# Norm Inflation for Benjamin-Bona-Mahony Equation in Fourier Amalgam and Wiener Amalgam Spaces with Negative Regularity 

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#### Abstract

We consider the Benjamin-Bona-Mahony (BBM) equation of the form $u_{t}+u_{x}+u u_{x}-$ $u_{x x t}=0,(x, t) \in \mathcal{M} \times \mathbb{R}$ where $\mathcal{M}=\mathbb{T}$ or $\mathbb{R}$. We establish norm inflation (NI) with infinite loss of regularity at general initial data in Fourier amalgam and Wiener amalgam spaces with negative regularity. This strengthens several known NI results at zero initial data in $H^{s}(\mathbb{T})$ established by BonaDai (2017) and the ill-posedness result established by Bona-Tzvetkov (2008) and Panthee (2011) in $H^{s}(\mathbb{R})$. Our result is sharp with respect to the local well-posedness result of Banquet-Villamizar-Roa (2021) in modulation spaces $M_{s}^{2,1}(\mathbb{R})$ for $s \geqslant 0$.


Keywords: BBM equation; ill-posedness; Fourier amalgam spaces; Wiener amalgam spaces; FourierLebesgue spaces; modulation spaces

MSC: 35Q53; 35R25 (primary); 42B35 (secondary)

## 1. Introduction

We study strong ill-posedness for the Benjamin-Bona-Mahony (BBM) equation of the form

$$
\left\{\begin{array}{l}
u_{t}+u_{x}+u u_{x}-u_{x x t}=0  \tag{1}\\
u(x, 0)=u_{0}(x)
\end{array}\right.
$$

where $u: \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$ unknown function and $\mathcal{M}=\mathbb{T}$ or $\mathbb{R}$. The BBM (1) can be written as

$$
\begin{equation*}
i u_{t}=\varphi\left(D_{x}\right) u+\frac{1}{2} \varphi\left(D_{x}\right) u^{2}, \quad u(x, 0)=u_{0}(x) \tag{2}
\end{equation*}
$$

where $\varphi(\xi)=\frac{\xi}{1+\xi^{2}}, D_{x}=\frac{1}{i} \partial_{x}$ and $\varphi\left(D_{x}\right)$ is the Fourier multiplier operator defined by

$$
\mathcal{F}\left[\varphi\left(D_{x}\right) u\right](\xi)=\varphi(\xi) \widehat{\mathcal{u}}(\xi)
$$

This BBM (1) model is the regularized counterpart of the Korteweg-de Vries (KdV) equation. This is extensively studied in the literature; see [1-5]. BBM equation (1) is wellsuited for modeling wave propagation on star graphs; see [6]. This model gave a good description of the propagation of surface water waves in a channel; see [5].

The aim of this paper is to establish the following strong ill-posedness (norm inflation at general initial data with infinite loss of regularity) for (1) in Fourier amalgam $\widehat{w}_{s}^{p, q}(\mathcal{M})$ and Wiener amalgam $W_{s}^{p, q}(\mathcal{M})$ spaces (to be defined in Section 2). We recall that

$$
\hat{w}_{s}^{p, q}(\mathcal{M})=\left\{\begin{array}{l}
\mathcal{F} L_{s}^{q}(\mathcal{M}) \text { (Fourier-Lebesgue spaces) if } p=q \\
M_{s}^{2, q}(\mathcal{M}) \text { (modulation spaces) if } p=2 \\
M_{s}^{2,2}(\mathcal{M})=W_{s}^{2,2}(\mathcal{M})=H^{s}(\mathcal{M}) \text { (Sobolev spaces) if } p=q=2 \\
\mathcal{F} L_{s}^{q}(\mathcal{M})=M_{s}^{p, q}(\mathcal{M})=W_{s}^{p, q}(\mathcal{M}) \text { if } \mathcal{M}=\mathbb{T}^{d} .
\end{array}\right.
$$

These time-frequency spaces are proven to be very fruitful in handling various problems in analysis and have gained prominence in nonlinear dispersive PDEs, e.g., [7-15]. We now state our main theorem.

Theorem 1. Assume that $1 \leqslant p, q \leqslant \infty, s<0$ and let

$$
X_{s}^{p, q}(\mathcal{M})=\left\{\begin{array}{l}
\widehat{w}_{s}^{p, q}(\mathbb{R}) \text { or } W_{s}^{2, q}(\mathbb{R}) \text { for } \mathcal{M}=\mathbb{R} \\
\mathcal{F} L_{s}^{q}(\mathbb{T}) \text { for } \mathcal{M}=\mathbb{T}
\end{array}\right.
$$

Then, norm inflation with infinite loss of regularity occurs to (1) everywhere in $X_{s}^{p, q}(\mathcal{M})$, i.e., for any $u_{0} \in X_{s}^{p, q}(\mathcal{M}), \theta \in \mathbb{R}$ and $\varepsilon>0$, there exists a smooth $u_{0, \epsilon} \in X_{s}^{p, q}(\mathcal{M})$ and $T>0$ satisfying

$$
\left\|u_{0}-u_{0, \epsilon}\right\|_{X_{s}^{p, q}}<\varepsilon, \quad 0<T<\epsilon
$$

such that the corresponding smooth solution $u_{\epsilon}$ to (1) with data $u_{0, \epsilon}$ exists on $[0, T]$ and

$$
\left\|u_{\epsilon}(T)\right\|_{X_{\theta}}>\frac{1}{\varepsilon}
$$

In particular, for any $T>0$, the solution map $X_{s}^{p, q}(\mathcal{M}) \ni u_{0} \mapsto u \in C\left([0, T], X_{\theta}^{p, q}(\mathcal{M})\right)$ for (1) is discontinuous everywhere in $X_{s}^{p, q}(\mathcal{M})$ for all $\theta \in \mathbb{R}$.

In [3], Bona and Tzvetkov proved that (1) is globally well-posed in $H^{s}(\mathbb{R})$ for $s \geqslant 0$. Moreover, they also proved that (1) is ill-posed for $s<0$ in the sense that the solution map $u_{0} \mapsto u(t)$ is not $C^{2}$ from $H^{s}(\mathbb{R})$ to $C\left([0, T], H^{s}(\mathbb{R})\right)$. Later, in [16], Panthee proved that it is discontinuous at the origin from $H^{s}(\mathbb{R})$ to $\mathcal{D}^{\prime}(\mathbb{R})$. Recently, Bona and Dai, in [17], established norm inflation for (1) at zero initial data in $\dot{H}^{s}(\mathbb{T})$ for $s<0$. We note that Theorem 1 also holds for the corresponding homogeneous $\dot{X}_{s}^{p, q}(\mathcal{M})$ spaces; see Remark 1. The particular case of Theorem 1 strengthens these results by establishing the infinite loss of regularity at every initial datum in $H^{s}(\mathcal{M})$ for $s<0$. In [18] (Theorem 1.7), Banquet and Villamizar-Roa proved that (1) is locally well-posed in $M_{s}^{2,1}(\mathbb{R})$ for $s \geqslant 0$. Thus, the particular case of Theorem 1 complements this result by establishing sharp, strong ill-posedness in $M_{s}^{2,1}(\mathbb{R})$ for $s<0$. To the best of the authors' knowledge, there is no well-posedness result for (1) in Fourier amalgam $\widehat{w}_{s}^{p, q}(p \neq 2)$ (except in $\mathcal{F} L^{1}(\mathcal{M})$; see Corollary 1 ) or in $W^{p, q}$ (except in $\left.H^{s}\right)$ spaces. The infinite loss of regularity for (1) is initiated in the present paper and thus Theorem 1 is new.

We use a Fourier analytic approach to prove Theorem 1. This approach dates back to Bejenaru and Tao [19] to obtain ill-posedness for quadratic NLS and further developed by Iwabuchi in [20]. Later, Kishimoto [21] established norm inflation (NI) for NLS on a special domain (special domain: $\mathbb{R}^{d_{1}} \times \mathbb{T}^{d_{2}}, d=d_{1}+d_{2}$ and with non-linearity: $\sum_{j=1}^{n} v_{j} u^{\rho_{j}}(\bar{u})^{\sigma_{j}-\rho_{j}}$ where $v_{j} \in \mathbb{C}, \sigma_{j} \in \mathbb{N}, \rho_{j} \in \mathbb{N} \cup\{0\}$ with $\sigma_{j} \geqslant \max \left(\rho_{j}, 2\right)$ ) and Oh [22] established NI at general initial data for cubic NLS. Recently, this approach has been used to obtain strong ill-posedness for NLW in [15,23]. We refer to [21] (Section 2) for a detailed discussion of this approach.

We now briefly comment on and outline the proof of Theorem 1. We first justify the convergence of a series of Picard terms, the smooth solutions to (1), in Wiener algebra $\mathcal{F} L^{1}$ (see Corollary 1). This is possible since the linear BBM propagator is unitary on $\mathcal{F} L^{1}$ and the bilinear operator for the nonlinearity in (2) is bounded (see Lemma 1). Then, (1)
experiences NI at general initial data because (with appropriately chosen initial data close to the given data) one Picard term dominates, in $X_{s}^{p, q}$-norm, the rest of the Picard iterate terms in the series for $s<0$ and also this term becomes arbitrarily large (see (16)-(18)). To this end, we perturb general initial data $u_{0}$ by $\phi_{0, N}$. Here, $\phi_{0, N}$ is defined on the Fourier side by a scalar (depends on $N$ ) multiplication of the characteristic function on the union of two intervals obtained by translation of $[-1,1]$ by $\pm N$ and so the size of support of $\phi_{0, N}$ remains uniform. Specifically, we set

$$
\mathcal{F} \phi_{0, N}=R \chi_{I_{N}}
$$

where $I_{N}=[-N-1,-N+1] \cup[N-1, N+1]$ with $N \gg 1, R=R(N) \gg 1$ (to be chosen later) and

$$
u_{0, N}=u_{0}+\phi_{0, N} .
$$

Eventually, this $u_{0, N}$ will play the role of $u_{0, \epsilon}$ in Theorem 1. Similarly, $\phi_{0, N}$ was used by Bona and Tzvetkov to establish that the solution map fails to become $C^{2}$ in [3] and also by Panthee [16] to conclude that, in fact, the solution map is discontinuous. In [3], the size of the support of $\phi_{0, N}$ on the Fourier side was allowed to vary as $N \rightarrow \infty$ with a normalizing constant to ensure that $\left\|\phi_{0, N}\right\|_{H^{s}} \sim 1$, whereas in [16], $\mathcal{F} \phi_{0, N}$ is taken as $\chi_{I_{N}}$, which implies $\left\|\phi_{0, N}\right\|_{H^{s}} \rightarrow 0$ as $N \rightarrow \infty$. To establish NI with infinite loss of regularity, we multiply $R=R(N) \gg 1$ with Panthee's choice of $\phi_{0, N}$ to ensure that the second Picard iterates $U_{2}(t)\left[u_{0, N}\right]$ have the desired property (as mentioned above) and reduce the analysis when considering a single term on the $\ell^{q}$-norm:

$$
\left\|\langle n\rangle^{\theta} f(n)\right\|_{\ell_{n}^{q}(n=1)}=2^{(s-\theta) / 2}\left\|\langle n\rangle^{s} f(n)\right\|_{\ell_{n}^{q}(n=1)} \quad \text { for all } \theta \in \mathbb{R}
$$

as done in NLW case in [23]. We note that finite loss of regularity of NLW was initiated by Lebeau in [24] and infinite loss of regularity for NLS, via a geometric optics approach, by Carles et al. in [25].

The rest of the paper is organized as follows. In Section 2, we recall the definitions of the time-frequency spaces. In Section 3, we establish power series expansion of the solution in $\mathcal{F} L^{1}$, by establishing $\widehat{w}_{s}^{p, q}$-estimates of the Picard terms for general data. In Section 4, we first prove various estimates of the Picard terms with particular choices of data, and this enables us to conclude the proof of Theorem 1.

## 2. Function Spaces

The notation $A \lesssim B$ means $A \leqslant c B$ for some constant $c>0$, whereas $A=B$ means $c^{-1} A \leqslant B \leqslant c A$ for some $c \geqslant 1$. Let $\mathcal{F}$ denote the Fourier transform and $\langle\cdot\rangle^{s}=$ $\left(1+|\cdot|^{2}\right)^{s / 2}, s \in \mathbb{R}$. Here, $\widehat{\mathcal{M}}$ denotes the Pontryagin dual of $\mathcal{M}$, i.e., $\widehat{\mathcal{M}}=\mathbb{R}$ if $\mathcal{M}=\mathbb{R}$ and $\widehat{\mathcal{M}}=\mathbb{Z}$ if $\mathcal{M}=\mathbb{T}$. $\mathcal{S}^{\prime}(\mathcal{M})$ denotes the space of tempered distributions; see, e.g., [26] (Part II) for details. The Fourier-Lebesgue space $\mathcal{F} L_{s}^{q}(\mathcal{M})(1 \leqslant q \leqslant \infty, s \in \mathbb{R})$ is defined by

$$
\mathcal{F} L_{s}^{q}(\mathcal{M})=\left\{f \in \mathcal{S}^{\prime}(\mathcal{M}): \mathcal{F} f\langle\cdot\rangle^{s} \in L^{q}(\widehat{\mathcal{M}})\right\}
$$

In the 1980s, Feichtinger [27] introduced the modulation spaces $M_{s}^{p, q}(\mathcal{M})$ and Wiener amalgam spaces $W_{s}^{p, q}(\mathcal{M})$ using shrot-time Fourier transform (STFT) (STFT is also known as windowed Fourier transform and is closely related to Fourier-Wigner and Bargmann transform. See, e.g., [28] (Lemma 3.1.1) and [28] (Proposition 3.4.1)). The STFT of a $f \in \mathcal{S}^{\prime}(\mathcal{M})$ with respect to a window function $0 \neq g \in \mathcal{S}(\mathcal{M})$ is defined by

$$
V_{g} f(x, y)=\int_{\mathcal{M}} f(t) \overline{T_{x} g(t)} e^{-2 \pi i y \cdot t} d t,(x, y) \in \mathcal{M} \times \widehat{\mathcal{M}}
$$

whenever the integral exists. Here, $T_{x} g(t)=g\left(t x^{-1}\right)$ is the translation operator on $\mathcal{M}$. We define modulation $M_{s}^{p, q}(\mathcal{M})$ and Wiener amalgam spaces $W_{s}^{p, q}(\mathcal{M})$, for $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$, by the norms:

$$
\|f\|_{M_{s}^{p, q}}=\| \| V_{g} f(x, y)\left\|_{L^{p}(\mathcal{M})}\langle y\rangle^{s}\right\|_{L^{q}(\widehat{\mathcal{M}})} \quad \text { and } \quad\|f\|_{W_{s}^{p, q}(\mathcal{M})}=\| \| V_{g} f(x, y)\langle y\rangle^{s}\left\|_{L^{q}(\widehat{\mathcal{M}})}\right\|_{L^{p}(\mathcal{M})} .
$$

The definition of the modulation space is independent of the choice of the particular window function; see [28] (Proposition 11.3.2(c)). There is also equivalent characterization of these spaces via frequency uniform decomposition (which is quite similar to Besov spaces-where decomposition is dyadic). To do this, let $\rho \in \mathcal{S}(\mathbb{R}), \rho: \mathbb{R} \rightarrow[0,1]$ be a smooth function satisfying $\rho(\xi)=1$ if $|\xi| \leqslant \frac{1}{2}$ and $\rho(\xi)=0$ if $|\xi| \geqslant 1$. Set $\rho_{n}(\xi)=\rho(\xi-n)$ and $\sigma_{n}(\xi)=\frac{\rho_{n}(\xi)}{\sum_{\ell \in \mathbb{Z}^{d}} \rho_{\ell}(\xi)}, n \in \mathbb{Z}$. Then, define the frequency-uniform decomposition operators by

$$
\square_{n}=\mathcal{F}^{-1} \sigma_{n} \mathcal{F}
$$

It is known [7] (Proposition 2.1), [27] that

$$
\|f\|_{M_{s}^{p, q}(\mathcal{M})}=\| \| \square_{n} f\left\|_{L_{x}^{p}(\mathcal{M})}\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})} \quad \text { and } \quad\|f\|_{W_{s}^{p, q}(\mathcal{M})}=\| \| \square_{n} f \cdot\langle n\rangle^{s}\left\|_{\ell_{n}^{q}(\mathbb{Z})}\right\|_{L_{x}^{p}(\mathcal{M})} .
$$

Recently, in [29], Oh and Forlano introduced Fourier amalgam spaces $\widehat{w}_{s}^{p, q}(\mathcal{M})(1 \leqslant$ $p, q \leqslant \infty, s \in \mathbb{R}):$

$$
\widehat{w}_{s}^{p, q}(\mathcal{M})=\left\{f \in \mathcal{S}^{\prime}(\mathcal{M}):\|f\|_{\widehat{w}_{s}^{p, q}}=\| \| \chi_{n+Q_{1}}(\tilde{\xi}) \mathcal{F} f(\xi)\left\|_{L_{\tilde{\xi}}^{p}(\widehat{\mathcal{M}})}\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})}<\infty\right\}
$$

where $Q_{1}=\left(-\frac{1}{2}, \frac{1}{2}\right]$. The homogeneous spaces $\dot{X}_{s}^{p, q}(\mathcal{M})$ corresponding to the above spaces can be defined by replacing the Japanese brackets $\langle\cdot\rangle^{s}$ with $|\cdot|^{s}$ in their definitions.

## 3. Local Well-Posedness in Wiener Algebra $\mathcal{F} L^{1}$

The integral version of (2) is given by

$$
\begin{equation*}
u(t)=U(t) u_{0}-\frac{i}{2} \int_{0}^{t} U(t-\tau) \varphi\left(D_{x}\right) u^{2}(\tau) d \tau \tag{3}
\end{equation*}
$$

where $\mathcal{F} U(t) \varphi\left(D_{x}\right) u(\xi)=e^{i t \varphi(\xi)} \varphi(\xi) \mathcal{F} u(\xi)$ and $U(t) u_{0}(x)=\mathcal{F}^{-1}\left(e^{\left.i t \varphi(\xi) \mathcal{F} u_{0}(\xi)\right)(x) \text { is the }, ~(\xi)}\right.$ unique solution to the linear problem

$$
i u_{t}=\varphi\left(D_{x}\right) u, \quad u(x, 0)=u_{0}(x) ; \quad(x, t) \in \mathcal{M} \times \mathbb{R} .
$$

Let us define the operator $\mathcal{N}$ given by

$$
\mathcal{N}(u, v)(t)=\int_{0}^{t} U(t-\tau) \varphi\left(D_{x}\right)(u v)(\tau) d \tau
$$

Definition 1 (Picard iteration). For $u_{0} \in L^{2}\left(\mathbb{R}^{d}\right)$, define $U_{1}\left[u_{0}\right](t)=U(t) u_{0}$ and for $k \geqslant 2$

$$
U_{k}\left[u_{0}\right](t)=-\frac{i}{2} \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\ k_{1}+k_{2}=k}} \mathcal{N}\left(U_{k_{1}}\left[u_{0}\right], U_{k_{2}}\left[u_{0}\right]\right)(t)
$$

Lemma 1. Let $1 \leqslant p, q \leqslant \infty, s, t \in \mathbb{R}$. Then, we have

$$
\begin{align*}
& \left\|U(t) u_{0}\right\|_{\widehat{w}_{s}^{p, q}}=\left\|u_{0}\right\|_{\widehat{w}_{s}^{p, q}}  \tag{1}\\
& \|\mathcal{N}(u, v)(t)\|_{\widehat{w}_{s}^{p, q}} \leqslant \int_{0}^{t}\|u(\tau)\|_{\mathcal{F} L^{1}}\|v(\tau)\|_{\widehat{w}_{s}^{p, q}} d \tau \leqslant t\|u\|_{L \infty\left((0, t), \mathcal{F} L^{1}\right)}\|v\|_{L^{\infty}\left((0, t), \hat{w}_{s}^{p, q}\right)} .
\end{align*}
$$

Proof. Note that

$$
\left\|U(t) u_{0}\right\|_{\widehat{w_{s}^{p, q}}}=\| \| \chi_{n+Q_{1}}(\xi) e^{i t \varphi(\xi)} \mathcal{F} u_{0}(\xi)\left\|_{L_{\zeta}^{p}(\widehat{\mathcal{M}})}\left(1+|n|^{2}\right)^{s / 2}\right\|_{\ell_{n}^{q}(\mathbb{Z})}=\left\|u_{0}\right\|_{\widehat{w_{s}^{p, q}}} .
$$

Using the fact that $|\varphi| \leqslant 1$, we have

$$
\begin{aligned}
\|\mathcal{N}(u, v)(t)\|_{\widehat{w}_{s}^{p, q}} & =\| \| \chi_{n+Q_{1}}(\xi) \int_{0}^{t} e^{i(t-\tau) \varphi(\xi)} \varphi(\xi)(\mathcal{F} u * \mathcal{F} v)(\xi)(\tau) d \tau\left\|_{L_{\xi}^{p}(\widehat{\mathcal{M}})}\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})} \\
& \leqslant\left\|\int_{0}^{t}\right\| \chi_{n+Q_{1}}(\xi) e^{i(t-\tau) \varphi(\xi)} \varphi(\xi)(\mathcal{F} u * \mathcal{F} v)(\xi)(\tau)\left\|_{L_{\xi}^{p}(\widehat{\mathcal{M}})} d \tau\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})} \\
& \leqslant\left\|\int_{0}^{t}\right\| \chi_{n+Q_{1}}(\xi)(\mathcal{F} u * \mathcal{F} v)(\xi)(\tau)\left\|_{L_{\xi}^{p}(\widehat{\mathcal{M}})} d \tau\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})} \\
& \leqslant \int_{0}^{t}\| \| \mathcal{F} u(\xi)(\tau)\left\|_{L_{\tilde{\zeta}}^{1}(\widehat{\mathcal{M}})}\right\| \chi_{n+Q_{1}(\xi) \mathcal{F} v(\xi)(\tau)\left\|_{L_{\xi}^{p}(\widehat{\mathcal{M}})}\langle n\rangle^{s}\right\|_{\ell_{n}^{q}(\mathbb{Z})} d \tau}=\int_{0}^{t}\|u(\tau)\|_{\mathcal{F} L^{1}}\|v(\tau)\|_{\widehat{w_{s}^{p}}} d \tau .
\end{aligned}
$$

Lemma 2 (See [21]). Let $\left\{b_{k}\right\}_{k=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$
b_{k} \leqslant C \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\ k_{1}+k_{2}=k}} b_{k_{1}} b_{k_{2}} \quad \forall k \geqslant 2 .
$$

Then, we have $b_{k} \leqslant b_{1} C_{0}^{k-1}$, for all $k \geqslant 1$, where $C_{0}=\frac{2 \pi^{2}}{3} C b_{1}$.
Lemma 3. There exists $c>0$ such that for all $t>0$ and $k \geqslant 2$, we have

$$
\left\|U_{k}\left[u_{0}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \leqslant(c t)^{k-1}\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k-1}\left\|u_{0}\right\|_{\widehat{w}_{s}^{p, q}} .
$$

Proof. Let $\left\{b_{k}\right\}$ be a sequence of nonnegative real numbers such that

$$
b_{1}=1 \quad \text { and } \quad b_{k}=\frac{1}{k-1} \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\ k_{1}+k_{2}=k}} b_{k_{1}} b_{k_{2}} \quad \forall k \geqslant 2 .
$$

By Lemma 2, we have $b_{k} \leqslant c_{0}^{k-1}$ for some $c_{0}>0$. In view of this, it is enough to prove the following claim:

$$
\left\|U_{k}\left[u_{0}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \leqslant b_{k} t^{k-1}\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k-1}\left\|u_{0}\right\|_{\hat{w}_{s}^{p, q}} .
$$

By Definition 1, Lemma 1 and using the fact that $\frac{|\xi|}{1+\xi^{2}} \leqslant 1$, we have

$$
\begin{equation*}
\left\|U_{k}\left[u_{0}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \leqslant \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\ k_{1}+k_{2}=k}} \int_{0}^{t}\left\|U_{k_{1}}\left[u_{0}\right](\tau)\right\|_{\mathcal{F} L^{1}}\left\|U_{k_{2}}\left[u_{0}\right](\tau)\right\|_{\hat{w}_{s}^{p, q}} d \tau \tag{4}
\end{equation*}
$$

Thus, we have

$$
\left\|U_{2}\left[u_{0}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \leqslant t\left\|U\left[u_{0}\right]\right\|_{L^{\infty}\left((0, t), \mathcal{F} L^{1}\right)}\left\|U\left[u_{0}\right]\right\|_{L^{\infty}\left((0, t), \hat{w}_{s}^{p, q}\right)}=t\left\|u_{0}\right\|_{\mathcal{F} L^{1}}\left\|u_{0}\right\|_{\hat{w}_{s}^{p, q}}
$$

Hence, the claim is true for $k=2$ as $b_{2}=1$. Assume that the result is true up to the label $(k-1)$. Then, from (4), we obtain

$$
\begin{aligned}
\left\|U_{k}\left[u_{0}\right](t)\right\|_{\widehat{w}_{s}^{p, q}}^{p} & \leqslant \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\
k_{1}+k_{2}=k}} b_{k_{1}} b_{k_{2}} \int_{0}^{t} \tau^{k_{1}-1}\left\|u_{0}\right\|_{\mathcal{F}^{1}}^{k_{1}} \times \tau^{k_{2}-1}\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k_{2}-1}\left\|u_{0}\right\|_{\widehat{w}_{s}^{p, q}} d \tau \\
& =b_{k} t^{k-1}\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k-1}\left\|u_{0}\right\|_{\widehat{w}_{s}^{p, q}} .
\end{aligned}
$$

Thus, the claim is true at the level $k$. This completes the proof.
Corollary 1. If $0<T \ll M^{-1}$, then for any $u_{0} \in \mathcal{F} L^{1}$ with $\left\|u_{0}\right\|_{\mathcal{F}^{1}} \leqslant M$, there exists a unique solution $u \in C\left([0, T], \mathcal{F} L^{1}(\mathcal{M})\right)$ to the integral equation (3) associated with (2), given by

$$
\begin{equation*}
u=\sum_{k=1}^{\infty} U_{k}\left[u_{0}\right] \tag{5}
\end{equation*}
$$

which converges absolutely in $C\left([0, T], \mathcal{F} L^{1}(\mathcal{M})\right)$.
Proof. Define

$$
\Psi(u)(t)=U(t) u_{0}-\frac{i}{2} \mathcal{N}(u, u)(t) .
$$

By Lemma 1, we have

$$
\begin{aligned}
\|\Psi(u)\|_{C\left([0, T], \mathcal{F} L^{1}\right)} & \leqslant\left\|u_{0}\right\|_{\mathcal{F} L^{1}}+T\|u\|_{C\left([0, T], \mathcal{F} L^{1}\right)}^{2} \\
\|\Psi(u)-\Psi(v)\|_{C\left([0, T], \mathcal{F} L^{1}\right)} & \lesssim T \max \left(\|u\|_{C\left([0, T], \mathcal{F} L^{1}\right)},\|v\|_{C\left([0, T], \mathcal{F} L^{1}\right)}\right)\|u-v\|_{C\left([0, T], \mathcal{F} L^{1}\right)} .
\end{aligned}
$$

Then, considering the ball

$$
B_{2 M}^{T}=\left\{\phi \in C\left([0, T], \mathcal{F} L^{1}\right):\|\phi\|_{C\left([0, T], \mathcal{F} L^{1}\right)} \leqslant 2 M\right\}
$$

with $T M \ll 1$, we find a fixed point of $\Psi$ in $B_{2 M}^{T}$ and hence a solution to (3). This completes the proof of the first part of the lemma. For the second part, we note that in view of Lemma 3, the series (5) converges absolutely if $0<T \ll M^{-1}$. Then, for $\epsilon>0$, there exists $j_{1}$ such that for all $j \geqslant j_{1}$, one has

$$
\begin{equation*}
\left\|u-u_{j}\right\|_{C\left([0, T], \mathcal{F} L^{1}\right)}<\epsilon \tag{6}
\end{equation*}
$$

where

$$
u=\sum_{k=1}^{\infty} U_{k}\left[u_{0}\right], \quad \text { and } \quad u_{j}=\sum_{k=1}^{j} U_{k}\left[u_{0}\right] .
$$

Note that $u, u_{j} \in B_{2 M}^{T}$ for all $j$ as $0<T \ll M^{-1}$. Using the continuity of $\Psi$ on $B_{2 M}^{T}$, we find $j_{2}$ such that for all $j \geqslant j_{2}$

$$
\begin{equation*}
\left\|\Psi(u)-\Psi\left(u_{j}\right)\right\|_{C\left([0, T], \mathcal{F} L^{1}\right)}<\epsilon . \tag{7}
\end{equation*}
$$

Note that

$$
\begin{aligned}
u_{j}-\Psi\left(u_{j}\right) & =\sum_{k=1}^{j} U_{k}\left[u_{0}\right]-U(t) u_{0}+\frac{i}{2} \mathcal{N}\left(u_{j}, u_{j}\right) \\
& =\sum_{k=2}^{j} U_{k}\left[u_{0}\right]+\frac{i}{2} \sum_{1 \leqslant k_{1}, k_{2} \leqslant j} \mathcal{N}\left(U_{k_{1}}\left[u_{0}\right], U_{k_{2}}\left[u_{0}\right]\right) \\
& =\frac{i}{2} \sum_{k=j+1}^{2 j} \sum_{\substack{1 \leqslant k_{1}, k_{2} \leqslant j \\
k_{1}+k_{2}=k}} \mathcal{N}\left(U_{k_{1}}\left[u_{0}\right], U_{k_{2}}\left[u_{0}\right]\right)=-\sum_{k=j+1}^{2 j} U_{k, j}\left[u_{0}\right]
\end{aligned}
$$

where we set

$$
U_{k, j}\left[u_{0}\right]=-\frac{i}{2} \sum_{\substack{1 \leqslant k_{1}, k_{2} \leqslant j \\ k_{1}+k_{2}=k}} \mathcal{N}\left(U_{k_{1}}\left[u_{0}\right], U_{k_{2}}\left[u_{0}\right]\right) .
$$

Note that $U_{k, j}$ has a lower number of terms in the sum above compared to that of $U_{k}$. Hence, proceeding as in the proof of Lemma 3, one achieves the same estimates for $U_{k, j}$. Thus, using $0<T \ll M^{-1}$,

$$
\begin{aligned}
\left\|u_{j}-\Psi\left(u_{j}\right)\right\|_{C\left([0, T], \mathcal{F} L^{1}\right)} & \leqslant \sum_{k=j+1}^{2 j}\left\|U_{k, j}\left[u_{0}\right]\right\|_{C\left([0, T], \mathcal{F} L^{1}\right)} \\
& \leqslant \sum_{k=j+1}^{2 j} c^{k-1} T^{k-1}\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k} \\
& \leqslant M \sum_{k=j+1}^{\infty}(c T M)^{k-1} \leqslant 2 M(c M T)^{j} .
\end{aligned}
$$

Then, there exists $j_{3}$ such that for $j \geqslant j_{3}$, one has

$$
\begin{equation*}
\left\|u_{j}-\Psi\left(u_{j}\right)\right\|_{C\left([0, T], \mathcal{F} L^{1}\right)}<\epsilon . \tag{8}
\end{equation*}
$$

Therefore, from (6)-(8), one has

$$
\|u-\Psi(u)\|_{C\left([0, T], F L^{1}\right)}<3 \epsilon .
$$

Thus, $u$ is the required fixed point for $\Psi$.

## 4. Proof of Theorem 1

We first prove NI with infinite loss of regularity at general data in $\mathcal{F} L^{1}(\mathcal{M}) \cap X_{s}^{p, q}(\mathcal{M})$. Subsequently, for general data in $X_{s}^{p, q}(\mathcal{M})$, we use the density of $\mathcal{F} L^{1}(\mathcal{M}) \cap X_{s}^{p, q}(\mathcal{M})$ in $\mathcal{X}_{s}^{p, q}(\mathcal{M})(s<0)$. Thus, let us begin with $u_{0} \in \mathcal{F} L^{1}(\mathcal{M}) \cap X_{s}^{p, q}(\mathcal{M})$. Now, define $\phi_{0, N}$ on $\mathcal{M}$ via the following relation

$$
\begin{equation*}
\mathcal{F} \phi_{0, N}(\xi)=R \chi_{I_{N}}(\xi) \quad(\xi \in \widehat{\mathcal{M}}) \tag{9}
\end{equation*}
$$

where $I_{N}=[-N-1,-N+1] \cup[N-1, N+1]$ and $N \gg 1, R \gg 1$ to be chosen later. Note that

$$
\begin{equation*}
\left\|\phi_{0, N}\right\|_{\hat{w}_{s}^{p, q}} \sim R N^{s} . \tag{10}
\end{equation*}
$$

Let us set

$$
\begin{equation*}
u_{0, N}=u_{0}+\phi_{0, N} \tag{11}
\end{equation*}
$$

Lemma 4 (See Lemma 3.6. in [21]). There exists $C>0$ such that for $u_{0}$ satisfying (9) and $k \geqslant 1$, we have

$$
\left|\operatorname{supp} \mathcal{F} U_{k}\left[\phi_{0, N}\right](t)\right| \leqslant C^{k}, \quad \forall t \geqslant 0
$$

4.1. Estimates in $\widehat{w}_{s}^{p, q}(\mathcal{M})$

Lemma 5. Let $u_{0}$ be given by (9), $s \leqslant 0$ and $1 \leqslant p, q \leqslant \infty$. Then, there exists $C$ such that
(1) $\left\|u_{0, N}-u_{0}\right\|_{\widehat{w}_{s}^{p, q}} \lesssim R N^{s}$
(2) $\left\|U_{1}\left[u_{0, N}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \lesssim 1+R N^{s}$
(3) $\left\|U_{2}\left[u_{0, N}\right](t)-U_{2}\left[\phi_{0, N}\right](t)\right\|_{\widehat{w}_{s}^{p, q}} \lesssim t R$
(4) $\left\|U_{k}\left[\vec{u}_{0, N}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \lesssim C^{k} R^{k} t^{k-1}$.

Proof. (1) follows from (10). By Lemma 1 and (10), we have $\left\|U_{1}\left[\phi_{0, N}\right](t)\right\|_{\hat{w}_{s}^{p, q}}=\left\|\phi_{0, N}\right\|_{\hat{w}_{s}^{p, q}} \sim$ $R N^{s}$. Then, (2) follows by using triangle inequality. By Lemma 3 and (10), we obtain

$$
\begin{aligned}
\left\|U_{k}\left[\phi_{0, N}\right](t)\right\|_{\widehat{w}_{s}^{p, q}} & \leqslant \sup _{\xi \in \widehat{\mathcal{M}}}\left|\mathcal{F} U_{k}\left[\phi_{0, N}\right](t, \xi)\right| \mu_{\widehat{\mathcal{M}}}\left(\operatorname{supp} \mathcal{F} U_{k}\left[\phi_{0, N}\right](t)\right)^{1 / p}\left\|\langle n\rangle^{s}\right\|_{\ell q}\left(n \in \operatorname{supp} \mathcal{F} U_{k}\left[\phi_{0, N}\right](t)\right) \\
& \lesssim(c t)^{k-1} R^{k}\left\|\langle n\rangle^{s}\right\|_{\ell q}\left(n \in \operatorname{supp} \mathcal{F} U_{k}\left[\phi_{0, N}\right](t)\right)
\end{aligned}
$$

where $\mu_{\widehat{\mathcal{M}}}(A)$ denotes the $\widehat{\mathcal{M}}$-measure of the set $A$. Since $s \leqslant 0$, for any bounded set $D \subset \mathbb{R}$, we have

$$
\left\|\langle n\rangle^{s}\right\|_{\ell q}(n \in D) \leqslant\left\|\langle n\rangle^{s}\right\|_{\ell q}\left(n \in B_{D}\right)
$$

where $B_{D} \subset \mathbb{R}$ is the interval centered at the origin with $|D|=\left|B_{D}\right|$. In view of this and Lemma 4, we obtain

$$
\left.\left\|\langle n\rangle^{s}\right\|_{\ell q\left(\operatorname{supp} \widehat{u}_{k}\left[\phi_{0, N}\right](t)\right)} \leqslant\left\|\langle n\rangle^{s}\right\|_{\ell( }\left(\left\{|n| \leqslant C^{k / d}\right\}\right)\right)
$$

Therefore,

$$
\begin{equation*}
\left\|U_{k}\left[\phi_{0, N}\right](t)\right\|_{\hat{w}_{s}^{p, q}} \leqslant C^{k} t^{k-1} R^{k} \tag{12}
\end{equation*}
$$

Now, observe that

$$
\begin{aligned}
I_{k}(t):= & U_{k}\left[u_{0, N}\right](t)-U_{k}\left[\phi_{0, N}\right](t) \\
& =\sum_{\substack{k_{1}, k_{2} \geq 1 \\
k_{1}+k_{2}=k}} \mathcal{N}\left(U_{k_{1}}\left[u_{0}+\phi_{0, N}\right], U_{k_{2}}\left[u_{0}+\phi_{0, N}\right]\right)-\mathcal{N}\left(U_{k_{1}}\left[\phi_{0, N}\right], U_{k_{2}}\left[\phi_{0, N}\right]\right) \\
& =\sum_{\substack{k_{1}, k_{2} \geq 1 \\
k_{1}+k_{2}=k}} \sum_{\left(\psi_{1}, \psi_{2}\right) \in \mathcal{C}} \mathcal{N}\left(U_{k_{1}}\left[\psi_{1}\right], U_{k_{2}}\left[\psi_{2 \sigma+1}\right]\right)
\end{aligned}
$$

where $\mathcal{C}=\left\{u_{0}, \phi_{0, N}\right\}^{2} \backslash\left\{\left(\phi_{0, N}, \phi_{0, N}\right)\right\}$. Observe that $\mathcal{C}$ has atleast one coordinate as $\vec{u}_{0}$. Using Lemma 1 and the proof of Lemma 3, it follows that

$$
\begin{aligned}
\left\|I_{k}(t)\right\|_{\hat{w}_{s}^{p, q}} & \lesssim \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\
k_{1}+k_{2}=k}} \sum_{\left(v_{1}, v_{2}\right) \in \mathcal{C}} \int_{0}^{t}\left\|U_{k_{1}}\left[v_{1}\right](\tau)\right\|_{\hat{w}_{s}^{p, q}}\left\|U_{k_{2}}\left[v_{2}\right](\tau)\right\|_{\mathcal{F}^{1}} d \tau \\
& \leqslant\left(2^{2}-1\right) 2\left\|u_{0}\right\|_{\widehat{w}_{s}^{p, q}}\left(\left\|u_{0}\right\|_{\mathcal{F} L^{1}}^{k-1}+\left\|\phi_{0, N}\right\|_{\mathcal{F} L^{1}}^{k-1} \int_{0}^{t} \tau^{k-2} d \tau \sum_{\substack{k_{1}, k_{2} \geqslant 1 \\
k_{1}+k_{2}=k}} b_{k_{1}} b_{k_{2}}\right. \\
& \leqslant 12 b_{k} t^{k-1} R^{k-1}\left\|u_{0}\right\|_{\hat{w}_{s}^{p, q}} \leqslant C^{k} t^{k-1} R^{k-1}\left\|u_{0}\right\|_{\hat{w}_{s}^{p, q}}
\end{aligned}
$$

as $R \gg 1$. Note that (3) is the particular case $k=2$ and (4) follows using the above and (12).

Lemma 6. Let $u_{0}$ be given by (9), $1 \leqslant p \leqslant \infty, s \in \mathbb{R}$ and $0<T \ll 1$, and then we have

$$
\left\|U_{2}\left[\phi_{0, N}\right](T)\right\|_{\hat{w}_{s}^{p, q}} \geqslant\| \| \chi_{n+Q_{1}}(\xi) \mathcal{F} U_{2}\left[\phi_{0, N}\right](T)(\xi)\left\|_{L_{\xi}^{p}\langle n\rangle^{s}}\right\|_{\ell q(n=1)} \gtrsim R^{2} T
$$

Proof. For notational convenience, we write

$$
\Gamma_{\xi}=\left\{\left(\xi_{1}, \xi_{2}\right): \xi_{1}+\xi_{2}=\xi\right\} \quad \text { and } \quad \Phi=c\left(-\varphi(\xi)+\varphi\left(\xi_{1}\right)+\varphi\left(\xi_{2}\right)\right) .
$$

Using the symmetry of set $\Gamma_{\tilde{\xi}}$, we have

$$
\begin{align*}
\mathcal{F} U_{2}\left[u_{0}\right](T)(\xi) & =\int_{0}^{T} e^{i c(T-t) \varphi(\xi)} \varphi(\xi)\left(\mathcal{F} U_{1}(t) u_{0} * \mathcal{F} U_{1}(t) u_{0}\right) d t \\
& =\int_{0}^{T} e^{i c(T-t) \varphi(\xi)} \varphi(\xi)\left[e^{i c t \varphi} \mathcal{F} u_{0} * e^{i c t \varphi} \mathcal{F} u_{0}\right](\xi) d t  \tag{13}\\
& =e^{i c T \varphi(\xi)} \varphi(\xi) R^{2} \int_{0}^{T} \int_{\Gamma_{\tilde{\xi}}} e^{i t \Phi} \chi_{I_{N}}\left(\xi_{1}\right) \chi_{I_{N}}\left(\xi_{2}\right) d \Gamma_{\xi} d t .
\end{align*}
$$

Note that, with $\xi_{1}+\xi_{2}=\xi$, one has

$$
\Phi\left(\xi, \xi_{1}, \xi_{2}\right)=c \frac{\xi \xi_{1} \xi_{2}\left(\xi^{2}-\xi_{1} \xi_{2}+3\right)}{\left(1+\xi_{1}^{2}\right)\left(1+\xi_{2}^{2}\right)\left(1+\xi^{2}\right)}
$$

and so for $\xi \in\left[\frac{1}{2}, 1\right]$ and $\xi_{1}, \xi_{2} \in I_{N}$, we have $|\Phi| \sim 1$. Hence, $|t \Phi| \ll 1$ for $0<t \ll 1$. Thus,

$$
\operatorname{Re} \int_{0}^{T} e^{i t \Phi} d t \geqslant \frac{T}{2} .
$$

Moreover, note that $|\varphi(\xi)| \gtrsim 1$ for $\xi \in\left[\frac{1}{2}, 1\right]$. Thus, we have for $\xi \in\left[\frac{1}{2}, 1\right] \subset I_{N}+I_{N}$

$$
\begin{equation*}
\left|\mathcal{F} U_{2}\left[u_{0}\right](T)(\xi)\right| \gtrsim R^{2} T \int_{\Gamma_{\tilde{\xi}}} \chi_{I_{N}}\left(\xi_{1}\right) \chi_{I_{N}}\left(\xi_{2}\right) d \Gamma_{\xi}=R^{2} T \chi_{I_{N}} * \chi_{I_{N}}(\xi) \geqslant R^{2} T \chi_{[-1,1]} \tag{14}
\end{equation*}
$$

as $\chi_{a+[-1,1]} * \chi_{b+[-1,1]} \geqslant \chi_{a+b+[-1,1]}$. The above pointwise estimate immediately gives the desired estimate:

$$
\left\|U_{2}\left[\phi_{0, N}\right](T)\right\|_{\widehat{w_{s}^{p, q}}} \geqslant\| \| \chi_{n+Q_{1}}(\xi) \mathcal{F} U_{2}\left[\vec{\phi}_{0, N}\right](T)(\xi)\left\|_{L_{\tilde{\xi}}^{p}\left(\left[\frac{1}{2}, 1\right] \cap \widehat{\mathcal{M}}\right)}\langle n\rangle^{s}\right\|_{\ell q(n=1)} \gtrsim T R^{2}
$$

provided $0<T \ll 1$.
4.2. Estimates in $W_{s}^{2, q}(\mathbb{R})$

Lemma 7 (inclusion). Let $p, q, q_{1}, q_{2} \in[1, \infty]$ and $s \in \mathbb{R}$. Then,
(1) $\|f\|_{W_{s}^{2, q}} \leqslant\|f\|_{\widehat{w}_{s}^{2, q}}$ if $q \leqslant 2$
(2) $\|f\|_{W_{s}^{p, q_{1}}} \lesssim\|f\|_{W_{s}^{p, q_{2}}}$ if $q_{1} \geqslant q_{2}$

Proof. (1) is a consequence of Minkowski inequality and Plancherel theorem, whereas (2) follows from the fact that $\ell^{q_{2}} \hookrightarrow \ell^{q_{1}}$ if $q_{1} \geqslant q_{2}$.

Lemma 8. Let $u_{0}$ be given by (9), $s \leqslant 0$ and $1 \leqslant p \leqslant \infty$. Then, there exists $C$ such that
(1) $\left\|u_{0, N}-u_{0}\right\|_{W_{s}^{2, q}} \lesssim R N^{s}$
(2) $\left\|U_{1}\left[u_{0, N}\right](t)\right\|_{W_{s}^{2, q}} \lesssim 1+R N^{s}$
(3) $\left\|U_{2}\left[u_{0, N}\right](t)-U_{2}\left[\phi_{0, N}\right](t)\right\|_{W_{s}^{2, q}} \lesssim t R$
(4) $\left\|U_{k}\left[\vec{u}_{0, N}\right](t)\right\|_{W_{s}^{2, q}} \lesssim C^{k} R^{k} t^{k-1}$

Proof. By Lemma 7, we have

$$
\left\|\vec{u}_{0, N}-\vec{u}_{0}\right\|_{W_{s}^{2, q}} \lesssim \begin{cases}\left\|\vec{u}_{0, N}-\vec{u}_{0}\right\|_{\widehat{w}_{s}^{2, q}} \lesssim R N^{s} & \text { for } q \in[1,2] \\ \left\|\vec{u}_{0, N}-\vec{u}_{0}\right\|_{W_{s}^{2,2}} \lesssim R N^{s} & \text { for } q \in(2, \infty]\end{cases}
$$

using Lemma 5 (1). Similarly, the other estimates also follow from Lemmata 5.
Lemma 9. Let $u_{0}$ be given by (9), $1 \leqslant p \leqslant \infty, s \in \mathbb{R}$ and $0<T \ll 1$, then we have

$$
\left\|U_{2}\left[\phi_{0, N}\right](T)\right\|_{W_{s}^{2, q}} \geqslant\| \| \mathcal{F}^{-1} \sigma_{n} \mathcal{F} U_{2}\left[\phi_{0, N}\right](T)(\xi)\langle n\rangle^{s}\left\|_{\ell q(n=1)}\right\|_{L_{\xi}^{2}} \gtrsim R^{2} T .
$$

Proof. Note that using Plancherel theorem and (14), we have

$$
\begin{aligned}
\left\|U_{2}\left[\phi_{0, N}\right](T)\right\|_{W_{s}^{2, q}} & \geqslant\| \| \mathcal{F}^{-1} \sigma_{n} \mathcal{F} U_{2}\left[\phi_{0, N}\right](T)(x)\langle n\rangle^{s}\left\|_{\ell q}\left(n=e_{1}\right)\right\|_{L_{x}^{2}} \\
& =2^{s / 2}\left\|\sigma_{e_{1}} \mathcal{F} U_{2}\left[\vec{\phi}_{0, N}\right](T)(\xi)\right\|_{L_{\xi}^{2}} \gtrsim R^{2} T .
\end{aligned}
$$

This completes the proof.
Proof of Theorem 1. We first consider the case $\mathcal{X}_{s}^{p, q}=\widehat{w}_{s}^{p, q}$. If the initial data $u_{0, N}$ satisfy (11), Corollary 1 guarantees the existence of the solution to (3) and the power series expansion in $\mathcal{F} L^{1}$ up to time $T R \ll 1$ (as $R \gg 1$ ). By Lemma 5, we obtain

$$
\begin{equation*}
\sum_{k=3}^{\infty}\left\|U_{k}\left[u_{0, N}\right](T)\right\|_{\hat{w}_{s}^{p, q}} \lesssim T^{2} R^{3} \tag{15}
\end{equation*}
$$

provided $T R \ll 1$. Note that

$$
\left\|u_{N}(T)\right\|_{\widehat{w}_{\theta}^{p, q}} \geqslant\| \| \chi_{n+Q_{1}} \mathcal{F} u_{N}(T)\left\|_{L^{p}}\langle n\rangle^{\theta}\right\|_{\ell(n=1)} \sim_{\theta, s}\| \| \chi_{n+Q_{1}} \mathcal{F} u_{N}(T)\left\|_{L^{p}}\langle n\rangle^{s}\right\|_{\ell(n=1)}
$$

Using Corollary 1 and triangle inequality, we have

$$
\begin{aligned}
& \left\|u_{N}(T)\right\|_{\widehat{w}_{\theta}^{p, q}} \\
& \gtrsim\left\|\left\|\chi_{n+Q_{1}} \mathcal{F} U_{2}\left[u_{0, N}\right](T)\right\|_{L^{p}}\langle n\rangle^{s}\right\|_{\ell(n=1)}-c\left(\| \| \chi_{n+Q_{1}} \mathcal{F} U_{1}\left[u_{0, N}\right](T)\left\|_{L^{p}}\langle n\rangle^{s}\right\|_{\ell^{q}(n=1)}\right. \\
& \left.\quad+\sum_{k=3}^{\infty}\| \| \chi_{n+Q_{1}} \mathcal{F} U_{k}\left[u_{0, N}\right](T)\left\|_{L^{p}}\langle n\rangle^{s}\right\|_{\ell^{q}(n=1)}\right) \\
& \gtrsim\left\|\left\|\chi_{n+Q_{1}} \mathcal{F} U_{2}\left[u_{0, N}\right](T)\right\|_{L^{p}}\langle n\rangle^{s}\right\|_{\ell(n=1)}-c\left\|U_{1}\left[\vec{u}_{0, N}\right](T)\right\|_{\widehat{w}_{s}^{p, q}}-c \sum_{k=3}^{\infty}\left\|U_{k}\left[\vec{u}_{0, N}\right](T)\right\|_{\hat{w}_{s}^{p, q}} .
\end{aligned}
$$

Let $m \in \mathbb{N}$. In order to ensure $\left\|u_{N}(T)\right\|_{\widehat{w}_{\theta}^{p, q}} \gtrsim\left\|U_{2}\left[\vec{u}_{0, N}\right](T)\right\|_{\widehat{w}_{s}^{p, q}} \gg m$, we rely on the conditions

$$
\left\|\left\|\chi_{n+Q_{1}} \mathcal{F} U_{2}\left[u_{0, N}\right](T)\right\|_{L^{2}}\langle n\rangle^{s}\right\|_{\ell q(n=1)} \gg\left\{\begin{array}{l}
\left\|U_{1}\left[\vec{u}_{0, N}\right](T)\right\|_{\widehat{w}_{s}^{p, q}}  \tag{16}\\
\sum_{k=2}^{\infty}\left\|U_{k}\left[\vec{u}_{0, N}\right](T)\right\|_{\widehat{w}_{s}^{p, q}} \\
m .
\end{array}\right.
$$

Thus, to establish NI with infinite loss of regularity at $u_{0}$ in $\widehat{w}_{s}^{p, q}$, we claim that it is enough to have the following:
(1) $C R N^{s}<1 / m$
(2) $T R \ll 1$
(3) $T R^{2} \gg m$
(4) $T R^{2} \gg T^{2} R^{3} \Leftrightarrow(2)$
(5) $0<T \ll 1$
as $N \rightarrow \infty$. Note that (1) ensures $\left\|u_{0}-u_{0, N}\right\|_{\hat{w}_{s}^{p, q}}<1 / m$, whereas (2) ensures the convergence of the infinite series in view of Lemma 5. In order to use Lemma 6, we need (4). In order to prove (17), in view of Lemma 6 and (15), we need (4). Condition (3) implies (18) using Lemma 5 (3) and Lemma 6. In order to prove (16), we need (1) and (3) by using Lemma 5 (2) and Lemma 6. Thus, it follows that

$$
\left\|u_{0}-u_{0, N}\right\|_{\widehat{w}_{s}^{p, q}}<1 / m \quad \text { and } \quad\left\|u_{N}(T)\right\|_{\hat{w}_{\theta}^{p, q}}>m
$$

Hence, the result is established. We shall now choose $R$ and $T$ as follows:

$$
R=N^{r} \quad \text { and } \quad T=N^{-\epsilon} .
$$

where $r, \epsilon$ are to be chosen below. Therefore, it is enough to check

$$
C R N^{s}=C N^{r+s}<1 / m, \quad T R=N^{-\epsilon+r} \ll 1, \quad T R^{2}=N^{-\epsilon+2 r} \gg m, \quad T=N^{-\epsilon} \ll 1 .
$$

Thus, we only need to achieve:

- $r+s<0$
- $-\epsilon+r<0$
- $-\epsilon+2 r>0$
- $\epsilon>0$
and take $N$ large enough. Let us concentrate on the choice of $\epsilon>0$ first. Note that the second and third conditions in the above are equivalent to

$$
r<\epsilon<2 r .
$$

To make room for $\epsilon$, we must have $r>0$. Thus, $r$ must satisfy

$$
0<r<-s
$$

where the latter condition comes from the first condition. Thus, it is enough to choose

$$
r=-\frac{s}{3}, \quad \epsilon=-\frac{s}{2}
$$

which will satisfy all the above four conditions. Hence, the result follows.
For the case $\mathcal{X}_{s}^{p, q}=W_{s}^{2, q}$, we use same argument as above. Note that using Lemmata 8 and 9, we have

$$
\begin{aligned}
& \left\|u_{N}(T)\right\|_{W_{\theta}^{2, q}} \\
& \geqslant\| \| \square_{n} u_{N}(T)\langle n\rangle^{\sigma}\left\|_{\ell q(n=1)}\right\|_{L^{2}} \sim_{\theta, s}\| \| \square_{n} u_{N}(T)\langle n\rangle^{s}\left\|_{\ell q(n=1)}\right\|_{L^{2}} \\
& \gtrsim\left\|\left\|\square_{n} U_{2}\left[u_{0, N}\right](T)\langle n\rangle^{s}\right\|_{\ell q(n=1)}\right\|_{L^{2}}-c\left\|U_{1}\left[u_{0, N}\right](T)\right\|_{W_{s}^{2, q}}-c \sum_{k=3}^{\infty}\left\|U_{k}\left[\vec{u}_{0, N}\right](T)\right\|_{W_{s}^{2, q}} \\
& \gtrsim\left\|\left\|\square_{n} U_{2}\left[u_{0, N}\right](T)\langle n\rangle^{s}\right\|_{\ell q(n=1)}\right\|_{L^{2}} \gg m .
\end{aligned}
$$

and $\left\|\vec{u}_{0, N}-\vec{u}_{0}\right\|_{W_{s}^{2, q}}<1 / m$ provided that we choose $R, N, T$ as in the case of $\widehat{w}_{s}^{p, q}$

Remark 1. It is easy to check that our proof of the main results will work even if we replace the weight $\langle\cdot\rangle^{s}$ by $|\cdot|^{s}$ in the function spaces involved. Since the analysis will be similar, we omit the details. We simply note that as $\langle n\rangle^{s}=|n|^{s}$ for large $n$, we have $\left\|\phi_{0, N}\right\|_{\dot{\tilde{w}}_{s}^{p, q}}=R N^{s}$, where $\phi_{0, N}$ is as in (9). Moreover, it should work with any weight $n \mapsto(\omega(n))^{s}(s<0)$ that is decreasing in $|n|$ and behaves as $|n|^{s}$ as $n \rightarrow \infty$.

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