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A Note on the Paired-Domination Subdivision Number of Trees

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Abstract: For a graph G with no isolated vertex, let $\gamma_{pr}(G)$ and $\mathrm{sd}_{\gamma_{pr}}(G)$ denote the paired-domination and paired-domination subdivision numbers, respectively. In this note, we show that if T is a tree of order $n \geq 4$ different from a healthy spider (subdivided star), then $\mathrm{sd}_{\gamma_{pr}}(T) \leq \min\{\frac{\gamma_{pr}(T)}{2} + 1, \frac{n}{2}\}$, improving the $(n-1)$-upper bound that was recently proven.

Keywords: paired-domination number; paired-domination subdivision number

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1. Introduction

Throughout the paper, $G = (V, E)$ is a simple connected graph with vertex set $V = V(G)$ of *order* $n = |V|$ and edge set $E(G) = E$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = |N_G(v)|$. When no confusion arises, we will delete the subscript G in N_G and $\deg_G$. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*. A stem is said to be *strong* if it is adjacent to at least two leaves. A *healthy spider* S_q for $q \geq 2$ is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex. The center vertex of a healthy spider will be called a *head*. Let P_n and C_n be the *path* and *cycle* of order n. The *diameter* of G, denoted by $diam(G)$, is the maximum value among minimum distances between all pairs of distinct vertices of G. A *matching* in a graph G is a set of pairwise non-intersecting edges, while a *perfect matching* in G is a matching that covers each vertex.

A *dominating set* of G is a subset S of V such that every vertex in $V - S$ has at least one neighbor in S. A subset S of V is a *paired-dominating set* of G, abbreviated PD-set, if S is a dominating set and the subgraph induced by the vertices of S contains a perfect matching. The *paired-domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a PD-set of G. If S is a PD-set with a perfect matching M, then two vertices u and v are said to be *partners (or paired)* in S if the edge $uv \in M$. We call a PD-set of minimum cardinality a $\gamma_{pr}(G)$*-set*. Note that every graph G without isolated vertices has a PD-set since the endvertices of any maximal matching in G form such a set. Paired-domination was introduced by Haynes and Slater [1] and is studied, for example, in [2–7]. For more details on paired-domination, we refer the reader to the recent book chapter [8].

As an application, in the design of networks for example, it is essential to study the effect that some modifications on the graph that have on the graph parameters. These modifications can be deletion or addition of vertices, deletion or addition of edges. We refer the reader to chapter 7 of [9] when the graph parameter is the domination number. Fink et al. [10], were the first to study the bondage number of G defined to be the minimum number of edges whose removal increases the domination number of G, while Kok and Mynhardt [11] were the first to study the reinforcement number of G defined to be the minimum number of edges which must be added to G in order to decrease the domination number of G. In [12], Velammal studied another kind of modification where the goal is find the minimum number of edges to be subdivided (each edge in G is subdivided at most once) in order to increase the domination number. For more details, see [13–17].

Our main purpose in this paper is to study of the *paired-domination subdivision number* of trees. This parameter was introduced by Favaron et al. in [18] and defined as follows. The *paired-domination subdivision number* $\mathrm{sd}_{\gamma_{pr}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the paired-domination number of G. Observe that since the paired-domination subdivision number of the complete graph K_2 remains unchanged when its only edge is subdivided, we will assume that the graph G has order at least 3. It is worth noting that it has recently been shown by Amjadi and Chellali [19] that the problem of computing the paired-domination subdivision number is NP-hard for bipartite graphs. The paired-domination subdivision number has been further studied by several authors (see [20–22]).

In [18], Favaron et al. have given some conditions for a graph (including trees) to have a small paired-domination subdivision number that we summarize by the following results.

Proposition 1 ([18])**.** *For every graph G of order $n \geq 3$, if $\gamma_{pr}(G) = 2$, then $1 \leq \mathrm{sd}_{\gamma_{pr}}(G) \leq 3$.*

Proposition 2 ([18])**.** *If G contains either a strong stem or adjacent stems, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2$.*

Proposition 3 ([18])**.** *If a connected graph G contains a path $v_1v_2v_3v_4v_5$ in which $\deg(v_i) = 2$ for $i = 2, 3, 4$, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq 4$.*

It should also be noted that Favaron et al. [18] conjectured that $\mathrm{sd}_{\gamma_{pr}}(G) \leq n - 1$ for all connected graphs of order n. In connection with this conjecture, Egawa et al. [20] proved that for every connected graph G of order $n \geq 4$, $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2n - 5$. Moreover, if further G has an edge uv such that u and v are not partners in any $\gamma_{pr}(G)$-set, then $\mathrm{sd}_{\gamma_{pr}}(G) \leq n - 1$. The conjecture has recently been settled in the affirmative in [22]. Restricted to the class of trees, we observe that for healthy spiders S_q with $q \geq 2$ or paths P_3, $\mathrm{sd}_{\gamma_{pr}}(T) = n - 1$.

In this note, we improve the $(n-1)$-upper bound on the paired-domination subdivision number for all trees T of order $n \geq 4$ different from a healthy spider by providing an upper bound on it in terms of the paired-domination number. More precisely, we will mainly show the following.

Theorem 1. *Let T be a tree of order $n \geq 4$ different from a healthy spider. Then $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{\gamma_{pr}(T)}{2} + 1$.*

In addition, we will also show that if T is a tree of order $n \geq 4$ different from a healthy spider, then its paired-domination subdivision number is at most $\frac{n}{2}$. Before giving the proof of our results, it is necessary to recall the following two useful results.

Proposition 4 ([18])**.** *Let G be a connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is obtained from G by subdividing the edge e, then $\gamma_{pr}(G') \geq \gamma_{pr}(G)$.*

Proposition 5 ([18])**.** *For $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{pr}}(P_n) = \mathrm{sd}_{\gamma_{pr}}(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 4 \\ 4 & \text{if} \quad n \equiv 1 \pmod 4 \\ 3 & \text{if} \quad n \equiv 2 \pmod 4 \\ 2 & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

We close this section by mentioning that the paired-domination number of a path P_n of order $n \geq 2$ is $2\lceil \frac{n}{4} \rceil$ (see [8]).

2. Proof of Theorem 1

For non-negative integers t_1, t_2 where $t_1 \geq 1$, let F_{t_1,t_2} be the tree obtained from a path $v_1v_2v_3v_4$ by adding t_1 pendant paths $v_1u_2^iu_1^i$, t_1 pendant paths $v_4w_2^iw_1^i$, and t_2 pendant paths $v_2z_2^iz_1^i$ (see Figure 1). Let $\mathcal{F}$ be the family of all trees F_{t_1,t_2} . The set $P = \{v_1, u_2^1, v_4, w_2^1\} \cup \{u_1^i, u_2^i, w_1^i, w_2^i \mid 2 \leq i \leq t_1\} \cup \{z_1^i, z_2^i \mid 1 \leq i \leq t_2\}$ is a PD-set of F_{t_1,t_2} and so $\gamma_{pr}(F_{t_1,t_2}) \leq 4t_1 + 2t_2$. One the other hand, if D is a $\gamma_{pr}(F_{t_1,t_2})$-set, then to paired-dominate the leaves of F_{t_1,t_2}, we must have $|D \cap \{v_1, u_1^i, u_2^i \mid 1 \leq i \leq t_1\}| \geq 2t_1$, $|D \cap \{v_4, w_1^i, w_2^i \mid 1 \leq i \leq t_1\}| \geq 2t_1$ and $|D \cap \{v_2, z_1^i, z_2^i \mid 1 \leq i \leq t_2\}| \geq 2t_2$ implying that $\gamma_{pr}(F_{t_1,t_2}) \geq 4t_1 + 2t_2$. Thus $\gamma_{pr}(F_{t_1,t_2}) = 4t_1 + 2t_2$.

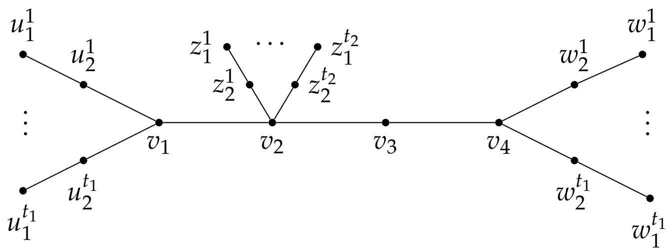

Figure 1. The graph F_{t_1,t_2}.

Lemma 1. *If $T \in \mathcal{F}$, then* $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n(T)-2}{2} = 1 + \frac{\gamma_{pr}(T)}{2}$.

Proof. Let $T = F_{t_1,t_2}$, and let T' be the tree obtained from T by subdividing the edge v_2v_1 with new vertex x, the edges $v_1u_2^i, u_2^iu_1^i$ with new vertices x_i, y_i respectively, for each i, and the edge $v_2z_2^j$ with new vertex a_j for each j, if $t_2 \geq 1$. Clearly the number of subdivided edges is $2t_1 + t_2 + 1 = \frac{n(T)-2}{2}$. Let D' be a $\gamma_{pr}(T')$-set. To paired-dominate each leaf u_1^j, we must have $|D' \cap \{u_1^j, u_2^j, y_j\}| \geq 2$ for each $1 \leq j \leq t_1$; to paired-dominate each leaf z_1^j we must have $|D' \cap \{z_1^j, z_2^j, a_j\}| \geq 2$ for each $1 \leq j \leq t_2$; and to paired-dominate the leaves $w_1^1, \ldots, w_1^{t_1}$ we may assume that $v_4, w_2^1, w_2^2, \ldots, w_2^{t_1}, w_1^2, \ldots, w_1^{t_1} \in D'$. Moreover, to paired-dominate the vertex v_1, we must have $|D' \cap \{x_1, \ldots, x_{t_1}, v_1, v_2, x\}| \geq 2$. Therefore $\gamma_{pr}(T') = |D'| \geq 4t_1 + 2t_2 + 2 > \gamma_{pr}(T)$. Hence $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n(T)-2}{2} = 1 + \frac{\gamma_{pr}(T)}{2}$. □

Now we are ready to start the proof of Theorem 1.

Proof of Theorem 1. If $\mathrm{diam}(T) \leq 3$, then clearly $\gamma_{pr}(T) = 2$ and by Proposition 2 we have $\mathrm{sd}_{\gamma_{pr}}(G) \leq 2 = \frac{\gamma_{pr}(T)}{2} + 1$. Hence, let $\mathrm{diam}(T) \geq 4$. Note that $\gamma_{pr}(T) \geq 4$. If T has a strong stem or adjacent stems, then the result follows from Proposition 2. Hence, we can assume that T has no strong stem or adjacent stems. If $\mathrm{diam}(T) = 4$ and $v_1v_2v_3v_4v_5$ is a diametral path in T, then since T is not a subdivided star, we must have $\deg_T(v_3) \geq 3$ and v_3 is a stem, which is a contradiction. Hence, we can assume that $\mathrm{diam}(T) \geq 5$. Let $v_1v_2v_3 \ldots v_k$ be a diametral path in T such that $\deg_T(v_3)$ is as small as possible. We consider two cases.

Case 1. $\mathrm{diam}(T) \in \{5, 6\}$. Root T at v_4, and consider the following subcases.

Subcase 1.1. v_4 is not a stem and $\deg_T(v_3) = 2$.

By the choice of the diametral path, we deduce that for each child w of v_4, the maximal subtree rooted at w is a either path P_2, P_3 or a healthy spider (if $\mathrm{diam}(T) = 6$). Let H be the forest of $T - v_4$ where each of its components is a healthy spider. Since $deg_T(v3) = 2$, note that H is empty if $\mathrm{diam}(T) = 5$. Now, let $v_4u_i^1u_i^2u_i^3$ be the (pendant) paths in T such that $\deg_T(u_i^1) = \deg_T(u_i^2) = 2$ and $\deg_T(u_i^3) = 1$ for each $i \in \{1, \ldots, r\}$, and let $v_4z_i^1z_i^2$ be the paths in T (if any) such that $\deg_T(z_i^1) = 2$ and $\deg_T(z_i^2) = 1$ for each $i \in \{1, \ldots, s\}$. Assume, without loss of generality, that $u_1^1 = v_3$. Let T' be the tree obtained from T by subdividing the edges v_4v_3, v_3v_2 with new vertices x, y, respectively, the edge $v_4u_i^1$ with a new vertex $u_i'^1$ for each $2 \leq i \leq r$ and the edge $v_4z_j^1$ with a new vertex $z_j'^1$ for each

$j \in \{1, \ldots, s\}$. Let D be a $\gamma_{pr}(T')$-set. To paired-dominate each leaf u_i^3 in T', we must have that $|D \cap \{u_i^1, u_i^2, u_i^3\}| \geq 2$ for each $i \in \{2, \ldots, r\}$, to paired-dominate $u_1^3 = v_1$ we must have $|D \cap \{u_1^2, u_1^3, y\}| \geq 2$, and to paired-dominate each leaf z_j^2 in T' we must have $|D \cap \{z_j^1, z_j^2, z'^1_j\}| \geq 2$ for each j. Also, to paired-dominate vertex x, we may assume that v_4 and x are partners in D'. It follows that $\gamma_{pr}(T') \geq \gamma_{pr}(H) + 2r + 2s + 2$. A similar argument shows that $\gamma_{pr}(T) \geq \gamma_{pr}(H) + 2r + 2s$. Moreover, the equality in the last inequality is attained since each PD-set of H can be extended to a PD-set of T by adding the set $\{z_j^1, z_j^2 \mid 1 \leq j \leq s\} \cup \{u_i^1, u_i^2 \mid 1 \leq i \leq r\}$. Thus $\gamma_{pr}(T) = \gamma_{pr}(H) + 2r + 2s$, and therefore $\gamma_{pr}(T') > \gamma_{pr}(T)$. It follows that $\mathrm{sd}_{\gamma_{pr}}(T) \leq r + s + 1$, and hence $\mathrm{sd}_{\gamma_{pr}}(T) \leq r + s + 1 \leq \frac{\gamma_{pr}(T)}{2} + 1$.

Subcase 1.2. v_4 is not a stem and $\deg_T(v_3) \geq 3$.

By assumption, for each child w of v_4, the maximal subtree rooted at w is either a healthy spider or a path P_2. Let $w_1, \ldots, w_r$ be the children of v_4 such that T_{w_i} is a healthy spider with head w_i, and let $w_i^1, \ldots, w_i^{\ell_i}$ be the children of w_i and let y_i^j be the leaf neighbor of w_i^j for each i, j. Also, let $v_4 z_i^1 z_i^2$ be the paths in T (if any) such that $\deg_T(z_i^1) = 2$ and $\deg_T(z_i^2) = 1$ for each $i \in \{1, \ldots, t\}$. Without loss of generality, let $w_1 = v_3$. Let T' be the tree obtained from T by subdividing the edge $v_4 v_3$ with vertex x, the edges $w_i w_i^1, \ldots, w_i w_i^{\ell_i}$ with vertices $w'^1_i, \ldots, w'^{\ell_i}_i$, respectively, and the edge $v_4 z_j^1$ with vertex z'^1_j for each j. Let D be a $\gamma_{pr}(T')$-set. To paired-dominate each leaf y_i^j in T', we must have $|D \cap \{w_i^j, y_i^j, w'^j_i\}| \geq 2$ for each i, j; to paired dominate each leaf z_j^2 we must have $|D \cap \{z_j^1, z_j^2, z'^1_j\} \geq 2$ for each $j \in \{1, \ldots, t\}$ and to paired dominate vertex x we may assume that v_4 and x are partners in D. Hence $\gamma_{pr}(T') \geq \sum_{i=1}^{r} 2\ell_i + 2t + 2$. A similar argument as above shows that $\gamma_{pr}(T) \geq \sum_{i=1}^{r} 2\ell_i + 2t$. Moreover, the equality in the last inequality is attained since $\{v_3, v_2\} \cup \{z_j^1, z_j^2 \mid 1 \leq j \leq t\} \cup (\cup_{i=2}^{r} \{w_i^j, y_i^j \mid 1 \leq j \leq \ell_i\}) \cup \{w_1^j, y_1^j \mid 2 \leq j \leq \ell_1\}$ is a PD-set of T. Thus $\gamma_{pr}(T) = \sum_{i=1}^{r} 2\ell_i + 2t$, and therefore $\gamma_{pr}(T') > \gamma_{pr}(T)$. It follows that $\mathrm{sd}_{\gamma_{pr}}(T) \leq \sum_{i=1}^{r} \ell_i + t + 1$, and hence $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{\gamma_{pr}(T)}{2} + 1$.

Subcase 1.3. v_4 is a stem.

Let w be a leaf neighbor of v_4. By assumption, w is the unique leaf adjacent to v_4 and v_4 is not adjacent to any stem. Hence, T has diameter 6. First let there be a path $v_4 u_3 u_2 u_1$ in T such that $\deg_T(u_2) = \deg_T(u_3) = 2$ and $\deg_T(u_1) = 1$. Without loss of generality, we may assume that $u_3 = v_3$. Let T' be the tree obtained from T by subdividing the edges $v_4 w, v_4 v_3, v_3 v_2, v_2 v_1$ with new vertices u, x, y, z, respectively, and let D' be a PD-set of T'. It is easy to see that $|D' \cap \{v_4, v_3, v_2, v_1, u, w, x, y, z\}| \geq 6$. If $v_4 \notin D'$ or $v_4 \in D'$ and its partner belongs to $\{u, x\}$, then $(D' \setminus \{v_4, v_3, v_2, v_1, u, w, x, y, z\}) \cup \{v_2, v_3, w, v_4\}$ is a PD-set of T smaller than D'. If $v_4 \in D'$ and its partner does not belong to $\{u, x\}$, then $(D' \setminus \{v_3, v_2, v_1, u, w, x, y, z\}) \cup \{v_2, v_3\}$ is a PD-set of T smaller than D'. Hence, $\gamma_{pr}(T') > \gamma_{pr}(T)$, and thus $\mathrm{sd}_{\gamma_{pr}}(T) \leq 4$. Now let D be a $\gamma_{pr}(T)$-set. To paired-dominate v_1 and v_7, we must have $|D \cap \{v_1, v_2, v_3\}| \geq 2$ and $|D \cap \{v_6, v_5, v_7\}| \geq 2$, respectively. Moreover, to paired-dominate w, we have $|D \cap \{w, v_4\}| \geq 1$. Since $\gamma_{pr}(T)$ is even, we have $\gamma_{pr}(T) \geq 6$. Consequently, $\mathrm{sd}_{\gamma_{pr}}(T) \leq 4 \leq \frac{\gamma_{pr}(T)}{2} + 1$ as desired. Therefore, we can assume that T has no such a path $v_4 u_3 u_2 u_1$ in T such that $\deg_T(u_2) = \deg_T(u_3) = 2$ and $\deg_T(u_1) = 1$. Thus for any child $v \neq w$ of v_4, the maximal subtree T_v is a healthy spider. Since $\mathrm{diam}(T) = 6$, we deduce that v_4 has at least two children whose maximal subtrees are healthy spiders. Let $v_3 = w_1, \ldots, w_r$ be the children of v_4 such that T_{w_i} is a healthy spider with head w_i. Suppose that $v_2 = w_1^1, \ldots, w_1^{\ell}$ are the children of w_1 and y_1^j is the leaf adjacent to w_1^j. Let T' be the tree obtained from T by subdividing the edges $v_4 w, v_4 v_3$ with new vertices x, y, respectively, the edges $w_1 w_1^j, y_1^j w_1^j$ with new vertices x_j and y_j, respectively for each $j \in \{1, \ldots, \ell\}$. Clearly the number of subdivided edges is $2\deg_T(w_1)$. Let D' be a $\gamma_{pr}(T')$-set. To paired-dominate $y_1^1, \ldots, y_1^{\ell}$, we may assume that $w_1^1, \ldots, w_1^{\ell}, y_1, \ldots, y_{\ell} \in D'$. Also to paired-dominate the vertices w, v_3, we must have $|D' \cap \{w, x, v_4, y, v_3, x_1, \ldots, x_\ell\}| \geq 4$. Now, if $v_4 \notin D'$ or $v_4 \in D'$ and its partner belongs to $\{x, y\}$, then $(D' \setminus \{w, x, v_4, y, v_3, x_1, \ldots, x_\ell, y_1, \ldots, y_\ell\}) \cup \{y_1^1, \ldots, y_\ell^1, v_4, v_3\}$ is a PD-set of T smaller than D'. If $v_4 \in D'$ and its partner does not belong to $\{x, y\}$, then $(D' \setminus \{w, x, y, v_3, x_1, \ldots, x_\ell, y_1, \ldots, y_\ell\}) \cup \{y_1^1, \ldots, y_\ell^1\}$ is a PD-set of T smaller than D'. In either case, we deduce that $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2\deg_T(w_1)$. Now let D be a $\gamma_{pr}(T)$-set and let T_{w_1}, T_{w_2} be the components of $T - \{v_4w_1, v_4w_2\}$ containing w_1 and w_2, respectively. To paired-dominate the leaves of T_{w_1} and T_{w_2} we must have $|D \cap V(T_{w_1})| \geq 2\deg_T(w_1) - 2$ and $|D \cap V(T_{w_2})| \geq 2\deg_T(w_2) - 2 \geq 2\deg_T(w_1) - 2$. Also to paired-dominate w we must have $|D \cap \{w, v_4\}| \geq 1$. Since $|D|$ is even, we deduce that $|D| \geq 4\deg_T(w_1) - 2$. This implies that $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2\deg_T(w_1) \leq \frac{\gamma_{pr}(T)}{2} + 1$.

Case 2. $\mathrm{diam}(T) \geq 7$. We consider two subcases.

Subcase 2.1 $\deg_T(v_3) = 2$.

If T is a path, then the result follows from Proposition 5 and the exact value of the paired-domination number of a path given at the end of Section 1. Hence, we assume that T is not a path, and thus $\gamma_{pr}(T) > 4$. If $\deg_T(v_4) = 2$, then $v_5v_4v_3v_2v_1$ is a path in T such that $\deg(v_2) = \deg(v_3) = \deg(v_4) = 2$ and the result follows from Proposition 3. Hence, assume that $\deg_T(v_4) \geq 3$. If v_4 is a stem, then using an argument similar to that described in Subcase 1.3, we can see that $\mathrm{sd}_{\gamma_{pr}}(G) \leq 4 \leq \frac{\gamma_{pr}(T)}{2} + 1$. Thus, we assume that v_4 is not a stem. Hence, each component of $T - v_4$ is of order at least 2. Moreover, since $\mathrm{diam}(T) \geq 7$, one component of $T - v_4$ different from the one containing v_3, must have order at least four and diameter at least three. Root T at v_4 and let $w_1, \ldots, w_r$ be the children of v_4 with depth at least three, $u_1, \ldots, u_s$ be the children of v_4 with depth two, and $z_1, \ldots, z_t$ be the children of v_4 with depth one, if any. We can assume, without loss of generality, that $v_3 = u_1$ and $v_5 = w_1$. Let T' be the tree obtained from T by subdividing the edges v_4v_3, v_3v_2, v_2v_1 with new vertices x, y, z, respectively, the edge v_4u_i with new vertex x_i for each $i \in \{2, \ldots, s\}$, the edge v_4z_i with new vertex a_i for each $i \in \{1, \ldots, t\}$ and each edge v_4w_i with new vertex y_i for all $i \in \{1, \ldots, r\}$. We note that all edges incident to v_4 are subdivided and the number of subdivided edges is $\deg_T(v_4) + 2$. Let D' be a PD-set of T' and F the set of all edges in $\{v_4u_2, , \ldots, v_4u_s, v_4w_1, \ldots, v_4w_r\}$ whose subdivision vertices belong to D'. Let T_1 be the tree obtained from T by subdividing only the edges in F. Clearly, to paired-dominate vertices v_1, v_2, v_3, x, y, z in T', we must have $|D' \cap \{v_1, v_2, v_3, v_4, x, y, z\}| \geq 4$. Now, if $v_4 \notin D'$ or $v_4 \in D'$ and its partner is not x, then $(D' - \{v_1, v_2, v_3, x, y, z\}) \cup \{v_3, v_2\}$ is a PD-set of T_1 smaller than D' and thus $\gamma_{pr}(T') > \gamma_{pr}(T_1) \geq \gamma_{pr}(T)$. If $v_4 \in D'$ and its partner is x, then $(D' - \{v_1, v_2, v_3, v_4, x, y, z\}) \cup \{v_3, v_2\}$ is a PD-set of T_1 smaller than D' and thus $\gamma_{pr}(T') > \gamma_{pr}(T_1) \geq \gamma_{pr}(T)$. We deduce that $\mathrm{sd}_{\gamma_{pr}}(T) \leq \deg_T(v_4) + 2$. Now let D be a $\gamma_{pr}(T)$-set. To paired-dominate the leaves in each $T_{w_i}, T_{u_j}, T_{z_\ell}$ we must have $|D \cap V(T_{w_i})| \geq 2$ and $|D \cap V(T_{u_j})| \geq 2$ for each i, j, and $|D \cap (\{v_4\} \cup (\cup_{m=1}^{t} V(T_{z_m}))| \geq 2t$. Assume that $\mathrm{diam}(T) \geq 9$. Then to paired-dominate v_{k-4}, we must have $|D \cap N(v_{k-4})| \geq 1$. Hence, $|D| \geq 2\deg_T(v_4) + 1$. But since $|D|$ is even, it follows that $|D| \geq 2\deg_T(v_4) + 2$. Therefore, $\mathrm{sd}_{\gamma_{pr}}(T) \leq \deg_T(v_4) + 2 \leq \frac{\gamma_{pr}(T)}{2} + 1$. Hence, we can assume in the sequel that $\mathrm{diam}(T) \in \{7, 8\}$. Now, consider the following situations.

(2.1.1) v_4 has a child w_i with depth 2 and degree at least three.
Then we must have $|D \cap V(T_{w_i})| \geq 4$, implying that $|D| \geq 2\deg_T(v_4) + 2$, and the desired result is obtained as above.

(2.1.2) $\deg_T(v_5) \geq 3$.
Let w be a neighbor of v_5 such that $w \notin \{v_4, v_6\}$. To paired-dominate w, we must have $|D \cap N[w]| \geq 1$, and thus $|D| \geq 2\deg_T(v_4) + 1$. Since $|D|$ is even, we have $|D| \geq 2\deg_T(v_4) + 2$, and the result follows as above.

(2.1.3) $\deg_T(v_{k-2}) \geq 3$. Then we have $|D \cap V(T_{v_{k-2}})| \geq 4$ implying that $|D| \geq 2\deg_T(v_4) + 2$, and the result follows as above.

(2.1.4) $\mathrm{diam}(T) = 8$.
If $\deg_T(v_6) \geq 3$ and w is a neighbor of v_6 such that $w \notin \{v_5, v_7\}$, then the result follows as in item 2. Hence, assume that $\deg_T(v_6) = 2$. By above item we can

assume that $\deg_T(v_5) = 2$. Since T is not a path, we must have $\deg_T(v_4) \geq 3$. By item (2.1.3), we may assume that $\deg_T(v_3) = \deg_T(v_7) = 2$. In this case, one can see that $\gamma_{pr}(T) \geq 6$ and $\mathrm{sd}_{\gamma_{pr}}(T) \leq 4$ (Proposition 3), and thus the desired result is obtained.

(2.1.5) $\mathrm{diam}(T) = 7$.
By items 2 and 3, we may assume that $\deg_T(v_5) = \deg_T(v_6) = 2$ and the result follows as item (2.1.4).

Subcase 2.2. $\deg(v_3) \geq 3$.

By the choice of diametral path in which $\deg(v_3)$ is as small as possible, there is no path $v_4u_1u_2u_3$ in T such that $\deg(u_1) = \deg(u_2) = 2$ and $\deg(u_3) = 1$. Thus the component of $T - v_3v_4$ containing v_3 is a healthy spider with head v_3. Similarly we may assume that the component of $T - v_{k-2}v_{k-3}$ containing v_{k-2} is a healthy spider with head v_{k-2}. By the choice of the diametral path, $\deg(v_3)$ is as small as possible and we have $\deg_T(v_3) \leq \deg_T(v_{k-2})$. Let $N_T(v_3) \setminus \{v_4\} = \{u_1 = v_2, \ldots, u_s\}$ and let u_i' be the leaf neighbor of u_i for each i. Suppose first that v_4 is a stem and let w be the leaf neighbor of v_4. Let T' be the tree obtained from T by subdividing the edges v_4w, v_4v_3 with new vertices x, y, respectively, and the edges v_3u_i, u_iu_i' with new vertices x_i, y_i, respectively, for each $i \in \{1, \ldots, s\}$. Clearly the number of subdivided edges is $2\deg_T(v_3)$. Let D' be a PD-set of T'. Without loss of generality, we can assume that $u_1, \ldots, u_s, y_1, \ldots, y_s \in D'$, where each u_i is paired with y_i. Moreover, to paired-dominate the vertices w, v_3, we must have $|D' \cap \{w, x, v_4, y, v_3, x_1, \ldots, x_s\}| \geq 4$. Let S be the set of all subdivision vertices. If $v_4 \notin D'$ or $v_4 \in D'$ and the partner of v_4 is in $\{x, y\}$, then $(D' \setminus (S \cup \{w\})) \cup \{v_3, v_4, u_1', \ldots, u_s'\}$ is a PD-set of T smaller than D'. If $v_4 \in D'$ and the partner of v_4 is not in $\{x, y\}$, then $(D' \setminus (S \cup \{w, v_3\})) \cup \{u_1', \ldots, u_s'\}$ is a PD-set of T smaller than D'. In either case, we obtain $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2\deg_T(v_3)$. Now let D be a $\gamma_{pr}(T)$-set and let T_{v_3} and $T_{v_{k-2}}$ be the components of $T - \{v_3v_4, v_{k-2}v_{k-3}\}$ containing v_3 and v_{k-2}, respectively. To paired-dominate the leaves of T_{v_3} and $T_{v_{k-2}}$ we must have $|D \cap V(T_{v_3})| \geq 2\deg_T(v_3) - 2$ and $|D \cap V(T_{v_{k-2}})| \geq 2\deg(v_{k-2}) - 2$. Also to paired-dominate w we must have $|D \cap \{w, v_4\}| \geq 1$. Since $|D|$ is even, we have $|D| \geq 4\deg_T(v_3) - 2$. Therefore, $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2\deg_T(v_3) \leq \frac{\gamma_{pr}(T)}{2} + 1$, as desired.

Suppose now that v_4 is not a stem. Then each component of $T - v_4$ is of order at least two. If $T \in \mathcal{F}$, then the result follows from Lemma 1. Hence, we assume that $T \notin \mathcal{F}$. Since $\mathrm{diam}(T) \geq 7$ and $\deg_T(v_3) \leq \deg_T(v_{k-2})$, we deduce that $|V(T_{v_{k-3}})| \geq |V(T_{v_3})| + 1$. On the other hand, since $T \notin \mathcal{F}$, either one of the components of $T - v_4$ that does not contain neither v_3 nor v_5 has order at least three or $|V(T_{v_{k-3}})| \geq |V(T_{v_3})| + 2$. Let $N(v_4) = \{w_1 = v_3, w_2 \ldots, w_r\}$. Let T' be the tree obtained from T by subdividing the edges v_4w_i with vertices z_i for $1 \leq i \leq r$, the edges v_3u_i, u_iu_i' with vertices x_i, y_i, respectively, for each $1 \leq i \leq s$. Note that the number of subdivided edges is $\deg_T(v_4) + 2\deg_T(v_3) - 2$. Let D' be a PD-set of T' and let F be the set of all edges incident with v_4 whose subdivision vertices belong to D'. Let T_2 be the tree obtained from T by subdividing only the edges in F. Without loss of generality, assume that $u_1, \ldots, u_s, y_1, \ldots, y_s \in D$, where each u_i is paired with y_i. Also, to paired-dominate vertex v_3, we must have $|D' \cap \{v_4, z_1, v_3, x_1, \ldots, x_s\}| \geq 2$. Let $W = \{z_1, x_1, \ldots, x_s, y_1, \ldots, y_s\}$. If $v_4 \notin D'$ or $v_4 \in D'$ and its partner is z_1, then $(D' \setminus (W \cup \{v_4, v_3\})) \cup \{v_3, u_2', \ldots, u_s'\}$ is a PD-set of T_2 smaller than D' and so $\gamma_p(T') > \gamma_p(T_2) \geq \gamma_p(T)$. If $v_4 \in D'$ and its partner is not z_1, then $(D' \setminus (W \cup \{v_3\})) \cup \{u_1', u_2', \ldots, u_s'\}$ is a PD-set of T_2 smaller than D' and again $\gamma_p(T') > \gamma_p(T_2) \geq \gamma_p(T)$. Consequently, $\mathrm{sd}_{\gamma_{pr}}(G) \leq \deg_T(v_4) + 2\deg_T(v_3) - 2$. Now let D be a $\gamma_{pr}(T)$-set. As seen above, we can see that for each child w_i of v_4 with depth two we have $|D \cap V(T_{w_i})| \geq 2\deg_T(w_i) - 2$. In particular, $|D \cap V(T_{v_3})| \geq 2\deg_T(v_3) - 2$. Similarly, $|D \cap V(T_{v_{k-2}})| \geq 2\deg_T(v_{k-2}) - 2 \geq 2\deg_T(v_3) - 2$. Moreover, if $q_1, ..., q_t$ are the children of v_4 with depth one (if any), then to paired-dominate the leaf neighbors of $q_1, ..., q_t$, we must have $|D \cap (\{v_4\} \cup (\cup_{i=1}^{t} V(T_{q_i}))| \geq 2t$.

Assume that $\mathrm{diam}(T) \geq 9$. Then to paired-dominate v_{k-4}, we must have $|D \cap N(v_{k-4})| \geq 1$. Hence $|D| \geq 4\deg_T(v_3) - 4 + 2(\deg_T(v_4) - 2) + 1 = 4\deg_T(v_3) + 2\deg_T(v_4)$

-7. But since $|D|$ is even, it follows that $|D| \geq 4\deg_T(v_3) + 2\deg_T(v_4) - 6$. Therefore, $\mathrm{sd}_{\gamma_{pr}}(T) \leq \deg_T(v_4) + 2\deg_T(v_3) - 2 \leq \frac{\gamma_{pr}(T)}{2} + 1$. Hence, we can assume in the sequel that $\mathrm{diam}(T) \in \{7, 8\}$. Now, consider the following situations.

(2.2.1) v_4 has a child $w_i \neq v_3$ with depth 2 and degree at least three.
Then we must have $|D \cap V(T_{w_i})| \geq 4$, implying that $|D| \geq 4\deg_T(v_3) + 2\deg_T(v_4) - 6$, and the desired result is obtained as above.

(2.2.2) $\deg_T(v_5) \geq 3$.
Let w be a neighbor of v_5 such that $w \notin \{v_4, v_6\}$. To paired-dominate w, we must have $|D \cap N[w]| \geq 1$, and thus $|D| \geq 4\deg_T(v_3) + 2\deg_T(v_4) - 7$. Since $|D|$ is even, we have $|D| \geq 4\deg_T(v_3) + 2\deg_T(v_4) - 6$, and the result follows as above.

(2.2.3) $\deg_T(v_3) < \deg_T(v_{k-2})$. Then we have $|D \cap V(T_{v_{k-2}})| \geq 2\deg_T(v_{k-2}) - 2 \geq 2\deg(v_3)$ implying that $|D| \geq 4\deg_T(v_3) + 2\deg_T(v_4) - 6$, and the result follows as above.

(2.2.4) $\mathrm{diam}(T) = 8$.
If $\deg_T(v_6) \geq 3$ and w is a neighbor of v_6 such that $w \notin \{v_5, v_7\}$, then the result follows as in item 2. Hence, assume that $\deg_T(v_6) = 2$. By above item we can assume that $\deg_T(v_5) = 2$. If $\deg_T(v_4) = 2$, then the result follows from Proposition 3. Thus, let $\deg_T(v_4) \geq 3$. Note that since v_4 is not a stem and according to the first item and the choice of diametral path, every subtree rooted at a child of v_4 different from v_3 and v_5 is a path P_2. Moreover, by the third item we may assume that $\deg_T(v_3) = \deg_T(v_7)$. In this case, one can see that $\gamma_{pr}(T) \geq 10$ and $\mathrm{sd}_{\gamma_{pr}}(T) \leq 4$ (for instance we can subdivide edges v_4v_5, v_5v_6, v_6v_7 and one edge incident with v_4 different from v_3v_4 and v_4v_5). Therefore, the desired result is obtained.

(2.2.5) $\mathrm{diam}(T) = 7$.
Since $T \notin \mathcal{F}$, we must have $\deg_T(v_3) < \deg_T(v_6)$ or $\deg_T(v_5) \geq 3$. In either case, the result follows by above items.

This completes the proof. □

The following upper bound on the paired domination number of a tree has been presented by Chellali and Haynes in [2].

Theorem 2 ([2]). *If T is a tree of order $n \geq 3$ with s stems, then $\gamma_{pr}(T) \leq \frac{n+2s-1}{2}$.*

According to Theorems 1 and 2, we obtain the following upper bound on the paired-domination subdivision number of a tree.

Corollary 1. *If T is a tree of order $n \geq 4$ with s stems different from a healthy spider, then $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n+2s+3}{4}$.*

Applying Theorem 1 and Corollary 1, we get the following result.

Corollary 2 ([20]). *If T is a tree of order $n \geq 4$ different from a healthy spider, then $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{n}{2}$.*

Proof. We first observe that if T has a strong stem, then by Proposition 2, $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2 \leq n/2$. Hence we assume that T has no strong stem. Let s be the number of stems in T and let t be the number of vertices that are neither leaves nor stems. Note that $t \neq 1$ since T is different from a healthy spider. Now, if $t = 0$, then $s = n/2$ and thus T has adjacent stems. By Proposition 2, $\mathrm{sd}_{\gamma_{pr}}(T) \leq 2 \leq n/2$. Hence we can assume that $t \geq 2$. Clearly, $s = \frac{n-t}{2}$. If $t \geq 3$, then $s \leq \frac{n-3}{2}$, and by Corollary 1, we obtain $\mathrm{sd}_{\gamma_{pr}}(T) \leq n/2$. Therefore, let $t = 2$. Thus $n \geq 6$ and is even. Let x and y be the two vertices of T that are neither leaves nor stems. Then $V(T) - \{x, y\}$ in which each stem is paired with its unique leaf neighbor, is a PD-set of D and so $\gamma_{pr}(T) \leq n - 2$. It follows from Theorem 1 that $\mathrm{sd}_{\gamma_{pr}}(T) \leq \frac{\gamma_{pr}(T)}{2} + 1 \leq n/2$ and the proof is complete. □

Let $H_1 = S_m$ and $H_2 = S_m$ be two healthy spiders with $m \geq 2$ feet each and centers x and y, respectively. Let T_m be the tree obtained from H_1 and H_2 by adding the edge xy (see Figure 2). It is not hard to see that $n(T_m) = 4m + 2$, $\gamma_{pr}(T_m) = 4m$ and $\mathrm{sd}_{\gamma_{pr}}(T_m) = 2m + 1$. Therefore the bounds of Theorem 1 and Proposition 2 are sharp.

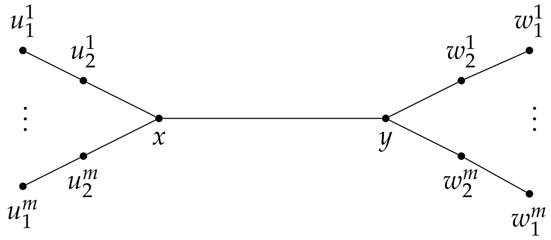

Figure 2. A tree T_m with $\mathrm{sd}_{\gamma_{pr}}(T_m) = \frac{\gamma_{pr}(T_m)}{2} + 1 = n(T_m)/2$.

Let T'_m be the tree obtained from T_m by subdividing the edge xy with a subdivision vertex u and adding a new vertex v and a new edge uv (see Figure 3). It is not hard to see that $n(T'_m) = 4m + 4$, $\gamma_{pr}(T'_m) = 4m + 2$ and $\mathrm{sd}_{\gamma_{pr}}(T'_m) = 2m + 2$. Therefore the bounds of Theorem 1 and Proposition 2 are sharp for any tree in the family $\mathcal{T} = \{T_m, T'_m \mid m \geq 2\}$.

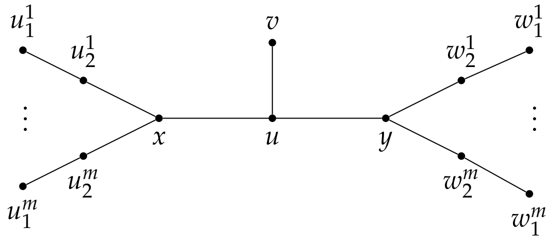

Figure 3. A tree T'_m with $\mathrm{sd}_{\gamma_{pr}}(T'_m) = \frac{\gamma_{pr}(T'_m)}{2} + 1 = n(T'_m)/2$.

We conclude this paper with two conjectures.

Conjecture 1. *For any connected graph G of order $n \geq 4$ different from a healthy spider,* $\mathrm{sd}_{\gamma_{pr}}(G) \leq \frac{\gamma_{pr}(G)}{2} + 2$.

Conjecture 2. *For any connected graph G of order $n \geq 7$ different from a healthy spider,* $\mathrm{sd}_{\gamma_{pr}}(G) \leq \frac{n}{2}$.

If G is the graph obtained from C_5 by adding a pendant edge at one vertex, then we have $\mathrm{sd}_{\gamma_{pr}}(C_5) = \mathrm{sd}_{\gamma_{pr}}(G) = 4$. Therefore, the condition $n \geq 7$ is necessary to establish the second conjecture.

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