorname{csch}(\xi))} \right)^2 \right) \tag{59}$$

Family-III

$$a_0 = \frac{3\delta}{16a\omega}, a_2 = -\frac{3\delta}{32a\omega}, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{32a\omega}, k = -\frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \tag{60}$$

Put (60) in (47),

$$U_{133} = \frac{3\delta}{16a\omega} - \frac{3\delta}{32a\omega} \left(\left((\coth(\xi) + \operatorname{csch}(\xi))^2 + \left(\frac{1}{(\coth(\xi) + \operatorname{csch}(\xi))} \right)^2 \right) \right) \tag{61}$$

$$V_{133} = \left(\frac{\omega}{k} \right) \left(\frac{3\delta}{16a\omega} - \frac{3\delta}{32a\omega} \left(\left((\coth(\xi) + \operatorname{csch}(\xi))^2 + \left(\frac{1}{(\coth(\xi) + \operatorname{csch}(\xi))} \right)^2 \right) \right) \right) \tag{62}$$

$$C = -1, B = 0, A = 1$$

Family-I

$$a_0 = \frac{3\delta}{8a\omega}, a_2 = -\frac{3\delta}{8a\omega}, a_1 = 0, b_1 = 0, b_2 = 0, k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \tag{63}$$

Put (63) in (47),

$$U_{141} = \frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \tanh^2(\xi) \tag{64}$$

$$V_{141} = \left(\frac{\omega}{k} \right) \left(\frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \tanh^2(\xi) \right) \tag{65}$$

Family-II

$$a_0 = \frac{3\delta}{8a\omega}, a_2 = 0, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{8a\omega}, k = -\frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \tag{66}$$

Put (66) in (47),

$$U_{142} = \frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\tanh(\xi)} \right)^2 \tag{67}$$

$$V_{142} = \left(\frac{\omega}{k}\right) \left(\frac{3\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\tanh(\xi)}\right)^2\right) \tag{68}$$

Family-III

$$a_0 = \frac{3\delta}{16a\omega}, a_2 = -\frac{3\delta}{32a\omega}, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{32a\omega}, k = \frac{\sqrt{\delta}}{4\sqrt{a}\sqrt{\omega}} \tag{69}$$

Put (69) in (47),

$$U_{143} = \frac{3\delta}{16a\omega} - \frac{3\delta}{32a\omega} (\text{Tanh}[\xi])^2 - \frac{3\delta}{32a\omega} \left(\frac{1}{\text{Tanh}[\xi]}\right)^2 \tag{70}$$

$$V_{143} = \left(\frac{\omega}{k}\right) \left(\frac{3\delta}{16a\omega} - \frac{3\delta}{32a\omega} (\text{Tanh}[\xi])^2 - \frac{3\delta}{32a\omega} \left(\frac{1}{\text{Tanh}[\xi]}\right)^2\right) \tag{71}$$

C = A = 1/2, B = 0.

Family-I

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = -\frac{3\delta}{8a\omega}, a_1 = 0, b_1 = 0, b_2 = 0, k = -\frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{72}$$

Put (72) in (47),

$$U_{151} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\tan(\xi) + \sec(\xi))^2 \tag{73}$$

$$V_{151} = \left(\frac{\omega}{k}\right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\tan(\xi) + \sec(\xi))^2\right) \tag{74}$$

Family-II

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = 0, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{8a\omega}, k = \frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \tag{75}$$

Put (75) in (47),

$$U_{152} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\tan(\xi) + \sec(\xi)}\right)^2 \tag{76}$$

$$V_{152} = \left(\frac{\omega}{k}\right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\tan(\xi) + \sec(\xi)}\right)^2\right) \tag{77}$$

Family-III

$$a_0 = \frac{\delta}{16a\omega}, a_2 = -\frac{3\delta}{32a\omega}, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{32a\omega}, k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \tag{78}$$

Put (78) in (47),

$$U_{153} = \frac{\delta}{16a\omega} - \frac{3\delta}{32a\omega} \left((\text{Sec}[\xi] + \text{Tan}[\xi]) + \left(\frac{1}{\tan(\xi) + \sec(\xi)} \right)^2 \right) \quad (79)$$

$$V_{153} = \left(\frac{\omega}{k} \right) \left(\frac{\delta}{16a\omega} - \frac{3\delta}{32a\omega} \left((\text{Sec}[\xi] + \text{Tan}[\xi]) + \left(\frac{1}{\tan(\xi) + \sec(\xi)} \right)^2 \right) \right) \quad (80)$$

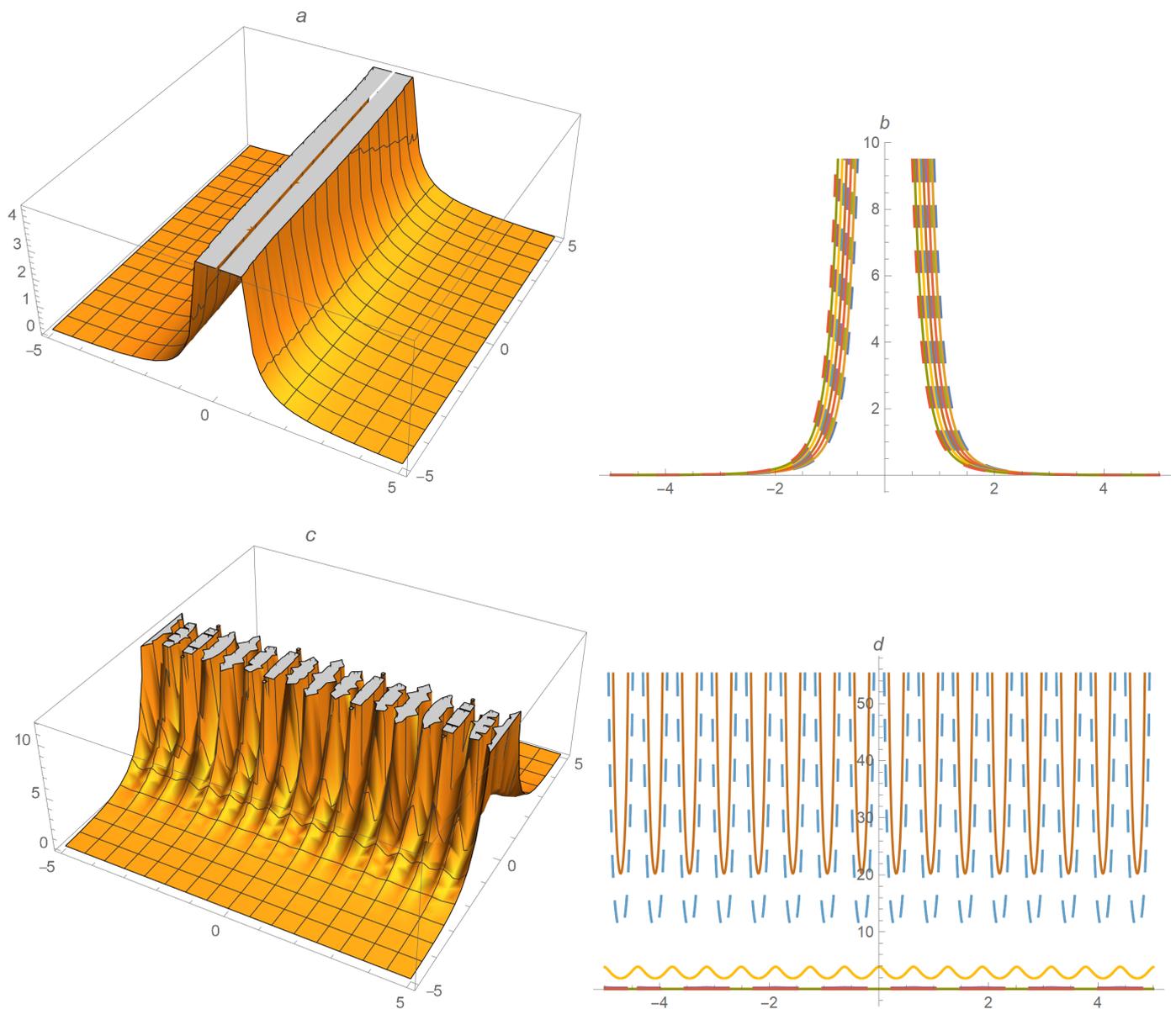


Figure 4. The profile of solutions U_{143} (a,b) and V_{143} (c,d) with $\alpha = 1, \delta = 0.002, a = 0.1, \omega = 0.5$ and $\alpha = 1, \delta = -0.3, a = 0.01, \omega = 0.3$ respectively.

$$C = A = -1/2, B = 0,$$

Family-I

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = -\frac{3\delta}{8a\omega}, a_1 = 0, b_1 = 0, b_2 = 0, k = \frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \quad (81)$$

Put (81) in (47),

$$U_{161} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\sec(\xi) - \tan(\xi))^2 \quad (82)$$

$$V_{161} = \left(\frac{\omega}{k}\right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\sec(\xi) - \tan(\xi))^2\right) \quad (83)$$

Family-II

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = 0, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{8a\omega}, k = -\frac{\sqrt{\delta}}{\sqrt{a}\sqrt{\omega}} \quad (84)$$

Put (84) in (47),

$$U_{162} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\sec(\xi) - \tan(\xi)}\right)^2 \quad (85)$$

$$V_{162} = \left(\frac{\omega}{k}\right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\sec(\xi) - \tan(\xi)}\right)^2\right) \quad (86)$$

Family-III

$$a_0 = \frac{\delta}{16a\omega}, a_2 = -\frac{3\delta}{32a\omega}, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{32a\omega}, k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \quad (87)$$

Put (87) in (47),

$$U_{163} = \frac{\delta}{16a\omega} - \frac{3\delta}{32a\omega} ((\sec(\xi) - \tan(\xi))^2 + \left(\frac{1}{\sec(\xi) - \tan(\xi)}\right)^2) \quad (88)$$

$$V_{163} = \left(\frac{\omega}{k}\right) \left(\frac{\delta}{16a\omega} - \frac{3\delta}{32a\omega} ((\sec(\xi) - \tan(\xi))^2 + \left(\frac{1}{\sec(\xi) - \tan(\xi)}\right)^2)\right) \quad (89)$$

C = A = -1, B = 0.

Family-I

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = -\frac{3\delta}{8a\omega}, a_1 = 0, b_1 = 0, b_2 = 0, k = \frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \quad (90)$$

Put (90) in (47),

$$U_{171} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\cot^2(\xi)) \quad (91)$$

$$V_{171} = \left(\frac{\omega}{k}\right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} (\cot^2(\xi))\right) \quad (92)$$

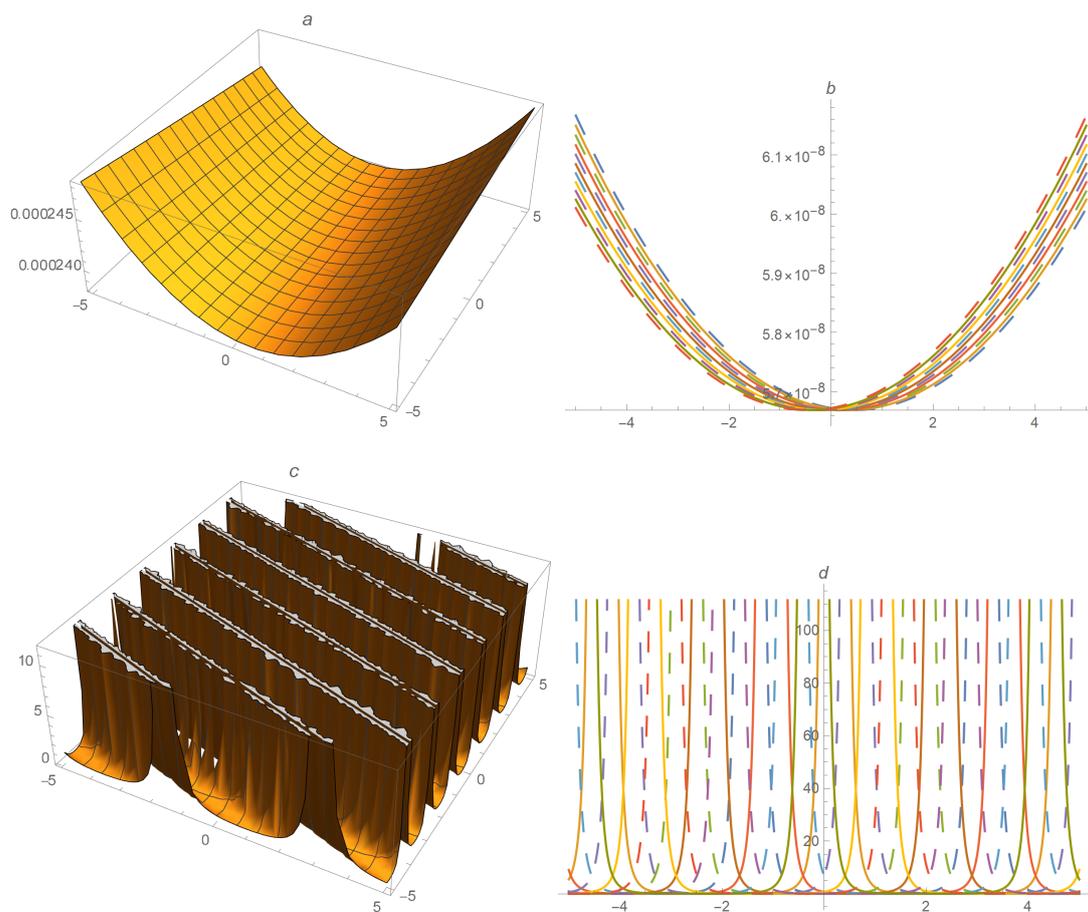


Figure 5. The profile of solutions U_{163} (a,b) and V_{163} (c,d) with $\alpha = 1, \delta = 0.002, a = 2.1, \omega = 0.5$ and $(\alpha = 1, \delta = 2, a = 2.1, \omega = 0.5)$ respectively.

Family-II

$$a_0 = -\frac{\delta}{8a\omega}, a_2 = 0, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{8a\omega}, k = -\frac{\sqrt{\delta}}{2\sqrt{a}\sqrt{\omega}} \tag{93}$$

Put (93) in (47),

$$U_{172} = -\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\cot(\xi)} \right)^2 \tag{94}$$

$$V_{172} = \left(\frac{\omega}{k} \right) \left(-\frac{\delta}{8a\omega} - \frac{3\delta}{8a\omega} \left(\frac{1}{\cot(\xi)} \right)^2 \right) \tag{95}$$

Family-III

$$a_0 = \frac{\delta}{16a\omega}, a_2 = -\frac{3\delta}{32a\omega}, a_1 = 0, b_1 = 0, b_2 = -\frac{3\delta}{32a\omega}, k = \frac{\sqrt{\delta}}{4\sqrt{a}\sqrt{\omega}} \tag{96}$$

Put (96) in (47),

$$U_{173} = \frac{\delta}{16a\omega} + \frac{3\delta}{32a\omega} (\cot^2(\xi) - \left(\frac{1}{\cot(\xi)} \right)^2) \tag{97}$$

$$V_{173} = \left(\frac{\omega}{k}\right) \left(\frac{\delta}{16a\omega} + \frac{3\delta}{32a\omega}(\cot^2(\zeta)) - \left(\frac{1}{\cot(\zeta)}\right)^2\right) \tag{98}$$

For C = 0

$$a_0 = 0, a_2 = 0, a_1 = 0, b_1 = \frac{1}{2}(-3)ABk^2, b_2 = \frac{1}{2}(-3)A^2k^2, \omega = \frac{\delta}{aB^2k^2} \tag{99}$$

Put (99) in (47),

$$U_{18} = \frac{1}{2}(-3)ABk^2 \left(\frac{1}{\exp(\frac{B\zeta}{B}) - A}\right) - \frac{1}{2}(-3)A^2k^2 \left(\frac{1}{\exp(\frac{B\zeta}{B}) - A}\right)^2 \tag{100}$$

$$V_{18} = \left(\frac{\omega}{k}\right) \left(\frac{1}{2}(-3)ABk^2 \left(\frac{1}{\exp(\frac{B\zeta}{B}) - A}\right) - \frac{1}{2}(-3)A^2k^2 \left(\frac{1}{\exp(\frac{B\zeta}{B}) - A}\right)^2\right) \tag{101}$$

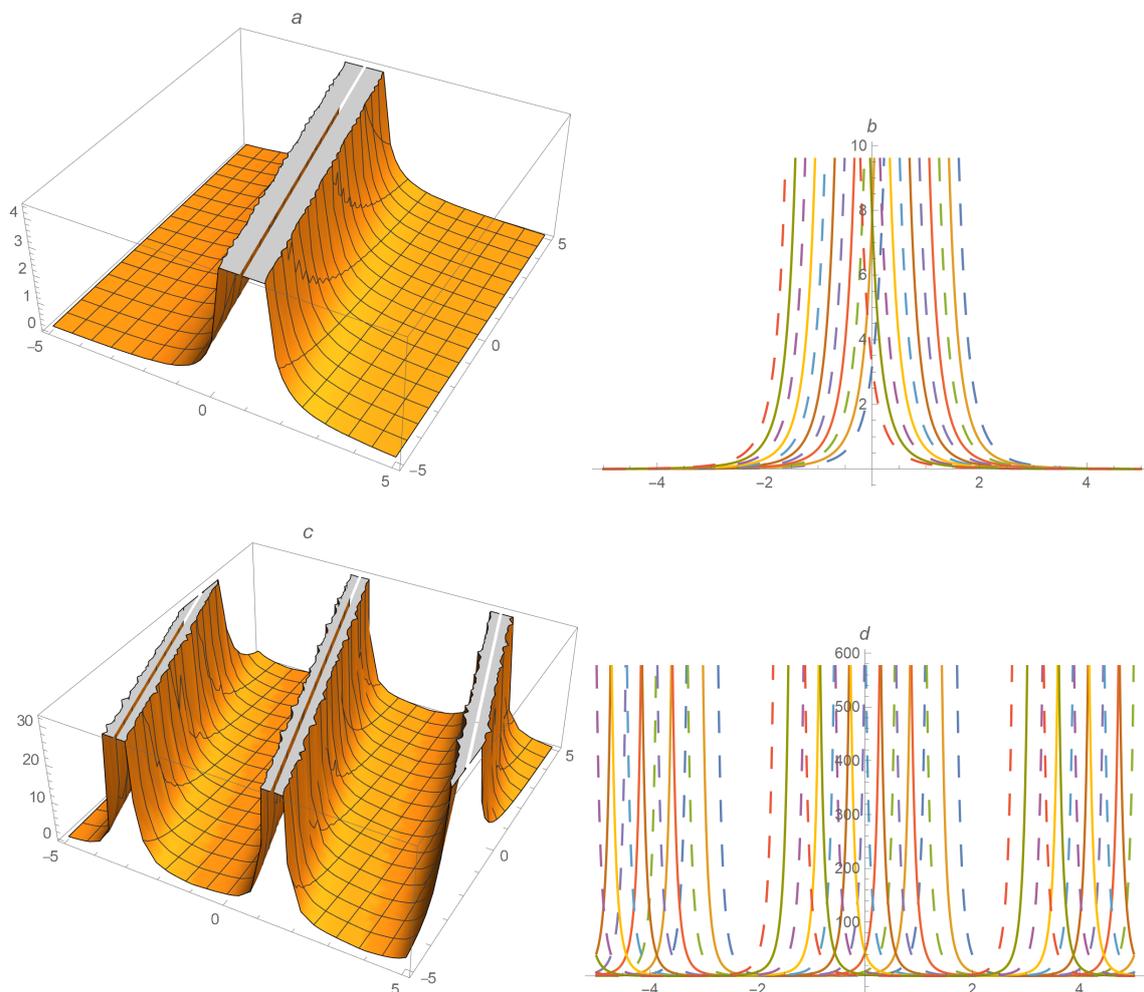


Figure 6. The profile of solutions U_{173} (a,b) and V_{173} (c,d) with $\alpha = 1, \delta = 0.002, a = 2.1, \omega = 0.5$ and $\alpha = 1, \delta = 0.1, a = 0.1, \omega = 0.5$ respectively.

5. Results and Discussion

The mathematical methods emphasize the wave solutions of Equation (1). In the derived solutions, parameters A_1, A_{-1}, A_2 and A_{-2} received various specific values due to these exact solutions being converted into different solitary wave solutions in different

forms, such as hyperbolic, trigonometric and rational functions (Figures 1–6). Currently several methods have been utilized to solve Equation (1) throughout the research literature [40–48]. Moreover, our investigated solutions are likely similar to other solutions in different research articles. Our solutions U_6 and V_6 in (35) and (36) are likely similar to the solutions U_{41} in Equation (82) and U_{42} in Equation (83), respectively, in [50]. Furthermore, our solutions U_{141} and V_{141} in (64) and (65) are similar in form to the solutions u_2 and v_2 in Equation (4.39), respectively, in [54]. Our solutions U_8 and V_9 in (41) and (42) are similar in form to the solutions $u_{23(\eta)}$ and $u_{214(\eta)}$ in Equation (50) and Equation (51), respectively, in [55]. The remainder of our derived solutions are novel and have not yet been reported in any research literature. Hence, our employed schemes are simple and useful for solving many other nonlinear problems in applied sciences.

6. Conclusions

In this study, we have proposed two novel techniques, namely, an improved form of simple equation and a modified form of F-expansion are utilized to construct exact solutions of the nonlinear fractional time space (2+1)-dimensional breaking soliton equation, via the properties of the modified Riemann-Liouville derivative. Multifarious transformation has been operated to renovate the fractional order differential equations. The constructed results have extensive potential to comprehend the interior configurations of the usual manifestations that arise in physics, mathematics and in other different fields. Hence, it is worth declaring that the execution of our techniques is extremely steady and well-organized for fractional differential equations.

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