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Global Dynamics for an Age-Structured Cholera Infection Model with General Infection Rates

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Abstract: This paper studies the global dynamics of a cholera model incorporating age structures and general infection rates. First, we explore the existence and point dissipativeness of the orbit and analyze the asymptotical smoothness. Then, we perform rigorous mathematical analysis on the existence and local stability of equilibria. Based on the uniform persistence, we further investigate the global behavior of the cholera infection model. The results of theoretical analysis are well confirmed by numerical simulations. This research generalizes some known results and provides deeper insights into the dynamics of cholera propagation.

Keywords: age structure; cholera infection model; general infection rates; stability analysis

MSC: 92D30; 34K20

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1. Introduction

Cholera is a serious infectious disease that is caused by the bacterium *Vibrio cholera*. Due to the cholera toxin produced by the bacterium *Vibrio cholera*, it is characterized by severe symptoms, including acute diarrhea, vomiting, hypotension and a weak pulse. Without proper medical treatment, it can cause dehydration and death within hours. This disease peaks in summer and its propagation among humans depends on direct person-to-person contact, as well as indirect contact through contaminated food and water [1,2]. Due to the lack of clean food and water, cholera can spread quickly in regions with poor sanitation conditions and has long been a threat to the public health of human society. In 2018, it was estimated that there were 2.9 million burden cases worldwide, with a death toll of around 95,000, which corresponds to dozens of countries and regions [3].

A mathematical model of cholera propagation was first proposed to study cholera infection in 1973 around the Mediterranean region [4]. Henceforward, there have been numerous studies on the dynamics of cholera infection models. Tien and Earn [2] established a water-borne infectious disease model including multiple propagation paths. Control strategies, such as vaccination, were also considered in cholera models in Posny et al. [5] to inhibit the propagation of epidemics. Recently, by combining the cholera infection around aquatic regions, as well as the interaction between the bacteriophage and the cholera bacterium, researchers constructed a refined cholera infection model and provided reasonable cholera control strategies [6]. Considering environmental uncertainties and stochastic factors, researchers also studied a cholera system with respect to the *Itô* stochastic differential equation and confirmed the decisive effect of the stochastic basic reproduction number on the system [7].

Age structure, incorporating the age structure of the pathogen the infection age of individuals, is a significant characteristic in the cholera model [8–11]. A cholera model with bilinear incidence rates including two age structures was introduced and discussed in the work of Brauer et al. [12] and was further investigated in the work of Wang and Zhang [13]. The relative compactness of the orbits and the uniform persistence of the system were explored in [13]. The local stability of disease-free equilibrium and endemic equilibrium was analyzed in [13] and global stability was studied in [12]. Furthermore, a cholera transmission model incorporating vaccination age was analyzed in [14]. Actually, incidence rates are influenced by the complicated connections between susceptible individuals and the infected individuals/pathons. Various nonlinear incidence rates have been considered by researchers [15–19].

Inspired by the above works, we aim to discuss an age-structured cholera model. At time t, let $S(t)$ and $i(t,a)$ stand for the number of susceptible individuals and infected individuals with infection age a, with $p(t,b)$ representing the quantity of aquatic cholera pathogens at the age of b. Then, the infectivity of infected individuals and the total infectivity of the cholera pathogen at time t can be measured by $J(t) = \int_0^\infty k(a)i(t,a)da$ and $Q(t) = \int_0^\infty q(b)p(t,b)db$, in which kernel functions denote the infectivity of infected individuals and pathogens at corresponding ages. In this manuscript, we consider the following cholera model, taking general incidence rates into account, which is a generation of the model in Brauer et al. [12].

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu S(t) - S(t)f(J(t)) - S(t)g(Q(t)), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b), \end{cases} \tag{1}$$

with boundary conditions

$$\begin{aligned} &i(t,0) = S(t)f(J(t)) + S(t)g(Q(t)), \ \ t > 0, \\ &p(t,0) = \int_0^\infty \xi(a)i(t,a)da := P(t), \ \ t > 0, \end{aligned} \tag{2}$$

and the initial condition

$$X_0 := (S(0), i(0,\cdot), p(0,\cdot)) = (S_0, i_0(\cdot), p_0(\cdot)) \in \chi_+, \tag{3}$$

where $\chi_+ = \mathbb{R}_+ \times \mathcal{L}^1_+(0,\infty) \times \mathcal{L}^1_+(0,\infty)$ is a functional space equipped with the norm

$$\|(\ell, \varphi, \phi)\|_{\chi_+} = |\ell| + \int_0^\infty |\varphi(a)|da + \int_0^\infty |\phi(b)|db.$$

In model (1), $\Lambda \in \mathbb{R}_+$ denotes the recruitment of the susceptible, and $\mu \in \mathbb{R}_+$ represents the natural death rate of individuals. $\gamma(b)$ describes the removal rate of pathogens at age b and $\xi(a)$ describes the pathogen shedding rate of an infected patient with infection age a. $\delta(a) = \mu + \delta_1(a) + \delta_2(a)$, where $\delta_1(a)$ is the disease-related death rate and $\delta_2(a)$ accounts for the recovery rate of infected individuals at infection age a. $S(t)f(J(t))$ and $S(t)g(Q(t))$ represent the direct and indirect transmission of cholera. For system (1), we make the following assumptions.

Assumption 1. *(I)* *The functions $\delta(a)$, $\gamma(b)$, $\xi(a)$, $k(a)$, $q(b) \in \mathcal{L}^\infty_+(0,+\infty)$ are bounded, integrable and Lipschitz-continuous. Denote $\bar{r} = ess.\sup_{a\in\mathbb{R}_+} r(a)$ and $\underline{r} = ess.\inf_{a\in\mathbb{R}_+} r(a)$ as essential upper and lower bound of $r(a)$ for $a \in \mathbb{R}_+$.*

(II) *There exists one positive constant $\mathfrak{a}$ satisfying $i(t,a) = 0$ for $a \in [\mathfrak{a}, +\infty)$.*

(III) *$f(\ell)$ and $g(\ell)$ are Lipschitz-continuous on $\mathbb{R}_+$ with $f(0) = g(0) = 0$, $\frac{f(\ell)}{\ell} \geq f'(\ell) \geq 0$, $\frac{g(\ell)}{\ell} \geq g'(\ell) \geq 0$ and $f''(\ell) \leq 0, g''(\ell) \leq 0$, for $\ell \in \mathbb{R}_+$.*

In this paper, for an age-infection model, we analyze the qualitative behavior by means of the Lyapunov functional method [20–22]. By considering the routes of the spread from the pathogen to the susceptible group and from the infected group to the susceptible group spread with generalized infection functions, we form a unified theoretical structure to present the propagation features of the epidemic. The basic reproduction number $\Re_0$ is defined as the threshold value, determining whether the epidemic dies out or not. Specifically, the cholera epidemic withers away if $\Re_0 < 1$, whereas if $\Re_0 > 1$, the disease persists at the endemic level.

The plan of this article is as follows. We give some preliminaries in the next section. In Section 3, we explore the existence and local stability of equilibria. In Section 4, we construct Lyapunov functionals to discuss the global stability of equilibria. In Section 5, we perform numerical simulations. Section 6 presents brief conclusions and a discussion.

2. Preliminaries

2.1. Existence and Uniqueness of Solutions

The standard theory for age-dependent models [8,11] can be applied to establish the existence and uniqueness of solutions for system (1) with boundary conditions (2) and initial condition (3). For this, we introduce the following Banach spaces

$$\begin{aligned}
\chi =& \mathbb{R} \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}),\\
\chi_0 =& \mathbb{R} \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}),\\
\chi_+ =& \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}),\\
\chi_{0+} =& \chi_+ \cap \chi_0 = \mathbb{R}_+ \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}).
\end{aligned}$$

In order to formulate system (1) as an abstract Cauchy problem [23], we define the following linear operator, where $Dom(\digamma_0) = \mathbb{R} \times \{0\} \times w^{1,1}(0,\infty) \times \{0\} \times w^{1,1}(0,\infty)$,

$$\digamma_0 : \; Dom(\digamma_0) \subset \chi \to \chi,$$

$$\digamma_0 \begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mu\phi_1 \\ \begin{pmatrix} -\varphi_1(0) \\ -\delta(\cdot)\varphi_1 - \varphi_1' \end{pmatrix} \\ \begin{pmatrix} -\varphi_2(0) \\ -\gamma(\cdot)\varphi_2 - \varphi_2' \end{pmatrix} \end{pmatrix},$$

and the nonlinear operator

$$\digamma : \; \overline{Dom(\digamma_0)} \subset \chi \to \chi,$$

$$\digamma \begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \Lambda - \phi_1 f(\int_0^\infty k(a)\varphi_1(a)da) - \phi_1 g(\int_0^\infty q(b)\varphi_2(b)db) \\ \begin{pmatrix} \phi_1 f(\int_0^\infty k(a)\varphi_1(a)da) + \phi_1 g(\int_0^\infty q(b)\varphi_2(b)db) \\ 0 \end{pmatrix} \\ \begin{pmatrix} \int_0^\infty \xi(a)\varphi_1(a)da \\ 0 \end{pmatrix} \end{pmatrix}.$$

Similarly to the proof process in [24,25], we can verify that operator $\digamma_0$ is a Hille–Yosida operator [23].

Let $u(t) = (S(t), (0, i(t,\cdot))^T, (0, p(t,\cdot))^T)^T \in \chi_{0+}$. System (1) can be expressed by the following abstract cauchy problem:

$$\begin{cases} \dfrac{du(t)}{dt} = \digamma_0 u(t) + \digamma(u(t)), \quad \forall t \geq 0, \\ u(0) = u_0 \in \chi_0 \bigcap \chi_{0+}. \end{cases}$$

Let $(S(t), (0, i(t,\cdot))^T, (0, p(t,\cdot))^T)^T \in \chi_{0+}$. We have the following theorem by [23,26]:

Theorem 1. *There exists one unique determined semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$ on χ_{0+} such that for any X_0, there exists one unique continuous map $\mathcal{U} \in C([0,\infty], \chi_{0+})$, acting as an integrated solution of the Cauchy problem, that is*

$$\begin{cases} \int_0^t \mathcal{U}(s)X_0 ds \in Dom(\digamma_0), & \forall t \geq 0, \\ \mathcal{U}(t)X_0 = X_0 + \digamma_0 \int_0^t \mathcal{U}(s)X_0 ds + \int_0^\infty \digamma(\mathcal{U}(s)X_0)ds, & \forall t \geq 0. \end{cases}$$

2.2. Point Dissipativeness

Let

$$\begin{aligned} \Xi := \{ & (S(t), i(t,a), p(t,b)) \in \chi_{0+} | \\ & S(t) + \int_0^\infty i(t,a)da \leq \frac{\Lambda}{\min\{\mu, \underline{\delta}\}}, \int_0^\infty p(t,b)db \leq \frac{\Lambda\overline{\xi}}{\min\{\mu, \underline{\delta}\}\underline{\gamma}} \}. \end{aligned}$$

Then we have the following proposition:

Theorem 2. *Ξ is a positive invariant set under the semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$. Moreover, the semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$ is point-dissipative and attracts all the positive solutions of system (1) in χ_{0+}.*

Proof. From the first equation of (1), we have $\frac{dS(t)}{dt} \leq \Lambda - \mu S(t)$. Due to $S(0) \leq \frac{\Lambda}{\mu}$, we have $S(t) \leq \frac{\Lambda}{\mu}$. Note that

$$\begin{aligned} \frac{d}{dt}\int_0^\infty i(t,a)da &= \int_0^\infty (-\frac{\partial}{\partial a}i(t,a) - \delta(a)i(t,a))da \\ &= -i(t,a)|_0^\infty - \int_0^\infty \delta(a)i(t,a)da \\ &\leq i(t,0) - \int_0^\infty \delta(a)i(t,a)da \\ &= S(t)f(J(t)) + S(t)g(Q(t)) - \int_0^\infty \delta(a)i(t,a)da. \end{aligned}$$

Combining the first equation of (1), one yields

$$\begin{aligned} \frac{d}{dt}\left(S(t) + \int_0^\infty i(t,a)da\right) &\leq \Lambda - \mu S(t) - \int_0^\infty \delta(a)i(t,a)da \\ &\leq \Lambda - \mu S(t) - \underline{\delta}\int_0^\infty i(t,a)da \\ &\leq \Lambda - \min\{\mu, \underline{\delta}\}\left(S(t) + \int_0^\infty i(t,a)da\right). \end{aligned}$$

Since $S(0) + \int_0^\infty i(0,a)da \leq \frac{\Lambda}{\min\{\mu,\underline{\delta}\}}$, we have $S(t) + \int_0^\infty i(t,a)da \leq \frac{\Lambda}{\min\{\mu,\underline{\delta}\}}$. Thus, it follows that

$$\begin{aligned} \frac{d}{dt}\left(\int_0^\infty p(t,b)db\right) &= \int_0^\infty \left[-\frac{\partial}{\partial b}p(t,b) - \gamma(b)p(t,b)\right]db \\ &\leq p(t,0) - \int_0^\infty \gamma(b)p(t,b)db \\ &\leq \int_0^\infty \xi(a)i(t,a)da - \underline{\gamma}\int_0^\infty p(t,b)db \\ &\leq \frac{\Lambda\overline{\xi}}{\min\{\mu, \underline{\delta}\}} - \underline{\gamma}\int_0^\infty p(t,b)db. \end{aligned}$$

This implies that $\int_0^\infty p(t,b)db \leq \frac{\Lambda\overline{\xi}}{\min\{\mu,\underline{\delta}\}\underline{\gamma}}$.

Hence, $\mathcal{U}(t)\Xi \subset \Xi$ and this implies that Ξ is a positively invariant set and attracts all positive solutions of (1). □

From Theorem 2, we obtain the following result.

Proposition 1. *If $X_0 \in \chi_+$ and $\|X_0\|_\chi \leq B$ with some sufficiently large constant B, then for $t \in \mathbb{R}_+$, we have the following propositions*

(i) $0 \leq S(t)$, $\int_0^\infty i(t,a)da$, $\int_0^\infty p(t,b)db \leq B$;
(ii) $i(t,0) \leq (\bar{k}f'(0) + \bar{q}g'(0))B^2$, $p(t,0) \leq \bar{\xi}B$.

Proof. From the boundedness of system (1), we can find a constant B such that proposition (i) holds. Due to Assumption 1 (III), we further have

$$\begin{aligned} i(t,0) =& S(t)f(J(t)) + S(t)g(Q(t)) \\ \leq& f'(0)S(t)J(t) + g'(0)S(t)Q(t) \\ \leq& f'(0)S(t)\bar{k}\int_0^\infty i(t,a)da + g'(0)S(t)\bar{q}\int_0^\infty p(t,b)db \\ \leq& (f'(0)\bar{k} + g'(0)\bar{q})B^2 \end{aligned}$$

and

$$p(t,0) = \int_0^\infty \xi(a)i(t,a)da \leq \bar{\xi}\int_0^\infty i(t,a)da \leq \bar{\xi}B.$$

This completes the proof. □

2.3. Asymptotical Smoothness and Global Attractor

From Equations (2) and (3), using the method presented in [8] to integrate the second and the third equations in (1) along the characteristic lines $t - a = const.$, we have

$$i(t,a) = \begin{cases} i(t-a,0)\varpi_1(a), & 0 \leq a \leq t, \\ i(0,a-t)\dfrac{\varpi_1(a)}{\varpi_1(a-t)}, & 0 \leq t \leq a, \end{cases} \tag{4}$$

and

$$p(t,b) = \begin{cases} p(t-b,0)\varpi_2(b), & 0 \leq b \leq t, \\ p(0,b-t)\dfrac{\varpi_2(b)}{\varpi_2(b-t)}, & 0 \leq t \leq b, \end{cases} \tag{5}$$

where

$$\varpi_1(a) = e^{-\int_0^a \delta(\tau)\mathrm{d}\tau} \text{ and } \varpi_2(b) = e^{-\int_0^b \gamma(\tau)\mathrm{d}\tau} \tag{6}$$

denote the fraction at which an infected cell and virus survive up to age a and b.

In order to explore the existence of an attractor, we first analyze the asymptotical smoothness of semiflow $\mathcal{U}(t)$. For this, we present the following proposition.

Proposition 2. *The functions $J(t)$, $Q(t)$ and $P(t)$ are Lipschitz-continuous.*

Proof. Here we give the proof of $J(t)$ being Lipschitz-continuous. From Assumption 1, there exists a positive constant M_k such that $|k(a+l) - k(a)| \leq M_k l$. Then, combining Proposition 1, it holds that

$$\begin{aligned} &|J(t+l) - J(t)| \\ \leq& \int_0^l k(a)i(t+l-a,0)\varpi_1(a)\mathrm{d}a + \left|\int_l^\infty k(a)i(t+l,a)\mathrm{d}a - \int_0^\infty k(a)i(t,a)\mathrm{d}a\right| \\ \leq& \bar{k}(\bar{k}f'(0) + \bar{q}g'(0))B^2 l + \left|\int_l^\infty k(a)i(t+l,a)\mathrm{d}a - \int_0^\infty k(a)i(t,a)\mathrm{d}a\right|. \end{aligned}$$

Let $o = a - l$; we have

$$
\begin{aligned}
&|J(t+l) - J(t)| \\
\leq& \bar{k}(\bar{k}f'(0) + \bar{q}g'(0))B^2 l + \left| \int_0^\infty k(o+l)i(t+l, o+l)\mathrm{d}o - \int_0^\infty k(a)i(t,a)\mathrm{d}a \right| \\
=& \bar{k}(\bar{k}f'(0) + \bar{q}g'(0))B^2 l + \left| \int_0^\infty \left(k(a+l)\frac{\varpi_1(a+l)}{\varpi_1(a)} - k(a) \right) i(t,a)\mathrm{d}a \right| \\
=& \bar{k}(\bar{k}f'(0) + \bar{q}g'(0))B^2 l + \left| \int_0^\infty k(a+l)\left(e^{-\int_a^{a+l}\delta(\tau)\mathrm{d}\tau} - 1 \right) i(t,a)\mathrm{d}a \right| \\
&+ \left| \int_0^\infty (k(a+l) - k(a))i(t,a)\mathrm{d}a \right| \\
\leq& \bar{k}(\bar{k}f'(0) + \bar{q}g'(0))B^2 l + \bar{k}\bar{\delta}lB + M_k lB := M_J.
\end{aligned}
$$

Hence, $J(t)$ is Lipschitz-continuous with the coefficient M_J. Through similar verification, the functions $Q(t)$ and $P(t)$ are Lipschitz-continuous with coefficients M_Q and M_P. □

For the asymptotical smoothness of the semiflow, the following lemma [27] is necessary.

Lemma 1. *The semiflow $\mathcal{U} : \mathbb{R}_+ \times \chi_+ \to \chi_+$ is asymptotically smooth if there are maps Ψ, $\Theta : \mathbb{R}_+ \times \chi_+ \to \chi_+$ such that $\mathcal{U}(t,x) = \Psi(t,x) + \Theta(t,x)$ and the following holds for any bounded closed set $\mathbb{B} \subset \chi_+$, which is forward invariant under $\mathcal{U}$:* (i) $\lim\limits_{t\to\infty} \mathrm{diam}\Theta(t,\mathbb{B}) = 0$; (ii) *There exists $t_{\mathbb{B}} \geq 0$ such that $\Psi(t,\mathbb{B})$ has compact closure for each $t \geq t_{\mathbb{B}}$.*

For condition (ii) of Lemma 1, we introduce the following lemma [27].

Lemma 2. *A set $\mathbb{B} \in \mathcal{L}^1_+(0,\infty)$ has a compact closure if the following conditions hold:* (i) $\sup\limits_{f\in\mathbb{B}} \int_0^\infty f(\ell)\mathrm{d}\ell < \infty$; (ii) $\lim\limits_{r\to\infty} \int_r^\infty f(\ell)\mathrm{d}\ell \to 0$ *uniformly in $f \in \mathbb{B}$*; (iii) $\lim\limits_{h\to 0^+} \int_0^\infty |f(\ell+h) - f(\ell)|\mathrm{d}\ell \to 0$ *uniformly in $f \in \mathbb{B}$*; (iv) $\lim\limits_{h\to 0^+} \int_0^h f(\ell)\mathrm{d}\ell \to 0$ *uniformly in $f \in \mathbb{B}$.*

Based on Lemmas 1 and 2, we investigate the asymptotical smoothness.

Theorem 3. *The semiflow $\mathcal{U}$ generated by (1) is asymptotically smooth.*

Proof. Define the maps Ψ and Θ such that $\mathcal{U} = \Psi + \Theta$, with

$$
\begin{cases}
\Psi(t,x_0) = (S(t), \check{i}(t,\cdot), \check{p}(t,\cdot)), \\
\Theta(t,x_0) = (0, \check{\varphi}_i(t,\cdot), \check{\varphi}_p(t,\cdot)),
\end{cases}
$$

where

$$
\begin{aligned}
\check{i}(t,a) &= \begin{cases} i(t-a,0)\varpi_1(a), & 0 \leq a \leq t, \\ 0, & 0 \leq t \leq a, \end{cases} \\
\check{p}(t,b) &= \begin{cases} p(t-b,0)\varpi_2(b), & 0 \leq b \leq t, \\ 0, & 0 \leq t \leq b, \end{cases} \\
\check{\varphi}_i(t,a) &= \begin{cases} 0, & 0 \leq a \leq t, \\ i(0,a-t)\dfrac{\varpi_1(a)}{\varpi_1(a-t)}, & 0 \leq t \leq a, \end{cases} \\
\check{\varphi}_p(t,b) &= \begin{cases} 0, & 0 \leq b \leq t, \\ p(0,b-t)\dfrac{\varpi_2(b)}{\varpi_2(b-t)}, & 0 \leq t \leq b. \end{cases}
\end{aligned}
$$

Firstly, we show that map Θ satisfies condition (i) of Lemma 1. For $X_0 \in \Gamma$ satisfying $\|X_0\|_\chi \le r$, letting $a-t=\epsilon_1$ and $b-t=\epsilon_2$, we have

$$\begin{aligned}
&\|\Theta(t,X_0)\|_\chi \\
=&\int_t^\infty \left| i(0,a-t)\frac{\varpi_1(a)}{\varpi_1(a-t)}\right| \mathrm{d}a + \int_t^\infty \left| p(0,b-t)\frac{\varpi_2(b)}{\varpi_2(b-t)}\right| \mathrm{d}b \\
=&\int_0^\infty \left| i(0,\epsilon_1)e^{-\int_{\epsilon_1}^{\epsilon_1+t}\delta(\tau)\mathrm{d}\tau}\right| \mathrm{d}\epsilon + \int_0^\infty \left| p(0,\epsilon_2)e^{-\int_{\epsilon_2}^{\epsilon_2+t}\gamma(\tau)\mathrm{d}\tau}\right| \mathrm{d}\epsilon \\
\le& e^{-\underline{\delta}t}\int_0^\infty |i(0,\epsilon_1)|\mathrm{d}\epsilon + e^{-\underline{\gamma}t}\int_0^\infty |p(0,\epsilon_2)|\mathrm{d}\epsilon \\
\le& e^{-\min\{\underline{\delta},\underline{\gamma}\}t}r, \qquad t\in\mathbb{R}_+.
\end{aligned}$$

This shows that $\|\Theta(t,X_0)\|_\chi \to 0$ as $t\to\infty$, which indicates that $\|\Theta(t,X_0)\|_\chi$ approaches 0 with uniform exponential speed. Thus, the proof of Lemma 1 (i) is completed.

Then, we verify that Lemma 2 holds. Using Proposition 1, we can verify that conditions (i), (ii) and (iv) of Lemma 2 hold since

$$0 \le \check{i}(t,a) \le S(t-a)[f'(0)J(t-a)+g'(0)Q(t-a)]\varpi_1(a) \le [f'(0)\bar{k}+g'(0)\bar{q}]B^2 e^{-\bar{\delta}a}.$$

Finally, we focus on condition (iii) of Lemma 2. For sufficiently small $h\in(0,t)$, we have

$$\int_0^\infty |\check{i}(a+h,t)-\check{i}(a,t)|\mathrm{d}a \le \zeta_1+\zeta_2+\zeta_3, \tag{7}$$

where

$$\begin{aligned}
\zeta_1 =& \int_0^{t-h} |S(t-a-h)f(J(t-a-h))\varpi_1(a+h) - S(t-a)f(J(t-a))\varpi_1(a)|\mathrm{d}a, \\
\zeta_2 =& \int_0^{t-h} |S(t-a-h)g(Q(t-a-h))\varpi_1(a+h) - S(t-a)g(Q(t-a))\varpi_1(a)|\mathrm{d}a, \\
\zeta_3 =& f'(0)\int_{t-h}^t |S(t-a)J(t-a)\varpi_1(a)|\mathrm{d}a + g'(0)\int_{t-h}^t |S(t-a)Q(t-a)\varpi_1(a)|\mathrm{d}a \\
\le& (f'(0)\bar{k}+g'(0)\bar{q})B^2 h := \zeta_{3M}.
\end{aligned}$$

Note that

$$\begin{aligned}
\zeta_1 =& \int_0^{t-h} |S(t-a-h)f(J(t-a-h))(\varpi_1(a+h)-\varpi_1(a))\mathrm{d}a \\
&+ \int_0^{t-h} S(t-a-h)|f(J(t-a-h))-f(J(t-a))|\varpi_1(a)\mathrm{d}a \\
&+ \int_0^{t-h} |S(t-a-h)-S(t-a)|f(J(t-a))\varpi_1(a)\mathrm{d}a \\
\le& f'(0)\bar{k}B^2\int_0^{t-h} |\varpi_1(a+h)-\varpi_1(a)|\mathrm{d}a \\
&+ f'(0)\int_0^{t-h} S(t-a-u)|J(t-a-h)-J(t-a)|\varpi_1(a)\mathrm{d}a \\
&+ f'(0)\int_0^{t-h} J(t-a)|S(t-a-h)-S(t-a)|\varpi_1(a)\mathrm{d}a.
\end{aligned} \tag{8}$$

Let $M_S = \Lambda+\mu B+(\bar{k}f'(0)+\bar{q}g'(0))B^2$ be the Lipschitz coefficient of $S(t)$. Then, the following holds:

$$\zeta_1 \le f'(0)\bar{k}B^2 h + f'(0)BM_J h^2 + f'(0)\bar{k}BM_s h^2 := \zeta_{1M}. \tag{9}$$

Similarly, we have

$$\zeta_2 \le g'(0)\bar{q}B^2h + g'(0)BM_Qh^2 + g'(0)\bar{q}BM_sh^2 := \zeta_{2M}. \tag{10}$$

Combining equations (9) and (10) with (7), we obtain

$$\int_0^\infty |\check{i}(a+h,t) - \check{i}(a,t)|\mathrm{d}a \le \zeta_{1M} + \zeta_{2M} + \zeta_{3M}.$$

ζ_{iM}, $i = 1,2,3$, does not rely on the initial condition X_0. Thus, Lemma 2 holds. Hence, $\check{i}(t,a)$ remains in a pre-compact subset in $\mathcal{L}^1_+(0,\infty)$, and so does $\check{p}(t,b)$. We thus accomplish the proof. □

Based on the above preparations, the following results hold due to Theorem 3.4.6 of Hale [28].

Theorem 4. *The semi-flow $\mathcal{U}(t)$ has a global attractor $\mathcal{A}$ in χ_+, which attracts all bound subsets of χ_+.*

3. Existence and Local Stability of Equilibria

3.1. Equilibria and Basic Reproductive Number

System (1) possesses two equilibria at most in Θ. Besides the infection-free equilibrium $E^0 = (S_0, 0, 0)$ with $S_0 = \Lambda/\mu$, there possibly exists an infection equilibrium $E^* = (S^*, i^*(a), p^*(b))$ in Θ, satisfying the following equations

$$\begin{cases} \Lambda = \mu S^* + S^* f(J^*) + S^* g(Q^*), \\ \dfrac{\partial i^*(a)}{\partial a} = -\delta(a) i^*(a), \\ \dfrac{\partial p^*(b)}{\partial b} = -\gamma(b) p^*(b), \\ i^*(0) = S^* f(J^*) + S^* g(Q^*), \\ p^*(0) = \displaystyle\int_0^\infty \xi(a) i^*(a) da, \end{cases} \tag{11}$$

where $J^* = \int_0^\infty k(a) i^*(a) da$ and $Q^* = \int_0^\infty q(b) p^*(b) db$.

From the second and third equations of system (11), we have

$$i^*(a) = i^*(0)\varpi_1(a), \quad p^*(b) = p^*(0)\varpi_2(b).$$

Let

$$\Pi_1 = \int_0^\infty k(a)\varpi_1(a)da, \ \Pi_2 = \int_0^\infty q(b)\varpi_2(b)db \text{ and } \Pi_3 = \int_0^\infty \xi(a)\varpi_1(a)da. \tag{12}$$

We can further obtain

$$\begin{aligned} J^* &= \int_0^\infty k(a) i^*(0)\varpi_1(a) da = \int_0^\infty k(a)[S^* f(J^*) + S^* g(Q^*)]\varpi_1(a) da \\ &= [S^* f(J^*) + S^* g(Q^*)]\Pi_1 \end{aligned} \tag{13}$$

and

$$\begin{aligned} Q^* &= \int_0^\infty q(b) p^*(0)\varpi_2(b) db = \int_0^\infty \xi(a) i^*(a) da \int_0^\infty q(b)\varpi_2(b) db \\ &= \Pi_2 \int_0^\infty \xi(a) i^*(0)\varpi_1(a) da = \Pi_2\Pi_3(S^* f(J^*) + S^* g(Q^*)) = \frac{\Pi_2\Pi_3}{\Pi_1} J^*. \end{aligned} \tag{14}$$

Thus, combining equations $S^* = \Lambda/(\mu + f(J^*) + g(Q^*))$, (13) and (14), we have

$$J^* = \frac{\Lambda\Pi_1[f(J^*) + g(Q^*)]}{\mu + f(J^*) + g(Q^*)} = \frac{\Lambda\Pi_1[f(J^*) + g(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)]}{\mu + f(J^*) + g(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)}.$$

Let $h(J) = \mu J + [J - \Lambda\Pi_1][f(J) + g(\frac{\Pi_2\Pi_3}{\Pi_1}J)]$. Then, we yield $h(0) = 0$, $h(\Lambda\Pi_1) = \mu\Lambda\Pi_1$ and

$$\begin{aligned} h'(0) &= \{\mu + [f(J) + g(\frac{\Pi_2\Pi_3}{\Pi_1}J)] + [J - \Lambda\Pi_1][f'(J) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(\frac{\Pi_2\Pi_3}{\Pi_1}J)]\}|_{J=0} \\ &= \mu[1 - \frac{\Lambda\Pi_1}{\mu}(f'(0) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(0))]. \end{aligned}$$

Define the basic reproduction number of system (1) as

$$\Re_0 = \frac{\Lambda\Pi_1}{\mu}(f'(0) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(0)). \tag{15}$$

When $\Re_0 > 1$, $h'(0) < 0$ and there exists at least one E^*. Then, we obtain

$$h'(J^*) = \mu + [f(J^*) + g(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)] + [J^* - \Lambda\Pi_1][f'(J^*) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)]$$

and

$$h''(J^*) = 2[f'(J^*) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)] + [J^* - \Lambda\Pi_1][f''(J^*) + \frac{\Pi_2\Pi_3}{\Pi_1}g''(\frac{\Pi_2\Pi_3}{\Pi_1}J^*)] > 0.$$

Thus, there exists one unique positive equilibrium E^*. This yields the following theorem.

Theorem 5. *System (1) always exists a disease-free steady state $E^0 = (S_0, 0, 0)$. Furthermore, another endemic steady state $E^* = (T^*, i^*(a), V^*)$ exists if $\Re_0 > 1$.*

3.2. Local Stability of Equilibria

The global asymptotical stability of equilibria is conducive to forecasting the trends of epidemics [29–35]. For this, we first focus on the local stability by exploring the corresponding characteristic equations.

Theorem 6. *The infection-free equilibrium is locally asymptotically stable when $\Re_0 < 1$. The infection equilibrium is locally asymptotically stable when $\Re_0 > 1$.*

Proof. The characteristic equation for the linearized part of system (1) with boundary conditions (2) on $(S_0, 0, 0)$ is

$$(\lambda + \mu)(-1 + S_0 f'(0)\pi_1(\lambda) + S_0 g'(0)\pi_2(\lambda)\pi_3(\lambda)) = 0, \tag{16}$$

where

$$\begin{aligned} \pi_1(\lambda) &= \int_0^\infty k(a)e^{-\lambda a}\varpi_1(a)da, \\ \pi_2(\lambda) &= \int_0^\infty q(b)e^{-\lambda b}\varpi_2(b)db, \\ \pi_3(\lambda) &= \int_0^\infty \xi(a)e^{-\lambda a}\varpi_1(a)da. \end{aligned}$$

Then, if $\mathfrak{R}_0 < 1$, all roots of the characteristic equation (16) have negative parts. If not, that is, if there exists a λ_0 such that $Re\lambda_0 \geq 0$, then

$$\begin{aligned} &-1 + S_0 f'(0)\pi_1(\lambda) + S_0 g'(0)\pi_2(\lambda)\pi_3(\lambda) \\ \leq &\frac{\Lambda\Pi_1}{\mu}[f'(0) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(0)] - 1 = \mathfrak{R}_0 - 1 < 0. \end{aligned}$$

This is a contradiction with equation (16). Thus, the infection-free equilibrium is locally asymptotically stable when $\mathfrak{R}_0 < 1$.

Similarly, for $\mathfrak{R}_0 > 1$, combining the linearization of the system on $(S^*, i^*(a), p^*(b))$, the corresponding characteristic equation of the linearization for system (1) is

$$(\lambda + \mu + f(J^*) + g(Q^*))/(\lambda + \mu) = S^* f'(J^*)\pi_1(\lambda) + S^* g'(Q^*)\pi_2(\lambda)\pi_3(\lambda). \tag{17}$$

Now we assume that system (17) has one characteristic root with a positive real root. Since $J^* = i^*(0)\Pi_1$, $Q^* = p^*(0)\Pi_2$, $\Pi_3 = \int_0^\infty \xi(a)\frac{i^*(a)}{i^*(0)}da$ and $p^*(0) = \int_0^\infty \xi(a)i^*(a)da$, we have

$$\begin{aligned} |S^* f'(J^*)\pi_1(\lambda) + S^* g'(Q^*)\pi_2(\lambda)\pi_3(\lambda)| &\leq \left| S^* \frac{f(J^*)}{J^*}\Pi_1(\lambda) + S^* \frac{g(Q^*)}{Q^*}\Pi_2(\lambda)\Pi_3(\lambda) \right| \\ &\leq \left| \frac{S^* f(J^*)}{i^*(0)} + \frac{S^* g(Q^*)}{p^*(0)}\Pi_3 \right| \\ &= 1. \end{aligned}$$

This is a contradiction with Equation (17). Thus, E^* is locally asymptotically stable when $\mathfrak{R}_0 > 1$. □

4. Global Stability of Equilibria

For the proof of the global attractiveness of equilibria, we apply the Lyapunov functional method. For the invariance principle, we have investigated the relative compactness of the orbits. For the well-posedness of Lyapunov functionals, the uniform persistence of system should also be discussed.

4.1. Uniform Persistence

In this section, we aim to investigate the uniform persistence of system (1). Define

$$M = \{(S, (0, i), (0, p)) \in \Xi : S(t) + \int_0^\infty i(t, a)da + \int_0^\infty p(t, b)db > 0\}$$

and $\partial M = \Xi \backslash M$.

Lemma 3. *The subsets M and ∂M are both positively invariant under the semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$ generated by system (1) on χ_{0+}, that is,*

$$\mathcal{U}(t)M \subset M, \mathcal{U}(t)\partial M \subset \partial M.$$

Moreover, for each $\xi \in \partial M, \mathcal{U}(t)\xi \to E^0$ as $t \to +\infty$.

Proof. Let $G(t) = \int_0^\infty i(t,a)da + \int_0^\infty p(t,b)db$. For any $\varsigma = (S(t), (0, i(t,a)), (0, p(t,b))) \in M$, we have

$$\begin{aligned}\frac{dG(t)}{dt} &= \int_0^\infty \left(-\delta(a)i(t,a) - \frac{\partial i(t,a)}{\partial a}\right)da + \int_0^\infty \left(-\gamma(b)p(t,b) - \frac{\partial p(t,b)}{\partial b}\right)db \\ &= -\int_0^\infty \delta(a)i(t,a)da + i(t,0) - \int_0^\infty \gamma(b)p(t,b)db + p(t,0) \\ &\geq -\overline{\delta}\int_0^\infty i(t,a)da - \overline{\gamma}\int_0^\infty p(t,b)db \\ &= -\max\{\overline{\delta}, \overline{\gamma}\}G(t).\end{aligned}$$

For any $\varsigma = (S(t), (0, i(t,a)), (0, p(t,b))) \in M$, we have $G(0) > 0$. Thus, $G(T) \geq G(0)e^{-\max\{\overline{\delta},\overline{\gamma}\}} > 0$ and then we have $\mathcal{U}(t)M \subset M$. Thus, M is positively invariant.

In the following, we try to prove that $\mathcal{U}(t)\partial M \subset \partial M$. For any $\varsigma = (S_0(t), (0, i_0(t,a)), (0, p_0(t,b))) \in \partial M$, we have

$$0 \leq \int_0^\infty i(t,a)da = \int_0^t i(t-a,0)\varpi_1(a)da + \int_t^\infty i(0,a-t)\frac{\varpi_1(a)}{\varpi_1(a-t)}da \leq 0$$

and

$$0 \leq \int_0^\infty p(t,b)db = \int_0^t p(t-b,0)\varpi_2(b)db + \int_t^\infty p(0,b-t)\frac{\varpi_2(b)}{\varpi_2(b-t)}db \leq 0.$$

Thus, $\int_0^\infty i(t,a)da = 0$ and $\int_0^\infty p(t,b)db = 0$. □

Then we obtain the following theorem by means of [36].

Theorem 7. *If $\mathfrak{R}_0 > 1$, then the semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$ is uniformly persistent with respect to the pair $(\partial M, M)$, that is, there exists $\epsilon > 0$, such that $\liminf_{t\to\infty} d(\mathcal{U}(t)\xi, \partial M) \geq \epsilon$, $\forall \xi \in M$.*

Proof. We need to verify that $W^s(E^0) \bigcap M = \emptyset$, where

$$W^s(E^0) = \{\xi \in \Omega : \lim_{t\to\infty} \mathcal{U}(t)\xi = E^0\}.$$

Suppose there exists $\xi_0 \in W^s(E^0) \bigcap M$. Then, there exists a t_1, such that

$$\int_0^\infty i(t_1,a)da + \int_0^\infty p(t_1,b)db > 0.$$

Since M is an invariant set, we have

$$\int_0^\infty i(t,a)da + \int_0^\infty p(t,b)db > 0,\ \forall t > t_1.$$

Since $\xi_0 \in W^s(E^0)$, we have $\lim_{t\to\infty} S(t) = S^0$. Thus, for $\epsilon_0 > 0$, there exists t_2, such that

$$S(t) > S^0 - \epsilon_0,\ \forall t \geq t_2.$$

Let $H(t) = \int_0^\infty \sigma(a)i(t,a)da + \int_0^\infty \varrho(b)p(t,b)db$, where

$$\begin{aligned}\sigma(a) &= \int_a^\infty [S_0 f'(0)k(u) + \varrho(0)\xi(u)]e^{-\int_a^u \delta(\tau)d\tau}du, \\ \varrho(b) &= \int_b^\infty S_0 g'(0)q(v)e^{-\int_b^v \gamma(\tau)d\tau}dv.\end{aligned}$$

Then we have

$$
\begin{aligned}
H'(t) = &\int_0^\infty \sigma(a)\frac{\partial i(t,a)}{\partial t}da + \int_0^\infty \varrho(b)\frac{\partial p(t,b)}{\partial b}db \\
=&\sigma(0)i(t,0) + \int_0^\infty i(t,a)[\sigma'(a) - \sigma(a)\delta(a)]da \\
&+ \varrho(0)p(t,0) + \int_0^\infty p(t,b)[\varrho'(b) - \varrho(b)\gamma(b)]db \\
=&\sigma(0)i(t,0) - \int_0^\infty i(t,a)[S_0 f'(0)k(a) - \varrho(0)\xi(a)]da \\
&+ \varrho(0)p(t,0) - \int_0^\infty p(t,b)S_0 g'(0)q(b)db.
\end{aligned}
$$

Since $p(t,0) = \int_0^\infty i(t,a)\xi(a)da$, $i(t,0) = Sf(J) + Sg(Q)$ and

$$
\sigma(0) = S_0 f'(0)\Pi_1 + \varrho(0)\Pi_3 = \frac{\Lambda}{\mu}\Pi_1[f'(0) + \frac{\Pi_2\Pi_3}{\Pi_1}g'(0)] = \Re_0,
$$

we further have for a sufficiently large t,

$$
\begin{aligned}
H'(t) =&\sigma(0)i(t,0) - S_0 f'(0)\int_0^\infty i(t,a)k(a)da - S_0 g'(0)\int_0^\infty p(t,b)q(b)db \\
=&\sigma(0)[Sf(J) + Sg(Q)] - S_0 f'(0)J - S_0 g'(0)Q \\
=&[\sigma(0)Sf(J) - S_0 f'(0)J] + [\sigma(0)Sg(Q) - S_0 g'(0)Q] \\
\geq&[\sigma(0)(S_0 - \epsilon_0)f(J) - S_0 f'(0)J] + [\sigma(0)(S_0 - \epsilon_0)g(Q) - S_0 g'(0)Q] \\
=&S_0[\sigma(0)(1 - \frac{\epsilon_0}{S_0})f(J) - f'(0)J] + S_0[\sigma(0)(1 - \frac{\epsilon_0}{S_0})g(Q) - g'(0)Q] \\
\geq&0.
\end{aligned}
$$

This indicates that $H(t)$ is a non-decreasing function for a sufficiently large t. Hence, for a sufficiently large t, $H(t) > 0$, which prevents the orbits from converging to E^0 as $t \to +\infty$. This contradicts $\xi_0 \in W^s(E^0)$. □

4.2. Global Stability of the Infection-Free Equilibrium

This subsection explores the global stability of the infection-free equilibrium E^0.

Theorem 8. *E^0 is globally asymptotically stable when $\Re_0 < 1$.*

Proof. Define the Liapunov function $L(t) = L_1(t) + L_2(t) + L_3(t)$, with

$$
L_1(t) = S(t) - S_0 - S_0\ln(\frac{S(t)}{S_0}),\ L_2(t) = \int_0^\infty \sigma(a)i(t,a)da,\ L_3(t) = \int_0^\infty \varrho(b)p(t,b)db.
$$

Then, calculating the derivatives of $L_i(t)$, $i = 1,2,3$, along the trajectories of system (1) gives

$$
\begin{aligned}
\frac{dL_1}{dt} =& -\frac{\mu}{S(t)}(S(t) - S_0)^2 - i(t,0) + S_0 f(J(t)) + S_0 g(Q(t)) \\
\leq& -\frac{\mu}{S(t)}(S(t) - S_0)^2 - i(t,0) \\
&+ S_0 f'(0)\int_0^\infty k(a)i(t,a)da + S_0 g'(0)\int_0^\infty q(b)p(t,b)db,
\end{aligned}
\tag{18}
$$

and

$$\begin{aligned}\frac{dL_2}{dt} &= \int_0^\infty \sigma(a)(-\delta(a)i(t,a) - \frac{\partial i(t,a)}{\partial a})da \\ &= -\int_0^\infty \sigma(a)\delta(a)i(t,a)da - \int_0^\infty \sigma(a)di(t,a) \\ &= \sigma(0)i(t,0) + \int_0^\infty i(t,a)(\sigma'(a) - \sigma(a)\delta(a))da,\end{aligned} \tag{19}$$

and similarly

$$\frac{dL_3}{dt} = \varrho(0)p(t,0) + \int_0^\infty p(t,b)(\varrho'(b) - \varrho(b)\gamma(b))db. \tag{20}$$

Since $\sigma'(a) = -[S_0 f'(0)k(a) + \varrho(0)\xi(a)] + \delta(a)\sigma(a)$ and $\varrho'(b) = -S_0 g'(0)q(b) + \gamma(b)\varrho(b)$, we further have

$$\begin{aligned}\frac{dL}{dt} =& \frac{dL_1}{dt} + \frac{dL_2}{dt} + \frac{dL_3}{dt} \\ \le& -\frac{\mu}{S(t)}(S(t) - S_0)^2 - i(t,0) + \sigma(0)i(t,0) \\ &+ \int_0^\infty i(t,a)[S_0 f'(0)k(a) + \sigma'(a) - \sigma(a)\delta(a) + \varrho(0)\xi(a)]da \\ &+ \int_0^\infty p(t,b)[S_0 g'(0)q(b) + \varrho'(b) - \varrho(b)\gamma(b)]db \\ =& -\frac{\mu}{S(t)}(S(t) - S_0)^2 + i(t,0)[\sigma(0) - 1].\end{aligned} \tag{21}$$

Thus, when $\Re_0 = \sigma(0) < 1$, $\frac{dL}{dt} \le 0$. The largest invariant set of $\{\frac{dL}{dt} = 0\}$ is singleton $\{E^0\}$. Hence, due to the invariance principle [37], E^0 is globally asymptotically stable when $\Re_0 < 1$. □

4.3. Global Stability of the Infection Equilibrium

In this subsection, we focus on the global stability of the infection equilibrium E^*. To this end, we introduce a function h defined by

$$h(z) = z - 1 - \ln z, \quad z \in R_+.$$

In order to ensure that $h\left(\frac{i(t,a)}{i^*(a)}\right)$ and $h\left(\frac{p(t,b)}{p^*(b)}\right)$ are well-defined, we have shown that $i(t,a)/i^*(a)$ and $p(t,b)/p^*(b)$ are bounded below and above through the above uniform persistence analysis. In the following, we prove the following result.

Theorem 9. *The infection equilibrium E^* is globally asymptotically stable when $\Re_0 > 1$.*

Proof. Define a Lyapunov function $V(t) = V_1(t) + V_2(t) + V_3(t)$, where

$$V_1(t) = S^* h(\frac{S}{S^*})i^*(0), \; V_2(t) = \int_0^\infty \Gamma(a)i^*(a)h(\frac{i}{i^*})di, \; V_3(t) = \frac{1}{\Pi_3}\int_0^\infty \Upsilon(b)p^*(b)h(\frac{p}{p^*})dp,$$

with

$$\begin{aligned}\Gamma(a) =& \frac{1}{\Pi_1}\int_a^\infty S^* f(J^*)k(u)e^{-\int_a^u \delta(\tau)d\tau}du + \frac{1}{\Pi_3}\int_a^\infty \Upsilon(0)\xi(u)e^{-\int_a^u \delta(\tau)d\tau}du, \\ \Upsilon(b) =& \frac{1}{\Pi_2}\int_b^\infty S^* g(Q^*)q(v)e^{-\int_b^v \gamma(\tau)d\tau}dv.\end{aligned}$$

Here, we make some preparations. Firstly, since $i_a^*(a) = -i^*(a)\delta(a)$, we have

$$\begin{aligned} i^*(a)\frac{d}{da}[\frac{i(t,a)}{i^*(a)} - 1 - \ln\frac{i(t,a)}{i^*(a)}] &= (1 - \frac{i^*(a)}{i(t,a)})\frac{i_a(t,a)i^*(a) - i(t,a)i_a^*(a)}{i^*(a)} \\ &= (1 - \frac{i^*(a)}{i(t,a)})[i_a(t,a) + i(t,a)\delta(a)]. \end{aligned}$$

Thus,

$$(1 - \frac{i^*(a)}{i(t,a)})i_a(t,a) = i^*(a)\frac{d}{da}[h(\frac{i(t,a)}{i^*(a)})] - \delta(a)[i(t,a) - i^*(a)]. \tag{22}$$

Similarly, we have

$$(1 - \frac{p^*(b)}{p(t,b)})p_b(t,b) = p^*(b)\frac{d}{db}[h(\frac{p(t,b)}{p^*(b)})] - \gamma(a)[p(t,b) - p^*(b)]. \tag{23}$$

Then, calculating the derivative of V_1 along system (1) gives

$$\begin{aligned} \frac{dV_1}{dt} =&[-\frac{\mu}{S}(S - S^*)^2 + S^*f(J^*) + S^*g(Q^*) - Sf(J) - Sg(Q) \\ &- \frac{S^*}{S}S^*f(J^*) - \frac{S^*}{S}S^*g(Q^*) + S^*f(J) + S^*g(Q)]i^*(0). \end{aligned} \tag{24}$$

Because of equation (22), we obtain

$$\begin{aligned} \frac{dV_2}{dt} &= -\int_0^\infty \Gamma(a)(1 - \frac{i^*(a)}{i(t,a)})[\frac{\partial i(t,a)}{\partial a} + \delta(a)i(t,a)]da \\ &= \int_0^\infty -\Gamma(a)i^*(a)\frac{d}{da}[h(\frac{i(t,a)}{i^*(a)})]da, \\ &= -\Gamma(a)i^*(a)h(\frac{i(t,a)}{i^*(a)})|_0^\infty + \int_0^\infty h(\frac{i(t,a)}{i^*(a)})[\Gamma'(a)i^*(a) + \Gamma(a)i_a^*(a)]da \\ &= -\Gamma(\infty)i^*(\infty)h(\frac{i(t,\infty)}{i^*(\infty)}) + \Gamma(0)i^*(0)h(\frac{i(t,0)}{i^*(0)}) \\ &\quad + \int_0^\infty h(\frac{i(t,a)}{i^*(a)})[\Gamma'(a)i^*(a) + \Gamma(a)i_a^*(a)]da. \end{aligned}$$

Due to the fact that $\Gamma'(a) = -\frac{1}{\Pi_1}S^*f(J^*)k(a) - \frac{1}{\Pi_3}\Upsilon(0)\xi(a) + \delta(a)\Gamma(a)$ and $i_a^*(a) = -i^*(a)\delta(a)$, we have

$$\frac{dV_2}{dt} \leq \Gamma(0)i^*(0)h(\frac{i(t,0)}{i^*(0)}) - \int_0^\infty i^*(a)h(\frac{i(t,a)}{i^*(a)})[\frac{1}{\Pi_1}S^*f(J^*)k(a) + \frac{1}{\Pi_3}S^*g(Q^*)\xi(a)]da.$$

Since

$$\Gamma(0)i^*(0) = \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)i^*(a)k(a)da + \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)da,$$

we further have

$$\begin{aligned} \frac{dV_2}{dt} \leq& \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)i^*(a)k(a)[h(\frac{i(t,0)}{i^*(0)}) - h(\frac{i(t,a)}{i^*(a)})]da \\ &+ \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)[h(\frac{i(t,0)}{i^*(0)}) - h(\frac{i(t,a)}{i^*(a)})]da. \end{aligned} \tag{25}$$

Similarly, we have

$$\frac{dV_3}{dt} \leq \frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)[h(\frac{p(t,0)}{p^*(0)}) - h(\frac{p(t,b)}{p^*(b)})]db. \tag{26}$$

We introduce

$$\begin{aligned}\mathcal{A}_0 :=& \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)k(a)i^*(a)[1 - \frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}]da \\ &+ \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)\xi(a)i^*(a)[1 - \frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}]da.\end{aligned}$$

Then we can verify that $\mathcal{A}_0 = 0$. Combining equations (24), (25) and (26), we can transfer $\frac{dV}{dt}$ as follows:

$$\frac{dV}{dt} = \frac{dV}{dt} + \mathcal{A}_0 \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5,$$

where

$$\begin{aligned}\mathcal{A}_1 :=& [-\frac{\mu}{S}(S - S^*)^2 - Sf(J) - Sg(Q)]i^*(0) \\ &+ \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)i^*(a)k(a)\frac{i(t,0)}{i^*(0)}da + \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)\frac{i(t,0)}{i^*(0)}da, \\ \mathcal{A}_2 :=& -\frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)\frac{i(t,a)}{i^*(a)}da + \frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)\frac{p(t,0)}{p^*(0)}db, \\ \mathcal{A}_3 :=& [S^*f(J^*) - \frac{S^*}{S}S^*f(J^*) + S^*f(J)]i^*(0) \\ &+ \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)i^*(a)k(a)[-\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)} + 1 - \frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}]da, \\ \mathcal{A}_4 :=& [S^*g(Q^*) - \frac{S^*}{S}S^*g(Q^*)]i^*(0) \\ &+ \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)\ln\frac{i(t,a)}{i^*(a)}da - \frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)\ln\frac{p(t,0)}{p^*(0)}db, \\ \mathcal{A}_5 :=& S^*g(Q)i^*(0) + \frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)[-\frac{p(t,b)}{p^*(b)} + \ln\frac{p(t,b)}{p^*(b)}]db \\ &+ \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)\xi(a)i^*(a)[1 - \frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)} - \ln\frac{i(t,0)}{i^*(0)}]da.\end{aligned}$$

Since $Sf(J) + Sg(Q) = i(t,a)$ and $i^*(a) = \varpi_1(a)i^*(0)$, we have

$$\begin{aligned}\mathcal{A}_1 =& -\frac{\mu}{S}(S - S^*)^2 i^*(0) - i(t,0)i^*(0) \\ &+ \frac{1}{\Pi_1}S^*f(J^*)i(t,0)\int_0^\infty \varpi_1(a)k(a)da + \frac{1}{\Pi_3}S^*g(Q^*)i(t,0)\int_0^\infty \varpi_1(a)\xi(a)da \\ =& -\frac{\mu}{S}(S - S^*)^2 i^*(0) \leq 0.\end{aligned} \tag{27}$$

Due to $\int_0^\infty \xi(a)i(t,a)da = p(t,0)$ and $\int_0^\infty \varpi_2(b)q(b)db = \Pi_2$, we obtain

$$\mathcal{A}_2 = -\frac{1}{\Pi_3}S^*g(Q^*)\int_0^\infty \xi(a)i(t,a)da + \frac{1}{\Pi_2\Pi_3}S^*g(Q^*)p(t,0)\int_0^\infty \varpi_2(b)q(b)db = 0. \tag{28}$$

Since $i^*(0) = \frac{1}{\Pi_1}\int_0^\infty k(a)\varpi_1(a)i^*(0)da = \frac{1}{\Pi_1}\int_0^\infty k(a)i^*(a)da = \frac{J^*}{\Pi_1}$, we obtain

$$\mathcal{A}_3 = \frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)k(a)i^*(a)[-h(\frac{S^*}{S}) - h(\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}) - h(\frac{i(t,a)}{i^*(a)}) + h(\frac{f(J)}{f(J^*)})],$$

and

$$\frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)k(a)i^*(a)h(\frac{i(t,a)}{i^*(a)})da = S^*f(J^*)\int_0^\infty \frac{k(a)i^*(a)}{\int_0^\infty k(a)\varpi_1(a)da}h(\frac{i(t,a)}{i^*(a)})da$$
$$=S^*f(J^*)i^*(0)\int_0^\infty \frac{k(a)i^*(a)}{J^*}h(\frac{i(t,a)}{i^*(a)})da \geq S^*f(J^*)i^*(0)h(\frac{J(t)}{J^*}).$$

Then, we have

$$\begin{aligned}\mathcal{A}_3 \leq &\frac{1}{\Pi_1}\int_0^\infty S^*f(J^*)k(a)i^*(a)[-h(\frac{S^*}{S}) - h(\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)})]da \\ &+ S^*f(J^*)i^*(0)[h(\frac{f(J)}{f(J^*)}) - h(\frac{J}{J^*})] \leq 0.\end{aligned} \tag{29}$$

Due to $p^*(0) = \int_0^\infty \xi(a)i^*(a)da$, $\Pi_2 = \int_0^\infty \varpi_2(b)q(b)db$ and

$$\int_0^\infty i^*(a)\xi(a)[1 - \frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}]da = \int_0^\infty i^*(a)\xi(a)da - p^*(0)\frac{1}{p(t,0)}\int_0^\infty \xi(a)i(t,a)da = 0,$$

we have

$$\begin{aligned}\mathcal{A}_4 =&\frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)[-h(\ln\frac{S^*}{S}) - h(\ln\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}) + \ln\frac{S}{S^*}]da \\ \leq&\frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)\ln\frac{S}{S^*}da.\end{aligned}$$

Thus, combining $\mathcal{A}_4$ and $\mathcal{A}_5$ yields

$$\begin{aligned}\mathcal{A}_4 + \mathcal{A}_5 \leq &\frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)\frac{g(Q)}{g(Q^*)}da \\ &+ \frac{1}{\Pi_3}\int_0^\infty S^*g(Q^*)i^*(a)\xi(a)[1 - \frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)} + \ln\frac{S}{S^*} - \ln\frac{i(t,0)}{i^*(0)}]da \\ &+ \frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)[\ln\frac{p(t,b)}{p^*(b)} - \frac{p(t,b)}{p^*(b)}]db.\end{aligned}$$

Then, due to $\frac{1}{\Pi_2}\int_0^\infty p^*(b)q(b)db = \frac{1}{\Pi_2}\int_0^\infty p^*(0)\varpi_2(b)q(b)db = \int_0^\infty i^*(a)\xi(a)da$ and

$$h(\frac{p(t,b)}{p^*(b)}) = \int_0^\infty \frac{q(b)p^*(b)}{\Pi_2 p^*(0)}h(\frac{p(t,b)}{p^*(b)})db \geq h\left(\frac{\int_0^\infty q(b)p(t,b)db}{\int_0^\infty q(b)p^*(b)db}\right) = h(\frac{Q}{Q^*}),$$

we further obtain

$$\begin{aligned}\mathcal{A}_4 + \mathcal{A}_5 \leq&\frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)\times \\ &[-h(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}) + h(\frac{g(Q)}{g(Q^*)}) - h(\frac{p(t,b)}{p^*(b)})]db \\ =&\frac{1}{\Pi_2\Pi_3}\int_0^\infty S^*g(Q^*)p^*(b)q(b)[-h(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)})]db \\ &+ \frac{1}{\Pi_3}S^*g(Q^*)p^*(0)[h(\frac{g(Q)}{g(Q^*)}) - h(\frac{p(t,b)}{p^*(b)})]db \\ \leq&0.\end{aligned} \tag{30}$$

From Equations (27)–(30), we have $\frac{dV}{dt} \leq 0$ and the largest invariant subset of set $\left\{ \frac{dV}{dt} = 0 \right\}$ is $\{E^*\}$. Due to the invariance principle [37], we conclude that E^* is globally asymptotically stable if it exists. $\square$

5. Numerical Simulations

In this section, as a special case for the age-infection model (1), we consider the following model:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu S(t) - \dfrac{S(t)\int_0^\infty k(a)i(t,a)da}{\int_0^\infty k(a)i(t,a)da + A} - S(t)\displaystyle\int_0^\infty q(b)p(t,b)db, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b), \end{cases} \tag{31}$$

with the initial condition (3) and the following boundary conditions

$$\begin{aligned} i(t,0) &= \frac{S(t)\int_0^\infty k(a)i(t,a)da}{\int_0^\infty k(a)i(t,a)da + A} + S(t)\int_0^\infty q(b)p(t,b)db, \ \ t > 0, \\ p(t,0) &= \int_0^\infty \xi(a)i(t,a)da, \ \ t > 0. \end{aligned}$$

Following (15), the basic reproduction number of system (31) is

$$\Re_1 = \frac{\Lambda}{A\mu}\Pi_1 + \frac{\Lambda}{\mu}\Pi_2\Pi_3.$$

From Theorems 8 and 9, we obtain the following corollary:

Corollary 1. *When $\Re_1 < 1$, model (31) generates unique infection-free equilibrium E_1^0, which is globally asymptotically stable. When $\Re_1 > 1$, model (31) has E_1^0 and a globally asymptotically stable infection equilibrium E_1^*.*

To verify the result, we perform numerical simulations. Following [6,7] and references therein, with some assumptions, we adopt the following coefficients, for $0 \leq a, b \leq 10$,

$$\Lambda = 1000, \ \ \mu = 10^{-5}, \ \ A = 10^5, \ \ \xi(a) = 1 + \sin\frac{(a-5)\pi}{10},$$

$$\delta(a) = 0.2\left(1 + \sin\frac{(a-5)\pi}{10}\right), \ \ \gamma(b) = 0.3\left(1 + \sin\frac{(b-5)\pi}{10}\right),$$

$$k(a) = k\left(1 + \sin\frac{(a-5)\pi}{10}\right), \ \ q(b) = q\left(1 + \sin\frac{(b-5)\pi}{10}\right).$$

Let $k = 10^{-5}$and observe the dynamical behavior of the model when q varies. Let $q = 10^{-4}$ decrease to $q = 10^{-10}$. The globally asymptotically stable E_1^* changes to be unstable and the epidemic is inhibited effectively, which can be seen in Figures 1 and 2.

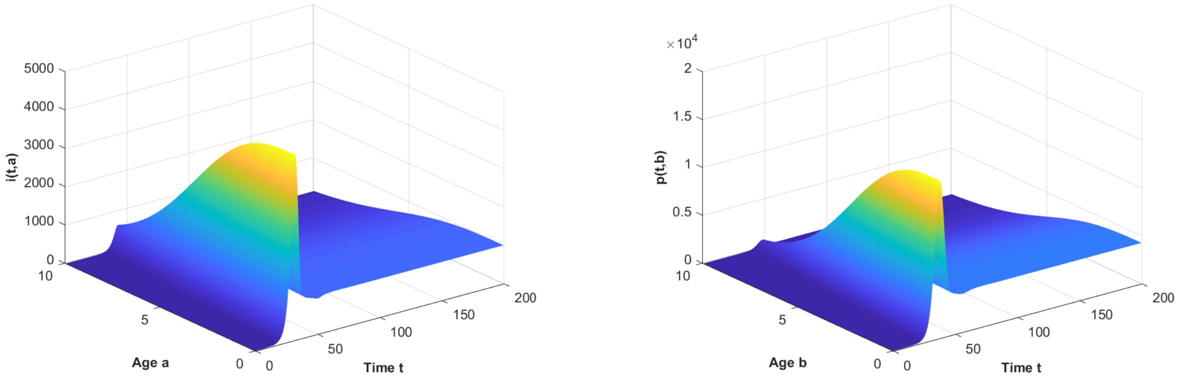

Figure 1. The long-term dynamical behavior of $i(t,a)$ and $p(t,b)$ as $q = 10^{-4}$.

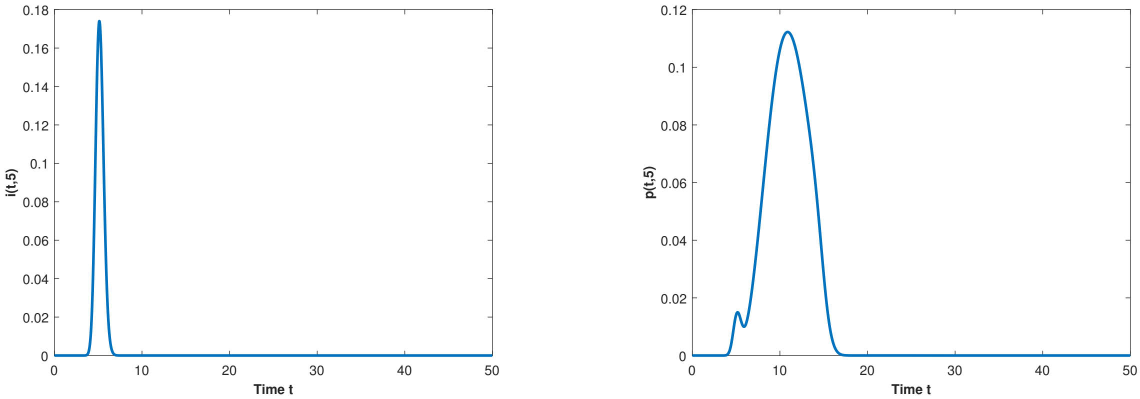

Figure 2. The long-term dynamical behavior of $i(t,a)$ and $p(t,b)$ for $a = b = 5$ as $q = 10^{-10}$.

6. Conclusions and Discussion

In this paper, an age-structured model of cholera infection was explored. By considering general infection functions, the discussion provided in this paper serves as a generalization and supplement to the work presented in F. Brauer et al. [12]. We applied the Lyapunov functional method to show that the global stability of equilibria are determined by the basic reproduction number $\Re_0$. The infection-free equilibrium is globally asymptotically stable if $\Re_0$ is less than one, whereas a globally asymptotically stable infection equilibrium emerges if $\Re_0$ is greater than one. This shows that both the direct contact with infected individuals and indirect pathogen infection have vital effects on cholera epidemics. It is significant to implement effective treatment for infected individuals and to clean pathogens from contaminated water in a timely fashion. More specifically, for the critical case when $\Re_0$ equals one, further bifurcation studies are needed.

In our model, vaccinated individuals and vaccination age have not been incorporated, which play vital effects on the spread of cholera. Furthermore, the immigration of infected individuals plays a significant role in the outbreak and infection of cholera. For the actual control and elimination of cholera, it is necessary to take into account the effects of vaccination and immigration [5,38]. Thus, our future work will consider these factors and focus on their effects on cholera transmission. In addition to qualitative analyses, tremendous amounts of works on numerical methods have been proposed and developed to deal with various epidemic models [39–41], which provide us with more aspects and methods to analyze in relation to this model.

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