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The Stability Analysis of A-Quartic Functional Equation

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Abstract: In this paper, we study the general solution of the functional equation, which is derived from additive–quartic mappings. In addition, we establish the generalized Hyers–Ulam stability of the additive–quartic functional equation in Banach spaces by using direct and fixed point methods.

Keywords: additive–quartic functional equation; Hyers–Ulam stability; fixed point method

MSC: 39B52; 39B82

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1. Introduction

The concept of stability for various functional equations arises when one replaces a functional equation by an inequality, which acts as a perturbation of the equation. The first stability problem of the functional equation was introduced by the mathematician S.M. Ulam [1] in 1940. Since then, this question has attracted the attention of many researchers. Note that the first solution to this question of Ulam was given by D.H. Hyers [2] in 1941 in the case of approximately additive mappings. Thereafter, Hyers' result was generalized by Aoki [3] and improved for additive mappings, and subsequently improved by Rassias [4] for linear mappings by allowing the Cauchy difference to be unbounded.

During the last eight decades, the stability problem of various functional equations was studied and established by several mathematicians for different kinds of mappings in various spaces, including random normed spaces and fuzzy Banach spaces [5,6], etc. For various other results on the stability of functional equations, see [7–17]. Most of the proofs of stability problems in the sense of Hyers–Ulam have used Hyers' direct method. The exact solution of the functional equation is explicitly obtained as the limit of a sequence, which starts from the given approximate solution.

In 2003, Radu [18] introduced a new method, called the fixed point alternative method, to investigate the existence of exact solutions and error estimations and established that a fixed point alternative method is more essential to the solution of the Ulam stability problem for approximate homomorphisms. Subsequently, some authors [19,20] applied the fixed alternative method to investigate the Hyers–Ulam stability of several functional equations in various directions [21,22]. To further explore the oscillation theory of functional differential equations, we refer the readers to [23–28].

In 2020, C. Park et al. [29] obtained the general solution and proved the Hyers–Ulam stability of the following quadratic–multiplicative functional equation of the form

$$f(st-uv)+f(sv+tu)=[f(s)+f(u)][f(t)+f(v)]$$

by using the direct method and the fixed point method. In the same year, Abasalt et al. [30] established the system of functional equations defining a multi m-Jensen mapping to a single equation. Using a fixed point theorem, they studied the generalized Hyers–Ulam stability of such an equation. Moreover, they proved that the the multi m-Jensen mappings are hyperstable.

Recently, Badora et al. [31] studied the Ulam stability of some functional equations using the Banach limit. They also illustrated the results with the examples of the linear functional equation in single variable and the Cauchy equation. In addition, Karthikeyan et al. [32] discussed the solution in vector spaces, proved the Ulam–Hyers stability of the quartic functional equation originating from the sum of the medians of a triangle in fuzzy normed space by using both direct and fixed point methods, and proved the Ulam–Hyers stability of the considered functional equation in paranormed spaces using both direct and fixed point methods. For more, see also [33–35].

One of the most famous functional equations is the additive functional equation:

$$h(w+x)=h(w)+h(x). \tag{1}$$

In 1821, it was first solved by A. L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field. Every solution of the additive functional Equation (1) is called an additive function.

In [13], Lee et al. considered the following functional equation:

$$h(2w+x)+h(2w-x)=4h(w+x)+4h(w-x)+24h(w)-6h(x). \tag{2}$$

It is said to be a quartic functional equation because the quartic function $h(x)=ax^4$ is a solution of the functional Equation (2).

Based on the above investigations, the main purpose of this paper is to prove the general solution of the additive–quartic functional equation of the form

$$\begin{aligned}
&h(\eta w_1+\eta^2 w_2+\eta^3 w_3)+h(-\eta w_1+\eta^2 w_2+\eta^3 w_3)+h(\eta w_1-\eta^2 w_2+\eta^3 w_3)+h(\eta w_1+\eta^2 w_2-\eta^3 w_3)\\
&=2\Big[h(\eta w_1+\eta^2 w_2)+h(\eta^2 w_2+\eta^3 w_3)+h(\eta w_1+\eta^3 w_3)+h(\eta w_1-\eta^2 w_2)+h(\eta^2 w_2-\eta^3 w_3)\\
&+h(\eta w_1-\eta^3 w_3)\Big]-2\Big[\eta^4(h(w_1)+h(-w_1))+\eta^8(h(w_2)+h(-w_2))+\eta^{12}(h(w_3)+h(-w_3))\Big]\\
&-\Big[\eta(h(w_1)-h(-w_1))+\eta^2(h(w_2)-h(-w_2))+\eta^3(h(w_3)-h(-w_3))\Big]
\end{aligned} \tag{3}$$

in different cases, where η is a fixed real number. It is easy to see that $h(w)=aw$ and $h(w)=aw^4$ satisfies the functional Equation (3). Moreover, by using the direct and fixed point method, we prove the generalized Hyers–Ulam stability of the additive–quartic functional Equation (3) in Banach spaces.

2. General Solution of the Additive Functional Equation (3) (When *h* Is Odd)

In this section, the authors investigate the general solution of the additive functional Equation (3).

Lemma 1. *Let W and X be real vector spaces. If an odd mapping $h : W \to X$ satisfies (3), then h is additive.*

Proof. Let $h : W \to X$ be a function which satisfies the functional Equation (3). Setting (w_1, w_2, w_3) by $(0,0,0)$ in (3), we get $h(0) = 0$. Replacing (w_1, w_2, w_3) by $(w,0,0)$ in (3), we get

$$h(\eta w) = \eta h(w), \tag{4}$$

for all $w \in W$. Replacing (w_1, w_2, w_3) by $(0, w, 0)$ in (3), we obtain

$$h(\eta^2 w) = \eta^2 h(w), \tag{5}$$

for all $w \in W$. Replacing (w_1, w_2, w_3) by $(0,0,w)$ in (3), we get

$$h(\eta^3 w) = \eta^3 h(w), \tag{6}$$

for all $w \in W$. In general, by using (4), (5) and (6), for any positive integer a, we have

$$h(aw) = ah(w), \tag{7}$$

for all $w \in W$. One can easily verify from (7) that

$$h\left(\frac{w}{a}\right) = \frac{1}{a} h(w), \tag{8}$$

for all $w \in W$. Replacing (w_1, w_2, w_3) by $\left(\frac{w}{\eta}, \frac{x}{\eta^2}, 0\right)$ in (3), we get

$$2h(w+x) + h(-w+x) + h(w-x) = 2h(w+x) + 2h(x-w) + 2h(-x) + 2h(w), \tag{9}$$

for all $w, x \in W$. Using oddness of h in (9), we get

$$0 = 2h(x-w) - 2h(x) + 2h(w), \tag{10}$$

for all $w, x \in W$. Setting w by $-w$ in (10), we have

$$2h(x+w) - 2h(x) + 2h(-w) = 0, \tag{11}$$

for all $w, x \in W$. Using oddness of h in (11), we get

$$h(w+x) = h(w) + h(x), \tag{12}$$

for all $w, x \in W$. □

3. General Solution of the Functional Equation (3) (When h Is Even)

In this section, we study the general solution of the quartic functional Equation (3) for an even case.

Lemma 2. *Assume that W and X are real vector spaces. If an even mapping $h : W \to X$ satisfies the functional equation*

$$h(2w+x) + h(2w-x) = 4h(w+x) + 4h(w-x) + 24h(w) - 6h(x), \tag{13}$$

for all $w, x \in W$ if only if $h : W \to X$ satisfies the functional equation

$$\begin{aligned}
&h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) + h(-\eta w_1 + \eta^2 w_2 + \eta^3 w_3) + h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) + h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) \\
&= 2\Big[h(\eta w_1 + \eta^2 w_2) + h(\eta^2 w_2 + \eta^3 w_3) + h(\eta w_1 + \eta^3 w_3) + h(\eta w_1 - \eta^2 w_2) + h(\eta^2 w_2 - \eta^3 w_3) \\
&+ h(\eta w_1 - \eta^3 w_3)\Big] - 2\Big[\eta^4(h(w_1) + h(-w_1)) + \eta^8(h(w_2) + h(-w_2)) + \eta^{12}(h(w_3) + h(-w_3))\Big] \\
&- \Big[\eta(h(w_1) - h(-w_1)) + \eta^2(h(w_2) - h(-w_2)) + \eta^3(h(w_3) - h(-w_3))\Big]
\end{aligned} \tag{14}$$

for all $w_1, w_2, w_3 \in W$.

Proof. Let $h : W \to X$ satisfy the functional Equation (2). Setting (w, x) by $(0, 0, 0)$, we get $h(0) = 0$. Replacing (w, x) by $(0, x)$, we get

$$h(-x) = h(x) \tag{15}$$

for all $x \in W$. Setting (w, x) by $(w, 0)$ and (w, w) in (2), we obtain

$$h(2w) = 16h(w) \ \ and \ \ h(3w) = 81h(w) \tag{16}$$

for all $w \in W$. In general, for any positive integer m, such that

$$h(mw) = m^4 h(w) \tag{17}$$

for all $w \in W$. Replacing w and x by $\eta w_1 + \eta^2 w_2$ and $\eta w_1 - \eta^2 w_2$ in (2), respectively, we have

$$h(3\eta w_1 + \eta^2 w_2) + h(\eta w_1 + 3\eta^2 w_2) = 64h(\eta w_1) + 64h(\eta^2 w_2) + 24h(\eta w_1 + \eta^2 w_2) - 6h(\eta w_1 - \eta^2 w_2) \tag{18}$$

for all $w_1, w_2 \in W$. Replacing ηw_1 and $\eta^2 w_2$ by $\eta w_1 + \eta^2 w_2$ and $2\eta^2 w_2$ in (2), respectively, we get

$$4h(\eta w_1 + \eta^2 w_2) + 4h(\eta w_1) = h(\eta w_1 + 3\eta^2 w_2) + h(\eta w_1 - \eta^2 w_2) + 6h(\eta w_1 + \eta^2 w_2) - 24h(\eta^2 w_2) \tag{19}$$

for all $w_1, w_2 \in W$. Interchanging ηw_1 and $\eta^2 w_2$ in (19), we obtain

$$4h(\eta^2 w_2 + 2\eta w_1) + 4h(\eta^2 w_2) = h(\eta^2 w_2 + 3\eta w_1) + h(\eta^2 w_2 - \eta w_1) + 6h(\eta^2 w_2 + \eta w_1) - 24h(\eta w_1) \tag{20}$$

for all $w_1, w_2 \in W$. Adding (19) and (20) and using (18), we get

$$\begin{aligned}
&4h(\eta w_1 + \eta^2 w_2) + 4h(\eta w_1) + 4h(\eta^2 w_2 + 2\eta w_1) + 4h(\eta^2 w_2) = h(\eta^2 w_2 + 3\eta w_1) + h(\eta^2 w_2 - \eta w_1) \\
&+ 6h(\eta^2 w_2 + \eta w_1) - 24h(\eta w_1) + h(\eta w_1 + 3\eta^2 w_2) + h(\eta w_1 - \eta^2 w_2) + 6h(\eta w_1 + \eta^2 w_2) - 24h(\eta^2 w_2)
\end{aligned}$$

for all $w_1, w_2 \in W$. Now, using Equation (18), we get

$$\begin{aligned}
&4h(\eta w_1 + \eta^2 w_2) + 4h(\eta^2 w_2 + 2\eta w_1) = 64h(\eta w_1) + 64h(\eta^2 w_2) + 24h(\eta w_1 + \eta^2 w_2) \\
&- 6h(\eta w_1 - \eta^2 w_2) + 12h(\eta w_1 + \eta^2 w_2) + h(\eta w_1 - \eta^2 w_2) + h(\eta^2 w_2 - \eta w_1) - 28h(\eta w_1) - 28h(\eta^2 w_2)
\end{aligned}$$

for all $w_1, w_2 \in W$. Again using $h(-w) = h(w)$, we have

$$h(\eta w_1 + 2\eta^2 w_2) + h(\eta^2 w_2 + 2\eta w_1) = 9h(\eta w_1) + 9h(\eta^2 w_2) + 9h(\eta w_1 + \eta^2 w_2) - h(\eta w_1 - \eta^2 w_2) \tag{21}$$

for all $w_1, w_2 \in W$. Replacing $z = \eta w_1$ and $x = \eta^3 w_3$ in (2), we have

$$h(2\eta w_1 + \eta^3 w_3) + h(2\eta w_1 - \eta^3 w_3) = 24h(\eta w_1) - 6h(\eta^3 w_3) + 4h(\eta w_1 + \eta^3 w_3) + 4h(\eta w_1 - \eta^3 w_3) \tag{22}$$

for all $w_1, w_2 \in W$. Substituting $z = \eta^2 w_2$ and $x = \eta^3 w_3$ in (2), we get

$$h(2\eta^2 w_2 + \eta^3 w_3) + h(2\eta^2 w_2 - \eta^3 w_3) = 24h(\eta^2 w_2) - 6h(\eta^3 w_3) + 4h(\eta^2 w_2 + \eta^3 w_3) + 4h(\eta^2 w_2 - \eta^3 w_3) \tag{23}$$

for all $w_1, w_2 \in W$. Adding (22) and (23), we obtain

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) + 9h(2\eta^2 w_2 - \eta^3 w_3) \\
&= 36h(\eta w_1 + \eta^3 w_3) + 36h(\eta w_1 - \eta^3 w_3) + 36h(\eta^2 w_2 + \eta^3 w_3) + 36h(\eta^2 w_2 - \eta^3 w_3) \\
&+ 216h(\eta w_1) + 216h(\eta^2 w_2) - 108h(\eta^3 w_3),
\end{aligned} \tag{24}$$

for all $w_1, w_2, w_3 \in W$. Interchanging $\eta w_1 = 2\eta w_1 + \eta^3 w_3$ and $\eta^2 x^2 = 2\eta^2 w_2 + \eta^3 w_3$ in Equation (21), we have

$$\begin{aligned}
&h(2\eta w_1 + 4\eta^2 w_2 + 3\eta^3 w_3) + h(4\eta w_1 + 2\eta^2 w_2 + 3\eta^3 w_3) = 9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) \\
&+ 9h(2\eta w_1 + 2\eta^2 w_2 + 2\eta^3 w_3) - h(2\eta w_1 - 2\eta^2 w_2),
\end{aligned} \tag{25}$$

for all $w_1, w_2, w_3 \in W$. Substituting ηw_1 and $\eta^2 w_2$ by $2\eta w_1 - \eta^3 w_3$ and $\eta^2 w_2 = 2\eta^2 w_2 - \eta^3 w_3$
in (21), we get

$$\begin{aligned}
&h(2\eta w_1 + 4\eta^2 w_2 - 3\eta^3 w_3) + h(4\eta w_1 + 2\eta^2 w_2 - 3\eta^3 w_3) = 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 - \eta^3 w_3) \\
&+ 9h(2\eta w_1 + 2\eta^2 w_2 - 2\eta^3 w_3) - h(2\eta w_1 - 2\eta^2 w_2),
\end{aligned} \tag{26}$$

for all $w_1, w_2, w_3 \in W$. Adding (25) and (26) and using (2), we get

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2r w_2 + \eta^3 w_3) = 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 - \eta^3 w_3) \\
&= 4h(\eta w_1 + 2\eta^2 w_2 + 3\eta^3 w_3) + 4h(\eta w_1 + 2\eta^2 w_2 - 3\eta^3 w_3) + 24h(\eta w_1 + 2\eta^2 w_2) \\
&- 6h(3\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 + 3\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 - 3\eta^3 w_3) + 24h(\eta w_1 + \eta^2 w_2) \\
&- 6h(3\eta^3 w_3) - 144h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) \\
&- 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) + 32h(\eta w_1 - \eta^2 w_2),
\end{aligned} \tag{27}$$

for all $w_1, w_2, w_3 \in W$. By Equations (24) and (27), we obtain

$$\begin{aligned}
&36h(\eta w_1 + \eta^3 w_3) + 36h(\eta w_1 - \eta^3 w_3) + 216h(\eta w_1) - 54h(\eta^3 w_3) + 36h(\eta^2 w_2 + \eta^3 w_3) \\
&+ 36h(\eta^2 w_2 - \eta^3 w_3) + 216h(\eta^2 w_2) - 54h(3\eta^3 w_3) = 4h(\eta w_1 + 2\eta^2 w_2 + 3\eta^3 w_3) \\
&+ 4h(\eta w_1 + 2\eta^2 w_2 - 3\eta^3 w_3) + 24h(\eta w_1 + 2\eta^2 w_2) - 6h(3\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 + \eta^3 w_3) \\
&+ 4h(2\eta w_1 + \eta^2 w_2 - 3\eta^3 w_3) + 24h(2\eta w_1 + \eta^2 w_2) - 6h(3\eta^3 w_3) - 144h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) \\
&- 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) + 32h(\eta w_1 - \eta^2 w_2),
\end{aligned} \tag{28}$$

for all $w_1, w_2, w_3 \in W$. Substituting ηw_1 and $\eta^2 w_2$ by $2\eta w_1 + \eta^3 w_3$ and $\eta^2 w_2 = 2\eta^2 w_2 - \eta^3 w_3$ in (21), we have

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) = h(2\eta w_1 + 4\eta^2 w_2 - \eta^3 w_3) + h(4\eta w_1 + 2\eta^2 w_2 + \eta^3 w_3) \\
&- 9h(2\eta w_1 + 2\eta^2 w_2) + h(2\eta w_1 - 2\eta^2 w_2 + 2\eta^3 w_3),
\end{aligned} \tag{29}$$

for all $w_1, w_2, w_3 \in W$. Putting $\eta w_1 = 2\eta w_1 - \eta^3 w_3$ and $\eta^2 w_2 = 2\eta^2 w_2 + \eta^3 w_3$ in (21), we have

$$\begin{aligned}
&9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) = h(2\eta w_1 + 4\eta^2 w_2 + \eta^3 w_3) \\
&+ h(4\eta w_1 + 2\eta^2 w_2 - \eta^3 w_3) - 9h(2\eta w_1 + 2\eta^2 w_2) + h(2\eta w_1 - 2\eta^2 w_2 - 2\eta^3 w_3),
\end{aligned} \tag{30}$$

for all $w_1, w_2, w_3 \in W$. Adding (29) and (30), we get

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) \\
&= h(2\eta w_1 + 4\eta^2 w_2 - \eta^3 w_3) + h(4\eta w_1 + 2\eta^2 w_2 + \eta^3 w_3) - 9h(2\eta w_1 + 2\eta^2 w_2) \\
&+ h(2\eta w_1 - 2\eta^2 w_2 + 2\eta^3 w_3) + h(2\eta w_1 + 4\eta^2 w_2 + \eta^3 w_3) + h(4\eta w_1 + 2\eta^2 w_2 - \eta^3 w_3) \\
&- 9h(2\eta w_1 + 2\eta^2 w_2) + h(2\eta w_1 - 2\eta^2 w_2 - 2\eta^3 w_3),
\end{aligned} \tag{31}$$

for all $w_1, w_2, w_3 \in W$. Using (2) in the above Equation (31), we get

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta^2 w_2 - \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) \\
&= 4h(\eta w_1 + 2\eta^2 w_2 + \eta^3 w_3) + 4h(\eta w_1 + 2\eta^2 w_2 - \eta^3 w_3) + 24h(\eta w_1 + 2\eta^2 w_2) \\
&- 6h(\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 + \eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 - \eta^3 w_3) + 24h(2\eta w_1 + \eta^2 w_2) \\
&- 6h(\eta^3 w_3) - 288h(\eta w_1 + \eta^2 w_2) \\
&+ 16h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) + 16h(\eta w_1 - \eta^2 w_2 - \eta^3 w_3),
\end{aligned} \tag{32}$$

for all $w_1, w_2, w_3 \in W$. Replacing $\eta^3 w_3$ by $3\eta^3 w_3$ in (32), we get

$$\begin{aligned}
&9h(2\eta w_1 + 3\eta^3 w_3) + 9h(2\eta^2 w_2 - 3\eta^3 w_3) + 9h(2\eta w_1 - 3\eta^3 w_3) + 9h(2\eta^2 w_2 + 3\eta^3 w_3) \\
&= 4h(\eta w_1 + 2\eta^2 w_2 + 3\eta^3 w_3) + 4h(\eta w_1 + 2\eta^2 w_2 - 3\eta^3 w_3) + 24h(\eta w_1 + 2\eta^2 w_2) \\
&- 6h(3\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 + 3\eta^3 w_3) + 4h(2\eta w_1 + \eta^2 w_2 - 3\eta^3 w_3) + 24h(2\eta w_1 + \eta^2 w_2) \\
&- 6h(3\eta^3 w_3) - 288h(\eta w_1 + \eta^2 w_2) \\
&+ 16h(\eta w_1 - \eta^2 w_2 + 3\eta^3 w_3) + 16h(\eta w_1 - \eta^2 w_2 - 3\eta^3 w_3),
\end{aligned} \tag{33}$$

for all $w_1, w_2, w_3 \in W$. Using (28) in (33), we have

$$\begin{aligned}
&9h(2\eta w_1 + \eta^3 w_3) + 9h(2\eta^2 w_2 - \eta^3 w_3) + 9h(2\eta w_1 - \eta^3 w_3) + 9h(2\eta^2 w_2 + \eta^3 w_3) \\
&= 36h(\eta w_1 + \eta^3 w_3) + 36h(\eta w_1 - \eta^3 w_3) + 216h(\eta w_1) - 54h(\eta^3 w_3) + 36h(\eta^2 w_2 + \eta^3 w_3) \\
&+ 36h(\eta^2 w_2 - \eta^3 w_3) + 216h(\eta^2 w_2) - 54h(\eta^3 w_3) + 144h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) \\
&+ 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) - 32h(w_1 - \eta^2 w_2) - 288h(w_1 + \eta^2 w_2) \\
&+ 16h(\eta w_1 - \eta^2 w_2 + 3\eta^3 w_3) + 16h(\eta w_1 - \eta^2 w_2 - 3\eta^3 w_3),
\end{aligned} \tag{34}$$

for all $w_1, w_2, w_3 \in W$. Substituting $\eta w_1 = \eta w_1 - \eta^2 w_2 + 3\eta^3 w_3$ and $\eta^2 w_2 = \eta w_1 - \eta^2 w_2 - 3\eta^3 w_3$ in (21), we have

$$\begin{aligned}
&9h(\eta w_1 - \eta^2 w_2 + 3\eta^3 w_3) + 9h(\eta w_1 - \eta^2 w_2 - 3\eta^3 w_3) \\
&= 81h(\eta w_1 - \eta^2 w_2 - \eta^3 w_3) + 81h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) \\
&- 144h(\eta w_1 - \eta^2 w_2) + 1296h(\eta^3 w_3),
\end{aligned} \tag{35}$$

for all $w_1, w_2, w_3 \in W$. Dividing $\left(\frac{16}{9}\right)$ on both sides of the last inequality, we get

$$\begin{aligned}
&16h(\eta w_1 - \eta^2 w_2 + 3\eta^3 w_3) + 16h(\eta w_1 - \eta^2 w_2 - 3\eta^3 w_3) \\
&= 144h(\eta w_1 - \eta^2 w_2 - \eta^3 w_3) + 144h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) \\
&- 256h(\eta w_1 - \eta^2 w_2) + 2304h(\eta^3 w_3),
\end{aligned} \tag{36}$$

for all $w_1, w_2, w_3 \in W$. Substituting (36) in (34), we obtain

$$\begin{aligned}
&9h(2\eta w_1 + 3\eta^3 w_3) + 9h(2\eta^2 w_2 - 3\eta^3 w_3) + 9h(2\eta w_1 - 3\eta^3 w_3) + 9h(2\eta^2 w_2 + 3\eta^3 w_3) \\
&= 36h(\eta w_1 + \eta^3 w_3) + 36h(\eta w_1 - \eta^3 w_3) + 216h(\eta w_1) - 54h(\eta^3 w_3) + 36h(\eta^2 w_2 + \eta^3 w_3) \\
&+ 36h(\eta^2 w_2 - \eta^3 w_3) + 216h(\eta^2 w_2) - 54h(\eta^3 w_3) + 144h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) \\
&+ 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) - 32h(w_1 - \eta^2 w_2) - 288h(w_1 + \eta^2 w_2) \\
&+ 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) + 144h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) + 2304h(\eta^3 w_3),
\end{aligned} \tag{37}$$

for all $w_1, w_2, w_3 \in W$. Replacing $\eta w_1 = 2\eta w_1 + 3\eta^3 w_3$ and $\eta^2 w_2 = 2\eta w_1 - 3\eta^3 w_3$ in (21), we have

$$\begin{aligned}
&h(6\eta w_1 - 3\eta^3 w_3) + h(6\eta w_1 + 3\eta^3 w_3) \\
&= 9h(2\eta w_1 + 3\eta^3 w_3) + 9h(2\eta w_1 - 3\eta^3 w_3) + 92h(4\eta w_1) - h(6\eta^3 w_3),
\end{aligned} \tag{38}$$

for all $w_1, w_2, w_3 \in W$. Substituting $\eta w_1 = 2\eta^2 w_2 - 3\eta^3 w_3$ and $\eta^2 w_2 = 2\eta^2 w_2 + 3\eta^3 w_3$ in (21), we get

$$\begin{aligned}
&h(6\eta^2 w_2 + 3\eta^3 w_3) + h(6\eta^2 w_2 - 3\eta^3 w_3) \\
&= 9h(2\eta^2 w_2 - 3\eta^3 w_3) + 9h(2\eta^2 w_2 + 3\eta^3 w_3) + 9h(4\eta^2 w_2) - h(6\eta^3 w_3),
\end{aligned} \tag{39}$$

for all $w_1, w_2, w_3 \in W$. Adding (38) and (39), we have

$$\begin{aligned}
&9h(2\eta w_1 + 3\eta^3 w_3) + 9h(2\eta^2 w_2 - 3\eta^3 w_3) + 9h(2\eta^2 w_2 - 3\eta^3 w_3) + 9h(2\eta^2 w_2 + 3\eta^3 w_3) \\
&= 324h(\eta w_1 + \eta^3 w_3) + 324h(\eta w_1 - \eta^3 w_3) + 1944h(\eta w_1) - 486h(\eta^3 w_3) \\
&+ 324h(\eta^2 w_2 + \eta^3 w_3) + 324h(\eta^2 w_2 - \eta^3 w_3) + 1944h(\eta^2 w_2) - 486h(\eta^3 w_3) - 2304h(\eta w_1) \\
&- 2304h(\eta^2 w_2) - 2592h(\eta^3 w_3),
\end{aligned} \tag{40}$$

for all $w_1, w_2, w_3 \in W$. From (37) and (40), the left-hand sides are equal, and we get

$$\begin{aligned}
&36h(\eta w_1 + \eta^3 w_3) + 36h(\eta^2 w_2 - \eta^3 w_3) + 216h(\eta w_1) - 54h(\eta^3 w_3) + 36h(\eta^2 w_2 - \eta^3 w_3) \\
&+ 216h(\eta^2 w_2) - 54h(\eta^3 w_3) + 144h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) + 144h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) \\
&- 32h(\eta w_1 - \eta^2 w_2) - 288h(\eta w_1 + \eta^2 w_2) + 144h(\eta w_1 - \eta^2 w_2 - \eta^3 w_3) \\
&+ 144h(\eta w_1 - \eta^2 w_2 + \eta^3 w_3) - 256h(\eta w_1 - \eta^2 w_2) + 2304h(\eta w_1 - \eta^2 w_2) + 2304h(\eta^3 w_3) \\
&= 324h(\eta w_1 + \eta^3 w_3) + 324h(\eta w_1 - \eta^3 w_3) + 1944h(\eta w_1) - 486h(\eta^3 w_3) \\
&+ 324h(\eta^2 w_2 + \eta^3 w_3) + 324h(\eta^2 w_2 - \eta^3 w_3) + 1944h(\eta^2 w_2) - 486h(\eta^3 w_3) - 2304h(\eta w_1) \\
&- 2304h(\eta^2 w_2) + 2592h(\eta^3 w_3),
\end{aligned} \tag{41}$$

for all $w_1, w_2, w_3 \in W$. From the resultant Equation (41), we get

$$\begin{aligned}
&h(\eta w_1 + \eta^2 w_2 + \eta^3 w_3) + h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) + h(\eta w_1 - \eta^2 w_2 - \eta^3 w_3) \\
&+ h(\eta w_1 + \eta^2 w_2 - \eta^3 w_3) = 2\Big(h(\eta w_1 + \eta^2 w_2) + h(\eta w_1 + \eta^3 w_3) + h(\eta^2 w_2 + \eta^3 w_3)\Big) \\
&+ 2\Big(h(\eta w_1 - \eta^2 w_2) + h(\eta w_1 - \eta^3 w_3) + h(\eta^2 w_2 - \eta^3 w_3)\Big) \\
&- 4\Big(h(\eta w_1) + h(\eta^2 w_2) + h(\eta^3 w_3)\Big),
\end{aligned} \tag{42}$$

for all $w_1, w_2, w_3 \in W$. Adding $\eta h(w_1) + \eta^2 h(w_2) + \eta^3 h(w_3)$ on the two sides of (42) and using the evenness of h, we obtain our desired result.

Conversely, $h : W \to X$ satisfies the functional Equation (3). Using the evenness of h in (3), we have

$$\begin{aligned}&h(\eta w_1+\eta^2w_2+\eta^3w_3)+h(-\eta w_1+\eta^2w_2+\eta^3w_3)+h(\eta w_1-\eta^2w_2+\eta^3w_3)\\&+h(\eta w_1+\eta^2w_2-\eta^3w_3)=2\Big(h(\eta w_1+\eta^2w_2)+h(\eta^2w_2+\eta^3w_3)+h(\eta w_1+\eta^3w_3)\Big)\\&+2\Big(h(\eta^2w_2-\eta w_1)+h(\eta^3w_3-\eta^2w_2)+h(\eta w_1-\eta^3w_3)\Big)\\&-4\eta^4h(w_1)-4\eta^8h(w_2)-4\eta^{12}h(w_3),\end{aligned} \tag{43}$$

for all $w_1, w_2, w_3 \in W$. Setting (w_1, w_2, w_3) by $(0, 0, 0)$ in (43), we get $h(0) = 0$. Replacing (w_1, w_2, w_3) by $(w, 0, 0)$, $(0, w, 0)$ and $(0, 0, w)$ in (43), we get

$$h(\eta w)=\eta^4h(w), h(\eta^2w)=\eta^8h(w) \text{ and } h(\eta^3w)=\eta^{12}h(w) \tag{44}$$

for all $w \in W$. It is easy to verify from (44) that

$$h\left(\frac{w}{a_i}\right)=\frac{1}{a_i}h(w),\ \ i=1,2,3 \tag{45}$$

for all $w \in W$. Replacing (w_1, w_2, w_3) by $\left(\frac{w}{\eta}, \frac{w}{\eta^2}, \frac{x}{\eta^3}\right)$ in (43), we get

$$\begin{aligned}&h(2w+x)+2h(x)+h(2w-x)=2h(2w)+\\&\qquad 4h(w+x)+2h(x-w)+2h(w-x)-8h(w)-4h(x),\end{aligned} \tag{46}$$

for all $w, x \in W$. Using the evenness of h in (46), we obtain

$$h(2w+x)+h(2w-x)=4h(w+x)+4h(w-x)+24h(w)-6h(x), \tag{47}$$

for all $w, x \in W$. Therefore, h is quartic. □

4. Stability Results for (3) (Direct Method)

In this section, we present the generalized Hyers–Ulam stability of the functional Equation (3). Throughout this section, let us consider W to be a normed space and X a Banach space. Define a mapping $A : W \to X$ defined by

$$\begin{aligned}A(w_1,w_2,w_3)=&\,h(\eta w_1+\eta^2w_2+\eta^3w_3)+h(-\eta w_1+\eta^2w_2+\eta^3w_3)\\&+h(\eta w_1-\eta^2w_2+\eta^3w_3)+h(\eta w_1+\eta^2w_2-\eta^3w_3)\\&-2\Big[h(\eta w_1+\eta^2w_2)+h(\eta^2w_2+\eta^3w_3)+h(\eta w_1+\eta^3w_3)\\&\qquad+h(\eta w_1-\eta^2w_2)+h(\eta^2w_2-\eta^3w_3)+h(\eta w_1-\eta^3w_3)\Big]\\&+2\Big[\eta^4(h(w_1)+h(-w_1))+\eta^8(h(w_2)+h(-w_2))+\eta^{12}(h(w_3)+h(-w_3))\Big]\\&+\Big[\eta(h(w_1)-h(-w_1))+\eta^2(h(w_2)-h(-w_2))+\eta^3(h(w_3)-h(-w_3))\Big].\end{aligned}$$

Lemma 3. *Let h be a solution of the quartic functional Equation (3), $j \in \{-1, 1\}$ and $\beta : W^3 \to [0, \infty)$ be a function such that*

$$\lim_{k\to\infty}\frac{\beta(\eta^{kj}w_1,\eta^{kj}w_2,\eta^{kj}w_3)}{\eta^{kj}}=0,$$

for all $w_1, w_2, w_3 \in W$. Moreover, let $A : W \to X$ be a function satisfying the inequality

$$\|A(w_1,w_2,w_3)\|\le\beta(w_1,w_2,w_3)$$

for all $w_1, w_2, w_3 \in W$. Then, there exists a unique additive function $B : W \to X$ such that

$$\|h(w) - B(w)\| \leq \frac{1}{2\eta} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\mu(\eta^k w)}{\eta^k},$$

where $\mu(w) = \beta(w, 0, 0)$. The mapping $B(w)$ is defined by

$$B(w) = \lim_{l \to \infty} \frac{h(\eta^l w)}{\eta^l}$$

for all $w \in W$.

Corollary 1. *Let ς and t be non-negative real numbers. Then, there exists a function $A : W \to X$ satisfying the inequality*

$$\|A(w_1, w_2, w_3)\| \leq \begin{cases} \varsigma, \\ \varsigma\left\{\sum_{i=1}^{3} \|w_i\|^t\right\}, \\ \varsigma\left\{\prod_{i=1}^{2} \|w_i\|^t + \sum_{i=1}^{3} \|w_i\|^{3t}\right\} \end{cases}$$

for all $w_1, w_2, w_3 \in W$. Then, there exists $B : W \to X$, which is a unique additive function such that

$$\|h(w) - B(w)\| \leq \begin{cases} \frac{\varsigma}{2|p-1|} \\ \frac{\varsigma\|z\|^t}{2|p-\eta^s|}; & t \neq 1 \\ \frac{\varsigma\|z\|^{3t}}{2|p-\eta^{3s}|}; & t \neq \frac{1}{3} \end{cases}$$

for all $w \in W$.

Theorem 1. *Let h be a solution of the quartic functional Equation (3) and $j \in \{-1, 1\}$. Let $\beta : W^3 \to [0, \infty)$ be a function such that*

$$\lim_{k \to \infty} \frac{\beta(\eta^{kj} w_1, \eta^{kj} w_2, \eta^{kj} w_3)}{\eta^{4kj}} = 0, \tag{48}$$

for all $w_1, w_2, w_3 \in W$ and let $Q : W \to X$ be a function fulfilling the inequality

$$\|Q(w_1, w_2, w_3)\| \leq \beta(w_1, w_2, w_3), \tag{49}$$

for all $w_1, w_2, w_3 \in W$. Then, there exists a unique additive function $B : W \to X$ such that

$$\|h(w) - R(w)\| \leq \frac{1}{4\eta^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\mu(\eta^k w)}{\eta^{4k}}, \tag{50}$$

for all $w \in W$, where $\mu(w) = \beta(w, 0, 0)$. The mapping $R(w)$ is defined by

$$R(w) = \lim_{l \to \infty} \frac{h(\eta^l w)}{\eta^{4l}}, \tag{51}$$

for all $w \in W$.

Proof. Consider that $j = 1$. Replacing (w_1, w_2, w_3) by $(w, 0, 0)$ in (49), we get

$$\left\|4\eta^4 h(w) - 4h(\eta w)\right\| \leq \beta(w, 0, 0), \tag{52}$$

for all $w \in W$. It follows from (51) that

$$\left\| \frac{h(\eta w)}{\eta^4} - h(w) \right\| \le \frac{1}{4\eta^4} \beta(w,0,0), \tag{53}$$

for all $w \in W$. Now, setting w by ηw and dividing by η^4 in (53), we arrive at

$$\left\| \frac{h(\eta^2 w)}{\eta^8} - \frac{h(\eta w)}{\eta^4} \right\| \le \frac{1}{4\eta^8} \beta(\eta z,0,0), \tag{54}$$

for all $w \in W$. Adding (53) and (54), we have

$$\left\| \frac{h(\eta^2 w)}{\eta^8} - h(w) \right\| \le \frac{1}{4\eta^4} \left[\beta(w,0,0) + \frac{\beta(\eta z,0,0)}{\eta^4} \right],$$

for all $w \in W$. In general, for any positive integer l, one can easily verify that

$$\left\| \frac{h(\eta^l w)}{\eta^{4l}} - h(w) \right\| \le \frac{1}{4\eta^4} \sum_{k=0}^{l-1} \frac{\mu(\eta^k w)}{\eta^{4k}},$$

for all $w \in W$ and

$$\left\| \frac{h(\eta^l w)}{\eta^{4l}} - h(w) \right\| \le \frac{1}{4\eta^4} \sum_{k=0}^{\infty} \frac{\mu(\eta^k w)}{\eta^{4k}}, \tag{55}$$

for all $w \in W$. To prove the convergence of the sequence $\left\{ \frac{h(\eta^l w)}{\eta^{4l}} \right\}$, replacing w by $\eta^c w$ and dividing η^{4c} in (55), we get

$$\left\| \frac{h(\eta^{l+c} w)}{\eta^{4(l+c)}} - \frac{h(\eta^c w)}{\eta^{4c}} \right\| \le \frac{1}{4\eta^4} \sum_{k=0}^{l-1} \frac{\mu(\eta^{k+c} w)}{\eta^{4(k+c)}} \to 0 \text{ as } c \to \infty, \tag{56}$$

for all $l, c > 0, w \in W$. Consequently, $\left\{ \frac{h(\eta^l w)}{\eta^{4l}} \right\}$ is a Cauchy arrangement. Since X is Banach space, there exists a mapping $R : W \to X$ to such an extent that

$$R(w) = \lim_{l \to \infty} \frac{h(\eta^l w)}{\eta^{4l}},$$

for all $w \in W$. Replacing $l \to \infty$ in (55), we see that (51) holds for $w \in W$. To prove that R satisfies (3), replacing (w_1, w_2, w_3) by $(\eta^c z, \eta^{2c} z, \eta^{3c} w)$ and partitioning η^{4c} in (49), we find

$$\frac{1}{\eta^{4c}} \left\| Q(\eta^c w, \eta^{2c} w, \eta^{3c} w) \right\| \le \frac{1}{\eta^{4c}} \beta(\eta^c w, \eta^{2c} w, \eta^{3c} w)$$

for all $w_1, w_2, w_3 \in W$. Let $c \to \infty$ in the above inequality and the value of $R(w)$, we find that $R(w_1, w_2, w_3) = 0$. Thus, R satisfies (3) for all $w_i \in W; i = 1,2,3$ to show that R is unique. Let $S(w)$ be another quartic mapping satisfying (3) and (51), such that

$$\begin{aligned} \|R(w) - S(w)\| &\le \frac{1}{\eta^{4c}} \{ \|R(\eta^c w) - h(\eta^c w)\| + \|h(\eta^c w) - S(\eta^c w)\| \} \\ &\le \frac{1}{4\eta^4} \sum_{l=0}^{\infty} \sum_{l=0}^{\infty} \frac{\mu(\eta^{l+c} w)}{\eta^{4(l+c)}} \to 0 \text{ as } c \to \infty, \end{aligned}$$

for all $w \in W$ and, also, R is unique. For $j = -1$, we obtain proof similar to that of Theorem 1. □

Corollary 2. *Let ς and t be non-negative real numbers. The mapping $A : W \to X$ satisfying the disparity*

$$\|A(w_1, w_2, w_3)\| \leq \begin{cases} \varsigma, \\ \varsigma\Big\{\sum_{i=1}^{3} \|w_i\|^t\Big\}, \\ \varsigma\Big\{\prod_{i=1}^{2} \|w_i\|^t + \sum_{i=1}^{3} \|w_i\|^{3t}\Big\} \end{cases} \tag{57}$$

for all $w_1, w_2, w_3 \in W$. Then, there exists a unique additive mapping $R : W \to X$ such that

$$\|h(w) - R(w)\| \leq \begin{cases} \frac{\varsigma}{4|\eta^4-1|} \\ \frac{\varsigma\|w\|^t}{4|\eta^4-\eta^t|}; \quad t \neq 4 \\ \frac{\varsigma\|w\|^{3t}}{4|\eta^4-\eta^{3t}|}; \quad t \neq \frac{4}{3} \end{cases} \tag{58}$$

for all $w \in W$.

5. Stability Result for (3) (Fixed Point Method)

In this section, we investigate the generalized Ulam–Hyers stability of the functional Equation (3) fixed point method.

Lemma 4. *Let $A : W \to X$ be mapping of this. There is a function $\beta : w^3 \to [0, \infty)$ with the conditions*

$$\lim_{k\to\infty} \frac{\beta(\eta_i^k w_1, \eta_i^k w_2, \eta_i^k w_3)}{\eta_i^k} = 0$$

where

$$\eta_i = \begin{cases} \eta, & i = 0; \\ \frac{1}{\eta}, & i = 1 \end{cases}$$

fulfilling the functional inequality

$$\|A(w_1, w_2, w_3)\| \leq \beta(w_1, w_2, w_3),$$

for all $w_1, w_2, w_3 \in W$. If there exists $J = J(i)$ such that the function

$$z \to \gamma(w) = \frac{1}{2}\beta\Big(\frac{w}{s}, 0, 0\Big)$$

has the property

$$\frac{\gamma(\eta_i w)}{\eta_i} = J\gamma(w),$$

for all $w \in W$. Then, there exists a unique additive function $R : W \to X$ fulfilling the functional Equation (3) and

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w)$$

for all $w \in W$.

Corollary 3. *Let ς and t be non-negative real numbers. Then, there exists a mapping $A : W \to X$, satisfying the disparity*

$$\|A(w_1, w_2, w_3)\| \leq \begin{cases} \varsigma, \\ \varsigma\Big\{\sum_{i=1}^{3} \|w_i\|^t\Big\}, \\ \varsigma\Big\{\prod_{i=1}^{2} \|w_i\|^t + \sum_{i=1}^{3} \|w_i\|^{3t}\Big\} \end{cases}$$

for all $w_1, w_2, w_3 \in W$. Then, there exists a unique additive mapping $R : W \to X$ such that

$$\|h(w) - R(w)\| \leq \begin{cases} \frac{\varsigma}{2|\eta-1|} \\ \frac{\varsigma\|w\|^t}{2|\eta-\eta^t|}; & t \neq 1 \\ \frac{\varsigma\|w\|^{3t}}{2|\eta-\eta^{3t}|}; & t \neq \frac{1}{3} \end{cases}$$

for all $w \in W$.

Theorem 2. *Let $Q : W \to X$ be a mapping for this. There is a function $\beta : w^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\beta(\eta_i^k w_1, \eta_i^k w_2, \eta_i^k w_3)}{\eta_i^{4k}} = 0 \tag{59}$$

where

$$\eta_i = \begin{cases} \eta, & i = 0; \\ \frac{1}{\eta}, & i = 1 \end{cases}$$

fulfilling the functional inequality

$$\|Q(w_1, w_2, w_3)\| \leq \beta(w_1, w_2, w_3), \tag{60}$$

for all $w_1, w_2, w_3 \in W$. If there exists $J = J(i)$ such that the function

$$w \to \gamma(w) = \frac{1}{4}\beta\left(\frac{w}{\eta}, 0, 0\right)$$

has the property

$$\frac{\gamma(\eta_i w)}{\eta_i^4} = J\gamma(w), \tag{61}$$

for all $w \in W$. Then, there exists $R : W \to X$ fulfills (3) and

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w), \tag{62}$$

for all $w \in W$.

Proof. Let e be the complete metric on Ω, with the property that

$$e(q, r) = \inf\{k \in (0, \infty) : \|q(w) - r(w)\| \leq k\gamma(w), w \in W\}.$$

It is easy to see that (Ω, e) is complete. Define $S : \Omega \to \Omega$ by $Sh(w) = \frac{1}{\eta_i^4}h(\eta_i w)$ for all $w \in W$. For $q, r \in \Omega$ and $w \in W$, we have

$$\begin{aligned} d(q, r) = k \Rightarrow & \|q(w) - r(w)\| \leq k\gamma(w), \\ \Rightarrow & \left\|\frac{q(\eta_i w)}{\eta_i^4} - \frac{r(\eta_i w)}{\eta_i^4}\right\| \leq \frac{1}{\eta_i^4}k\gamma(\eta_i w), \\ \Rightarrow & \|Sq(w) - Sr(w)\| \leq \frac{1}{\eta_i^4}k\gamma(\eta_i w), \\ \Rightarrow & \|Sq(w) - Sr(w)\| \leq Jk\gamma(w) \\ \Rightarrow & e(Sq(w), Sr(w)) \leq kJ. \end{aligned}$$

This implies $e(Sq, Sr) \leq Je(q, r)$. Hence, S is strictly contractive mapping on Ω with Lipschitz constant J. It is follows from (52) that

$$\|4\eta^4 h(w) - 4h(\eta w)\| \leq \beta(w, 0, 0), \tag{63}$$

for all $w \in W$. It follows from (63) that

$$\|\eta^4 h(w) - h(\eta w)\| \leq \frac{\beta(w,0,0)}{4}, \tag{64}$$

for all $w \in W$. Using the above condition, for $i = 0$, it reduces to

$$\left\|h(w) - \frac{h(\eta w)}{\eta^4}\right\| \leq \frac{1}{\eta^4}\gamma(w) \Rightarrow \|h(w) - Sh(w)\| \leq J\gamma(w),$$

for all $w \in W$. Hence, we get

$$e(Sh(w) - h(w)) \leq J = J^{1-i}, \tag{65}$$

for all $w \in W$. Replacing w by $\frac{w}{\eta}$ in (64), we have

$$\left\|\eta^4 g\left(\frac{w}{\eta} - h(w)\right)\right\| \leq \frac{1}{4}\beta\left(\frac{w}{\eta}, 0, 0\right), \tag{66}$$

for all $w \in W$. Using the above condition, for $i = 1$, we get

$$\left\|\eta^4 g\left(\frac{w}{\eta} - h(w)\right)\right\| \leq \gamma(w) \Rightarrow \|Sh(w) - h(w)\| \leq \gamma(w),$$

for all $w \in W$. Hence, we get

$$e(h(w) - Sh(w)) \leq \eta^4 = J^{1-i}, \tag{67}$$

for all $w \in W$. From (65) and (67), we can conclude

$$e(h(w) - Sh(w)) \leq J^{1-i} < \infty, \tag{68}$$

for all $w \in W$. It follows a fixed point $R : S \to \Omega$, such that

$$R(w) = \lim_{k\to\infty} \frac{h(\eta_i^k w)}{\eta_i^{4k}}, \tag{69}$$

for all $w \in W$. In order to prove that $R : Z \to X$ satisfies the quartic functional Equation (3), the evidence is similar to that of Theorem 1. Since R is a unique fixed point S on the set $\Delta = \{g \in \Omega / e(g, R) < \infty\}$, R is the unique function such that

$$e(g, R) \leq \frac{1}{1-J} e(g, Sg),$$

that is,

$$e(g, R) \leq \frac{J^{1-i}}{1-J}$$

implies that

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w),$$

for all $w \in W$. $\square$

Corollary 4. *Let ς and t be non-negative real numbers. If a function $Q : W \to X$ fulfilling the functional inequality*

$$\|Q(w_1, w_2, w_3)\| \leq \begin{cases} \varsigma, \\ \varsigma\left\{\sum_{i=1}^{3} \|w_i\|^t\right\}, \\ \varsigma\left\{\prod_{i=1}^{2} \|w_i\|^t + \sum_{i=1}^{3} \|w_i\|^{3t}\right\} \end{cases} \tag{70}$$

for all $w_1, w_2, w_3 \in W$. Then, there exists $R : W \to X$ such that

$$\|h(w) - R(w)\| \leq \begin{cases} \frac{\varsigma}{4|\eta^4-1|} \\ \frac{\varsigma\|w\|^t}{4|\eta^4-\eta^t|}; \quad t \neq 4 \\ \frac{\varsigma\|w\|^{3t}}{4|\eta^4-\eta^{3t}|}; \quad t \neq \frac{4}{3} \end{cases} \tag{71}$$

for all $w \in W$.

Proof. Setting

$$\beta(w_1, w_2, w_3) \leq \begin{cases} \varsigma, \\ \varsigma\left\{\sum_{i=1}^{3} \|w_i\|^t\right\}, \\ \varsigma\left\{\prod_{i=1}^{3} \|w_i\|^t + \sum_{i=1}^{3} \|w_i\|^{3t}\right\} \end{cases}$$

for all $w_1, w_2, w_3 \in W$. Now,

$$\frac{\beta(\eta_i^k w_1, \eta_i^k w_2, \eta_i^k w_3)}{\eta_i^{4k}} = \begin{cases} \frac{\varsigma}{\eta_i^{4k}}, \\ \frac{\varsigma}{\eta_i^{4k}}\left\{\sum_{i=1}^{3} \|\eta_i w_i\|^t\right\}, \\ \frac{\varsigma}{\eta_i^{4k}}\left\{\prod_{i=1}^{3} \|\eta_i w_i\|^t + \sum_{i=1}^{3} \|\eta_i w_i\|^{3t}\right\}, \end{cases} = \begin{cases} \to 1 \quad \text{as} \quad k \to \infty \\ \to 1 \quad \text{as} \quad k \to \infty \\ \to 1 \quad \text{as} \quad k \to \infty. \end{cases}$$

Hence, Equation (59) holds. Since we have

$$\gamma(w) = \frac{1}{4}\left(\frac{w}{\eta}, 0, 0\right)$$

and

$$\gamma(w) = \frac{1}{4}\beta\left(\frac{w}{\eta}, 0, 0\right) = \begin{cases} \frac{\varsigma}{4} \\ \frac{\varsigma\|w\|^t}{4q^t} \\ \frac{\varsigma\|w\|^{3t}}{4q^t} \end{cases}.$$

In addition,

$$\frac{1}{\eta_i^4}\gamma(\eta_i w) = \left\{ \begin{array}{l} \frac{1}{\eta_i^4}\frac{\varsigma}{4} \\ \frac{1}{\eta_i^4}\frac{\varsigma\|w\|^t\eta_i^t}{4\eta^t} \\ \frac{1}{\eta_i^4}\frac{\varsigma\|w\|^{3t}\eta_i^t}{4\eta^{3t}} \end{array} \right\} = \left\{ \begin{array}{l} \eta_i^{-4}\gamma(w) \\ \eta_i^{t-4}\gamma(w) \\ \eta_i^{3t-4}\gamma(w) \end{array} \quad \forall w \in W. \right\}.$$

Therefore, the inequality (61) holds for the following cases:

Case 1: Let $J = \eta^{-4}$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{\eta^{-4}}{1-\eta^{-4}} = \frac{\varsigma}{4(\eta^4-1)}, \quad \text{for} \quad i = 0.$$

Case 2: Let $J = \eta^4$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{1}{1-\eta^{-4}}\frac{\varsigma}{4} = \frac{\varsigma}{4(1-\eta^4)} \quad \text{for} \quad i = 0$$

Case 3: Let $J = \eta^{t-4}$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{\eta^{t-4}}{1-\eta^{t-4}}\frac{\varsigma\|w\|^t}{4\eta^t} = \frac{\varsigma\|w\|^t}{4(\eta^4 - \eta^t)} \quad \text{for} \quad t < 4, i = 0.$$

Case 4: Let $J = \eta^{4-t}$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{1}{1-\eta^{4-t}}\frac{\varsigma\|w\|^t}{4\eta^t} = \frac{\varsigma\|w\|^t}{4(\eta^t - \eta^4)} \quad \text{for} \quad t > 4, i = 1.$$

Case 5: Let $J = \eta^{3t-4}$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{\eta^{3t-4}}{1-\eta^{3t-4}}\frac{\varsigma\|w\|^{3t}}{4\eta^{3t}} = \frac{\varsigma\|w\|^{3t}}{4(\eta^4 - \eta^{3t})} \quad \text{for} \quad t < \frac{4}{3}, i = 0.$$

Case 6: Let $J = \eta^{4-3t}$

$$\|h(w) - R(w)\| \leq \frac{J^{1-i}}{1-J}\gamma(w) = \frac{1}{1-\eta^{4-3t}}\frac{\varsigma\|w\|^{3t}}{4\eta^{3t}} = \frac{\varsigma\|w\|^{3t}}{4(4^{3t} - \eta^4)} \quad \text{for} \quad t > \frac{4}{3}, i = 1.$$

Hence, the proof is completed. □

6. Conclusions

In this paper, we have introduced the mixed-type additive–quartic functional Equation (3) and have obtained the general solution of the mixed-type additive–quartic functional Equation (3). Furthermore, we have proven the generalized Hyers–Ulam stability for the mixed-type additive–quartic functional Equation (3) in Banach space using direct and fixed point methods.

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