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A 2-arc Transitive Hexavalent Nonnormal Cayley Graph on A_{119}

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Abstract: A Cayley graph $\Gamma = \mathsf{Cay}(G, S)$ is said to be normal if the base group G is normal in $\mathsf{Aut}\Gamma$. The concept of the normality of Cayley graphs was first proposed by M.Y. Xu in 1998 and it plays a vital role in determining the full automorphism groups of Cayley graphs. In this paper, we construct an example of a 2-arc transitive hexavalent nonnormal Cayley graph on the alternating group A_{119}. Furthermore, we determine the full automorphism group of this graph and show that it is isomorphic to A_{120}.

Keywords: simple group; nonnormal Cayley graph; arc-transitive graph; automorphism group

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1. Introduction

Throughout this paper, all graphs are assumed to be finite and undirected.

For a graph Γ, we use $V\Gamma$, $E\Gamma$, $Arc\Gamma$ and $\mathsf{Aut}\Gamma$ to denote the vertex set, edge set, arc set and full automorphism group of the graph Γ, respectively. A graph Γ is said to be *arc-transitive* if the full automorphism group $\mathsf{Aut}\Gamma$ acts transitively on $Arc\Gamma$. We use $\mathsf{val}\Gamma$ to denote the valency of the Γ, and we say Γ is a cubic, tetravalent, pentavalent or hexavalent graph, meaning $\mathsf{val}\Gamma = 3, 4, 5$ or 6.

Let G be a finite group with identity element 1 and S (say Cayley subset) a subset of G such that $1 \notin S$ and $S = S^{-1} := \{x^{-1} \mid x \in S\}$. Define the *Cayley graph* $\mathsf{Cay}(G, S)$, that is, the Cayley graph of G with respect to the Cayley subset S as the graph with vertex set G such that $g, h \in G$ are adjacent if and only if $hg^{-1} \in S$. It is easy to see that the valency of $\mathsf{Cay}(G, S)$ is $|S|$. As we all know, $\mathsf{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$. On the other hand, letting $R(G)$ be the right regular representation of G and letting $\mathsf{AutCay}(G, S)$ be the full automorphism group of $\mathsf{Cay}(G, S)$, there are clearly $R(G) \leq \mathsf{AutCay}(G, S)$, and $R(G)$ acts transitively on the vertices of $\mathsf{Cay}(G, S)$. Then, the graph $\mathsf{Cay}(G, S)$ is vertex-transitive, and G (or $R(G)$) can be viewed as a regular subgroup of $\mathsf{AutCay}(G, S)$. Conversely, a connected graph Γ is isomorphic to a Cayley graph of a group G if and only if the full automorphism group $\mathsf{Aut}\Gamma$ contains a subgroup which acts regularly on $V\Gamma$ and the subgroup is isomorphic to G (see [1]). A Cayley graph $\Gamma = \mathsf{Cay}(G, S)$ is said to be a *normal Cayley graph* if the base group G is normal in $\mathsf{Aut}\Gamma$; otherwise, Γ is said to be a *nonnormal Cayley graph* (see [2]).

The study about Cayley graphs on finite non-abelian simple groups has always attracted much attention because of Cayley graphs with high levels of symmetry; for example, vertex-transitivity, edge-transitivity and arc-transitivity are widely used in the design of interconnection networks. For more detailed applications, we recommend that readers refer to [3,4]. Let G be a finite non-abelian simple group, and let $\Gamma = \mathsf{Cay}(G, S)$ be a connected arc-transitive Cayley graph on G. The main motivation for classifying 2-arc-transitive nonnormal Cayley graphs comes from the fact that Fang, Ma and Wang [5] proved all but finitely that many locally primitive Cayley graphs of valency $d \leq 20$ or a prime number of the finite non-abelian simple groups are normal. In [5] (Problem 1.2), they proposed the following problem: classify nonnormal locally primitive Cayley graphs (note that 2-arc-transitive graphs must be locally primitive) of finite simple groups with valency $d \leq 20$ or a prime number. To solve this problem, we should study each valency $d \leq 20$ or a prime number. In the case where Γ is a cubic graph (3-valent), Li [6] proved that Γ must be normal, except for seven exceptions. On the basis of Li's result, Xu et al. [7,8] proved that Γ must be normal, except for two exceptions on A_{47}. In the case where Γ is a tetravalent graph (4-valent), Fang et al. in [9] proved that most of such Γ are normal, except for Cayley graphs on a list of G. Further, Fang et al. in [10] proved that Γ are normal when Γ is 2-transitive, except for two graphs on M_{11}. In the case where Γ is a pentavalent graph (5-valent), Zhou and Feng [11] proved that all 1-transitive Cayley Γ of simple groups are normal. Ling and Lou in [12] gave an example of a 2-transitive pentavalent nonnormal Cayley graph on A_{39}. Therefore, the next natural problem is to study the case of the 6-valent. However, there are no known nonnormal examples of hexavalent 2-arc-transitive Cayley graphs on finite simple groups.

The aim of this paper is to construct a nonnormal example of a connected 2-arc transitive hexavalent Cayley graph on a finite non-abelian simple group. Our main result is the following theorem.

Theorem 1. *There exists a nonnormal example of a connected 2-arc-transitive hexavalent Cayley graph on the alternating group A_{119}, and the full automorphism group of this graph is isomorphic to the alternating group A_{120}.*

2. Preliminaries

In this section, we give some necessary preliminary results which are used in later discussions.

Let G be a finite group and let H be a subgroup of G. Then we have the following result (see [13] (Ch. I, 1.4)).

Lemma 1. *Let G be a group and let H be a subgroup of G. Let $\mathrm{N}_G(H)$ be the normalizer of H in G, and let $\mathrm{C}_G(H)$ be the centralizer of H in G. Then, $\mathrm{N}_G(H)/\mathrm{C}_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathsf{Aut}(H)$ of H.*

We next introduce the definition of a Sabidussi coset graph. Let G be a group, $g \in G \backslash H$ such that $g^2 \in H$, and let H be a core-free subgroup of G. Define the *Sabidussi coset graph* $\mathsf{Cos}(G, H, g)$ of G with respect to the core-free subgroup H as the graph with vertex set $[G : H]$ (the set of cosets of H in G) such that Hx and Hy are adjacent if and only if $yx^{-1} \in HgH$. The following lemma follows from [14], and it can be easily proved by the definition of the coset graphs (see [15] (Theorem 3) for example).

Lemma 2. *Let G be a group, $g \in G \backslash H$ such that $g^2 \in H$, and let H be a core-free subgroup of G. Let $\Gamma = \mathsf{Cos}(G, H, g)$ be a Sabidussi coset graph of G with respect to H. Then, Γ is G-arc-transitive and the following holds:*

(1) The valency of the graph Γ is equal to $|H : H \cap H^g|$.
(2) Γ is a connected graph if and only if $\langle H, g \rangle = G$.
(3) If G contains a subgroup R is regular on $V\Gamma$, then $\Gamma \cong \mathsf{Cay}(R, S)$, where $S = R \cap HgH$.

Conversely, if Σ is an X-arc-transitive graph, then Σ is isomorphic to a Sabidussi coset graph $\mathsf{Cos}(X, X_v, g)$, where $g \in \mathrm{N}_X(X_{vw})$ is a 2-element such that $g^2 \in X_v$, and $v \in V\Sigma$, $w \in \Sigma(v)$.

Proof. Let Σ be an X-arc-transitive graph. Let $v \in V\Sigma$ be a vertex of Σ and $w \in \Sigma(v)$. Since Σ is X-arc-transitive, there is g such that $v^g = w$. For each $x \in X$, define $\varphi : Hx \longrightarrow v^x$. Then we can verify that φ is a graph isomorphic from Σ to $\mathsf{Cos}(X, X_v, g)$. Since Σ is undirected, we have $g^2 \in X_v$. Hence, $(H \cap Hg)^g = H \cap Hg$. Thus, we can choose a 2-element g satisfying $g \in \mathrm{N}_X(X_{vw})$. $\square$

Let $t_1 \geq 0$ and $t_2 \geq 0$ be two integers. We denote by the $\{2,3\}$-group the finite group of the order $2^{t_1}3^{t_2}$. Following the definition of relevant objects in [16] (Theorem 3.1), we have the following lemma, which is about the stabilizers of arc-transitive hexavalent graphs. For the structure of the received stabilizers, see the proof in [16] (Page 926).

Lemma 3. *Let s be a positive integer, and let Γ be a connected hexavalent (G,s)-transitive graph for some $G \leq \mathrm{Aut}\Gamma$. Let $v \in V\Gamma$. Then $s \leq 4$ and one of the following statements holds:*

(1) *For $s = 1$, the stabilizer G_v is a $\{2,3\}$-group.*
(2) *For $s = 2$, the stabilizer $G_v \cong \mathrm{PSL}(2,5)$, $\mathrm{PGL}(2,5)$, A_6 or S_6.*
(3) *For $s = 3$, the stabilizer $G_v \cong \mathrm{D}_{10} \times \mathrm{PSL}(2,5)$, $F_{20} \times \mathrm{PGL}(2,5)$, $\mathsf{A}_5 \times \mathsf{A}_6$, $\mathsf{S}_5 \times \mathsf{S}_6$. $(\mathrm{D}_{10} \times \mathrm{PSL}(2,5)) \cdot \mathbb{Z}_2$ with $\mathrm{D}_{10} \cdot \mathbb{Z}_2 = F_{20}$ and $\mathrm{PSL}(2,5) \cdot \mathbb{Z}_2 = \mathrm{PGL}(2,5)$, or $(\mathsf{A}_5 \times \mathsf{A}_6) \rtimes \mathbb{Z}_2$ with $\mathsf{A}_5 \rtimes \mathbb{Z}_2 = \mathsf{S}_5$ and $\mathsf{A}_6 \rtimes \mathbb{Z}_2 = \mathsf{S}_6$.*
(4) *For $s = 4$, the stabilizer $G_v \cong \mathbb{Z}_5^2 \rtimes \mathrm{GL}(2,5) = \mathrm{AGL}(2,5)$.*

3. A 2-arc Transitive Hexavalent Nonnormal Cayley Graph on A_{119}

In this section, we construct a connected 2-arc transitive hexavalent nonnormal Cayley graph on A_{119} and determine its full automorphism group. In fact, if $\Gamma := \mathsf{Cay}(G,S)$ is a Cayley graph of a non-abelian simple group G, then G is core free in X, where $G \leq X \leq \mathrm{Aut}\Gamma$. Let $v \in V\Gamma$ and $H = X_v$. Suppose that $|H| = n$. Then by Lemma 3, n may be 60, 120, etc. Consider the action of X on the set of $[X:G]$ by right multiplication; then, $X \lesssim \mathsf{S}_n$. So, we may construct the nonnormal Cayley graph in S_n, where $n = 60$, 120, etc. The following example is really the case where we construct $n = 120$.

Construction 1. *Let G be the alternating group on the set $\{2,3,\ldots,120\}$. Then, $G \cong \mathsf{A}_{119}$. Let $H = \langle a,b \rangle < X := \mathsf{A}_{120}$ (the alternating group on $\{1,2,\ldots,120\}$), where the following holds:*

$$
\begin{aligned}
a = \ & (1\,2\,4\,3)(5\,13\,12\,17)(6\,14\,11\,18)(7\,15\,10\,19)(8\,16\,9\,20)(21\,61\,101\,81)\\
& (22\,62\,102\,82)(23\,63\,103\,83)(24\,64\,104\,84)(25\,65\,105\,85)(26\,66\,106\,86)\\
& (27\,67\,107\,87)(28\,68\,108\,88)(29\,69\,109\,89)(30\,70\,110\,90)(31\,71\,111\,91)\\
& (32\,72\,112\,92)(33\,73\,113\,93)(34\,74\,114\,94)(35\,75\,115\,95)(36\,76\,116\,96)\\
& (37\,77\,117\,97)(38\,78\,118\,98)(39\,79\,119\,99)(40\,80\,120\,100)(41\,56\,59\,46)\\
& (42\,55\,57\,48)(43\,54\,60\,45)(44\,53\,58\,47)(49\,51\,52\,50),
\end{aligned}
$$

$$
\begin{aligned}
b = \ & (1\,21\,41)(2\,22\,42)(3\,23\,43)(4\,24\,44)(5\,25\,45)(6\,26\,46)(7\,27\,47)\\
& (8\,28\,48)(9\,29\,49)(10\,30\,50)(11\,31\,51)(12\,32\,52)(13\,33\,53)(14\,34\,54)\\
& (15\,35\,55)(16\,36\,56)(17\,37\,57)(18\,38\,58)(19\,39\,59)(20\,40\,60)(61\,85\,110)\\
& (62\,86\,112)(63\,87\,109)(64\,88\,111)(65\,95\,106)(66\,93\,108)(67\,96\,105)\\
& (68\,94\,107)(69\,81\,113)(70\,82\,114)(71\,83\,115)(72\,84\,116)(73\,100\,119)\\
& (74\,99\,117)(75\,98\,120)(76\,97\,118)(77\,91\,101)(78\,89\,102)(79\,92\,103)\\
& (80\,90\,104).
\end{aligned}
$$

Take $x \in X$ as follows:

$$
\begin{aligned}
x = \ & (1\,79)(2\,80)(3\,60)(4\,58)(5\,113)(6\,64)(7\,114)(8\,63)(9\,112)(10\,111)(11\,12)\\
& (13\,47)(15\,73)(16\,106)(19\,43)(20\,41)(21\,118)(22\,120)(23\,24)(25\,50)(26\,49)\\
& (27\,68)(28\,66)(30\,62)(32\,61)(33\,42)(34\,44)(35\,103)(36\,101)(37\,107)(38\,45)\\
& (40\,75)(48\,108)(53\,54)(55\,115)(56\,116)(57\,119)(59\,117)(65\,109)(67\,110)\\
& (69\,102)(70\,104)(71\,72)(76\,105)(81\,89)(82\,93)(84\,99)(85\,98)(87\,91)(88\,94)\\
& (90\,100)(96\,97).
\end{aligned}
$$

Define $\Sigma = \mathsf{Cos}(X,H,x)$.

Lemma 4. *The graph $\Sigma = \mathsf{Cos}(X, H, x)$ in Construction 1 is a connected 2-arc-transitive graph and isomorphic to the nonnormal hexavalent Cayley graph $\mathsf{Cay}(G, S)$ of G, determined by $S = \{x_1, x_1^{-1}, x_2, x_3, x_4, x_5\}$ with the following:*

$$\begin{aligned}
x_1 = \;& (2\ 3\ 38\ 36\ 95\ 101\ 45\ 60\ 80\ 77)(4\ 37\ 97\ 103\ 46\ 35\ 96\ 107\ 58\ 78)\\
& (5\ 90\ 106\ 66\ 51\ 28\ 16\ 100\ 113\ 74)(6\ 18\ 64\ 76\ 98\ 109\ 14\ 65\ 85\ 105)\\
& (7\ 92\ 114\ 73\ 68\ 49\ 52\ 26\ 27\ 15)(8\ 20\ 31\ 41\ 63\ 75\ 59\ 83\ 117\ 40)\\
& (9\ 88\ 104\ 11\ 69\ 93\ 120\ 54\ 21\ 81\ 115\ 23\ 56\ 91\ 111\ 71)\\
& (10\ 87\ 116\ 24\ 55\ 89\ 118\ 53\ 22\ 82\ 102\ 12\ 70\ 94\ 112\ 72)\\
& (13\ 34\ 30\ 17\ 62\ 44\ 47\ 67\ 86\ 110)(19\ 32)(29\ 42\ 48\ 99\ 119\ 39\ 57\ 84\ 108\ 33)\\
& (43\ 61),\\
x_2 = \;& (2\ 97)(4\ 99)(5\ 73)(6\ 74)(7\ 8)(9\ 41)(10\ 88)(11\ 78)(13\ 58)(14\ 96)\\
& (15\ 60)(16\ 95)(17\ 66)(18\ 65)(19\ 52)(20\ 50)(22\ 40)(23\ 27)(24\ 32)(25\ 29)\\
& (26\ 33)(28\ 39)(31\ 36)(35\ 38)(43\ 116)(44\ 77)(45\ 62)(46\ 48)(47\ 61)(49\ 107)\\
& (51\ 105)(53\ 110)(54\ 75)(55\ 109)(56\ 76)(63\ 120)(64\ 119)(67\ 91)(68\ 92)\\
& (69\ 103)(70\ 94)(71\ 104)(72\ 93)(79\ 113)(80\ 86)(81\ 112)(83\ 111)(87\ 114)\\
& (89\ 90)(98\ 117)(100\ 118)(101\ 102),\\
x_3 = \;& (3\ 39)(4\ 37)(5\ 106)(6\ 105)(7\ 50)(8\ 49)(9\ 35)(10\ 45)(11\ 36)\\
& (12\ 47)(13\ 113)(14\ 114)(15\ 16)(18\ 118)(19\ 28)(20\ 56)(21\ 92)(23\ 91)\\
& (26\ 120)(27\ 94)(29\ 30)(31\ 107)(32\ 108)(33\ 112)(34\ 110)(38\ 97)(40\ 98)\\
& (41\ 42)(43\ 102)(44\ 101)(51\ 87)(52\ 85)(53\ 117)(54\ 96)(57\ 89)(58\ 90)\\
& (59\ 116)(60\ 115)(62\ 80)(63\ 67)(64\ 72)(65\ 69)(66\ 73)(68\ 79)(71\ 76)\\
& (75\ 78)(81\ 82)(83\ 109)(84\ 111)(93\ 119)(99\ 104)(100\ 103),\\
x_4 = \;& (2\ 20)(3\ 7)(4\ 12)(5\ 9)(6\ 13)(8\ 19)(11\ 16)(15\ 18)(21\ 49)(23\ 84)\\
& (24\ 101)(25\ 112)(26\ 28)(27\ 110)(29\ 94)(30\ 60)(31\ 96)(32\ 59)(33\ 85)\\
& (34\ 120)(35\ 87)(36\ 118)(38\ 53)(40\ 55)(42\ 76)(44\ 74)(45\ 119)(46\ 117)\\
& (47\ 48)(50\ 64)(51\ 102)(54\ 67)(56\ 65)(57\ 108)(58\ 106)(62\ 104)(63\ 82)\\
& (66\ 116)(68\ 114)(69\ 86)(71\ 88)(73\ 97)(75\ 99)(77\ 105)(78\ 80)(79\ 107)\\
& (81\ 103)(89\ 91)(90\ 115)(92\ 113)(98\ 109)(100\ 111),\\
x_5 = \;& (2\ 111)(3\ 89)(4\ 32)(5\ 21)(6\ 118)(7\ 23)(8\ 117)(9\ 95)(11\ 93)\\
& (13\ 99)(14\ 27)(15\ 100)(16\ 28)(17\ 106)(18\ 20)(19\ 105)(22\ 73)(24\ 75)\\
& (29\ 112)(30\ 69)(33\ 108)(34\ 80)(35\ 107)(36\ 79)(37\ 38)(39\ 101)(40\ 103)\\
& (41\ 49)(42\ 53)(44\ 59)(45\ 58)(47\ 51)(48\ 54)(50\ 60)(56\ 57)(61\ 116)\\
& (62\ 64)(63\ 114)(65\ 94)(66\ 96)(67\ 102)(68\ 104)(71\ 90)(72\ 110)(77\ 84)\\
& (78\ 82)(85\ 120)(86\ 119)(87\ 88)(91\ 109)(97\ 113)(98\ 115).
\end{aligned}$$

Proof. Let $\Delta := \{1, 2, \ldots, 120\}$. Then, X has a natural action on Δ. By Magma [17], $\langle H, x \rangle = X$, and so the graph Σ is connected by Lemma 2 (2). Furthermore, by Magma [17], we have that H is regular on Δ. However, G is the stabilizer of point 1 in X. Hence, X has a factorization $X = GH = HG$ with $G \cap H = 1$. Therefore, G is regular on $[X : H]$. By Lemma 2 (3), Σ is isomorphic to a Cayley graph of $G = \mathsf{A}_{119}$. Additionally, by the computation of Magma [17] (for the Magma code, see Appendix A), we have $\frac{|H|}{|H \cap H^x|} = 6$. Hence, Lemma 2 (1) implies that Σ is a hexavalent graph. Since $H \cong \mathrm{PGL}(2,5)$, Lemma 3 implies that Σ is 2-arc transitive. Since X is a non-abelian simple group, G is not normal in $X \leq \mathsf{Aut}\Sigma$. It follows that Σ is nonnormal. Let x_1, x_2, x_3, x_4, x_5 and S be defined as in this lemma. By the computation of Magma [17] (for the Magma code, see Appendix B), we have $G \cap (HxH) = S$. Thus, by Lemma 2 (3), we have that Σ is isomorphic to $\mathsf{Cay}(G, S)$. This completes the proof of the lemma. □

In the next lemma, we show that the full automorphism group $\mathsf{Aut}\Sigma$ is isomorphic to alternating group A_{120}.

Lemma 5. *The full automorphism group $\mathsf{Aut}\Sigma$ of the 2-arc-transitive hexavalent graph $\Sigma = \mathsf{Cos}(X, H, x)$ in Construction 1 is isomorphic to alternating group A_{120}.*

Proof. Let $\mathsf{A} = \mathsf{Aut}\Sigma$. Assume first that the full automorphism group A is quasiprimitive on $V\Sigma$. Let N be a minimal normal subgroup of A. Then, N is transitive on $V\Sigma$. It implies that N is insoluble. Thus, N is isomorphic to $T_1 \times T_2 \times \cdots \times T_d = T^d$, where $T_i \cong T$ for each $1 \leq i \leq d$, T is a non-abelian simple group, and $d \geq 1$. Let p be the largest prime factor of the order of A_{119}. Then, $p > 5$ and $p^2 \not| \, |\mathsf{A}_{119}|$. Since N is transitive on $V\Sigma$ and $|V\Sigma| = |\mathsf{A}_{119}|$, we have that p divides $|N|$. Assume that $d \geq 2$. Then, p^d divides $|N|$. However, by Lemma 3, the order of the stabilizer A_v divides $2^7 \cdot 3^3 \cdot 5^3$, and so $|\mathsf{A}|$ divides $2^7 \cdot 3^3 \cdot 5^3 \cdot |\mathsf{A}_{119}|$ which is divisible by p^d, a contradiction. Hence, we have $d = 1$ and $N = T \trianglelefteq \mathsf{A}$. Let $C = \mathsf{C}_{\mathsf{A}}(T)$ be the centralizer of T in A. Then, $C \trianglelefteq N_{\mathsf{A}}(T) = \mathsf{A}$ and $CT = C \times T$. If $C \neq 1$, since A is quasiprimitive on $V\Sigma$, this implies that C is transitive on $V\Sigma$. It implies that p divides $|C|$. Therefore, p^2 divides $|CT|$, which divides $|\mathsf{A}|$, and so we have that p^2 divides $|\mathsf{A}|$, a contradiction. Hence, $C = 1$, and $\mathsf{A} \leq \mathsf{Aut}(T)$ is almost simple.

Since $T \cap X \trianglelefteq X \cong \mathsf{A}_{120}$, it follows that $T \cap X = 1$ or X. If $T \cap X = 1$, then since $\frac{|\mathsf{A}|}{|X|} \mid 2^4 \cdot 3^2 \cdot 5^2$, we have $|T| \mid 2^4 \cdot 3^2 \cdot 5^2$; note that $p > 5$, $p \mid |T|$, a contradiction. Thus, $T \cap X = X$, and so $X \leq T$. It follows that $|T : X|$ divides $|\mathsf{A} : X|$, which divides $2^4 \cdot 3^2 \cdot 5^2$. By [18] (pp. 135–136), we can conclude that $T = X \cong \mathsf{A}_{120}$. Thus, $\mathsf{A} \leq \mathsf{Aut}(T) \cong \mathsf{S}_{120}$. If $\mathsf{A} \cong \mathsf{S}_{120}$, then $|\mathsf{A}_v| = \frac{|\mathsf{A}|}{|G|} = 240$, a contradiction to Lemma 3. Hence, $\mathsf{A} \cong \mathsf{A}_{120}$.

Now assume that the full automorphism group A is not quasiprimitive on $V\Sigma$. Then there is a minimal normal subgroup M of A that acts nontransitively on $V\Sigma$. Since $M \cap X \trianglelefteq X$, we have $M \cap X = 1$ or X. For the latter case $M \cap X = X$, we have $X \leq M$, and so M is transitive on $V\Sigma$, a contradiction. For the former case, $M \cap X = 1$, then we have that $|M|$ divides $\frac{|\mathsf{A}|}{|X|}$, which divides $2^4 \cdot 3^2 \cdot 5^2$.

Assume that M is insoluble. Since $|M|$ divides $2^4 \cdot 3^2 \cdot 5^2$, and the simple groups A_5, A_6, $\mathrm{PSp}(4,3)$ are the only $\{2,3,5\}$-factor non-abelian simple groups (see [19] (Table 1), and note that the definition of the $\{2,3,5\}$-group is similar to $\{2,3\}$-group); by checking the orders of these groups, it is easy to figure out $M \cong \mathsf{A}_5$ or A_5^2 or A_6. Then since $|M| \cdot |\mathsf{A}_{120}| = |M| \cdot |X| = |L| = |V\Sigma| \cdot |L_v| = |\mathsf{A}_{119}| \cdot |L_v|$, we have $|L_v| = 2^5 \cdot 3^2 \cdot 5^2$ or $2^7 \cdot 3^3 \cdot 5^3$ or $2^6 \cdot 3^3 \cdot 5^2$, a contradiction to the description of the orders of the stabilizers in Lemma 3.

Assume that M is soluble. Then $M \cong \mathbb{Z}_2^r$ or $\mathbb{Z}_3^s$ or $\mathbb{Z}_5^l$, where $1 \leq r \leq 4$, $1 \leq s \leq 2$ and $1 \leq l \leq 2$. Let $L = MX$. Then $L = M{:}X$, a split expansion of M by X. Further, we have $L/\mathsf{C}_L(M) \lesssim \mathsf{Aut}(M) \cong \mathrm{GL}(r,2)$ or $\mathrm{GL}(s,3)$ or $\mathrm{GL}(l,5)$. We note that M is a subgroup of $\mathsf{C}_L(M)$. If $M = \mathsf{C}_L(M)$, then we have $L/\mathsf{C}_L(M) = L/M \cong X \cong \mathsf{A}_{120} \lesssim \mathrm{GL}(r,2)$ or $\mathrm{GL}(s,3)$ or $\mathrm{GL}(l,5)$. However, for each $1 \leq r \leq 4$, $1 \leq s \leq 2$ and $1 \leq l \leq 2$, $\mathrm{GL}(r,2)$, $\mathrm{GL}(s,3)$ or $\mathrm{GL}(l,5)$ has no subgroup isomorphic to the alternating group A_{120}. Hence, we have $M < \mathsf{C}_L(M)$ and $1 \neq \mathsf{C}_L(M)/M \trianglelefteq L/M \cong \mathsf{A}_{120}$. It implies that $\mathsf{A}_{120} \cong \mathsf{C}_L(M)/M$; then $|\mathsf{C}_L(M)| = |M| \cdot |X| = |L|$ since $\mathsf{C}_L(M) \trianglelefteq L$, we have $\mathsf{C}_L(M) = L = MX$, and X centralizes M. Hence, $L = M \times X$. Then $L_v/X_v = L_v/L_v \cap X \cong L_vX/X \cong L/X \cong M$. Thus, $L_v \cong X_v.M$. Note that with the order of the stabilizers given in Lemma 3, we conclude $M \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$. In the case where $M \cong \mathbb{Z}_3$, we have $|L_v| = |X_v| \cdot |M| = 360$, then $L_v \cong \mathsf{A}_6$, $\mathsf{A}_6 \cong \mathrm{PGL}(2,5).\mathbb{Z}_3$, but there is no normal subgroup which is isomorphic to $\mathrm{PGL}(2,5)$ in A_6, a contradiction. In the case where $M \cong \mathbb{Z}_5$, we have $|L_v| = |X_v| \cdot |M| = 600$, then $L_v \cong \mathrm{D}_{10} \times \mathrm{PSL}(2,5)$, $\mathrm{D}_{10} \times \mathrm{PSL}(2,5) \cong \mathrm{PGL}(2,5).\mathbb{Z}_5$; by [17], there is no normal subgroup with order 120 in $\mathrm{D}_{10} \times \mathrm{PSL}(2,5)$, so clearly, $\mathrm{PGL}(2,5) \ntrianglelefteq \mathrm{D}_{10} \times \mathrm{PSL}(2,5)$, which also leads to a contradiction. This completes the proof of the lemma. □

Proof of Theorem 1. Now we are ready to prove our main Theorem 1. Let $\Sigma = \mathsf{Cos}(X, H, x)$ be the graph as in Construction 1. Then, Lemma 4 shows that Σ is a connected 2-arc-transitive graph and isomorphic to a nonnormal hexavalent Cayley graph $\mathsf{Cay}(G, S)$, with $G \cong \mathsf{A}_{119}$. This proves the statement of the former part of Theorem 1. The next Lemma 5 shows that the full automorphism group $\mathsf{Aut}\Sigma$ of the graph Σ is isomorphic to alternating group A_{120}. This proves the statement of the latter part of Theorem 1, and so completes the proof of Theorem 1. □

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Appendix A. Magma Codes Used in Computing the Valency $\frac{|H|}{|H \cap H^x|}$

```
val:=function(H,x);
m:=Order(H)/Order(H meet H^x);
return M;
end function;
```

Appendix B. Magma Codes Used in Computing the Elements of $G \cap (HxH)$

```
elt:=function(a,b,x);
X:=Alt(120);
G:=Stabilizer(X,1);
H:=sub<X|a,b>;
M:=[];
for m in H do
  for n in H do
   if 1^(m*x*n) eq 1 then
    if not m*x*n in M then
     Append(~M,m*x*n);
    end if;
   end if;
  end for;
end for;
return M;
end function;
```

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