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Canonical Coordinates and Natural Equation for Lorentz Surfaces in \mathbb{R}_1^3

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Abstract: We consider Lorentz surfaces in \mathbb{R}_1^3 satisfying the condition $H^2 - K \neq 0$, where K and H are the Gaussian curvature and the mean curvature, respectively, and call them Lorentz surfaces of general type. For this class of surfaces, we introduce special isotropic coordinates, which we call canonical, and show that the coefficient F of the first fundamental form and the mean curvature H , expressed in terms of the canonical coordinates, satisfy a special integro-differential equation which we call a natural equation of the Lorentz surfaces of a general type. Using this natural equation, we prove a fundamental theorem of Bonnet type for Lorentz surfaces of a general type. We consider the special cases of Lorentz surfaces of constant non-zero mean curvature and minimal Lorentz surfaces. Finally, we give examples of Lorentz surfaces illustrating the developed theory.

Keywords: Lorentz surfaces; pseudo-Euclidean space; canonical coordinates; natural equations

MSC: primary 53B30; secondary 53A35



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1. Introduction

The question of describing surfaces with a prescribed mean or Gaussian curvature in the Euclidean 3-space \mathbb{R}^3 and also in the other Riemannian space forms have been the subject of an intensive study. In particular, the geometry of spacelike or timelike surfaces in the Minkowski 3-space \mathbb{R}_1^3 has been of wide interest. For example, a Kenmotsu-type representation formula for spacelike surfaces with prescribed mean curvature was obtained by Akutagawa and Nishikawa in [1]. In [2], Gálvez et al. obtained a representation for spacelike surfaces in \mathbb{R}_1^3 using the Gaussian map and the conformal structure given by the second fundamental form. Magid proved that the Gauss map and the mean curvature of a timelike surface satisfy a system of partial differential equations and found a Weierstrass representation formula for timelike surfaces in \mathbb{R}_1^3 [3]. Timelike surfaces in \mathbb{R}_1^3 with prescribed Gaussian curvature and Gauss map are studied in [4], where a Kenmotsu-type representation for such surfaces is given. This representation is used to classify the complete timelike surfaces with positive constant Gaussian curvature in terms of harmonic diffeomorphisms between simply connected Lorentz surfaces and the universal covering of the de Sitter Space.

On the other hand, it is known that the minimal Lorentz surfaces in \mathbb{R}_1^3 , \mathbb{R}_1^4 , and \mathbb{R}_2^4 can be parametrized by special isothermal coordinates, called *canonical*, such that the main invariants (the Gaussian curvature and the normal curvature) of the surface satisfy a system of partial differential equations called a *system of natural PDEs*. The geometry of the corresponding minimal surface is determined by the solution of this system of natural PDEs.

In [5], canonical coordinates for the class of minimal Lorentz surfaces in the Minkowski space \mathbb{R}_1^4 are introduced, and the following system of natural PDEs is obtained:

$$\begin{aligned} \sqrt[4]{K^2 + \varkappa^2} \Delta^h \ln \sqrt[4]{K^2 + \varkappa^2} &= 2K; \\ \sqrt[4]{K^2 + \varkappa^2} \Delta^h \arctan \frac{\varkappa}{K} &= 2\varkappa; \end{aligned} \quad K^2 + \varkappa^2 \neq 0, \tag{1}$$

where K is the Gaussian curvature, \varkappa is the curvature of the normal connection (the normal curvature), and Δ^h is the hyperbolic Laplace operator in \mathbb{R}_1^2 .

Similar results are obtained for minimal Lorentz surfaces in the pseudo-Euclidean space with neutral metric \mathbb{R}_2^4 in [6,7]. The corresponding system of PDEs has the following form:

$$\begin{aligned} \sqrt[4]{|K^2 - \varkappa^2|} \Delta^h \ln \sqrt[4]{|K^2 - \varkappa^2|} &= 2K; \\ \sqrt[4]{|K^2 - \varkappa^2|} \Delta^h \ln \left| \frac{K + \varkappa}{K - \varkappa} \right| &= 4\varkappa; \end{aligned} \quad K^2 - \varkappa^2 \neq 0. \tag{2}$$

The minimal Lorentz surfaces in \mathbb{R}_1^3 can also be considered as surfaces in \mathbb{R}_1^4 or \mathbb{R}_2^4 , in which cases $\varkappa = 0$. Thus, systems (1) and (2) are reduced to one PDE:

$$\sqrt{|K|} \Delta^h \ln \sqrt{|K|} = 2K; \quad K \neq 0, \tag{3}$$

which is the natural equation of minimal Lorentz surfaces in \mathbb{R}_1^3 . Of course, the results in this case can be directly obtained. In [8], canonical coordinates are introduced for minimal Lorentz surfaces in \mathbb{R}_1^3 and equations equivalent to (3) are derived.

Thus, the following natural question arises: *How to generalize the concepts of canonical coordinates and natural equation for a wider class of Lorentz surfaces in \mathbb{R}_1^3 than that of the minimal ones?* The class of Weingarten Lorentz surfaces in \mathbb{R}_1^3 with different real principal curvatures (which is equivalent to $H^2 - K > 0$, where K and H are the Gaussian curvature and the mean curvature, respectively) is considered in [9]. Canonical principal coordinates are introduced for this class of surfaces, and a natural nonlinear partial differential equation is derived, which is equivalent to (3) in the case of a minimal surface.

The main purpose of this article is to generalize the notion of canonical coordinates and to find a natural PDE for all Lorentz surfaces satisfying $H^2 - K \neq 0$ (we call them surfaces of the general type). We introduce special isotropic coordinates (which we call *canonical*) for these surfaces and obtain a natural integro-differential equation. The natural equation for the class of minimal Lorentz surfaces is given by

$$\sqrt{|K|} (\ln \sqrt{|K|})_{uv} = K; \quad K \neq 0.$$

It can be reduced to (3) by changing the isotropic coordinates with isothermal ones. This shows that the newly obtained results for an arbitrary Lorentz surface of a general type in \mathbb{R}_1^3 generalize the known results for the case of a minimal Lorentz surface.

In Section 2, we give some basic formulas for Lorentz surfaces in \mathbb{R}_1^3 parametrized by arbitrary isotropic coordinates. We present the Gauss and Codazzi equations in terms of these coordinates and formulate the fundamental theorem of Bonnet type.

In Section 3, we introduce the notion of canonical isotropic coordinates for the class of Lorentz surfaces of general type ($H^2 - K \neq 0$) in \mathbb{R}_1^3 . We prove existence and uniqueness theorems for these coordinates and give the relation between the canonical coordinates and the natural parameters of the isotropic curves of the surface.

In Section 4, we consider Lorentz surfaces of general type parametrized by canonical coordinates and show that the coefficient F of the first fundamental form and the mean curvature H of such surface satisfy the following integro-differential equation

$$\frac{FF_{uv} - F_u F_v}{F} = \left(\varepsilon_1 + \int_{v_0}^v F(u, s) H_u(u, s) ds \right) \left(\varepsilon_2 + \int_{u_0}^u F(s, v) H_v(s, v) ds \right) - F^2 H^2,$$

$\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$. We call it the natural equation of the Lorentz surfaces in \mathbb{R}_1^3 and prove a fundamental theorem of Bonnet type. We consider in detail the special cases of a Lorentz surface with a non-zero constant mean curvature and a minimal Lorentz surface.

In Section 5, we give examples of different types of Lorentz surfaces and their canonical coordinates in \mathbb{R}_1^3 .

2. Preliminaries

Let \mathbb{R}_1^3 be the standard three-dimensional pseudo-Euclidean space in which the indefinite inner scalar product is given by the formula:

$$\langle a, b \rangle = -a_1b_1 + a_2b_2 + a_3b_3.$$

Let $\mathcal{M} = (\mathcal{D}, x)$ be a Lorentz surface in \mathbb{R}_1^3 , where $\mathcal{D} \subset \mathbb{R}^2$ and $x : \mathcal{D} \rightarrow \mathbb{R}_1^3$ is an immersion. The coefficients of the first fundamental form of \mathcal{M} are denoted as usually by E, F, G and L, M, N denote the coefficients of the second fundamental form. Then, the Gaussian curvature K and the mean curvature H of \mathcal{M} are given by the formulas (see [10,11]):

$$K = \frac{LN - M^2}{EG - F^2}; \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

In a neighbourhood of each point of \mathcal{M} , there exist isotropic coordinates (u, v) such that $E = G = 0$ [10]. Such parameters are also called null coordinates [12]. It can easily be seen that, if (u, v) and (\tilde{u}, \tilde{v}) are two different pairs of isotropic coordinates in a neighbourhood of a fixed point, then they are related either by $u = u(\tilde{u}), v = v(\tilde{v})$, or $u = u(\tilde{v}), v = v(\tilde{u})$.

Furthermore, we consider a surface \mathcal{M} parametrized by isotropic coordinates and, without loss of generality, we assume that $F > 0$. Then, the formulas for K and H take the following form:

$$K = \frac{M^2 - LN}{F^2}; \quad H = \frac{M}{F}. \tag{4}$$

Consider the tangent vector fields $X = x_u, Y = x_v$ and denote by l the unit normal vector field $l = \frac{x_u \times x_v}{|x_u \times x_v|}$ such that $\{X, Y, l\}$ be a positively oriented frame field in \mathbb{R}_1^3 . Since \mathcal{M} is parametrized by isotropic coordinates, we have:

$$X^2 = Y^2 = 0; \quad l^2 = 1; \quad \langle X, Y \rangle = F; \quad \langle X, l \rangle = \langle Y, l \rangle = 0.$$

Hence, we get

$$X_v = Y_u; \quad \langle X_u, X \rangle = \langle X_v, X \rangle = \langle Y_u, Y \rangle = \langle Y_v, Y \rangle = \langle l_u, l \rangle = \langle l_v, l \rangle = 0.$$

Using the last equalities, we obtain the following Frenet-type formulas for the frame field $\{X, Y, l\}$:

$$\begin{cases} X_u = \frac{F_u}{F}X & + Ll; \\ Y_u = & MI; \\ l_u = -\frac{M}{F}X - \frac{L}{F}Y; \end{cases} \quad \begin{cases} X_v = & MI; \\ Y_v = & \frac{F_v}{F}Y + NI; \\ l_v = -\frac{N}{F}X - \frac{M}{F}Y. \end{cases} \tag{5}$$

The integrability conditions of (5), considered as a system of PDE for the triple (X, Y, l) , imply the following Gauss equation:

$$\frac{FF_{uv} - F_uF_v}{F} = LN - M^2 \tag{6}$$

and the Codazzi equations

$$L_v = F \left(\frac{M}{F} \right)_u; \quad N_u = F \left(\frac{M}{F} \right)_v. \tag{7}$$

Note that (6) and (4) imply $K = -\frac{1}{F}(\ln F)_{uv}$, which is the Gauss's *Theorema Egregium* in the case of isotropic coordinates.

As it is well known, the Gauss and Codazzi equations are not only necessary, but also sufficient conditions for the existence of a solution to the PDE system (5). This gives us a fundamental Bonnet-type theorem for Lorentz surfaces in \mathbb{R}_1^3 . The proof is analogous to that of the classical theorem for surface in \mathbb{R}^3 (see ([13])).

Theorem 1. *Let \mathcal{M} be a Lorentz surface in \mathbb{R}_1^3 parametrized by isotropic coordinates. Then, the coefficients F, L, M, N of the first and the second fundamental forms of \mathcal{M} give a solution to the Gauss and Codazzi Equations (6) and (7). If $\hat{\mathcal{M}}$ is obtained from \mathcal{M} by a proper motion in \mathbb{R}_1^3 , then $\hat{\mathcal{M}}$ generates the same solution to (6) and (7).*

Conversely, if the functions $F, L, M,$ and N satisfy Equations (6) and (7), then, at least locally, there exists a unique (up to a proper motion in \mathbb{R}_1^3) Lorentz surface \mathcal{M} parametrized by isotropic coordinates, such that the given functions are the coefficients of the first and the second fundamental forms of \mathcal{M} .

3. Canonical Isotropic Coordinates of Lorentz Surfaces in \mathbb{R}_1^3

In the present section, we will show that the Lorentz surfaces satisfying $H^2 - K \neq 0$ admit special isotropic coordinates, which we will call canonical. It follows from the Codazzi Equation (7) and the second equality of (4) that

$$L_v = FH_u; \quad N_u = FH_v. \tag{8}$$

Integrating the last equalities, we obtain

$$L = L(u, v_0) + \int_{v_0}^v F(u, s)H_u(u, s) ds; \quad N = N(u_0, v) + \int_{u_0}^u F(s, v)H_v(s, v) ds. \tag{9}$$

Now, we will try to choose the isotropic coordinates in such a way that $L(u, v_0)$ and $N(u_0, v)$ have the simplest form.

First, let us find the transformation formulas for the coefficients of the first and the second fundamental forms under changes of the isotropic coordinates. There are two possible cases:

1. If

$$u = u(\tilde{u}); \quad v = v(\tilde{v}),$$

then, we have $\tilde{F} = Fu'v'$, which implies $u'v' > 0$, since we have assumed at the beginning that $\tilde{F} > 0$ and $F > 0$. In this case, the orientation of the surface does not change, i.e., $\tilde{l} = l$. Then, for the coefficients $\tilde{F}, \tilde{L}, \tilde{M}, \tilde{N}$, we have

$$\tilde{F} = Fu'v'; \quad \tilde{L} = Lu'^2; \quad \tilde{M} = Mu'v'; \quad \tilde{N} = Nv'^2. \tag{10}$$

2. In the case $u = u(\tilde{v}); v = v(\tilde{u})$, it is sufficient to consider only the change of coordinates:

$$u = \tilde{v}; \quad v = \tilde{u},$$

since the general case is reduced to this and the previous one. In this case, the orientation of the surface changes, i.e., $\tilde{l} = -l$, and we have

$$\tilde{F} = F; \quad \tilde{L} = -N; \quad \tilde{M} = -M; \quad \tilde{N} = -L. \tag{11}$$

The transformation formulas (10) and (11) show that, if $L = 0$ or $N = 0$ for some isotropic coordinates, then $\tilde{L} = 0$ or $\tilde{N} = 0$ for any isotropic coordinates. Furthermore, we consider surfaces satisfying $L \neq 0$ and $N \neq 0$ at least locally. It follows from (4) that

$$H^2 - K = \frac{LN}{F^2}, \tag{12}$$

which implies that the conditions $L \neq 0$ and $N \neq 0$ are equivalent to $H^2 - K \neq 0$.

We give the following definition.

Definition 1. A Lorentz surface \mathcal{M} in \mathbb{R}_1^3 is said to be of general type, if $H^2 - K \neq 0$.

The Lorentz surfaces of general type are naturally divided into two subclasses.

Definition 2. A Lorentz surface of general type in \mathbb{R}_1^3 is said to be of first kind (resp. second kind), if $H^2 - K > 0$ (resp $H^2 - K < 0$).

Remark 1. Since K and H are invariants of \mathcal{M} , the properties of a surface to be the general type, as well as the kind of the surface, are geometric—they do not depend on the local parametrization and are invariant under motions in \mathbb{R}_1^3 . It is known that the discriminant of the characteristic polynomial of the Weingarten map has the form $D = 4(H^2 - K)$ [11]. Hence, the surfaces of general type are those surfaces for which the Weingarten map has two different eigenvalues. Moreover, the surfaces of the first kind are those with real eigenvalues, and the surfaces of the second kind are those with complex eigenvalues.

Now, we will introduce special isotropic coordinates by the following:

Definition 3. Let $\mathcal{M} = (\mathcal{D}, x)$ be a Lorentz surface of general type in \mathbb{R}_1^3 parametrized by isotropic coordinates (u, v) , such that $F > 0$, and $x_0 = x(u_0, v_0)$ be a point of \mathcal{M} . We call (u, v) canonical coordinates with initial point x_0 , if the coefficients of the second fundamental form satisfy the conditions:

$$L(u, v_0) = \varepsilon_1; \quad N(u_0, v) = \varepsilon_2, \tag{13}$$

where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$.

Remark 2. In the general case, condition (13) for the coordinates to be canonical depends on the choice of the initial point (u_0, v_0) . If (u_1, v_1) is another point in the same neighbourhood, from (9), it is obvious that $L(u, v_0) \neq L(u, v_1)$ and $N(u_0, v) \neq N(u_1, v)$ in general. In the special case of a surface with constant mean curvature H , L is a function of u and N is a function of v because of (8). Hence, for the class of surfaces with constant mean curvature, the canonicity of the coordinates does not depend on the choice of the initial point (u_0, v_0) .

Theorem 2. If $\mathcal{M} = (\mathcal{D}, x)$ is a Lorentz surface of general type in \mathbb{R}_1^3 and x_0 is a fixed point, then we can introduce canonical coordinates with initial point x_0 .

Proof. Let x_0 be a fixed point of \mathcal{M} and (u, v) be isotropic coordinates in a neighbourhood of x_0 such that $F > 0$ and $x_0 = x(u_0, v_0)$. Consider the change of the coordinates $u = u(\tilde{u})$ and $v = v(\tilde{v})$. According to (10) and Definition 3, the new coordinates (\tilde{u}, \tilde{v}) are canonical if and only if

$$L(u, v_0)u'^2 = \tilde{L}(\tilde{u}, \tilde{v}_0) = \pm 1; \quad N(u_0, v)v'^2 = \tilde{N}(\tilde{u}_0, \tilde{v}) = \pm 1. \tag{14}$$

The signs on the right-hand sides of (14) are chosen to coincide with the signs of L and N , respectively. Thus, we obtain ordinary differential equations for $u(\tilde{u})$ and $v(\tilde{v})$ whose solutions have the following form:

$$\tilde{u} = \tilde{u}_0 + \int_{u_0}^u \sqrt{|L(s, v_0)|} ds; \quad \tilde{v} = \tilde{v}_0 + \int_{v_0}^v \sqrt{|N(u_0, s)|} ds, \tag{15}$$

where $(\tilde{u}_0, \tilde{v}_0)$ is arbitrarily chosen. It follows from (12) that $L \neq 0$ and $N \neq 0$, which imply $\tilde{u}' > 0$ and $\tilde{v}' > 0$. Hence, equalities (15) define new isotropic coordinates (\tilde{u}, \tilde{v}) satisfying the condition $\tilde{F} > 0$. Since (15) is equivalent to (14), then $\tilde{L}(\tilde{u}, \tilde{v}_0) = \pm 1$ and $\tilde{N}(\tilde{u}_0, \tilde{v}) = \pm 1$. Thus, (\tilde{u}, \tilde{v}) are canonical coordinates of \mathcal{M} with initial point x_0 . \square

Remark 3. If \mathcal{M} is a surface of first kind according to Definition 2, then (12) implies $LN > 0$. It follows from (11) that a change in the coordinates numeration leads to a change in the signs of L and N . Hence, surfaces of the first kind admit both canonical coordinates for which $L = N = 1$ and canonical coordinates for which $L = N = -1$.

If \mathcal{M} is of the second kind, then $LN < 0$, and hence (10) and (11) show that the signs of L and N do not change under changes of the isotropic coordinates. Therefore, the surfaces of the second kind can be divided into two subclasses: surfaces with canonical coordinates such that $L = 1$ and $N = -1$, and surfaces with canonical coordinates such that $L = -1$ and $N = 1$.

Now, we will discuss the question of uniqueness of the canonical coordinates.

Theorem 3. Let \mathcal{M} be a Lorentz surface of general type in \mathbb{R}_1^3 and (u, v) be canonical coordinates with initial point $x_0 = x(u_0, v_0)$ of \mathcal{M} . Then, (\tilde{u}, \tilde{v}) is another pair of canonical coordinates with the same initial point x_0 if and only if

$$\begin{aligned} u &= \delta\tilde{u} + c_1; & \text{or} & & u &= \delta\tilde{v} + c_1; \\ v &= \delta\tilde{v} + c_2; & & & v &= \delta\tilde{u} + c_2, \end{aligned} \tag{16}$$

where $\delta = \pm 1$, c_1 and c_2 are constants.

Proof. First, we consider the case $u = u(\tilde{u})$ and $v = v(\tilde{v})$. Equalities (10) imply that (\tilde{u}, \tilde{v}) are canonical coordinates if and only if $u'^2 = 1$, $v'^2 = 1$, and $u'v' > 0$. The last conditions are equivalent to the first pair of equalities in (16).

The case $u = u(\tilde{v})$ and $v = v(\tilde{u})$ reduces to the previous one by means of (11). \square

The meaning of the last theorem is that the canonical coordinates are uniquely determined up to a numeration, a sign and an additive constant.

At the end of this section, we will characterize the canonical coordinates in terms of the null curves lying on the considered surfaces. Recall that, if α is a null curve ($\alpha'^2 = 0$) in \mathbb{R}_1^3 , then $\alpha''^2 \geq 0$. The null curves satisfying $\alpha''^2 > 0$ are called *non-degenerate*. It is known that these curves admit a parametrization such that $\alpha''^2 = 1$ [11]. Such parameter is known in the literature as a *natural parameter* or *pseudo arc-length parameter*, since it plays a role similar to the role of the arc-length parameter for non-null curves [14].

Theorem 4. Let $\mathcal{M} = (\mathcal{D}, x)$ be a Lorentz surface in \mathbb{R}_1^3 parametrized by isotropic coordinates (u, v) , such that $F > 0$, and $x_0 = x(u_0, v_0)$ be a point of \mathcal{M} . Then, \mathcal{M} is of general type if and only if the null curves lying on \mathcal{M} are non-degenerate. The coordinates (u, v) of \mathcal{M} are canonical with initial point x_0 if and only if they are natural parameters of the null curves on \mathcal{M} passing through x_0 .

Proof. The Frenet formulas (5) imply $x_{uu}^2 = X_u^2 = L^2$ and $x_{vv}^2 = Y_v^2 = N^2$. Hence, the u -lines and v -lines are non-degenerate if and only if $L \neq 0$ and $N \neq 0$, which is equivalent to \mathcal{M} being of the general type. Moreover, u is a natural parameter ($x_{uu}^2 = 1$) of the null curve $x(u, v_0)$ passing through x_0 if and only if $L^2(u, v_0) = 1$. Analogously, v is a natural parameter ($x_{vv}^2 = 1$) of the null curve $x(u_0, v)$ passing through x_0 if and only if $N^2(u_0, v) = 1$. The last conditions are equivalent to $L(u, v_0) = \pm 1$ and $N(u_0, v) = \pm 1$, which means that the coordinates (u, v) of \mathcal{M} are canonical with initial point x_0 . \square

4. Natural Equation of Lorentz Surfaces of General Type in \mathbb{R}_1^3

In this section, we will consider the Gauss and Codazzi equations of a Lorentz surface of general type $\mathcal{M} = (\mathcal{D}, x)$ in \mathbb{R}_1^3 parametrized by canonical isotropic coordinates (u, v) with initial point $x_0 = x(u_0, v_0) \in \mathcal{M}$. In such case, the coefficients of the second fundamental form are expressed by the coefficient F of the first fundamental form, the mean curvature H , and the constants ε_1 and ε_2 (see Definition 3). It follows from (4), (9) and (13) that:

$$L = \varepsilon_1 + \int_{v_0}^v F(u, s)H_u(u, s) ds; \quad M = FH; \quad N = \varepsilon_2 + \int_{u_0}^u F(s, v)H_v(s, v) ds, \quad (17)$$

where $\varepsilon_1 = \pm 1$; $\varepsilon_2 = \pm 1$, the signs depending on the kind of surface. Substituting these expressions in the Gauss Equation (6), we obtain:

$$\frac{FF_{uv} - F_uF_v}{F} = \left(\varepsilon_1 + \int_{v_0}^v F(u, s)H_u(u, s) ds\right) \left(\varepsilon_2 + \int_{u_0}^u F(s, v)H_v(s, v) ds\right) - F^2H^2. \quad (18)$$

Consequently, F and H give a solution to the integro-differential Equation (18), which we call the natural equation of the Lorentz surfaces of general type in \mathbb{R}_1^3 . The converse is also true. Namely, the following Bonnet-type theorem holds:

Theorem 5. *Let $\mathcal{M} = (\mathcal{D}, x)$ be a Lorentz surface of general type in \mathbb{R}_1^3 and (u, v) be canonical isotropic coordinates with initial point $x_0 = x(u_0, v_0) \in \mathcal{M}$. Then, the coefficient F of the first fundamental form and the mean curvature H of \mathcal{M} give a solution to the natural Equation (18). If $\hat{\mathcal{M}}$ is obtained from \mathcal{M} by a proper motion in \mathbb{R}_1^3 , then $\hat{\mathcal{M}}$ generates the same solution to (18).*

Conversely, let $F > 0$ and H be functions of (u, v) defined in a neighbourhood of (u_0, v_0) and satisfying the natural Equation (18), where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. Then, at least locally, there exists a unique (up to a proper motion in \mathbb{R}_1^3) Lorentz surface of general type in \mathbb{R}_1^3 defined by $x = x(u, v)$ in canonical isotropic coordinates with initial point $x_0 = x(u_0, v_0)$, such that the given functions F and H are the non-zero coefficient of the first fundamental form and the mean curvature, respectively, and the signs of the corresponding coefficients L and N of the second fundamental form coincide with the signs of ε_1 and ε_2 .

Proof. We have already seen that the coefficient F and the mean curvature H of a surface with the given properties satisfy the natural equation. Now, we will prove the converse.

Given the functions F and H satisfying (18), we define functions $L, M,$ and N by equalities (17). Then, (18) implies that the quadruple F, L, M, N satisfies the Gauss Equation (6). Differentiating (17), we get that the Codazzi Equation (7) is also fulfilled. Applying Theorem 1, we get a Lorentz surface \mathcal{M} parametrized by isotopic coordinates whose coefficient of the first fundamental form is the given function F and the coefficients of the second fundamental form are the functions $L, M, N,$ defined by (17). Comparing (9) with (17), we obtain $L(u, v_0) = \varepsilon_1, N(u_0, v) = \varepsilon_2,$ which means that \mathcal{M} is of the general type and (u, v) are canonical isotropic coordinates. Comparing (4) with (17), we see that the mean curvature of \mathcal{M} is the given function H . Hence, the surface \mathcal{M} has the necessary properties.

Moreover, if $\hat{\mathcal{M}}$ is another surface with the same properties, then (17) is also valid for $\hat{\mathcal{M}}$. Thus, \mathcal{M} and $\hat{\mathcal{M}}$ have one and the same coefficients of the first and second fundamental forms. Hence, according to Theorem 1, $\hat{\mathcal{M}}$ is obtained from \mathcal{M} by a proper motion in \mathbb{R}_1^3 . \square

Now, we will consider the natural Equation (18) in the case of a surface with constant mean curvature H . In this case, Equation (18) takes the form:

$$\frac{FF_{uv} - F_uF_v}{F} = \varepsilon_1\varepsilon_2 - F^2H^2, \quad (19)$$

and (17) implies $L = \varepsilon_1, N = \varepsilon_2.$ Then, it follows from (12) that

$$H^2 - K = \frac{\varepsilon_1\varepsilon_2}{F^2}; \quad F = \frac{1}{\sqrt{|H^2 - K|}}. \quad (20)$$

If we rewrite (19) in the form

$$\frac{1}{F}(\ln F)_{uv} = \frac{\varepsilon_1 \varepsilon_2}{F^2} - H^2 \tag{21}$$

then, by use of (20), we obtain

$$\sqrt{|H^2 - K|}(\ln \sqrt{|H^2 - K|})_{uv} = K; \quad H^2 - K \neq 0. \tag{22}$$

We call (22) the *natural equation* of constant mean curvature Lorentz surfaces in \mathbb{R}_1^3 .

Thus, we can formulate the following Bonnet-type theorem for Lorentz surfaces of constant non-zero mean curvature.

Theorem 6. *Let $\mathcal{M} = (\mathcal{D}, x)$ be a Lorentz surface of general type in \mathbb{R}_1^3 with constant non-zero mean curvature H parametrized by canonical isotropic coordinates. Then, the Gaussian curvature K satisfies the natural Equation (22). If $\hat{\mathcal{M}}$ is obtained from \mathcal{M} by a proper motion in \mathbb{R}_1^3 , then $\hat{\mathcal{M}}$ generates the same solution to (22).*

Conversely, let H be a non-zero constant and K be a function of (u, v) satisfying the natural Equation (22). Then, at least locally, there exist (up to a proper motion in \mathbb{R}_1^3) exactly two Lorentz surfaces of general type in \mathbb{R}_1^3 parametrized by canonical isotropic coordinates, with the constant H as non-zero mean curvature and the function K as Gaussian curvature.

Proof. We have already seen that the mean curvature H and the Gaussian curvature K of a surface with the given properties satisfy Equation (22). Now, we will prove the converse.

Given the constant H and the function K satisfying (22), we define a function F and constants $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$ such that equalities (20) hold true. Then, equality (22) implies (21), which is equivalent to (19), the latter being the natural Equation (18) in case H is constant. Note that the function F is determined uniquely by (20), while, for the choice of ε_1 and ε_2 , we have two different options depending on the choice of signs. This means that, according to Theorem 5, we obtain two different Lorentz surfaces \mathcal{M}_1 and \mathcal{M}_2 parametrized by canonical isotropic coordinates, whose mean curvature is the given constant H and the coefficient of the first fundamental form is the given function F . Equalities (20) hold true for both \mathcal{M}_1 and \mathcal{M}_2 , and hence K is determined uniquely by F and H . Consequently, the Gaussian curvature of both \mathcal{M}_1 and \mathcal{M}_2 is the given function K .

The surface \mathcal{M}_2 cannot be obtained from \mathcal{M}_1 by a proper motion in \mathbb{R}_1^3 , since the proper motions preserve the signs of ε_1 and ε_2 , and the signs of these constants are different for \mathcal{M}_1 and \mathcal{M}_2 . If $\hat{\mathcal{M}}$ is another surface with the same properties, then equalities (20) also hold true for $\hat{\mathcal{M}}$. Hence, $\mathcal{M}_1, \mathcal{M}_2$, and $\hat{\mathcal{M}}$ have one and the same coefficients of the first fundamental form and equal mean curvatures. Moreover, the constants ε_1 and ε_2 for $\hat{\mathcal{M}}$ coincide with the constants for one of the two surfaces \mathcal{M}_1 or \mathcal{M}_2 . Thus, according to Theorem 5, $\hat{\mathcal{M}}$ can be obtained from one of the two surfaces \mathcal{M}_1 or \mathcal{M}_2 by a proper motion in \mathbb{R}_1^3 . □

Remark 4. *The two surfaces obtained in the last theorem are really different. We have already seen that \mathcal{M}_2 cannot be obtained from \mathcal{M}_1 by a proper motion in \mathbb{R}_1^3 . Furthermore, \mathcal{M}_2 cannot be obtained from \mathcal{M}_1 by coordinate change of the form $u = u(\tilde{u})$ and $v = v(\tilde{v})$, since such a change preserves the signs of ε_1 and ε_2 , according to (10). \mathcal{M}_2 cannot be obtained from \mathcal{M}_1 also by coordinate change of the form $u = u(\tilde{v})$ and $v = v(\tilde{u})$, since, in such case, the sign of H changes, according to (4) and (11), but the surfaces \mathcal{M}_1 and \mathcal{M}_2 have equal mean curvatures. Such a pair of surfaces is presented in Examples 4 and 5.*

Now, we will consider the case of a surface \mathcal{M} with zero mean curvature H , i.e., \mathcal{M} is a minimal surface in \mathbb{R}_1^3 . In this case, equality (22) takes the form:

$$\sqrt{|K|}(\ln \sqrt{|K|})_{uv} = K; \quad K \neq 0. \tag{23}$$

Let us point out that the Gaussian curvature K and the canonical coordinates are invariant under non-proper motions in \mathbb{R}_1^3 . Hence, in this case, the surface is determined uniquely by the solution of (23) up to an arbitrary motion. We have the following Bonnet-type theorem for minimal Lorentz surfaces in \mathbb{R}_1^3 .

Theorem 7. *Let $\mathcal{M} = (D, x)$ be a minimal Lorentz surface of general type in \mathbb{R}_1^3 parametrized by canonical isotropic coordinates. Then, the Gaussian curvature K of \mathcal{M} satisfies the natural Equation (23). If $\hat{\mathcal{M}}$ is obtained from \mathcal{M} by a motion in \mathbb{R}_1^3 , then $\hat{\mathcal{M}}$ generates the same solution to (22).*

Conversely, given a function K of (u, v) satisfying the natural Equation (23), there exists, at least locally, a unique (up to a motion in \mathbb{R}_1^3) minimal Lorentz surface of general type parametrized by canonical isotropic coordinates, such that its Gaussian curvature is the given function K .

Proof. The proof is similar to the proof of Theorem 6; therefore, we will not give it in detail, and we will only point out the difference. Again, given the function K , we obtain two different minimal surfaces \mathcal{M}_1 and \mathcal{M}_2 parametrized by canonical isotropic coordinates, having K as the Gaussian curvature and having different signs of ε_1 and ε_2 . Let $\tilde{\mathcal{M}}_1$ be a surface obtained from \mathcal{M}_1 by a non-proper motion in \mathbb{R}_1^3 . Then, the Gaussian curvature K and the signs of ε_1 and ε_2 of $\tilde{\mathcal{M}}_1$ are the same as those of \mathcal{M}_2 . Thus, equalities (20) imply that $\tilde{\mathcal{M}}_1$ and \mathcal{M}_2 have one and the same coefficient F of the first fundamental form. Hence, according to Theorem 5, \mathcal{M}_2 is obtained from $\tilde{\mathcal{M}}_1$ by a proper motion in \mathbb{R}_1^3 . Consequently, \mathcal{M}_2 is obtained from \mathcal{M}_1 by a non-proper motion in \mathbb{R}_1^3 . \square

5. Examples

In this section, we will consider examples of Lorentz surfaces of general type in \mathbb{R}_1^3 , illustrating the developed theory. First, we give an example of a minimal Lorentz surface.

Example 1. *Let us consider the surface in \mathbb{R}_1^3 , determined by the following parametrization:*

$$x = \frac{1}{6}(u^3 - v^3 + 3u - 3v, -u^3 + v^3 + 3u - 3v, 3u^2 - 3v^2). \tag{24}$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = \frac{1}{2}(u - v)^2; \quad L = 1; \quad M = 0; \quad N = 1.$$

The Gaussian curvature and the mean curvature of the surface defined above are expressed as follows:

$$K = -\frac{4}{(u - v)^4}; \quad H = 0. \tag{25}$$

Hence, the surface defined by (24) is a minimal Lorentz surface (of Enneper-type) parametrized by isotropic coordinates. Moreover, the coordinates (u, v) are canonical, since $L = N = 1$. The Gaussian curvature K is negative, so the surface is of first kind according to Definition 2. The function K given in (25) is a solution to the natural Equation (23).

Example 2. *Let us consider the surface in \mathbb{R}_1^3 , defined by*

$$x = \frac{1}{6}(u^3 - v^3 + 3u - 3v, -u^3 + v^3 + 3u - 3v, 3u^2 + 3v^2). \tag{26}$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = \frac{1}{2}(u + v)^2; \quad L = 1; \quad M = 0; \quad N = -1.$$

The Gaussian curvature and the mean curvature are expressed as follows:

$$K = \frac{4}{(u + v)^4}; \quad H = 0. \tag{27}$$

As in the previous example, (26) defines a minimal Lorentz surface of Enneper-type parametrized by canonical isotropic coordinates. In this example, the Gaussian curvature K is positive and hence the surface is of a second kind according to Definition 2. The function K given in (27) is also a solution to the natural Equation (23).

Now, we will give examples of surfaces with non-zero constant mean curvature.

Example 3. We consider the Lorentz sphere in \mathbb{R}_1^3 parametrized by isothermal coordinates (t, s) as follows:

$$x = (\sinh t \operatorname{sech} s, \cosh t \operatorname{sech} s, \tanh s).$$

Changing the coordinates with isotropic ones, we obtain:

$$x = (\sinh(u - v) \operatorname{sech}(u + v), \cosh(u - v) \operatorname{sech}(u + v), \tanh(u + v)). \tag{28}$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = 2 \operatorname{sech}^2(u + v); \quad L = 0; \quad M = 2 \operatorname{sech}^2(u + v); \quad N = 0.$$

The Gaussian curvature and the mean curvature are given by:

$$K = 1; \quad H = 1.$$

Hence, the surface defined by (28) is a Lorentz surface with non-zero constant mean curvature parametrized by isotropic coordinates. In this example, $H^2 - K = 0$, which means that the surface is not of a general type within the meaning of Definition 1. In this case, we cannot introduce canonical coordinates in the sense of Definition 3.

Example 4. Let us consider the cylinder in \mathbb{R}_1^3 , parametrized by isothermal coordinates (t, s) as follows:

$$x = (t, \cos s, \sin s).$$

Changing the coordinates with isotropic ones, we obtain:

$$x = (u - v, \cos(u + v), \sin(u + v)).$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = 2; \quad L = 1; \quad M = 1; \quad N = 1.$$

The Gaussian curvature and the mean curvature are given by:

$$K = 0; \quad H = \frac{1}{2}.$$

This is an example of a Lorentz surface with non-zero constant mean curvature parametrized by canonical isotropic coordinates. It corresponds to the trivial (zero) solution to Equation (22).

Example 5. Now, let us consider the hyperbolic Lorentz cylinder in \mathbb{R}_1^3 , parametrized by isothermal coordinates (t, s) as follows:

$$x = (\sinh s, \cosh s, t).$$

Changing the coordinates with isotropic ones, we obtain:

$$x = (\sinh(u - v), \cosh(u - v), u + v).$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = 2; \quad L = -1; \quad M = 1; \quad N = -1.$$

The Gaussian curvature and the mean curvature are given by:

$$K = 0; \quad H = \frac{1}{2}.$$

This is also an example of a Lorentz surface with non-zero constant mean curvature parametrized by canonical isotropic coordinates. It also corresponds to the trivial (zero) solution to Equation (22).

Comparing the results of the last two examples, we see that the two cylinders have equal constant mean curvatures and equal Gaussian curvatures. Hence, they give one and the same solution to the natural Equation (22). However, there is a difference in the signs of $\varepsilon_1 = L$ and $\varepsilon_2 = N$. These two cylinders form a pair of surfaces \mathcal{M}_1 and \mathcal{M}_2 as the ones described in the proof of Theorem 6.

Finally, we will consider a surface with non-constant mean curvature.

Example 6. Let us consider the hyperbolic Lorentz cone \mathcal{M} in \mathbb{R}_1^3 , parametrized by isothermal coordinates (t, s) as follows:

$$x = (e^{\frac{t}{2}} \sinh s, \sqrt{3} e^{\frac{t}{2}}, e^{\frac{t}{2}} \cosh s).$$

Changing the coordinates with isotropic ones, we obtain:

$$x = (e^{\frac{u+v}{2}} \sinh(u-v), \sqrt{3} e^{\frac{u+v}{2}}, e^{\frac{u+v}{2}} \cosh(u-v)).$$

The coefficients of the first and the second fundamental forms are:

$$E = G = 0; \quad F = 2e^{u+v}; \quad L = \frac{\sqrt{3}}{2} e^{\frac{u+v}{2}}; \quad M = -\frac{\sqrt{3}}{2} e^{\frac{u+v}{2}}; \quad N = \frac{\sqrt{3}}{2} e^{\frac{u+v}{2}}.$$

The Gaussian curvature and the mean curvature are given by:

$$K = 0; \quad H = -\frac{\sqrt{3}}{4} e^{-\frac{u+v}{2}}.$$

Hence, in this example, \mathcal{M} is a Lorentz surface with a non-constant mean curvature parametrized by isotropic coordinates. The coordinates (u, v) are not canonical, since $L(u, v_0) \neq \pm 1$ and $N(u_0, v) \neq \pm 1$.

We will introduce canonical isotropic coordinates (\tilde{u}, \tilde{v}) with initial point $(u_0, v_0) = (0, 0)$. Using formulas (15), we get:

$$\tilde{u} = \tilde{u}_0 + 2\sqrt{2}\sqrt[4]{3}(e^{\frac{u}{4}} - 1); \quad \tilde{v} = \tilde{v}_0 + 2\sqrt{2}\sqrt[4]{3}(e^{\frac{v}{4}} - 1).$$

To simplify the formulas, we choose $\tilde{u}_0 = \tilde{v}_0 = 2\sqrt{2}\sqrt[4]{3}$. Then,

$$\tilde{u} = 2\sqrt{2}\sqrt[4]{3} e^{\frac{u}{4}}; \quad \tilde{v} = 2\sqrt{2}\sqrt[4]{3} e^{\frac{v}{4}}; \quad u = 4 \ln \frac{\tilde{u}}{2\sqrt{2}\sqrt[4]{3}}; \quad v = 4 \ln \frac{\tilde{v}}{2\sqrt{2}\sqrt[4]{3}}.$$

Using the last equalities and formula (10), we express the coefficient of the first fundamental form \tilde{F} and the mean curvature \tilde{H} in terms of the canonical coordinates (\tilde{u}, \tilde{v}) as follows:

$$\tilde{F} = \frac{\tilde{u}^3 \tilde{v}^3}{1152}; \quad \tilde{H} = -\frac{48\sqrt{3}}{\tilde{u}^2 \tilde{v}^2}. \tag{29}$$

According to Theorem 5, the functions \tilde{F} and \tilde{H} given by (29) provide a solution to the natural Equation (18) in the case $\varepsilon_1 = \varepsilon_2 = 1$.

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