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New General Variants of Chebyshev Type Inequalities via Generalized Fractional Integral Operators

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Abstract: In this study, new and general variants have been obtained on Chebyshev's inequality, which is quite old in inequality theory but also a useful and effective type of inequality. The main findings obtained by using integrable functions and generalized fractional integral operators have generalized many existing results as well as iterating the Chebyshev inequality in special cases.

Keywords: chebyshev type inequalities; generalized fractional integral operators

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1. Introduction

In inequality theory, the most efficient known method of obtaining inequality is to use an existing equation. Inequalities can help compare quantities with this method and lead to the emergence of new problems for approximation theory. In addition, classical and analytical inequalities found by similar methods create the link between inequality theory and physics, statistics, economics and engineering sciences. Chebyshev's inequality obtained with the help of Chebyshev functional is one of the best examples of this situation. Chebyshev inequality was given by Chebyshev in [1] as follows.

$$|\Upsilon(\Psi,\Phi)| \leq \frac{1}{12}(\xi-\zeta)^2\|\Psi'\|_\infty\|\Phi'\|_\infty, \tag{1}$$

where $\Psi,\Phi : [\zeta,\xi] \to \mathbb{R}$ are absolutely continuous mappings whose derivatives Ψ', $\Phi' \in L_\infty[\zeta,\xi]$ and

$$T(\Psi,\Phi) = \frac{1}{\xi-\zeta}\int_\zeta^\xi \Psi(\tau)\Phi(\tau)d\tau - \left(\frac{1}{\xi-\zeta}\int_\zeta^\xi \Psi(\tau)d\tau\right)\left(\frac{1}{\xi-\zeta}\int_\zeta^\xi \Phi(\tau)d\tau\right), \tag{2}$$

which is called the Chebyshev functional, provided the integrals in (2) exist. Based on the Chebyshev functional, many new inequalities have been derived and used frequently in areas such as inequality theory and approximation theory. For several new results, generalizations, refinements and extensions can be found in [2–9].

Differentiation of functions is a tool commonly used by mathematicians in solving theoretical problems and generating solutions to real world problems. In classical analysis, the concept of differential has been used on the basis of integer order for a long time, but it has been understood that real world problems cannot be expressed only by systems of differential equations containing integer order differentials. As a result of this need, a new window has been discovered for fractional analysis and thus to fractional order derivatives and integral operators. In addition to the properties of kernels used in theirs presentation of fractional derivatives and integral operators, singularity and locality, they made a difference to classical analysis by generalizing the integer-order derivatives and integral operators. In addition, many real world problems that cannot be solved with classical analysis methods and concepts have also been solved (see the papers [10–12]). The existence of fractional analysis and the definition of new fractional integral and derivative operators revealed a similar situation for the inequality theory. Many inequalities have been generalized with the help of fractional integral operators and led to the construction of new approaches (see the papers [13–23]).

By giving some important concepts and some known definitions of fractional analysis, necessary literature background will be provided to obtain results.

Definition 1 (See [24])**.** *Diaz and Parigun have defined the $\kappa-$gamma function Γ_κ, as the generalization of the classical gamma function. This interesting special function have been given as:*

$$\Gamma_\kappa(\tau) = \lim_{n-\infty} \frac{n!\kappa^n (n\kappa)^{\frac{x}{\kappa}} - 1}{(\tau)_{n,\kappa}}, \kappa > 0.$$

It is shown that Mellin transform of the exponential function $e^{-\frac{t^\kappa}{\kappa}}$ is the $\kappa-$gamma function clearly presented by:

$$\Gamma_\kappa(\alpha) = \int_0^\infty e^{-\frac{t^\kappa}{\kappa}} \, t_1^{\alpha-1} \, dt_1.$$

Obviously, $\Gamma_\kappa(\tau+\kappa) = \tau\Gamma_\kappa(\tau)$, $\Gamma(\tau) = \lim_{\kappa\to 1}\Gamma_\kappa(\tau)$ and $\Gamma_\kappa(\tau) = k^{\frac{\tau}{\kappa}-1}\Gamma(\frac{\tau}{\kappa})$.

Definition 2 (See [12])**.** *Let us define the function*

$$\mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}(\tau) = \sum_{m=0}^{\infty} \frac{\sigma(m)}{\kappa\Gamma_\kappa(\rho\kappa m+\lambda)}\tau^m \qquad (\rho,\lambda > 0; |\tau| < \mathbb{R}),$$

where the coefficients $\sigma(m)$ for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is a bounded sequence of $\mathbb{R}^+$.

Definition 3 (See [25])**.** *For $\kappa > 0$, let $\Phi : [\zeta,\xi] \to \mathbb{R}$ be an increasing and monotone mapping that has derivative continuously such that $\Phi'(\tau)$ on (ζ,ξ). The left and right side generalized $\kappa-$fractional integrals of function Ψ with respect to function Φ on $[\zeta,\xi]$ are defined as following:*

$$\mathcal{J}^{\sigma,\kappa,\Phi}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = \int_\zeta^\tau \frac{\Phi'(t)}{(\Phi(\tau)-\Phi(t))^{1-\frac{\lambda}{\kappa}}} \mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}[\omega(\Phi(\tau)-\Phi(t))^\rho]\Psi(t)dt, \quad \tau > \zeta \tag{3}$$

and

$$\mathcal{J}^{\sigma,\kappa,\Phi}_{\rho,\lambda,\xi-;\omega}\Psi(\tau) = \int_\tau^\xi \frac{\Phi'(t)}{(\Phi(t)-\Phi(\tau))^{1-\frac{\lambda}{\kappa}}} \mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}[\omega(\Phi(t)-\Phi(\tau))^\rho]\Psi(t)dt, \quad \tau < \xi \tag{4}$$

where $\lambda,\rho > 0, \omega \in \mathbb{R}$.

Remark 1 (See [25])**.** *Some important special cases of the integral operators that is defined in Definition 3 can be concluded as:*

i *In case of $\kappa = 1$, the generalized operator in (3) reduces to generalized fractional integral of Ψ with respect to another function such as Φ on $[\zeta,\xi]$:*

$$\mathcal{J}^{\sigma,\Phi}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = \int_\zeta^\tau \frac{\Phi'(t)}{(\Phi(\tau)-\Phi(t))^{1-\lambda}} \mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(\Phi(\tau)-\Phi(t))^\rho]\Psi(t)dt, \quad \tau > \zeta.$$

ii *In case of $\Phi(t) = t$, the operator in (3) overlaps with the generalized $\kappa-$fractional integral of Ψ, this relation is seen as:*

$$\mathcal{J}^{\sigma,\kappa}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = \int_{\zeta}^{\tau} (\tau - t)^{\frac{\lambda}{\kappa}-1} \mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}[\omega(\tau - t)^{\rho}]\Psi(t)dt, \quad \tau > \zeta.$$

iii *In case of $\Phi(t) = ln(t)$, the operator in (3) coincides to generalized Hadamard $\kappa-$fractional integral of Ψ, this case can be shown as:*

$$\mathcal{H}^{\sigma,\kappa}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = \int_{\zeta}^{\tau} (ln\frac{\tau}{t})^{\frac{\lambda}{\kappa}-1} \mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}[\omega(ln\frac{\tau}{t})^{\rho}]\Psi(t)\frac{dt}{t}, \quad \tau > \zeta.$$

iv *In case of $\Phi(t) = \frac{t^{s+1}}{s+1}$, for $s \in \mathbb{R} - \{-1\}$, the operator in (3) reduces to generalized $(\kappa, s)-$fractional integral of Ψ, associated definition can be given as:*

$${}_{s}\mathcal{J}^{\sigma,\kappa}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = (1+s)^{1-\frac{\lambda}{\kappa}} \int_{\zeta}^{\tau} (\tau^{s+1} - t^{s+1})^{\frac{\lambda}{\kappa}-1}\, t^{s}\, \mathcal{F}^{\sigma,\kappa}_{\rho,\lambda}[\omega(\frac{\tau^{s+1} - t^{s+1}}{s+1})^{\rho}]\Psi(t)dt, \quad \tau > \zeta.$$

Remark 2. *By a similar argument, one can represent the same reductions for the integral operator that is defined in (4). We omit the details.*

Remark 3. *By setting $\kappa = 1$ and $\Phi(t) = t$ in Definition 3, it is obvious to see that the definition can be reduced to the generalized fractional integral operators that is established by Agarwal (see [26]) and Rania et al. (see [12]) as followings:*

$$\mathcal{J}^{\sigma}_{\rho,\lambda,\zeta+;\omega}\Psi(\tau) = \int_{\zeta}^{\tau} (\tau - t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(\tau - t)^{\rho}]\Psi(t)dt, \quad \tau > \zeta,$$

and

$$\mathcal{J}^{\sigma}_{\rho,\lambda,\xi-;\omega}\Psi(\tau) = \int_{\tau}^{\xi} (t - \tau)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t - \tau)^{\rho}]\Psi(t)dt, \quad \tau < \xi.$$

Remark 4. *Several well-known integral operators can be obtained by different special cases of Φ.*

Remark 5. *Under the special cases such that $\omega = 0$, $\lambda = \alpha$ and $\sigma(0) = 1$ in Definition 3, we can capture the generalized integral operator that defined by Akkurt et al. (see [27]).*

Remark 6. *Let we remember some other special cases under the conditions such that $\omega = 0$, $\lambda = \alpha$ and $\sigma(0) = 1$ in Definition 3:*

- If we take $\kappa = 1$, the definition turns into fractional integrals of function Ψ with respect to another function Φ (see [10]).
- If we set $\Phi(t) = t$, then the definition reduces to κ-fractional operators (see [15]).
- If we choose $\Phi(t) = ln(t)$ and $k = 1$, the definition coincides with the Hadamard fractional integrals (see [10]).
- If we select $\Phi(t) = \frac{t^{s+1}}{s+1}$, for $s \in \mathbb{R} - \{-1\}$, the definition overlaps with $(\kappa, s)-$fractional integral operators (see [17]).
- Finally, if we set $\Phi(t) = \frac{t^{s+1}}{s+1}$, for $s \in \mathbb{R} - \{-1\}$ and $\kappa = 1$, the definition reduces to the Katugampola fractional integral operators (see [28]).

The following new result have been given by Set et al. for Chebyshev type inequalities via conformable integrals and generalized fractional integral operators.

Theorem 1 (See [7]). *Let t be a positive valued mapping on $[0,\infty]$ and let Ψ and Φ be differentiable mappings on $[0,\infty]$. If $\Psi' \in L_{m_1}([0,\infty])$, $\Phi' \in L_{m_2}([0,\infty])$, $m_1 > 1$, $m_1^{-1} + m_2^{-1} = 1$, then for all $\tau > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $\theta > 0$, we have*

$$\begin{aligned}
& \Big|(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(\tau;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Psi\Phi)(\tau;p) + (\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(\tau;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Psi\Phi)(\tau;p) \\
- \quad & (\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Psi)(\tau;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Phi)(\tau;p) - (\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t\Psi)(\tau;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t\Phi)(\tau;p)\Big| \\
\leq \quad & ||\Psi'||_{m_1}||\Phi'||_{m_2}\int_0^\tau\int_0^\tau (\tau-\tau_1)^{(\beta-1)}(\tau-\rho)^{(\theta-1)}|\tau_1-\rho|t(\tau_1)t(\rho) \\
\times \quad & E^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}\Big(\omega(x-\tau_1)^\alpha;p\Big)E^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}\Big(\omega(x-\rho)^\lambda;p\Big)d\tau_1 d\rho \\
\leq \quad & ||\Psi'||_{m_1}||\Phi'||_{m_2}\tau(\epsilon^{\omega,\delta,q,r,c}_{0^+,\alpha,\beta,\sigma}t)(\tau;p)(\epsilon^{\omega,\delta,q,r,c}_{0^+,\lambda,\theta,p}t)(\tau;p).
\end{aligned}$$

The main motivation in producing inequality is to obtain new approaches, to find better boundaries to a known inequality, and to generalize existing inequalities in the literature. In accordance with this purpose, it is necessary to use integral operators whose results are more general and can obtain many operators for their special cases. In this sense, the main purpose of this study is to obtain new and general Chebyshev type inequalities by using generalized fractional integral operators, one of the important concepts of fractional analysis. Several special cases of our main findings have been provided.

2. Main Results

Theorem 2. *Assume that Ψ and Φ are mappings on $L^+[\zeta,\xi]$ that are synchronous on $[\zeta,\xi]$. Assume that $\varpi : [\zeta,\xi] \to \mathbb{R}$ is an increasing positive-valued mapping that has derivative on (ζ,ξ) continuously, then one has a new result that includes fractional integral operators as following:*

$$\begin{aligned}
& J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi\Phi)(\xi) \\
\geq \quad & \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{k}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Psi(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Phi(\xi).
\end{aligned} \tag{5}$$

Proof. By using the conditions such that Ψ and Φ are synchronously mappings on $[\zeta,\xi]$, we can write

$$\Big(\Psi(u)-\Psi(v)\Big)\Big(\Phi(u)-\Phi(v)\Big) \geq 0; \quad u,v \in [\zeta,\xi].$$

Simplfying this inequality, we get

$$\Psi(u)\Phi(u) + \Psi(v)\Phi(v) \geq \Psi(u)\Phi(v) + \Psi(v)\Phi(u).$$

To return the expression to an inequality that involves integral operator, firstly we multiply by $\frac{\varpi'(u)}{\Big(\varpi(\xi)-\varpi(u)\Big)^{1-\frac{\lambda}{k}}} F^{\sigma,\kappa}_{\rho,\lambda}\left[w\Big(\varpi(\xi)-\varpi(u)\Big)^{\rho}\right]$ and then apply integration to the statement with respect to u over ζ to ξ, these operations provide

$$
\begin{aligned}
&\int_{\zeta}^{\xi} \frac{\varpi'(u)}{\Big(\varpi(\xi)-\varpi(u)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(u)\Big)^{\rho}\Big]\Psi(u)\Phi(u)du \\
&+\int_{\zeta}^{\xi} \frac{\varpi'(u)}{\Big(\varpi(\xi)-\varpi(u)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(u)\Big)^{\rho}\Big]\Psi(v)\Phi(v)du \\
\geq\ & \Phi(v)\int_{\zeta}^{\xi} \frac{\varpi'(u)}{\Big(\varpi(\xi)-\varpi(u)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(u)\Big)^{\rho}\Big]\Psi(u)du \\
&+\Psi(v)\int_{\zeta}^{\xi} \frac{\varpi'(u)}{\Big(\varpi(\xi)-\varpi(u)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(u)\Big)^{\rho}\Big]\Phi(u)du.
\end{aligned}
$$

From this, we have

$$
\begin{aligned}
& J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi\Phi)(\xi)+\Psi(v)\Phi(v)\Big[\varpi(\xi)-\varpi(\zeta)\Big]^{\frac{\lambda}{\kappa}} \\
& \times F_{\rho,\lambda+\kappa}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big] \\
\geq\ & \Phi(v)J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi)(\xi)+\Psi(v)J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Phi)(\xi).
\end{aligned} \tag{6}
$$

After multiplying the inequality by $\frac{\varpi'(v)}{\Big(\varpi(\xi)-\varpi(v)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(v)\Big)^{\rho}\Big]$ and integrating the resulting inequality with respect to v over ζ to ξ, gives us

$$
\begin{aligned}
& J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi\Phi)(\xi)\Big[\varpi(\xi)-\varpi(\zeta)\Big]^{\frac{\lambda}{\kappa}} F_{\rho,\lambda+\kappa}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big] \\
& +\Big[\varpi(\xi)-\varpi(\zeta)\Big]^{\frac{\lambda}{\kappa}} F_{\rho,\lambda+\kappa}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big] \\
& \times\int_{\zeta}^{\xi} \frac{\varpi'(v)}{\Big(\varpi(\xi)-\varpi(v)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(v)\Big)^{\rho}\Big]\Psi(v)\Phi(v)dv \\
\geq\ & J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi)(\xi)\int_{\zeta}^{\xi} \frac{\varpi'(v)}{\Big(\varpi(\xi)-\varpi(v)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big]\Phi(v)dv \\
& +J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Phi)(\xi)\int_{\zeta}^{\xi} \frac{\varpi'(v)}{\Big(\varpi(\xi)-\varpi(v)\Big)^{1-\frac{\lambda}{\kappa}}} F_{\rho,\lambda}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(v)\Big)^{\rho}\Big]\Psi(v)dv,
\end{aligned}
$$

$$
\begin{aligned}
& J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi\Phi)(\xi)\Big[\varpi(\xi)-\varpi(\zeta)\Big]^{\frac{\lambda}{\kappa}} F_{\rho,\lambda+\kappa}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big] \\
\geq\ & J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Psi)(\xi)J_{\rho,\lambda,\zeta^{+};w}^{\sigma,\kappa,\varpi}(\Phi)(\xi).
\end{aligned}
$$

This completes the proof and we obtain the desired result as given in (5). □

Remark 7. *By proceeding a similar argument but now for any $\Psi,\Phi\in L^{-}[\zeta,\xi]$ that are synchronous on $[\zeta,\xi]$, one can easily obtain;*

$$
\begin{aligned}
& J_{\rho,\lambda,\xi^{-};w}^{\sigma,\kappa,\varpi}(\Psi\Phi)(\zeta) \\
\geq\ & \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F_{\rho,\lambda+\kappa}^{\sigma,\kappa}\Big[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\Big]\right]^{-1} J_{\rho,\lambda,\xi^{-};w}^{\sigma,\kappa,\varpi}(\Psi)(\zeta)J_{\rho,\lambda,\xi^{-};w}^{\sigma,\kappa,\varpi}(\Phi)(\zeta).
\end{aligned} \tag{7}
$$

Remark 8. *If we take $\kappa = 1$, $\lambda = 1$, $w = 0$, $\sigma(0) = 1$ and $\varpi(t) = t$ in 2 (or in Remark 7), then the aforementioned inequality reduces to the classical Chebyshev inequality.*

Theorem 3. *Assume that Ψ and Φ are mappings that defined on $L_1^+[\zeta,\xi] \bigcap L_2^+[\zeta,\xi]$ which are synchronous mappings on $[\zeta,\xi]$. Assume that $\varpi : [\zeta,\xi] \to \mathbb{R}$ is increasing and positive-valued mapping with continuous derivative on the open interval (ζ,ξ), then we have the following inequality;*

$$\begin{aligned} &\Big[\varpi_2(\xi) - \varpi_2(\zeta)\Big]^{\frac{\lambda_2}{\kappa_2}} F^{\sigma_2,\kappa_2}_{\rho_2,\lambda_2+\kappa_2}\Big[w_2\Big(\varpi_2(\xi) - \varpi_2(\zeta)\Big)^{\rho_2}\Big] J^{\sigma_1,\kappa_1,\varpi_1}_{\rho_1,\lambda_1,\zeta^+;w_1}(\Psi\Phi)(\xi) \\ &+\Big[\varpi_1(\xi) - \varpi_1(\zeta)\Big]^{\frac{\lambda_1}{\kappa_1}} F^{\sigma_1,\kappa_1}_{\rho_1,\lambda_1+\kappa_1}\Big[w_1\Big(\varpi_1(\xi) - \varpi_1(\zeta)\Big)^{\rho}\Big] J^{\sigma_2,\kappa_2,\varpi_2}_{\rho_2,\lambda_2,\zeta^+;w}(\Psi\Phi)(\xi) \\ &\geq J^{\sigma_1,\kappa_1,\varpi_1}_{\rho_1,\lambda_1,\zeta^+;w_1}(\Psi)(\xi) J^{\sigma_2,\kappa_2,\varpi_2}_{\rho_2,\lambda_2,\zeta^+;w}(\Phi)(\xi) + J^{\sigma_1,\kappa_1,\varpi_1}_{\rho_1,\lambda_1,\zeta^+;w_1}(\Phi)(\xi) J^{\sigma_2,\kappa_2,\varpi_2}_{\rho_2,\lambda_2,\zeta^+;w}(\Psi)(\xi). \end{aligned} \tag{8}$$

Proof. By writing σ_1 in place of σ, κ_1 in place of κ, ρ_1, λ_1 in place of ρ, λ and taking $\varpi_1 = \varpi$ in (6) and by applying multiplication by $\frac{\varpi_2'(v)}{\Big(\varpi_2(\xi)-\varpi_2(v)\Big)^{1-\frac{\lambda_2}{\kappa_2}}} F^{\sigma_2,\kappa_2}_{\rho_2,\lambda_2}\Big[w_2\Big(\varpi_2(\xi) - \varpi_2(v)\Big)^{\rho_2}\Big]$ and integrating both sides of the resulting inequality with respect to v between ζ and ξ gives us (8). □

Remark 9. *In case of $\sigma_1 = \sigma_2$, $\rho_1 = \rho_2$, $\lambda_1 = \lambda_2$, $\varpi_1 = \varpi_2$, $w_1 = w_2$, we can easily obtain Theorem 2.*

Theorem 4. *Assume that $\{\Psi_i\}_{i=1,2,\ldots,n}$ are positive-valued and increasing mappings on $L^+[\zeta,\xi]$, also assume that $\varpi : [\zeta,\xi] \to \mathbb{R}$ is an increasing positive-valued mapping with a continuous derivative on (ζ,ξ), then we have;*

$$\begin{aligned} &\left[J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Big(\prod_{i=1}^{n}\Psi_i\Big)(\xi)\right] \\ \geq\ &\left[\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\Big[w\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\rho}\Big]\right]^{1-n} \left[\prod_{i=1}^{n}[J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi_i)(\xi)\right]. \end{aligned} \tag{9}$$

Proof. We will use induction for the proof. Let we start with $n = 1$ ($n \in \mathbb{N}$), we can see that the (9) obviously satisfies for $n = 2$ (9) immediately be clear from (5), due to the synchronous properties of the mappings Ψ_1 and Ψ_2 on $[\zeta,\xi]$. Suppose that the statement of (9) holds for some $n \in \mathbb{N}$. By setting $\Psi = \prod_{i=1}^{n}\Psi_i$ and $\Phi = \Psi_{n+1}$. By considering the increasing properties of the mappings Ψ and Φ on $[\zeta,\xi]$, therefore (5) and the induction hypothesis for n yields.

$$\begin{aligned} &J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Big(\prod_{i=1}^{n}\Psi_i\Psi_{n+1}\Big)(\xi) \\ \geq\ &\left[\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\Big[w\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\rho}\Big]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Big(\prod_{i=1}^{n}\Psi_i\Big)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Big(\Psi_{n+1}\Big)(\xi) \\ \geq\ &\left[\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\Big[w\Big(\varpi(\xi) - \varpi(\zeta)\Big)^{\rho}\Big]\right]^{-n} \prod_{i=1}^{n+1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}\Big(\Psi_i\Big)(\xi). \end{aligned}$$

This completes the induction and the proof. □

Theorem 5. *Assume that for* $\Psi, \Phi : [0, \infty) \to \mathbb{R}$, $\Psi, \Phi \in L^+[\zeta, \xi]$, Ψ *is increasing mapping and* Φ *is differentiable such that* Φ' *bounded as* $m = \inf_{t \in [0,\infty)} \Phi'(t)$. *If* $\varpi : [\zeta, \xi] \to \mathbb{R}$ *is increasing positive-valued function with continuous derivative on* (ζ, ξ), *then we have;*

$$\begin{aligned}
&J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi\Phi)(\xi)\\
&\geq \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi)\\
&-\frac{m}{\left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]}\\
&\times J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) + m J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Psi)(\xi)
\end{aligned}$$

where $t(\chi) = \chi$ *is well-known identity function.*

Proof. Let us consider $P(x) = m\chi$ and $Q(\chi) = \Phi(\chi) - P(\chi)$. Here, we can say that Q is differentiable and increasing on $[0, \infty)$. Therefore, by using the inequality that is given in (5) and we can easily write:

$$\begin{aligned}
&J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi Q)(\xi)\\
&\geq \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q)(\xi)\\
&= \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi)\\
&- \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P)(\xi).
\end{aligned} \tag{10}$$

Since $J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P)(\xi) = m J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi)$ and $J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi P)(\xi) = m J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Psi)(\xi)$, then (10) implies:

$$\begin{aligned}
&J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi\Phi)(\xi) = J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi Q)(\xi) + J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi P)(\xi)\\
&\geq \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi)\\
&-\left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P)(\xi) + J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi P)(\xi)\\
&\geq \left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi)\\
&-\frac{m}{\left[\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\Big(\varpi(\xi)-\varpi(\zeta)\Big)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) + m J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Psi)(\xi).
\end{aligned}$$

The desired result is obtained. □

Theorem 6. *Assume that for* $\Psi, \Phi : [0, \infty) \to \mathbb{R}$, $\Psi, \Phi \in L^+[\zeta, \xi]$, Ψ *and* Φ *are differentiable such that* Ψ' *bounded as* $m_1 = \inf_{t \in [0,\infty)} \Psi'(t)$ *and* Φ' *bounded as* $m_2 = \inf_{t \in [0,\infty)} \Phi'(t)$. *If* $\varpi : [\zeta, \xi] \to \mathbb{R}$ *is increasing positive-valued function with continuous derivative on* (ζ, ξ), *then we have;*

$$
\begin{aligned}
&J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi\Phi)(\xi) \\
&\geq \left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi) \\
&- \frac{m_2}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) \\
&- \frac{m_1}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) \\
&+ \frac{m_1 m_2}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) \\
&+ m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Psi)(\xi) + m_1 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Phi)(\xi) - m_1 m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t^2)(\xi)
\end{aligned}
$$

where $t(\chi) = \chi$ is well-known identity function.

Proof. Let us consider $P_1(x) = m_1\chi$ and $Q_1(\chi) = \Psi(\chi) - P_1(\chi)$, also $P_2(\chi) = m_2(\chi)$ and $Q_2(\chi) = \Phi(\chi) - P_2(\chi)$. Let we remember from the hypothesis Q_1 and Q_2 are differentiable and increasing mappings on $[0, \infty)$, applying the inequality that is given (5) gives us:

$$
\begin{aligned}
&J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_1 Q_2)(\xi) \\
&\geq \left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_1)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_2)(\xi) \\
&\geq \left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]\right]^{-1} \\
&\times \left[J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) - J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P_1)(\xi)\right]\left[J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi) - J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P_2)(\xi)\right]
\end{aligned}
$$

$$
\geq \left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}} F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]\right]^{-1} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi) \quad (11)
$$

$$
\begin{aligned}
&- \frac{m_2}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) \\
&- \frac{m_1}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Phi)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) \\
&+ \frac{m_1 m_2}{\left[\left(\varpi(\xi)-\varpi(\zeta)\right)^{\frac{\lambda}{\kappa}}\right] F^{\sigma,\kappa}_{\rho,\lambda+\kappa}\left[w\left(\varpi(\xi)-\varpi(\zeta)\right)^{\rho}\right]} J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi) J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t)(\xi).
\end{aligned}
$$

Moreover,

$$
J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_1 P_2)(\xi) = m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(tQ_1)(\xi) = m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Psi)(\xi) - m_1 m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t^2)(\xi). \quad (12)
$$

Similarly,

$$
J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_2 P_1)(\xi) = m_1 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t\Phi)(\xi) - m_1 m_2 J^{\sigma,\kappa,v}_{\rho,\lambda,\zeta^+;w}(t^2)(\xi) \quad (13)
$$

and
$$J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P_1P_2)(\xi) = m_1m_2 J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(t^2)(\xi). \tag{14}$$

From the equality
$$\Psi\Phi = (Q_1 + P_1)(Q_2 + P_2) = Q_1Q_2 + Q_1P_2 + Q_2P_1 + P_1P_2,$$

we have
$$J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(\Psi\Phi)(\xi) = J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_1Q_2)(\xi) + J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_1P_2)(\xi) + J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(Q_2P_1)(\xi) + J^{\sigma,\kappa,\varpi}_{\rho,\lambda,\zeta^+;w}(P_1P_2)(\xi),$$

and this equality together with (11)–(14) implies the required result. □

Remark 10. *In case of* $m_1 = 0$, *we obtain Theorem 5.*

3. Conclusions

The main findings of our study are designed to prove Chebyshev type integral inequalities with the help of generalized fractional integral operators. The special cases of the results of Theorems 6, which constitute the main findings, have been presented as remarks, revealing that each main finding is a generalized Chebyshev type inequality. It is clear that these inequalities are reduced to Chebyshev's inequality in special cases, and it can be observed that our findings produce upper bounds for some divergent integrals by setting special selections of functions and parameters.

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