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On the Topology of Warped Product Pointwise Semi-Slant Submanifolds with Positive Curvature

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Abstract: In this paper, we obtain some topological characterizations for the warping function of a warped product pointwise semi-slant submanifold of the form $\Omega^n = N_T^l \times_f N_\phi^k$ in a complex projective space $\mathbb{C}P^{2m}(4)$. Additionally, we will find certain restrictions on the warping function f, Dirichlet energy function $\mathbb{E}(f)$, and first non-zero eigenvalue λ_1 to prove that stable l-currents do not exist and also that the homology groups have vanished in Ω^n. As an application of the non-existence of the stable currents in Ω^n, we show that the fundamental group $\pi_1(\Omega^n)$ is trivial and Ω^n is simply connected under the same extrinsic conditions. Further, some similar conclusions are provided for CR-warped product submanifolds.

Keywords: warped product submanifolds; complex projective spaces; homology groups; homotopy; sphere theorems; stable currents; kinetic energy

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1. Introduction and Main Results

A classical challenge in Riemannian geometry is to discuss the geometrical and topological structures of submanifolds. The stable currents and homology groups are the most important characterizations of the Riemannian submanifolds because they control the behavior of the topology of submanifolds. The notion of non-existence stable current and vanishing homology on pinching the second fundamental form was introduced by Lawson-Simons [1]. Xin proved in [2] as the following important form:

Theorem 1 ([1,2]). *Suppose Ω^n is a compact n-dimensional submanifold in a space form $\widetilde{\Omega}(c)$ of curvature $c \geq 0$. Suppose l, k is any positive integer, that is, $l + k = n$, and the inequality*

$$\sum_{A=1}^{l} \sum_{B=l+1}^{n} \Big\{ 2||h(e_A, e_B)||^2 - g\big(h(e_A, e_A), h(e_B, e_B)\big) \Big\} < lkc \tag{1}$$

holds for all orthonormal basis $\{e_i, \cdots, e_n\}$ of the tangent space $T\Omega^n$; then there are no stable l-currents in Ω^n and

$$\mathbb{H}_l(M^n, \mathbb{G}) = \mathbb{H}_{n-l=k}(\Omega^n, \mathbb{G}) = 0,$$

where $\mathbb{H}_i(\Omega^n, \mathbb{G})$ stands for i integral homology groups of Ω^n, while $\mathbb{G}$ is a finite abelian group with integer coefficients.

The generalized Poincaré conjecture for dimension $n \geq 5$ was proved by Smale [3] by using the nonexistence for the stable currents over compact submanifolds on a sphere. Then, Lawson and Simons obtained the striking sphere theorem in [1], in which they showed that an n-dimensional compact-oriented submanifold Ω^n in the unit sphere $\mathbb{S}^{n+k}$

is homeomorphic to the sphere $\mathbb{S}^n$ with $n \neq 3$, provided that the second fundamental form was bounded above by a constant that depends on the dimension n. Additionally, it was proved that Ω^3 is homotopic to the sphere $\mathbb{S}^3$. Using Theorem 1, Leung [4] proved that for a compact connected oriented submanifold Ω^n in the unit sphere $\mathbb{S}^{n+k}$ with $\|h(X,X)\|^2 < \frac{1}{3}$ thus Ω^n is homeomorphic to the sphere $\mathbb{S}^n$ in the case $n \neq 3$, and also that Ω^3 is homotopic to a sphere $\mathbb{S}^3$. More recently, geometric, topological, and differentiable rigidity theorems of the Riemannian submanifold connecting to parallel mean curvature in space forms such that $c + H^2 > 0$ have been obtained in terms of Ricci curvature in [5]. In some articles such as [3,6–17], several results have been derived on topological and differentiable structures of singular submanifolds and submanifolds with specific effective conditions for the second fundamental form, sectional curvatures, and Ricci curvatures.

However, very few topological obstructions to warped product submanifolds with positive sectional curvature are known; for example, Sahin et al. [13] verified some outcomes for the non-existence of the stable current and vanishing homology groups into a contact CR-warped product which immersed in a sphere with an odd dimension, by putting suitable restrictions on the Laplacian and the gradient of the warping function. Taking the benefits of the constant section curvature which could be zero or one, Sahin [13,14] extended this study on a class of CR-warped product in an Euclidean space and in the nearly Kaehler six-sphere. By assuming negative constant section curvature, Ali et al. [18–20] obtained various results on CR-warped product, especially on the complex hyperbolic spaces, and many structures about this subject remain open.

Therefore, we shall study the warped product pointwise semi-slant submanifolds of complex projective spaces where the constant sectional curvature $c = 4 > 0$ is positive. More specifically, our motivation comes from the studies of Sahin [21]. In that paper, Sahin investigated the warped product pointwise semi-slant submanifolds in a Kähler manifold, and also showed that the warped product pointwise semi-slant of form $N_T^l \times_f N_\phi^k$ is nontrivial. It was shown by the Ref. [21] that the warped product pointwise semi-slant submanifold $N_T^l \times_f N_\phi^k$ of Kähler manifold generalized the CR-warped products [22] and the angle ϕ is treated as a slant function. In this case, suppose $\mathbb{C}^* = \mathbb{C} - \{0\}$ and $\mathbb{C}_*^{m+1} = \mathbb{C}^{m+1} - \{0\}$. Additionally, assuming that the action $\mathbb{C}^*$ on $\mathbb{C}_*^{m+1}$ can be expressed using γ, which means $(z_0, z_1, \ldots, z_m) = (\gamma z_0, \gamma z_1, \ldots, \gamma z_m)$, then all equivalent classes set are produced from this idea are represented using $\mathbb{C}P^m$. If we denote with $\pi(z)$, the equivalent classes which contains z, then $\mathbb{C}_*^{m+1} \to \mathbb{C}P^m$ is a surjection, and it is well-known that $\mathbb{C}P^m$ endowed a complex structure derived by the complex construction of $\mathbb{C}^{m+1}$ with a Kähler metric such that the constant holomorphic sectional curvature is equal to 4 [23]. We can observe that the almost complex J on $\mathbb{C}P^m(4)$ is determined by the almost complex construction of $\mathbb{C}^{m+1}$ *via* the Hopf fibration. Let us now recall introduce the following Theorem 1.

Theorem 2. *Let $\Omega^n = N_T^l \times_f N_\phi^k$ be a compact warped product pointwise semi-slant submanifold with regard to the complex projective space $\mathbb{C}P^{2m}(4)$, which satisfies the following condition*

$$f\Delta f + (\csc^2\phi + \cot^2\phi + k)||\nabla f||^2 < \Big(3l - \frac{\|h_\mu\|^2}{k}\Big)f^2, \tag{2}$$

where ∇f and Δf are the gradient and the Laplacian of the warped function f, respectively. Then we have the following:

(a) The warped product submanifold Ω^n does not exist for any stable integral l-currents.

(b) The i integral homology groups of Ω^n with integer coefficients vanish; that is,

$$\mathbb{H}_l(\Omega^n, \mathbb{G}) = \mathbb{H}_k(\Omega^n, \mathbb{G}) = 0.$$

(c) The finite fundamental group $\pi_1(\Omega)$ is null, that is, $\pi_1(\Omega) = 0$. Moreover, Ω^n is a simply connected warped product manifold.

Remark 1. *To apply Theorem 2, suppose the slant function ϕ becomes globally constant, setting $\phi = \frac{\pi}{2}$ from [24]. Then, the pointwise slant submanifold N^k_ϕ turns into a totally real submanifold $N^k_\perp$. Thus, a warped product pointwise semi-slant submanifold $\Omega^n = N^l_T \times_f N^k_\phi$ turns to CR-warped products within a Kähler manifold of the type $\Omega^n = N^l_T \times_f N^k_\perp$ such that N^l_T, as well as $N^k_\perp$ are holomorphic and totally real submanifolds, respectively [22].*

Therefore, we deduce the following result from Theorem 2 and Remark 1 for the non-existence of stable integrable l-currents and homology groups in the CR-warped product submanifolds of the complex projective space $\mathbb{C}P^{2m}(4)$.

Corollary 1. *Let $\Omega^n = N^l_T \times_f N^q_\perp$ be a compact CR-warped product submanifold of the complex projective space $\mathbb{C}P^{2m}(4)$. In this case, the following conditions occur:*

$$f\Delta f + (1+k)\|\nabla f\|^2 < \Big(3l - \frac{\|h_\mu\|^2}{k}\Big) f^2. \tag{3}$$

Then we have the following:

(a) For the CR-warped product submanifold, Ω^n does not have any stable integral l-currents.

(b) The i integral homology groups of Ω^n with integer coefficients vanish; that is,

$$\mathbb{H}_l(\Omega^n, \mathbb{G}) = \mathbb{H}_k(\Omega^n, \mathbb{G}) = 0.$$

(c) The finite fundamental group $\pi_1(\Omega)$ is null, that is, $\pi_1(\Omega) = 0$. Moreover, Ω^n is a simply connected warped product manifold.

Other important motivation for our study comes from the Ref. [25], where some geometric mechanics on Riemannian manifolds were studied. From that study, we found that for a positive differentiable function φ ($\varphi \in \mathcal{F}(\Omega^n)$) defined at a compact Riemannian manifold Ω, the *Dirichlet energy* of that function φ is given as in see [25] (p. 41), as follows:

$$\mathbb{E}(\varphi) = \frac{1}{2}\int_{\Omega^n} ||\nabla\varphi||^2 dV \qquad 0 < E(\varphi) < \infty. \tag{4}$$

Using the Dirichlet energy Formula (4) for a compact manifold without a boundary, as well as Theorem 2, we give the next theorem:

Theorem 3. *Under similar suppositions as in Theorem 2 with satisfied pinching condition*

$$\mathbb{E}(f) < \frac{1}{(4\csc^2\phi + 2k)}\int_{\Omega^n} \Big(3l - \frac{\|h_\mu\|^2}{k}\Big) f^2 dV. \tag{5}$$

Thus, the following properties hold:

(a) For the warped product submanifold Ω^n, there are no stable integral l-currents.

(b) The i integral homology groups of Ω^n with integer coefficients vanished; that is,

$$\mathbb{H}_l(\Omega^n, \mathbb{G}) = \mathbb{H}_k(\Omega^n, \mathbb{G}) = 0,$$

(c) The finite fundamental group $\pi_1(\Omega)$ is null, that is, $\pi_1(\Omega) = 0$. Moreover, Ω^n is a simply connected warped product manifold.

Using the result of Theorem 3, we can now recall the next sphere theorem for the compact oriented CR-warped product submanifold of a complex projective space $\mathbb{C}P^{2m}(4)$ due to Chen [22], that is,

Corollary 2. *Let $M^n = N_T^p \times_f N_\perp^q$ be a compact CR-warped product submanifold at a complex projective space $\mathbb{C}P^{2m}(4)$ satisfying*

$$\mathbb{E}(f) < \left(\frac{1}{2(2+k)}\right) \int_{\Omega^n} \left(3l - \frac{\|h_\mu\|^2}{k}\right) f^2 dV. \tag{6}$$

Then, the following properties are satisfied:

(a) For the warped product submanifold Ω^n, there are no stable integral l-currents.
(b) The i integral homology groups of Ω^n with integer coefficients vanished; that is,

$$\mathbb{H}_l(\Omega^n, \mathbb{G}) = \mathbb{H}_k(\Omega^n, \mathbb{G}) = 0.$$

(c) The finite fundamental group $\pi_1(\Omega)$ is null, that is, $\pi_1(\Omega) = 0$. Moreover, Ω^n is simply connected warped product manifold.

Let Ω^n be an n-dimensional compact Riemannian manifold, and therefore, the Laplacian is a second-order quasilinear operator on Ω^n, given as

$$\Delta\varphi = -div(\nabla\varphi). \tag{7}$$

For such a Laplacian, we can found many applications in mathematics as well as in physics, and this is possible due to the eigenvalue problem of Δ. The corresponding Laplace eigenvalue equation is defined as follows: a real number λ is named eigenvalue if it is a non-vanishing function φ, which satisfies the following equation:

$$\Delta\varphi = \lambda\varphi, \quad \text{on} \quad \Omega^n, \tag{8}$$

with appropriate boundary conditions. Considering a Riemannian manifold Ω^n with no boundary, the first nonzero eigenvalue of Δ, defined as λ_1, includes variational properties (cf. [26]):

$$\lambda_1 = \inf\left\{ \frac{\int_\Omega ||\nabla\varphi||^2 dV}{\int_\Omega \|\varphi\|^2 dV} \Big| \; \varphi \in W^{1,2}(\Omega^n)\backslash\{0\}, \int_\Omega \varphi dV = 0 \right\}. \tag{9}$$

Inspired by the above characterization, using the first non-zero eigenvalue of the Laplace operator and the maximum principle for the first non-zero eigenvalue λ_1, we deduce the following:

Theorem 4. *Let $\Omega^n = N_T^l \times_f N_\phi^k$ be compact, oriented warped product pointwise semi-slant submanifolds of the complex projective space $\mathbb{C}P^{2m}(4)$; that is, f is a non-constant eigenfunction of the first non-zero eigenvalue λ_1. Assume that*

$$\lambda_1 < \left(\frac{\int_{\Omega^n} \left(3l - \frac{\|h_\mu\|^2}{k}\right) f^2 dV}{(2\csc^2\phi + k) \int_{\Omega^n} f^2 dV} \right) \tag{10}$$

holds. Then the properties (a), (b), and (c) of Theorem 2 are satisfied.

Remark 2. *Some important applications of this theory can be found for the singularity structure in liquid crystals, in the system in statistical mechanics with low dimensions, and physical phase transitions (see [27]). In addition, general relativity contains warped product manifolds as a model of space-times. There are two famous warped product spaces. One is the generalization of Robertson-Walker space-times, and the other is the standard static space-times [17,28–31]. General relativity depends heavily on the differential topological methods, especially in mathematical physics, and particularly regarding the way that the space-time homology is used in in quantum gravity [13,17]. On the other hand, the formulation of a theory which unifies quantum mechanics* *and the special theory of relativity, performed by Dirac nearly a century ago, required introduction of new mathematical and physical concepts which led to models that, on one hand, have been very successful in terms of the interpretation of physical reality but, on the other, still creates some challenges, both conceptual and computational. A central notion of relativistic quantum mechanics is a construct known as the Dirac operator. It may be defined as the result of factorization of a second-order differential operator in the Minkowski space. The eigenvalues of the Dirac operator on a curved spacetime are diffeomorphism-invariant functions of the geometry. They form an innite set of observables for general relativity. Some recent work suggests that they can be taken as variables for an invariant description of the gravitational field's dynamics. Because this paper is connected to both warped product manifold and homotopy-homology theory, its results can be used as physical applications.*

2. Some Important Background

This part includes some notations and definitions that is important to the work relay essentially on [18,21,22]. Suppose $\mathbb{C}P^m$ is a m-dimensional complex projective space among the Fubini-Study metric g_{FS} with J being its almost complex structure. In the case where the Levi-Civita connection is defined using $\widetilde{\nabla}$, the Fubini-Study metric is Kähler, that is, $\widetilde{\nabla}J = 0$. A Kähler manifold $\widetilde{\Omega}^m$ for a positive constant sectional curvature $c = 4 > 0$ is named a complex projective space $\mathbb{C}P^m(4)$ and can be endowed with the Fubini-Study metric g_{FS}. Therefore, the curvature tensor $\widetilde{R}$ of $\mathbb{C}P^m(4)$ is given as:

$$\begin{aligned}\widetilde{R}(U_1, U_2, V_1, V_2) =& g(U_1, V_2)g(U_2, V_1) - g(U_1, V_1)g(U_2, V_2) \\ &+ g(JU_1, V_2)g(JU_2, V_1) - g(JU_2, V_2)g(JU_1, V_1) \\ &+ 2g(U_1, JU_2)g(JV_1, V_2),\end{aligned} \tag{11}$$

for all $U_1, U_2, V_1, V_2 \in \mathfrak{X}(CP^m(4))$. Assume that Ω^n is an isometrically immersed to an almost Hermitian manifold $\widetilde{\Omega}^m$ among the induced metric g. The Gauss equation of the submanifold Ω^n is determined by:

$$\begin{aligned}\widetilde{R}(U_1, U_2, V_1, V_2) =& R(U_1, U_2, V_1, V_2) + g\big(h(U_1, V_1), h(U_2, V_2)\big) \\ &- g\big(h(U_1, V_2), h(U_2, V_1)\big),\end{aligned} \tag{12}$$

where $\widetilde{R}$ and R are curvature tensors at $\widetilde{M}^m$ and Ω^n, in the same order. The definition of mean curvature vector H of the orthonormal frame $\{e_1, e_2, \cdots e_n\}$ of the tangent space TM on Ω^n is given as

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_A, e_A), \tag{13}$$

where $n = \dim \Omega$.

$$h^r_{AB} = g(h(e_A, e_B), e_r), \quad ||h||^2 = \sum_{A,B=1}^{n} g(h(e_A, e_B), h(e_A, e_B)). \tag{14}$$

The gradient positive function φ defined on Ω^n and its squared norm is written as:

$$\nabla\varphi = \sum_{i=1}^{n} e_i(\varphi)e_i, \quad \text{and} \quad \|\nabla\varphi\|^2 = \sum_{i=1}^{n} \big((\varphi)e_i\big)^2. \tag{15}$$

We will provide some short definitions of different classes of submanifold Ω^n according to J conserves all tangent spaces of Ω^n, such that

(i) Ω^n is *holomorphic* submanifold if $J(T_x\Omega) \subseteq T_x\Omega$ [22].
(ii) Ω^n is named *totally real* submanifold in the case where $J(T_x\Omega) \subseteq T_x^{\perp}\Omega$ [22].
(iii) Combining (i) and (ii) such that $T\Omega = \mathcal{D}^T \oplus \mathcal{D}^{\perp}$, then Ω^n is a CR-submanifold [22].

(iv) In the case where the angle $\phi(X)$ is enclosed by JX and the tangent space is $T_x\Omega$ for any vector field X of Ω^n that is not equal to zero, it is a real-valued function such that $\phi : \Omega \to \mathbb{R}$, then Ω^n is called a pointwise slant submanifold (more details in [24]). In the same paper, the authors provided a necessary and sufficient condition for Ω^n to be a pointwise slant $T^2X = -\cos^2\phi X$, where T is tangential $(1,1)$ tensor field [24].

(v) If the tangent space $T\Omega$ is introduced as a decomposition in the form of $T\Omega = \mathcal{D}^T \oplus \mathcal{D}^\phi$ for $J(\mathcal{D}^T) \subseteq \mathcal{D}^T$ and pointwise slant distribution $\mathcal{D}^\phi$, then Ω^n is classified as a pointwise semi-slant submanifold [21]. For some examples of pointwise semi-slant submanifolds in a Kähler manifold, and related problems, we recommend the Ref. [18,21].

Regarding the above study, we give some remarks as follows.

Remark 3. *If we consider a slant function $\phi : \Omega^n \to R$ that is globally constant on Ω^n and $\phi = \frac{\pi}{2}$, thus, Ω^n is named a CR-submanifold.*

Remark 4. *In the case where a slant function is $\phi : \Omega^n \to (0, \frac{\pi}{2})$, then Ω^n is called a proper pointwise semi-slant submanifold.*

Remark 5. *The normal bundle $T^\perp\Omega$ of Ω is expressed as $T^\perp\Omega = F\mathcal{D}^\phi \oplus \mu$ with respect to invariant subspace μ, that is, $J(\mu) \subseteq \mu$.*

2.1. Warped Product Submanifolds

Warped product manifolds $\Omega^n = N_1^l \times_f N_2^k$ were originally initiated by Bishop and O'Neill [28], where N_1^l and N_2^k are two Riemannian manifolds and their Riemannain metrics are g_1 and g_2 in the same order. f is also a smooth function defined on N_1^l. The warped product manifold $\Omega^n = N_1^l \times_f N_2^k$ is the manifold $N_1^l \times N_2^k$ furnished by the Riemannian metric $g = g_1 + f^2 g_2$, and the function f is named a warping function of Ω^n. The following important consequences of the warped product manifolds are given in [28,29]. For all $U_1, U_2 \in \mathfrak{X}(TN_1)$ and $V_1, V_2 \in \mathfrak{X}(TN_2)$, where we have

$$\nabla_{V_1} U_1 = \nabla_{U_1} V_1 = \frac{(U_1 f)}{f} V_1. \tag{16}$$

$$\mathcal{R}(U_1, V_1)U_2 = \frac{\mathcal{H}^f(U_1, V_1)}{f} U_2, \tag{17}$$

where $\mathcal{H}^f$ is a Hessian tensor of f. Furthermore, we have

$$g(\nabla \ln f, X) = X(\ln f). \tag{18}$$

2.2. The Non-Trivial Warped Product Pointwise Semi-Slant Submanifolds

Based on the pointwise semi-slant submanifold definition, it is possible to define the warped product pointwise semi-slant submanifolds of a Kähler manifold as follows:

$$(i)\ N_\phi^k \times_f N_T^l, \quad and \quad (ii)\ N_T^l \times_f N_\phi^k.$$

We will consider the second type because the first type of $N_\phi^k \times_f N_T^l$ is trivial (see Theorem 4.1 in [21]). Additionally for the non-trivial case $N_T^l \times_f N_\phi^k$ with examples, see the Ref. [21]. This warped product pointwise semi-slant submanifold is interesting because it is a generalized CR-warped product [22]. The proofs of the main results are ready to be introduced as follows.

3. Proof of the Main Results

3.1. Proof of Theorem 2

Let $\Omega^n = N_T^l \times_f N_\phi^k$ be an $n = l + k-$dimensional warped product pointwise semi-slant submanifold with $\dim N_T^l = l = 2A$ and $\dim N_\phi^k = k = 2B$, where N_ϕ^k and N_T^l are integral manifolds of $\mathcal{D}^\phi$ and $\mathcal{D}$, in the same order. Then, $\{e_1, e_2, \cdots e_A, e_{A+1} = Je_1, \cdots e_{2A} = Je_A\}$ and $\{e_{2A+1} = e_1^*, \cdots e_{2A+B} = e_B^*, e_{2A+B+1} = e_{B+1}^* = \sec\phi Pe_1^*, \cdots e_{l+k} = e_k^* = \sec\phi Pe_B^*\}$ will be orthonormal frames of TN_T and TN_ϕ, in the same order. Therefore, the orthonormal basis of $F\mathcal{D}^\phi$ and μ are $\{e_{n+1} = \bar{e}_1 = \csc\phi Fe_1^*, \cdots e_{n+B} = \bar{e}_B = \csc\phi Fe_1^*, e_{n+B+1} = \bar{e}_{B+1} = \csc\phi\sec\phi FPe_1^*, \cdots e_{n+2B} = \bar{e}_{2B} = \csc\phi\sec\phi FPe_B^*\}$ and $\{e_{n+2B+1}, \cdots e_{2m}\}$, respectively. Then, we arrange the terms

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k}&\Big\{2||h(e_A,e_B)||^2 - g\big(h(e_B,e_B),h(e_A,e_A)\big)\Big\}\\&= \sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A,e_B^*),e_r\big)\\&\quad+\sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{||h(e_A,e_B)||^2 - g\big(h(e_B,e_B),h(e_A,e_A)\big)\Big\}.\end{aligned}$$

Then, from the Gauss Equation (12), we have

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k}&\Big\{2||h(e_A,e_B)||^2 - g\big(h(e_B,e_B),h(e_A,e_A)\big)\Big\}\\&= \sum_{A=1}^{l}\sum_{B=1}^{k} g\big(R(e_A,e_B)e_A,e_B\big)\\&\quad-\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\widetilde{R}(e_A,e_B)e_A,e_B\big)\\&\quad+\sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A,e_B^*),e_r\big)^2,\end{aligned} \tag{19}$$

Using the orthonormal frames $\{e_i\}_{1\le A\le p}$ as well as $\{e_B\}_{1\le B\le q}$ of N_T^l and N_ϕ^k, respectively, in (17), we derive

$$R(e_A,e_B)e_A = \frac{e_B}{f}\mathcal{H}^f(e_A,e_A).$$

Summing up, with respect to the orthonormal frame $\{e_B\}_{1\le B\le q}$ in addition to taking into account the adoption of the opposite of the usual sign convention for the Laplacian, one obtains:

$$\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(R(e_A,e_B)e_A,e_B\big) = -\frac{k}{f}\sum_{A=1}^{l} g\big(\nabla_{e_A}\nabla f, e_A\big). \tag{20}$$

Thus, from Equations (19) and (20), we derive

$$\sum_{A=1}^{l}\sum_{B=1}^{k}\left\{2||h(e_A,e_B)||^2 - g\big(h(e_B,e_B),h(e_A,e_A)\big)\right\}$$
$$= \sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A,e_B^*),e_r\big)^2 - \sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\widetilde{R}(e_A,e_B)e_A,e_B\big) - \frac{k}{f}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\nabla_{e_A}\nabla f,e_A\big). \tag{21}$$

Firstly, the term Δf for Ω^n is computed, which is originally the Laplacian of f.

$$\Delta f = -\sum_{i=1}^{n} g\big(\nabla_{e_i} gradf, e_i\big) = -\sum_{A=1}^{l} g\big(\nabla_{e_A} gradf, e_A\big) - \sum_{B=1}^{k} g\big(\nabla_{e_B} gradf, e_B\big).$$

The previous equation will be rewritten using components of N_ϕ^k for an adapted orthonormal frame. One obtains:

$$\Delta f = -\sum_{A=1}^{l} g\big(\nabla_{e_A} gradf, e_A\big) - \sum_{j=1}^{B} g\big(\nabla_{e_j} gradf, e_j\big) - \sec^2\phi \sum_{j=1}^{B} g\big(\nabla_{Te_j} gradf, Te_j\big).$$

It is noted that ∇ is a Levi-Civita connection on Ω^n, and N_T^l is also totally geodesic in M^n. It leads to $gradf \in \mathfrak{X}(TN_T)$, and then we have

$$\frac{\Delta f}{f} = -\frac{k}{f}\sum_{A=1}^{l} g\big(\nabla_{e_A} gradf, e_A\big) - k||\nabla(\ln f)||^2.$$

It is clear that the next equation is satisfied

$$-\frac{1}{f}\sum_{A=1}^{l} g\big(\nabla_{e_A} gradf, e_A\big) = \Delta(\ln f) + (k-1)||\nabla \ln f||^2. \tag{22}$$

This result, combined with (21) yields

$$\sum_{A=1}^{l}\sum_{B=1}^{k}\left\{2||h(e_A,e_B)||^2 - g\big(h(e_B,e_B),h(e_A,e_A)\big)\right\}$$
$$= \sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A,e_B^*),e_r\big)^2 - \sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\widetilde{R}(e_A,e_B)e_A,e_B\big) + k\Delta(\ln f) + k(k-1)||\nabla \ln f||^2. \tag{23}$$

At this point, suppose $X = e_A$ and $Z = e_B$ for $1 \leq A \leq l$ and $1 \leq B \leq k$, in the same order. Thus, by the use of the bilinear form h definition according to an orthonormal basis, we can write

$$\sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A, e_B^*), e_r\big)^2 = \sum_{r=n+1}^{n+2B}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A, e_B^*), e_r\big) + \|h_\mu\|^2.$$

In the previous equation, the first term at the right-hand side is a $F\mathcal{D}^\phi$-component, while the second term is a μ invariant subspace. From the viewpoint of an adapted orthonormal basis, vector fields of N_T^l and N_ϕ^k are summed up over the vector fields of N_T^l and N_ϕ^k. Then, using Lemma 5.2 from [21] and (Equation (5.8) of Lemma 5.3 in [21]), we conclude that

$$\begin{aligned}\sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A, e_B^*), e_r\big)^2 =& 2\big(\cot^2\phi + \csc^2\phi\big)\sum_{A=1}^{l}\sum_{B=1}^{k}\big(e_A \ln f)\big)^2 g(e_B^*, e_B^*)^2 \\ &+ 2\big(\cot^2\phi + \csc^2\phi\big)\sum_{A=1}^{l}\sum_{B=1}^{k}\big(Je_A \ln f)\big)^2 g(e_B^*, e_B^*)^2 \\ &+ \|h_\mu\|^2.\end{aligned}$$

Using the squared norm definition of the gradient function f (15) (ii), one obtains:

$$\sum_{r=n+1}^{2m}\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(h(e_A, e_B^*), e_r\big)^2 = k\big(\cot^2\phi + \csc^2\phi\big)||\nabla \ln f||^2 + \|h_\mu\|^2. \tag{24}$$

From (23) and (24), we get:

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k}&\Big\{2||h(e_A, e_B)||^2 - g\big(h(e_B, e_B), h(e_A, e_A)\big)\Big\} \\ &= k\Delta(\ln f) + k(k-1)||\nabla(\ln f)||^2 \\ &\quad + k\big(1 + 2\cot^2\phi\big)||\nabla(\ln f)||^2 + \|h_\mu\|^2 \\ &\quad - \sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\widetilde{R}(e_A, e_B)e_A, e_B\big).\end{aligned} \tag{25}$$

For the symmetry of the curvature tensor R, the following relation holds:

$$\sum_{A=1}^{l}\sum_{B=1}^{k} g\big(\widetilde{R}(e_A, e_B)e_A, e_B\big) = \sum_{A=1}^{l}\sum_{B=1}^{k} \widetilde{R}\big(e_A, e_B, e_A, e_B\big). \tag{26}$$

Next, we remark that the curvature tensor Formula (11) for the complex projective space $\mathbb{C}P^{2m}(4)$ is easily given as

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k} \widetilde{R}\big(e_A, e_B, e_A, e_B\big) = \sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{& g(e_A, e_B)g(e_B, e_A) - g(e_A, e_A)g(e_B, e_B) \\ &- g(Je_A, e_A)g(Je_B, e_B) \\ &+ 3g(Je_A, e_B)g(Je_B, e_A)\Big\}.\end{aligned} \tag{27}$$

As we know, if $e_A \in \mathfrak{X}(TN_T)$ and $e_B \in \mathfrak{X}(TN_\phi)$, then $g(e_A, e_B) = 0$, and $g(Je_A, e_A) = 0(resp, g(Je_B, e_B) = 0)$, using $Je_A \perp e_A(Je_B \perp e_B)$, respectively. Similarly, from

(Equation (2.6) in [21]), we derive that $g(Je_A, e_B) = g(Te_A + Fe_A, e_B) = 0$ for $Te_A \in \mathfrak{X}(TN_T)$ and $Fe_A \in \mathfrak{X}(FN_\phi)$. Thus, (27) implies that

$$\sum_{A=1}^{l}\sum_{B=1}^{k} \widetilde{R}(e_A, e_B, e_A, e_B) = -\sum_{A=1}^{l}\sum_{B=1}^{k} g(e_A, e_A)g(e_B, e_B) = -lk \tag{28}$$

Therefore, using (26) and (28), we finally get

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{2||h(e_A, e_B)||^2 - g\big(h(e_B, e_B), h(e_A, e_A)\big)\Big\}& \\ = k\Delta(\ln f) + k(k-1)\|\nabla(\ln f)\|^2 + \|h_\mu\|^2& \\ + k\big(\cot^2\phi + \csc^2\phi\big)\|\nabla(\ln f)\|^2 + lk.&\end{aligned} \tag{29}$$

Now, computing $\Delta \ln f$, we get:

$$\begin{aligned}\Delta(lnf) &= -\,div(\nabla(lnf)) = -div\Big(\frac{\nabla f}{f}\Big) \\ &= -\,g(\nabla(\frac{1}{f}), \nabla f) + \frac{1}{f}\Delta f \\ &= \frac{1}{f^2}\|\nabla f\|^2 + \frac{1}{f}\Delta f.\end{aligned} \tag{30}$$

Then, from (29) and (30), we find that

$$\begin{aligned}\sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{2||h(e_A, e_B)||^2 - g\big(h(e_B, e_B), h(e_A, e_A)\big)\Big\}& \\ = \frac{k\Delta f}{f} + \frac{k\|\nabla f\|^2}{f^2}\big(\cot^2\phi + \csc^2\phi + k\big)& \\ + lk + \|h_\mu\|^2.&\end{aligned} \tag{31}$$

Let the pinching condition (2) be satisfied. Then, from (31), we get

$$\sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{2||h(e_A, e_B)||^2 - g\big(h(e_B, e_B), h(e_A, e_A)\big)\Big\} < 4lk.$$

It well-known that constant sectional curvature for the complex projective spaces $\mathbb{C}P^{2m}(4)$ is equal to $c = 4$. Then, the last equation is implied:

$$\sum_{A=1}^{l}\sum_{B=1}^{k}\Big\{2||h(e_A, e_B)||^2 - g\big(h(e_B, e_B), h(e_A, e_A)\big)\Big\} < lkc. \tag{32}$$

Therefore, using Theorem 1, we reached our promised results (a) and (b). For the third part, let us assume that $\pi_1(\Omega) \neq 0$. From the compactness of Ω^n, it follows from the classical theorem of Cartan and Hadamard that there is a minimal closed geodesic in any non-trivial homotopy class in $\pi_1(\Omega)$, which leads to a contradiction. Therefore, $\pi_1(\Omega) = 0$. This is the third part of the theorem. If the finite fundamental group is null of any Riemannian manifold, this Riemannian manifold is simply connected. As a result, Ω^n is simply connected.

3.2. *Proof of Theorem 3*

In the case where Ω^n is a compact Riemannian manifold with no boundary, $\partial\Omega^n = \emptyset$, thus using [32], the divergence property $\int_{\Omega^n}(\Delta f)dV = 0$. Using this fact, we get

$$\begin{aligned} 0 &= \int_{\Omega^n} \Delta\left(\frac{f^2}{2}\right)dV \\ &= -\int_{\Omega^n} div\left(\nabla\left(\frac{f^2}{2}\right)\right)dV \\ &= -\int_{\Omega^n} div(f\nabla f)dV = -\int_{\Omega^n} g(\nabla f, \nabla f)dV + \int_{\Omega^n} f\Delta f dV, \end{aligned}$$

which implies that

$$\int_{\Omega^n} f\Delta f dV = \int_{\Omega^n} \|\nabla f\|^2 dV. \tag{33}$$

Using (3) with inequality (6), then it can be rewritten as:

$$\frac{1}{2}\int_{\Omega^n} \|\nabla f\|^2 dV < \left(\frac{1}{2(2\csc^2\phi + k)}\right)\int_{\Omega^n}\left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2 dV. \tag{34}$$

By using the trigonometric identities $1 + \cot^2\phi = \csc^2\phi$ and (33) in the above equation, we get

$$\int_{\Omega^n} f\Delta f dV + \left(\cot^2\phi + \csc^2\phi + k\right)\int_{\Omega^n} \|\nabla f\|^2 dV < \int_{\Omega^n}\left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2 dV.$$

It is equivalent to the following:

$$f\Delta f + \left(\cot^2\phi + \csc^2\phi + k\right)\|\nabla f\|^2 < \left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2. \tag{35}$$

Hence, using Theorem 3, we get the required results. This completes the proof of the Theorem.

3.3. *Proof of Theorem 4*

Assuming f is a non-constant warping function, by the use of the minimum principle on the first eigenvalue λ_1, one can obtain [26] (p. 186):

$$\lambda_1 \int_{\Omega^n} (f)^2 dV \leq \int_{\Omega^n} \|\nabla f\|^2 dV. \tag{36}$$

The equality holds if, and only if $\Delta f = \lambda_1 f$. On the other hand, if our assumption (10) holds, then using the equality in (36), we get

$$(2\csc^2\phi + k)\int_{\Omega^n} \|\nabla f\|^2 dV < \int_{\Omega^n}\left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2 dV.$$

Utilizing (33) and rearranging this with trignometric functions, we have

$$(\cot^2\phi + \csc^2\phi + k)\int_{\Omega^n} \|\nabla f\|^2 dV + \int_{\Omega^n} f\Delta f dV < \int_{\Omega^n}\left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2 dV. \tag{37}$$

Hence, we get the inequality

$$(\cot^2\phi + \csc^2\phi + k)\|\nabla f\|^2 + f\Delta f < \left(3l - \frac{\|h_\mu\|^2}{k}\right)f^2.$$

Therefore, the assertion follows from Theorem 2. The proof is completed.

Hamiltonian at the point $x \in M^n$ for the local orthonormal frame, is given as (see [25]):

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{n} p(e_i)^2. \tag{38}$$

Substituting $p = d\varphi$ into the previous equation, and since d is a differentiable operator, we use (15) to have:

$$H(d\varphi, x) = \frac{1}{2}\sum_{i=1}^{n} d\varphi(e_i)^2 = \frac{1}{2}\sum_{i=1}^{n} e_i(\varphi)^2 = \frac{1}{2}||\nabla\varphi||^2. \tag{39}$$

Using the previous equation leads to the next result from inequality (2), as the following:

Corollary 3. *Under the same assumption in Theorem 2, it satisfies the following inequality:*

$$H(df, x) < \frac{(3l - \|h_\mu\|^2)f^2}{k(4\csc^2\phi + 2k)} - \frac{f\Delta f}{2}, \tag{40}$$

where $H(df, x)$ is the Hamiltonian of the warping function f, so no stable integral l-currents exist in Ω^n and $\mathbb{H}_l(\Omega^n, \mathbb{G}) = \mathbb{H}_k(\Omega^n, \mathbb{G}) = 0$.

Proof. Combining the Hamiltonian formula (39) and inequality (2), we have the result. □

3.4. Proof of the Corollarys 1 and 2

The proof of Corollarys 1 and 2 can be obtained directly from the Theorems 2 and 3 by substituting $\phi = \frac{\pi}{2}$ to derive a totally real submanifold from a pointwise slant submanifold.

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