

Article

Epistemic Complexity of the Mathematical Object “Integral”

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Abstract: The literature in mathematics education identifies a traditional formal mechanistic-type paradigm in Integral Calculus teaching which is focused on the content to be taught but not on how to teach it. Resorting to the history of the genesis of knowledge makes it possible to identify variables in the mathematical content of the curriculum that have a positive influence on the appropriation of the notions and procedures of calculus, enabling a particularised way of teaching. Objective: The objective of this research was to characterise the anthology of the integral seen from the epistemic complexity that composes it based on historiography. Design: The modelling of epistemic complexity for the definite integral was considered, based on the theoretical construct “epistemic configuration”. Analysis and results: Formalising this complexity revealed logical keys and epistemological elements in the process of the theoretical constitution that reflected epistemological ruptures which, in the organisation of the information, gave rise to three periods for the integral. The characterisation of this complexity and the connection of its components were used to design a process of teaching the integral that was applied to three groups of university students. The implementation showed that a paradigm shift in the teaching process is possible, allowing students to develop mathematical competencies.



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1. Introduction

Integral Calculus teaching should be developed as a solid mathematical culture in which university students can qualitatively and quantitatively analyse different phenomena of the everyday environment to increase their abstraction and reasoning capacities. However, the literature on mathematics education shows that Integral Calculus teaching focuses on a formal mechanistic approach, which emphasises the content to be taught rather than how to teach it. It has identified students' difficulties in establishing connections for the integral that would enable them to be competent in its use and handling. For example, to resolve a precise problem situation, the integral may be used as an area, although in another context, it may be an antiderivative or a measure. These aspects make the integral a mathematically complex object. They favour presenting it from the articulation of the complexity which constitutes it, understood as a plurality of valuable meanings for the design and implementation of instructional processes, so as to improve them permanently.

The authors of [1,2] focused on the integral as a mathematical object of teaching, exposing the need for studies of the ontology of mathematical objects that allow us to characterise their complexity, given that they offer elements for the design and execution of instructional processes which are different from the traditional ones. Hence, this study maintains interest in this line of research by proposing two objectives: (1) to characterise the anthology of the integral analysed from the epistemic complexity that composes it, based on a historical–epistemological–hermeneutical study; and (2) to consider this complexity when guiding the instructional process with university students and determining whether the articulation of this complexity in some way allows us to overcome the learning problems identified and to achieve some change in the current teaching paradigm.

The onto-semiotic approach to mathematical cognition and instruction (hereafter, OSA [3]) was used as a theoretical support because it offers tools that enable the identification of the complexity of mathematical entities and the connection of the units in which this complexity erupts, following the model outlined in [4,5], through multiple meanings (partial meanings) described in terms of practices and epistemic configurations of the primary objects activated in these practices [6].

This manuscript is focused on responding to the first objective, related to formalising the complexity of the integral, sharing the position put forward in [4,7] and considering that, in studies on the ontology of a mathematical object, logical keys and epistemological elements are evident in the process of theoretical constitution, which not only allow us to better understand the concept, but also reveal characteristic aspects of the mathematical construction activity that must be taken into account for its comprehension. This component enabled the identification of epistemological breaks in the evolution of the integral which, in the organisation of the information, gave rise to three periods, each of which generated a global epistemic configuration, described in detail in the Results section.

The development of the second objective aimed to demonstrate whether the complexity identified for the integral is the origin of the various difficulties manifested in the teaching and learning processes of Integral Calculus referenced in the mathematics education literature, and whether knowing it allows for some kind of change in the current teaching paradigm. The Methodology section of this work describes how the proposal was developed, the results of which, due to their length and detail, can be consulted in [8,9].

2. Theoretical Background

The research interest is focused on the complexity of the integral. We cite some works, among them: for the definite integral [10,11]; for the integral in general [1,2,12–16]. Elements that allowed to identify a first classification for the epistemic complexity of the definite integral, ref. [10] describes four particular epistemic configurations of reference:

1. The geometric is used to determine the area under a curve and the abscissa axis and to calculate lengths, areas, and volumes in a static geometric context. Leibniz is considered as its main driver;
2. The result of the process of change frames all those cases in which the integral is necessary to solve situations in other sciences associated with non-static processes. Newton is presented as the promoter of this meaning;
3. The inverse of the derivate is from the original relationship between the derivate and integral. It is associated with the works of Newton in 1711 and Leibniz in 1710;
4. The approach to the limit is related to the formalisation initiated by Cauchy in 1825, which gave rise to a new definition of definite integral.

The author of [7] complemented the above four configurations by adding another two.

1. The algebraic is generated from the formalisation of the concepts taught by teachers, considering that they spend a large part of their time practicing the integration rules;
2. The generalized is framed by the need to expand the set of integrable functions after the foundation built by Cauchy in 1831.

It should be noted that the historical changes found are characterized by the solutions that is presented for the existing problem in a certain epistemic configuration at a certain moment in history. These changes may imply both the rupture of the epistemic configuration and its evolution to an inclusive or complementary one. From these parameters, [12] raises eight partial meaning, also shown as epistemic reference configurations, which complement or modify those exposed by the authors mentioned above:

1. Intuitive, related to the work of the Greeks in geometry;
2. Primitive, related to Newton and Leibniz's works to find the primitive of a function. Establishes the inverse relationship between derivatives and integrals from the Fundamental Theorem of Calculus;
3. Geometric, related to the resolution of clearly geometric problems in the Middle Ages;

4. Summation, related to the problem of the foundation of the calculus which began at the end of the 19th century;
5. Approximate, related to the intra- and extra-mathematical application of the definite integral;
6. Extra mathematics, related to the breadth of application possibilities for the integral to problem situation in different areas of knowledge;
7. Accumulated, related to the intuitive processes of “integration” produced in the medieval period from Newton’s work linked to dynamics on 1687, (change and movement);
8. Technological, related to the use of mathematical software to perform calculations on computers and the ability to use the appropriate software tools.

These eight configurations demonstrate the extension of the study, not only to the definite integral, but also tangentially involves the indefinite integral, without clarifying its origin, unchecking it, and defining it as an entity independent of the definite integral. Hence, the existence of certain limitations in each partial meaning is shown, which later became motivating situation for other more consistent “meanings”, here called “secondary”. For example, improper integrals are extension of the defined integrals, which were not contemplated in any of the previous configurations.

3. Theoretical Framework

In this research, we use the OSA as theoretical support because it offers theoretical tools that can help us reflect on the complexity of mathematical objects and the possibility of articulating them in the search for an explanation of how they arise [3]. In the OSA, the analysis of mathematical activity involves determining the types of entities involved in this activity. It distinguishes six main types: situations/problems, actions, language, concepts, properties, and arguments [6]. These objects relate to each other forming “configurations” that can be epistemic or cognitive, if seen from the perspective of the mathematical institution, or personal, if seen from the perspective of the subject who performs them [3]. A configuration is defined as networks of intervening and emergent objects of the systems of practices activated to solve problems. Systems of experiences and configurations are proposed as theoretical instruments to represent mathematical activity, which allows mathematical knowledge to be inferred [5]. The conception of epistemic configuration is one useful instrument for the analysis of mathematical writings and historical-epistemological studies of mathematical objects [3].

The notion of language use plays a significant protagonist in OSA, alongside the concept of institution; is considered the contextual mechanism that relativizes the ways of being, of existing of mathematical entities [3]. According to their use of language, the mathematical objects that interact in mathematical experiences and those that appear from them can be considered as from the perspective of being/existing, grouped into facets or dual dimensions (see [5]). For this work, we will use one of those dual dimensions: the unitary-systemic. According to this duality, mathematical objects can participate as unitary objects or as a system. When a mathematical entity is considered an object, a unitary perspective on it is being adopted. However, there are times when it is possible to adopt a systemic perspective on such an object, for example, when considering its component parts.

The Emergence of Mathematical Objects in OSA

Ref. [5] showed that the route by which mathematical entities emerge from experiences is complex, and differentiates at least two levels of occurrence. In the primary, representations, descriptions, propositions, procedures, problems and opinions (first objects) appear, established in “epistemic configurations”. Mathematical practice can be thought of metaphorically as “climbing a ladder”. The phase on which the practice is based forms a configuration of primary entities previously known. Whereas, the higher phase arrived at as a consequence of experience generates a new configuration of primary entities

in which one (or some) of those entities were not known previously; thus, new primary entities emerge as a consequence of mathematical practice [4]. On a second level, there is the emergence of a mathematical object (the integral, in our case) that can be characterised by different representations: antiderivatives, areas under curves, accumulation functions, among others, and which can have equivalent definitions, properties, theorems, etc. This second emergence is a consequence of the interactions of different aspects that implicitly or explicitly generate in the classroom a descriptive-realistic vision of mathematics that considers: (1) mathematical propositions and enunciations refer to properties of mathematical objects; (2) these objects attain a certain autonomous kind of existence of the subjects that they know them and the language they use to know them.

From this point of view, the integral object is located on the second level. It is the emergence of a universal reference related with diverse configurations of primary objects a certain let the mathematical conventions to be carried out in different contexts in which the integral has been interpreted as: an approximation of the limit of a Riemann sum, the inverse of the derivative, the result of a process of change, a summation, the result of a process of accumulation, as an driver that discoveries from a given function-other function (its primitive)-, that allows us to understand that the integral can be demarcated and embodied in various ways. According to OSA [4,5], the result is that it is considered an object, named an integral, which rescues the global reference title of all primary object configurations. Now, this global reference in the mathematical activity takes the form of a specific configuration of primary objects. Therefore, what can be done with this second-level object is determined by this configuration. Ref. [4] mentioned that in OSA, the entity that rescues the role of global reference can be seen as single to simplify the reasons and, equally, as multiple, simultaneously, as it metaphorically breaks down into a combination of first objects grouped in various configurations.

OSA's idea of the complexity of mathematical objects makes it possible to identify a diverse system of problem-solving practices in which the (secondary) mathematical object does not appear directly. What does appear are representations of the object (secondary), diverse meanings, propositions, properties, actions, procedures, and opinions that are applied to that mathematical object (epistemic configurations of primary objects). In other words, throughout history, different epistemic configurations of primary objects have been generated for the study of the (secondary) mathematical object, some of which have served to generalise the pre-existing ones.

4. Methodology

To determine the complexity of the integral, we carried out a study based on the interpretative-hermeneutic paradigm, setting off from recognising the difference between social and natural phenomena, seeking the greater complexity and the unfinished character of the former, which are always conditioned by the participation of the human being. Under this premise, we pursued finding, interpreting, and clarifying the development of the different notions for the integral.

4.1. Sample

Epistemic configurations proposed by two authors, here called *tertiaries* [11,12], who perform the first classification for the definite integral using theoretical tools provided by OSA.

4.2. Instruments

A template to build an epistemic configuration; primary, secondary, and tertiary bibliographic sources. Specialised software to systematise the information collected and matrices for data triangulation.

4.3. Procedure and Data Analysis

The modelling of the complexity of the definite integral proposed in [11,12] was considered as a starting point. In the epistemic configurations proposed by those authors, improper integrals were not shown as an extension of definite integrals, were not they clearly determined. Since the origin of these integrals is not considered in any configuration, it motivated us to delve into secondary sources on the history of mathematics: [17–30], to identify, characterise, and categorise their evolution, to determine in which configuration they fit or fail to complete or modify them. In this phase, the analysis instrument used was the format for the construction of an epistemic configuration; each of the six elements that comprise it was broken down, analysed and modified as the selected and systematised data were collected and analysed.

The classification of the systematised information showed the need to go deeper into the study, based on a historiography for the integral, where it was necessary to resort to secondary bibliographic sources, i.e., texts about the history of mathematics related in the theoretical background of this work. The triangulation of this information made it possible for us to understand different notions of the mathematical object integral, considering the context in which each one arose. Interpreted in a modern context, they allow to distinguish between definite, indefinite, and improper integrals, elements that lead to conclude that the tertiary sources, by focusing the research on the definite integral, did not consider historiographical aspects of the evolution of the concepts of the integral from its origins. between the components that modify the epistemic configurations proposed by [11,12], are, for example: Archimedes' theorem for the quadrature of the spiral or Fermat's two squares theorem, which influenced the works of Newton and Leibniz in the Middle Ages in the constitution of the concept of the integral, theoretical aspects needed to consult primary sources, also related in the theoretical background of this work.

Once those limitations were detected, the next step was to check, both in the secondary and primary sources, whether the type of problem addressed in each historical period involved the use of a clearly indefinite integral, or whether the result found was the one we know today as a consequence of the calculation of a definite integral, or an improper one, considering that, for each period, these differences were not yet known. The idea was to complete the modelling of the complexity of the integral already identified in the elements that make up the epistemic configurations created from the primary and secondary sources, rethinking what those mathematicians did, but this time from a modern perspective, which would allow us to identify the type of integral that was used, and whether it corresponded to finding a solution to the problematic situation identified for each period. In this phase of the research, due to the amount of information collected and aiming to purify it, it was necessary to follow the same method used in [31] for optimisation; [6] for the derivative, given that they studied the epistemic complexity of these mathematical objects also using the tools provided by OSA. The deductions obtained were subjected to a triangulation process, preserving the structure of the epistemic configuration tool, model abbreviated in Table 1, which allowed characterising the complexity of the integral by identifying three global periods, each of which generated a global epistemic configuration, described in detail in Section 5 of this paper.

Table 1. Global complexity for the integral.

Global Epistemic Configurations	Primary Configurations	
	[7]	[8]
GEC1 (Origins of the Integral)	Geometric	Primitive
		Intuitive
GEC2 (Integration as a support for recent Integral Calculus)	Result of a change process	Geometric
	Inverse of the derivative	
GEC3 (Formalisation of Integral Calculus)	Approach to the limit	Summation
	Generalised	Extra math
	Algebraic	Approximate
		Cumulative
		Technological

4.4. Validation of the Proposed Characterisation

Once this epistemic complexity was identified, we proposed to determine how it contributes to the instructional processes from the current programmes offered in three colleges of a university in the city of Bogotá: Finance, Cadastral Engineering, and Administration. To this end, we analysed the syllabus of these programmes, focusing on the subject Calculus 2, where integral calculus is taught. The purpose was to determine what part of this complexity was present and, if any, how it is articulated for the teaching of integral calculus and its applications. After analysing the three Calculus 2 programmes in detail, a restructuring of the part corresponding to the teaching of the integral was proposed, articulating the three global epistemic configurations proposed, redesigning activities that would allow the integral to be shown from different concepts, allowing reflection on its logical structure of production, construction and application. A representative sample of partial meanings was formulated, connected to each other, taking as a reference the complexity elaborated and proposed in this work. A sequence of tasks was designed and implemented with three groups of students from these colleges, with follow-up during three academic semesters, with the aim of observing some evidence in the students of the connection between partial meanings and their use when solving problems in different contexts, thus analysing their performance. In other words, to evidence the development of mathematical competencies in students when using integrals. To verify the benefits of this restructuring, the work we carried out with these groups was compared with the results obtained with a fourth group that continued to be taught the integral in traditional classes. Due to their length and details, the results found can be consulted in [8] and were expanded in [9].

5. Results and Implications

It presents the results of this work in two senses: (1) The complexity of the integral, the object of this paper, and (2) we mention some details related to the characterisation of the complexity of the integral and its articulation when planning and implementing a sequence of tasks with university students, since the details and results of this characterisation can be found in [8].

5.1. Results in Relation to the Complexity of the Integral

This study allowed to identify the existence of certain limitations in each of the partial meanings put forward by the tertiary authors, given that they only focused their interest on the definite integral, leaving aside elements that became motivating situations for other more consistent “meanings” involving indefinite and improper integrals, here called secondary, and which are detailed in the three global epistemic configurations that we

propose. Table 1 shows the synthesis of this globality, articulating the primary epistemic configurations within the global ones, allowing us to visualise the complexity of the integral.

From this classification, we modelled the complexity of the integral in three global epistemic configurations: (1) origins of the integral (GEC1); (2) the operation of integration to support the nascent Integral Calculus (GEC2); (3) formalization of the Integral Calculus (GEC3). It should be clarified that, within these global configurations, we distinguish in detail the appearance of definite, indefinite and improper integrals that allowed to locate the configurations proposed by the tertiary sources within one of the three global epistemic configurations established here.

In the validation of the pilot group, we identified in the students that this notorious complexity allows them to focus their attention not only on algorithms and techniques, but on the very complexity of the integral, allowing them to identify the different meanings and access the understanding of fundamental concepts of the Integral Calculus, such as: calculation of area between curves (with definite integrals), application of the convergence criteria for improper integrals; calculation of the centre of gravity of a body and the force of attraction of gravity; calculation of the area of a flat enclosure, calculation of the length of a curve, calculation of the volume and area of a solid of revolution, among others, enabling them to establish different relationships between these concepts, apply and extrapolate them to other situations that require the solution of new problems.

5.1.1. Origins of the Integral

Around 340-194 B.C the Athenian school tackled three problems related to measurement: doubling cube, trisecting an angle, and squaring the circle, all of them in a clearly intra-mathematical context. For reasons of space, we present the position of three mathematicians representing this school who worked to find a solution to those problems: Eudoxo around 340-330 B.C. created the exhaustion method, inscribing a succession of polygons in the non-rectilinear figure to be squared, choosing the sequence in such a way that the differences between the measure of the figure to be squared and the measure of each polygon form a sequence that satisfies the hypothesis of the previous proposition.

Euclid about 300 B.C., using the method proposed by Eudoxo, carried out measurements in which he compared known and unknown magnitudes, respecting the principle of homogeneity (one-dimensional, compares one segment with another taken as a reference unit. Two-dimensional, finds a square equivalent to any plane figure. Three-dimensional, finds a cube equivalent to any solid). In the first two propositions of Book XII, Euclid exposes the idea of decomposing-recomposing rectilinear plane figures to obtain their squareness. Archimedes considered Democritus as the first, who, following Euclidean approaches, established the formula for the volume of a cone or a pyramid correctly, “considering these solids as if they were formed by innumerable parallel layers” [32] (p. 23).

At the end of the third century B.C., Archimedes, retaining this form of reasoning, used strict proofs to find areas, volumes, centres of gravity of curves, surfaces, circles, spheres, conics, and spirals; he perfecting the exhaustive method. He combined the geometric with the laws of mechanics and the exhaustive method, a process that gave rise to the indivisible and infinitesimal, respectively. He positioned the method of exhaustion as an approximation between inscribed and circumscribed geometric figures of a given measurement that delimit the figure sought, so that the difference between them is so small that they are considered equivalent.

We found in this type of work, that the ancient Greeks, from purely geometric processes, implicitly used the integral as an operation, whose result was impossible to determine due to the theoretical limitations of this time [33]. That is, if we look at it in current terms, it would reflect the implicit use of the “indefinite integral”, but the results found showed a measure, a number; which, seen in current terms, is the equivalent of the application of a “definite integral”, and sometimes that of an “improper” one. For example, in the case of the quadrature of the Archimedes’ spiral—A curve that describes

a material point that moves with uniform speed along a ray that rotates with uniform angular speed around its end—We start from a succession of infinite layers that cover this area and, although the result is a number (which was considered a measure), such succession of layers, with the current look, can be understood as a succession of functions that converge to another function. Now, we find that the application of this implicit notion of integral is known today as indefinite integral, but the ancient Greeks used it to obtain a measurement (today we know that a measurement is obtained as a result of calculating a definite integral). Procedure that, translated to current notations, is practically the same as the Riemann integral, whose polar equation has the form $\rho = av$, where $a > 0$ y is a constant. We illustrate as an example the way Archimedes used the following theorem: *The area of the first cycle of a spiral is equal to one third of the area of the circumscribed circle* (Figure 1).

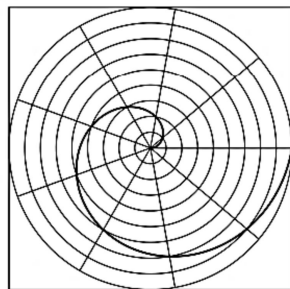


Figure 1. Quadrature of a spiral.

Demonstration 1. Let us consider a spiral with polar equation $\rho = av$. Let us calculate the area when the polar angle varies from 0 to 2π , i.e., from the first turn of the spiral. The radius of the circumscribed circle is $2\pi a$. To do this, we divide this circle into sectors of amplitude $v = \frac{2\pi}{n}$, from $v = \frac{2\pi k}{n}$ to $v = \frac{2\pi(k+1)}{n}$ for $k = 0, 1, \dots, n-1$. In each sector, we examine the spiral arc that remains within it and delimit the area corresponding to said spiral arc between the areas of two circular sectors. The largest circular sector area inscribed in each spiral arc is $2\left(\frac{a2\pi k}{n}\right) \cdot 2\left(\frac{2\pi}{n}\right)$, and the smallest circular sector area circumscribed in each spiral arc is $2\left(a2\pi(k+1)/n\right) \cdot (2\pi/n)$. [30] (p. 140).

In modern notation, the area, S , of the spiral verifies that: $\sum_{k=0}^{n-1} \frac{1}{2} \left(\frac{a2\pi k}{n}\right)^2 \frac{2\pi}{n} = \frac{4\pi^3 a^2}{n^3} \sum_{k=0}^{n-1} k^2 < S < \sum_{k=1}^n \frac{1}{2} \left(\frac{a2\pi k}{n}\right)^2 \frac{2\pi}{n} = \frac{4\pi^3 a^2}{n^3} \sum_{k=1}^n k^2$. Archimedes knew that $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$. Using this result, he wrote the above inequality in the form $4\pi^3 a^2 \frac{1}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) < S < 4\pi^3 a^2 \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)$. He took $k = \frac{1}{3}\pi(2\pi a)^2$, a third of the area of the circumscribed circle, subtracted k , the previous inequality, and carried out simple operations, obtaining: $K\left(-\frac{3}{2n} + \frac{1}{2n^2}\right) < S - K < K\left(\frac{3}{2n} + \frac{1}{2n^2}\right)$; because $\frac{1}{n^2} \leq \frac{1}{n}$, this becomes $-\frac{2k}{n} < S - K < \frac{2K}{n}$. Using the Archimedean axiom, the conclusion is that $S = K$. We observe that the proposition starts from an implicit, “indefinite” assumption, whose result is a number (implicitly, it is the result which, in current terms, corresponds to calculating a definite integral). The global epistemic configuration 1 (GEC1) associated with this period is summarised in Table 2.

Table 2. GEC1. Origins of the integral.

Components	Description
Problem situations	Relative measurement problem, in an intra-mathematical context, from three situations: (a) doubling the cube; (b) trisecting an angle; and (c) squaring the circle. Problem of measuring long distances (for example, the distance from the Moon to the Earth). Problems in which areas, volumes, centres of gravity of curves, surfaces, circles, spheres, conics, and spirals must be found.
Languages	Geometry.
Definitions	Commensurable and incommensurable magnitudes. Different types of curves. Elements of the curves.
Procedures	Basic processes for measuring: direct measurement; decompose, recompose and superimpose, respecting the principle of homogeneity. Method of exhaustion (the method of exhaustion inscribes a succession of polygons in the non-rectilinear figure to be squared, from an approximation between inscribed and circumscribed geometric figures of a known measure, which delimit the figure to be determined; this value is assumed as the “limit”). Method of reduction to absurdity.
Propositions	Principle of homogeneity: only magnitudes of the same dimension can be compared. Results obtained for specific cases of surface measurement and volumetric measurement (for example: the area of the first cycle of a spiral is equal to one-third of the area of the circumscribed circle).
Arguments	The argument consists of the correct application of the method of exhaustion to solve the problem, using integration as the fundamental operation.

Source: own creation.

The ancient Greeks conceived tangency as static and geometric, suitable for calculating the circumference but not a spiral. Hence, Archimedes established two ways of operating with infinity: the mechanical method, incorporating the indivisibles; and the method of exhaustion, infinitely small due to the presumption of the existence of a “limit”. In this configuration, one of the basic principles is that of homogeneity. However, it is a principle that presents problems because it is not always possible to fit a figure within another a fixed number of times.

5.1.2. Integration as a Support for Nascent Integral Calculus

Regarding the GEC1 configuration, we highlight the overcoming of the homogeneity principle. Here we quote a primary source, whose contributions generate a breakdown of the integration operation, which influenced other mathematicians of the time, establishing the concept of integral in a more general and abstract way, that is, as a new discipline. The example is in [34], located within the geometric algebra of the Greeks (explains how arithmetic operations can be done using ruler and compass), who broke with tradition by considering that any algebraic expression, for example, a^2 and b^3 represent segments, (for the ancient Greeks, a^2 and b^3 were area and volume, respectively).

On the use of letters in geometry

... Frequently, it is not essential to draw lines on the paper, designating each of them by a letter was enough. Therefore, to add line BD and GH, I name one a and the other b and write $a + b$. Therefore, I will write $a - b$ to indicate the subtraction of b from a . Additionally, I will inscribe ab to indicate the multiplication of one by the other; $\frac{a}{b}$ to divide a by b ; aa or a^2 to multiply a by itself; and a^3 to multiply this result one more time by a , and so on to infinity; in addition, $\sqrt{a^2 + b^2}$ is used to obtain the square root of $a^2 + b^2$; finally, $\sqrt[3]{ca^3 - b^3 + abb}$ obtains the cube root of $ca^3 - b^3 + abb$, and similarly for others. It should be noted that with a^2 and b^3 , and comparable terminologies, I do not generally conceive but simple lines, although I name them squares or cubes, expressions used in

algebra. Likewise, we must consider that all the parts of each line are articulated by an equal number of dimensions when the unit has not been determined in the statement of the problem. Thus, a^3 contains the same dimensions as abb or b^3 , these being the components of the line that I have named $\sqrt{ca^3 - b^3 + abb}$. The same does not happen, however, when the unit is determined, because it can always be assumed whatever the dimensions, etc. [34] (p. 66).

Thanks to Descartes' analytic geometry, there is a bridge between geometry and analysis, expanding the domain of the geometric curves, developing new methods to calculate tangents and areas. From these extensions, Kepler modified the method of exhaustion indicating: "any figure or body is represented in the form of a figure by a set of infinitely small parts" [24] (p. 170), and introduced concepts such as indivisible and infinitesimal, that allow us to develop techniques to calculate tangents or make quadratures in a heuristic way; contrary to Cavalieri, who kept integration as an operation reasoning in the Greek style, considering a plane figure made up of a set of lines, and a solid made up of an indefinite number of parallel plane fragments.

During the 16th century, the use of infinitesimal quantities was imposed in the solution of problems of calculation of tangents, areas and volumes. We highlight Fermat, Wallis, Pascal and Barrow as representatives of the era, because they present a conceptual and methodological disruption with the strictly geometric approach of Cavalieri, originating a progressive arithmetization that led to the implicit use of the limit. As an example, we show how Fermat calculated the quadrature of the hyperbola $y = x^{-2}$ for $x \geq a$ since they are essential elements of the current definite integral. To facilitate understanding, we will use modern terminology and notation (Figure 2).

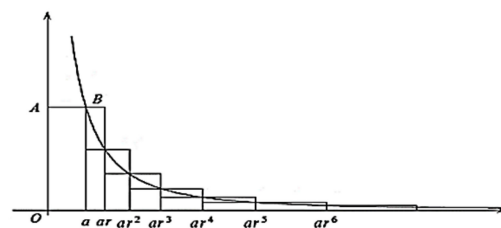


Figure 2. Quadrature of Fermat's hyperbola.

Let us choose a number $r > 1$ and consider the abscissa points $a, ar, ar^2, \dots$. The area of the inscribed rectangles (Figure 2) is $(ar - a)\frac{1}{(ar)^2} + (ar^2 - ar)\frac{1}{(ar^2)^2} + (ar^3 - ar^2)\frac{1}{(ar^3)^2} + \dots = \frac{r-1}{ar^2} \sum_{k=0}^{\infty} \frac{1}{r^k} = \frac{1}{ar}$. The area of the circumscribed rectangles is given by:

$$(ar - a)\frac{1}{(a)^2} + (ar^2 - ar)\frac{1}{(ar)^2} + (ar^3 - ar^2)\frac{1}{(ar^2)^2} + \dots = \frac{r-1}{a} \sum_{k=0}^{\infty} \frac{1}{r^k} = \frac{r}{a}$$

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Therefore, calling S the area under the curve, we have that $\frac{1}{ar} < S < \frac{r}{a}$. Since this inequality is valid for every $r > 1$, we conclude that $S = \frac{1}{a}$. We note that this value is precisely the area of the rectangle $OABa$. In these quadratures of Fermat, there are, from a current perspective, three essential aspects of the defined integral: (a) the division of the

area under the curve into infinitely small area elements, (b) approximation of the sum of those area elements by infinitesimal rectangles of height given by the analytical equation of the curve and (c) an attempt to express something similar to a limit of said sum when the number of elements increases indefinitely as they become infinitely small. This is in current notation: $F(x) = \int_0^a x^n dx = \frac{a^{n+1}}{n+1}$, for all n in rationals, whit $n \neq -1$.

In the same direction, we find Wallis arithmetizing Cavalieri's indivisibles, transforming the calculation of quadratures into the problematic of finding the area below the curve by a Cartesian equation. Let us see the way in which the area below the curve $y = x^k$ with $k = 1, 2, \dots$ on the section $[0, a]$ was calculated (Figure 3), since this process influenced the works of Newton between 1666–1676 and Leibniz between 1675–1695 that later would formalize the nascent infinitesimal calculation.

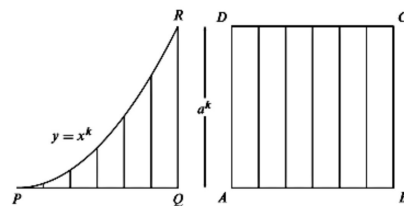


Figure 3. Comparing indivisibles.

Wallis considered the PQR region made up of infinitely many parallel vertical lines of length as equal to x^k . He divided the segment $PQ = AB = a$ into n parts $h = \frac{a}{n}$ long, where n tends infinity. The sum of these infinite lines is prototypical $0^k + h^k + (2h)^k + \dots + (nh)^k$. Equally, the area of the rectangle ABCD is $a^k + a^k + a^k + \dots + a^k = (nh)^k + (nh)^k + \dots + (nh)^k$, and the correspondence between the PQR and ABCD areas is $\frac{\text{areaPQR}}{\text{areaABCD}} = \frac{0^k + 1^k + 2^k + \dots + n^k}{n^k + n^k + \dots + n^k}$. In current terms, it can be summarised as: $\lim_{n \rightarrow \infty} \frac{f(0) + f(1) + \dots + f(n)}{f(n) + f(n) + \dots + f(n)} = \frac{1}{\sigma(f) + 1}$, assuming that said limit exists. This process is known as the Wallis interpolation method. This technique shows that the sums necessary to calculate quadratures can be performed arithmetically better than in terms of geometric ratios. This evidences the definitive disruption with the rigor of Greek geometry and with the Aristotelian tradition of avoiding infinity.

Using his incomplete induction method, Wallis generalised the results for finite sums and infinite series (today known as the intuitive use of passing to the limit). Wallis identified that what is considered static can become dynamic, thus defining four important elements in the conceptualisation of the definite integral: (a) the determination of the area of the rectangle as the product of the base by height; (b) the division of the area under the curve into infinitely small area elements (infinitesimal rectangles of height determined by the equation of the curve); (c) approximation to the numerical determination of the sum of those elements; and (d) an attempt to express the equivalent of what will be the limit of this sum when the number of elements increases indefinitely as they become infinitely small.

In 1647, Saint-Vincent, considering these four elements, derived an extension for the definite integral. He studied a generalisation of the notion of integral, as contemplated until then. Analysing the area under the hyperbola $y = \frac{1}{x}$, he showed that, if the relation of the successive lengths is constant (Figure 4), i.e., $\frac{A_1}{A_2} = \frac{A_2}{A_3}$, then the areas I, II, and III are equal.

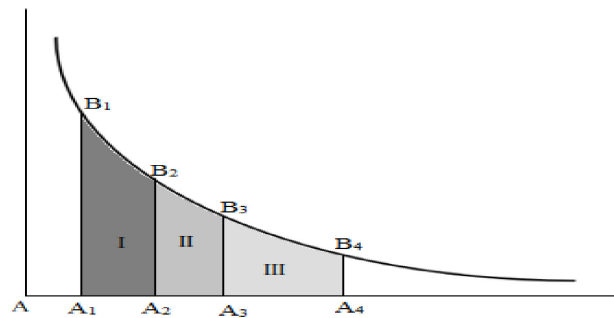


Figure 4. Relationship between successive lengths.

He demonstrated that if points are arranged according to a geometric progression on one of the asymptotes of a hyperbola, the areas cut under the curve because they are parallel to the other asymptote are equal, since the areas of the curvilinear trapeziums $A_1A_2B_1B_2$; $A_2A_3B_2B_3$, K are equal when the lengths $AA_1, AA_2, AA_3, AA_4, K$ are in geometric progression. Therefore, [35] studied in terms of areas the values of what, in current terms, is $F(x) = \int_a^x f(t)dt$, which represents an improper integral; he identified that the function F has restrictions, it must be defined and bounded in a finite interval $[a, b]$.

Subsequently, Barrow, following the direction outlined by Wallis, called “fundamental theorem” the inverse relationship between problems of tangents and squares, based on geometric methods. Ref. [36] mentions that Barrow inferred the use of elements that were later key to the precision of the fundamental theorem of the calculus, breaking with the integration operation, turning it into a new field of work, with definitions, properties and theorems that need to be considered. Barrow did this by representing two curves: $y = f(x)$ e $y = g(x)$, in Figure 5, the segment AD represents the abscissa axis where it takes values x . The quantity $g(x)$ represents the value of the area under the graph of f between point A y x . Given an abscissa point D , he tried to demonstrate that the slant of the tangent $y = g(x)$ at point $F(D, g(D))$ is equal to $f(D) = DE$.

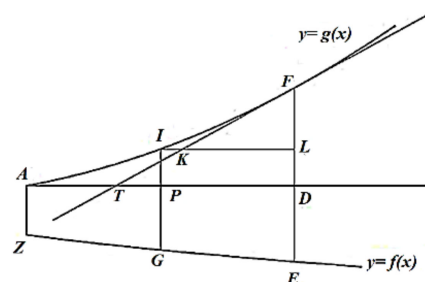


Figure 5. Fundamental theorem.

Demonstration 2. We draw a straight line FT through F that intersects at T line AD and such that, we want to prove that FT it is tangent to $y = g(x)$ at point F . The horizontal distance KL , from any point L on line EF to line FT is less than the distance IL from said point L to the curve $y = g(x)$. This proves that line FT is always below $y = g(x)$. We have that: $\frac{FL}{KL} = \frac{DF}{TD} = DE$. On the other hand: area $ADEZ = FD$; area $APGZ = PI = LD$; area $PDEG = FD - LD = FL$, since area $PDEG < \text{rectangle } PD \cdot DE$. It follows that $< PD \cdot DE$, then $DE > \frac{FL}{PD}$, and, therefore, $\frac{FL}{KL} > \frac{FL}{PD} \rightarrow KL < PD = IL$. We deduce that point K is below curve $y = g(x)$; thus, line FT is on one side of the curve. To complete the demonstration, it is necessary to repeat the reasoning, taking points on the right of EF . This proves that TF is tangential to $y = g(x)$ at D and its slope is $DE = f(D)$. In current terms, what Barrow proved is that: $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.

Following these reasoning's Newton (developing absolute theses) and Leibniz (developing relative theses) positioned the integration operation as a generalization of the

calculation of quadrature in the field of dynamic physics, establishing the inverse relationship between problems of tangents and quadrature's. Both adhered to the physical-mathematical model for the intellection of the natural world; they synthesized and established a systematic algorithmic instrument known as the Infinitesimal Calculus, the Newtonian equivalent of Leibnizian differential and integral calculus. The three main characteristics of this new calculation are: (1) They unified in two general concepts, the integral and the derived, the great variety of techniques and problems that were addressed with particular methods. (2) They developed symbolism and formal calculation rules that could be applied to algebraic and transcendent functions, independent of any geometric meaning that made the use of these general concepts almost automatic. (3) They recognized the fundamental inverse relationship between referral and integration.

Both Newton and Leibniz understood this new calculation differently; Newton used a mathematical calculation while Leibniz developed a logical calculation. On the one hand, Newton elaborated a "purely mathematical reduction" of the quantifiable relationships entity to entity; and, on the other, Leibniz articulated a "strictly logical construction" from minimal (primitive) concepts of expression. Leibnizian doctrine has a more coherent cut than Newtonian philosophy since it provides universal logical tools, which are independent of the object of analysis and thus achieve "absolutely necessary" legitimacy.

The dynamic physics of these two thinkers makes it possible to scrutinize the roots of a conceptual opposition that ended up becoming the confrontation of two divergent and representative worldviews. We believe there were two archetypal ways of conceiving "reality". For Leibniz, logical calculation is the possible construction of complex concepts from the primitive ones by virtue of reason. Leibniz never neglected consistency in the rational construction of his system, which always respects the demands of his own logical principles. His doctrine is crossed by a total commitment to the principle of sufficient reason. Leibniz elaborated a dynamic that facilitates communication between metaphysical and physical considerations, while Newton's new analysis is founded on the fairly use of infinitesimal magnitudes ("moments"; "indefinite" and "infinitely" small magnitudes created from a steady flow in a given time with Cartesian curve graphs (incorporated since equalities).

The history of mathematics evidences Newton's imprint on calculus and mathematical physics in the eighteenth century, generally judged as negative in comparison with Leibniz's achievement. The paradox according to which Leibnizian calculus made progress in the mathematization of the scheme of gravitation is frequently declared as a clear symbol of crisis in the Newtonian field. Ref. [37] (p. 292) states: "The Principia were to remain a fossilized classic, on the wrong side of the border between past and future in the application of mathematics to physics", since, when Newton used algorithms, he was accused of having developed gross notation, a preference for less general geometric proofs compared to the Leibnizian calculus. Ref. [37] (p. 285) mentions: "The Newtonian version of calculus, the fluxions and series method, was crude in notation and inelegant in methodology".

However, we find Newton who suddenly disapproved Descartes's canon of problem examination and construction by synthesizing (More information on: Galuzzi, M. "I marginalia di Newton alla seconda edizione latina della Geometria di Descartes ei problemi ad essi collegati", in: Belgioioso, G., Climino, G., Costabel, P. and Papuli, G. (eds.) Descartes: il Metodo e i Saggi. Istituto dell'Enciclopedia Italiana, Rome 1990, 387–417) it in the inverse method of fluxions. Through this method he was able to tackle the problem of "squaring curves". When considering a superficial t as generated by the flux of the ordinate and sliding at right angles to the abscissa z , he unstated that the percentage of flux superficial of the area is equal to the ordinate (he declared $\frac{\dot{t}}{z} = \frac{y}{1}$). In this way, Newton devised integration as anti-differentiation. His approach was to apply the technique to "equations that will define the ratio of $\dot{t}$ to $\dot{z}$ ". Consequently, an equation is obtained for $\dot{t}$ y $\dot{z}$, where "there will be two equations, the last, that will define the curve, and the first, that will define the area" [38] (p. 197). In Leibnizian terms, he constructed the first integral tables in the history of mathematics, giving importance to the inverse method.

Newton established methods corresponding to integration by parts and substitution. He called “method of series and fluxions” the techniques of expansion of series, determination of tangents and quadrature of curves. This method was a “new analysis” that extended to the themes that Descartes had abolished since his “*common analysis*”—For example mechanical curves— from the use of infinite series. Newton’s “new analysis” is a definitive disruption of the integration operation, making it a new branch of mathematics. Newton, by knowing this, according to the Pappusian canon, knew that this “new analysis” was secondary to the creation, and that it should be carried out in terms independent of algebraic criteria. Hence, Newton’s legacy to his followers was complex. Newton devoted his efforts to developing an elaborate algorithm collected in mutual ex-amination between Arithmetic universalis and the new analysis. He carried out to his followers the idea that the Greek classics were greater to modern mathematicians, and that the ancients had out of view heuristic geometric tools that could, be recovered by patiently examining the remaining texts.

From the early to the middle eighteenth century, Newtonian mathematics grew with thinkers for example Taylor, Stirling, Cotes, De Moivre, and Maclaurin who were dedicated to Newton’s mathematics. Taylor supplemented Newtonian mathematics because he thought that the new method of fluxions was only a specific situation of a bigger theory—the method of increments, that we now call finite difference calculus, as Newton had envisioned in *Methodus differentialis* (1711) (A critical edition can be found in Newton, I. *The Mathematical Papers*. Op. cit., 8, 244–255).

Taylor wrought characteristically with finite variances and found results in the new analysis using limit arguments, leaving finite additions tending to zero. The consequence with these methods was the expansion in Taylor series. In studying infinite series, it was hoped to address quadrature, which, in Leibniz’s words, was related to integration. Newton, working on *De quadratura*, set up a structure of research related to the finite integral which was completed by Roger Cotes in 1714, recorded in his works “*Logometria*” y “*Harmonia mensurarum, sive analysis & synthesis per rationum & angulorum mensuras pro-motæ*” on 1722. Following the Newtonian legacy, Maclaurin clearly mentions the synthetic method of fluxions, as the “precise and elegant” Newtonian method. His goal was to present this method as a relation to Archimedes’ method of exhaustion. He defined fluxion as “the speed with which a quantity flows, at any time limit while it is supposed to be generated (Ibid., [38], (p. 57)).” Maclaurin, agrees with Berkeley in identifying in Newton’s synthetic method of fluxions an ontological basis absent in Leibniz’s differential calculus, since infinitesimals do not have “a real existence”.

It was even mentioned that Newton’s “new analysis” was only a generality of the “Archimedean method”. Maclaurin, in his “*Treatise*”, ruled out doctrines entrenched in the Newtonians; he satisfied the need he felt to provide a firmly anchored target for estimates in the method of fluxions; he argued that the theorems of the calculus were not at all about “fictions” or “phantoms of defunct quantities” as Berkeley held, but were considered kinetic in the sense that Newton held when operating with fluxions and fluxions existing in nature” (Op. cit., [38] (pp. 122–123)).

Eighteenth-century European mathematics evolved with the approaches of Newton and generalised by Maclaurin, evidenced in a change of language, new lines of research and new values underlying mathematical practice. This period constituted a significant change for the nascent infinitesimal calculus. Newton had to face a German competitor (Leibniz), who arrived at results similar to his own, promoting a different view of mathematics (The situations involved in the polemic between Newton and Leibniz are pointed out by [26,30]. In general: Newton developed the system of series and fluxions between 1665 and 1669. Leibniz developed his differential and integral calculus around 1675, published articles from 1684.). Leibniz left to his disciples the choice to hold different approaches about the ontological question of the existence of the infinitesimals; he wanted to defend its usefulness as symbols in mathematical calculations. The global epistemic configuration 2 (GEC2) associated with this period is presented in Table 3.

Table 3. GEC2. The integration operation as a support for the nascent Integral Calculus.

Components	Description
Problem situations	- Heuristic procedures based on the Euclidean demonstration; - Develop techniques to calculate tangents or make quadratures; - Lack of a satisfactory arithmetic theory for immeasurable quantities; - Need to demonstrate; - Use of algebraic symbolism.
Languages	Geometric, arithmetic, analytical, algebraic.
Rules (definitions, propositions, procedures)	- Invention and the use of analytic geometry to solve quadrature, tangent, maximum and minimum problems; - Use of infinity; - Use of geometric algebra; - Derivatives, antiderivatives, integrals; - Introduction of the symbol ∞ to represent infinity; - Considering the infinitely small; - Creation of the infinitesimal calculus; - Generalisation of the calculus of quadrature by Newton and Leibniz, who handled the integration operation as an inverse relationship between problems of tangents and quadratures.
Arguments	Cavalieri maintained integration as an operation, reasoning that, in the Greek style: - It substitutes the evaluation of an infinite sum of infinitely small elements; - It generalises the calculation of quadratures, abandons the method of exhaustion; - It does not give an explicit definition of indivisible. Criticisms of Cavalieri's work involve aspects related to the continuum, infinity and its rhetorical exposition, extensive and intricate geometric reasoning, which make it difficult to read and understand. Conceptual and methodological disruption of Cavalieri's geometric approach give an arithmetization that lead to the implicit use of the limit. Analytics has become the appropriate method to replace geometric intuition in counting and measuring processes. Wallis: - Totally disrupted the rigor of Greek geometry and the Aristotelian tradition of avoiding infinity; - Transformed the quadrature calculation problem into the problem of finding the area under the curve; - Identified four elements that were important in the conceptualisation of the defined integral. Newton: - Positioned the integration operation as a generalisation of the calculation of quadratures in the field of dynamic physics; - Adhered to physical-mathematical models for the intellection of the natural world; - Elaborated procedures corresponding to integration-by-parts and substitution; - Provided the first known integral tables in the history of mathematics. Leibniz. - Postulated logical calculation as the possible construction of complex concepts from primitives by virtue of reason; - Developed logical calculations; - For the new calculation, he provided universal logical tools that are independent of the object of analysis, thus achieving "absolutely necessary" legitimacy. Newton and Leibniz synthesised and established a systematic algorithmic instrument known as infinitesimal calculus (Newtonian differential calculus and Leibnizian integral).

Table 3. Cont.

Components	Description
Relations	- Lack of rigor for techniques used in a heuristic way; - Implementation of general algorithms, in algebraic and non-geometric terms, giving rise to new infinitesimal calculations; - Dominant, implicit idea of the “<i>indefinite integral</i>” as an operator whose development focused on problems that gave rise to “<i>definite integrals</i>.” These two initiated others, discovered but not formalised in that period: “<i>improper integrals</i>”, which would later be formalised as “<i>improper integrals of the first and second species</i>”, arose by extending the notion of integral to unbounded intervals, and to unbounded functions on a bounded interval; - Newton (developing absolute theses) and Leibniz (developing relative theses) positioned the integration operation as a generalisation of the calculus of quadratures in the field of dynamic physics, establishing inverse relationships between problems of tangents and quadratures; - Newton and Leibniz adhered to the physical-mathematical model for the intellection of the natural world, synthesising and establishing a systematic algorithmic instrument known as <i>Infinitesimal Calculus</i> with the following characteristics: - The unification of two general concepts, <i>integral and derived</i>, and the great variety of techniques and problems that were approached with specific methods; - The development of symbolism and formal rules of calculus that could be applied to algebraic and transcendent functions, regardless of any geometric meaning; - Recognition of the fundamental inverse relationship between derivation and integration.

Source: own creation.

5.1.3. Integral Calculus Foundation

Ref. [39] mentions that Jacques Bernoulli suggested the name “integral” to Leibniz. This is an epistemologically significant fact, because “with the incorporation of a name to designate a specific operation, a notion that merits special treatment is being identified” (p. 38). The integral was no longer just a tool to solve the general problem of calculus of quadratures until it became a new concept with its own problems and methods.

Revolutionary changes generated by the “new analysis” proposed by Newton and Leibniz in the eighteenth-century mathematics have been presented in three periods [39]: a geometrical period, where geometrical situations and thoughts predominate; analytical or “algebraic”, started by Euler in 1740 and reaching the end of the century with Lagrange; and a third period starting in the nineteenth century with Cauchy’s writings.

By this time, only a few mathematicians noticed the change from the geometrical to the algebraic period. Ref. [39] mentions that since 1740, Euler was probably the first to think of calculus not as an algorithm for the study of curves or other geometrical objects (as in the works of Leibniz and Newton), but as the study of functions understood as “reasoned expressions composed of variables and constants” (p. 340).

These changes [40] he called the “degeometrization” of 18th century analysis, whose mathematical entities are now functions that can also be multivariate, of the form $f(x, y, z, \dots)$, caused by the study of orthogonal ([40]) trajectories and continuum mechanics ([41]). Approaches to analytical dynamics from the minimum action or from virtual velocities led Euler to deploy the calculation of variations ([40,42]). Given Euler’s importance in this process of de-geometrization of Leibnizian calculus, we share [43] position of giving this new representative theory the name of “Eulerian calculus” to differentiate it from Leibnizian calculus.

A characteristic of this stage was the lack of formalization of his theory due to rigor problems and consequent theoretical foundation, aspects that marked a new stage in the history of the integral, transforming it into the emerging Integral Calculus, developing and formalizing the concepts of integral defined and its extensions (improper integrals). Euler,

in a letter to Goldbach (1744), explains that before writing the manual on Infinitesimal Calculus, he considered that he had to develop a series of previous topics, related to the infinities necessary for the understanding of calculus. He developed progressive tract on differential and integral calculus in the guidelines of his *Introductio in analysin infinitorum* (1748), in *Institutiones calculi differentialis* (1755) and in *Institutiones calculi integralis* (1768–70). Experienced classes of functions, simple and multivariate, as symbolic expressions, the purpose of which was to establish their derivatives and integrals ([43,44]). These treatises have a double nature: a taxonomic nature, where he proposes a classification of functions, and another of an instrumental nature, where he presents the decomposition of polynomials as a product of simple factors (those corresponding to real roots) or double factors (corresponding to imaginary roots). [45] mentions that Euler devised methods of elimination and decomposition in simple fractions, proposing to eliminate any reference made to geometry in the study of variable quantities, through the concept of abstract or universal quantity.

D'Alembert, Lagrange and Laplace following the guidelines proposed by Euler worked on: classes of functions, instead of curves, surfaces, partial differential equations, calculus of variations, analytic mechanics and the algebraic representation of differential and integral calculus. They reached procedures and requirements to operate with symbols and not with geometrical properties, safeguarding the need to move calculus away from geometry (Op. cit., [34] (p. 319)). After Newton and Leibniz, mathematics advanced parallel to the analytical procedure applied in trigonometry, to the discovery of “partial differences” or “partial fluxions”, of the “calculus variationum”. Ref. [46] cites [47] with the advent of the principle of virtual velocities and its use in Lagrange’s *Mécanique analytique* in 1788. He shows how Lagrange uses these tools in astronomy.

On the other hand, mathematicians European continent acknowledged and approved Leibniz’s ideals, where mathematics could be understood as reasoning from symbol manipulation, regardless of metaphysical concerns and specific interpretations. Leibniz allowed his followers to retain different approaches, for example, ontological questions related to the presence of infinitesimals or the roots of negative numbers. He wanted to preserve the use of symbols when performing mathematical calculations. Leibniz urged his followers to ignore interpretative metaphysical questions when working mathematically. Guidelines applied by Euler, Lagrange and Laplace. Ref. [47] (p. 250) mentions that “a feature of the French academy’s growing commitment to analytical methods in physics in the course of the eighteenth century was to override the teleological metaphysics of rational mechanics” (p. 250).

Thus, the concept of function took center stage, the problem of series representation is related to the integration problem, facts that radically transformed infinitesimal calculus. We perceive this change thanks to [43] on the transformation of heat’s works; he extended the domain of functions beyond the continuous ones, and established the conditions that a function must fulfil to be represented in trigonometric series. One of those conditions was the integrability of the function over a given interval, which made it necessary to reconsider the concept of integral. For those times, the integral was considered a necessary solution tool, but it was not the main concept of study. Fourier provided the notation of the extremes of integration which, in modern notation, $f(x) = \frac{d}{dx} \left(\int_a^x f(t) dt \right)$ means that the main problem consisted in the asymptotic development of the function $\int_a^x f(t) dt$, (that is, $\int_a^x f(t) dt = x^n + k$), considered a variant of improper integration. The above refers to the good definition of the function, relating more to the improper integrals of the second kind. Ref. [48] states that with Fourier the integral is seen as the area under the curve, asking the question “how discontinuous can a function be to make it integrable?” (p. 66); however, we found that the formalization of the improper integral was discussed by DeMorgan, in 1830, with convergent series representing the integral $\int_u^{+\infty} x^\alpha e^{-x} dx$ for $u > 0$ arbitrary. Poisson approached the resolution of an improper integral by extending the complex plane, considering: $dx = -i(\cos z + isenz)dz$, deducing that $\int \frac{dx}{x} = [\log(-(\cos z + isenz))]_0^{(2n+1)\pi}$ [48] (pp. 70–71).

Cauchy adopted rigorous methods followed today, such as Cauchy's integral theorem, Cauchy-Riemann's conditions, or Cauchy's sequences. Through the concepts of limit, function, and convergence, he managed to position an analytical definition of an integral defined for continuous functions, proposed the current notation for this type of integrals, replacing the cumbersome Fourier notation $\int f(x) dx \left[\begin{smallmatrix} x = b \\ x = a \end{smallmatrix} \right]$ by $\int_a^b f(x) dx$; formalizing properties of the integral, expressed with the new notation as: (1) $\int_x^{x_0} f(x) dx = - \int_{x_0}^x f(x) dx$; (2) $\int_x^{x_0} (f + ig)(x) dx = \int_x^{x_0} f(x) dx + i \int_x^{x_0} g(x) dx$; and (3) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. With these contributions, Cauchy definitively separated the integral from the differential calculus, demonstrating the inverse relation of the derivative and the integral through the fundamental theorem of calculus in its first historical version, and defined it as a limit of sums.

Dirichlet is credited with the modern "formal" definition of a function. With the characteristic function of rational, he reflected on the relationship with the requirement that infinite points of discontinuity must meet in order for a function to be integrable; he established the false condition that for the function to be integrable, it is sufficient that the discontinuity points form a scattered set. However, Riemann, based on the Cauchy and Dirichlet conceptions, incorporated a definition of integral that welcomed highly discontinuous arbitrary functions. He defined an integral that generalized Cauchy's (Cauchy's Integral Theorem, also known as the Cauchy-Goursat theorem in complex analysis, is a statement concerning line integrals for holomorphic functions on the complex plane), gave a precise definition of the integral of a function defined in an interval that must be bounded and closed. This new integral allowed Volterra to demonstrate the presence of a bounded derivative that was not Riemann-integrable, imposing, in this way, a severe limitation to the Fundamental Theorem of Calculus for the Riemann integral, fact that originated a profound revision of the notion of integral. Hankel tried to generalize the integrability condition of the Riemann function in terms of the jump concept of a function, classifying the functions into integrable and non-integrable. [39] (p. 12) indicates that "Hankel's work initiates the conjunctivist approach to the integration theory that allows us to found modern integration theory" We cannot neglect, in reviewing this evolution of the integral, the analytical mechanics of the 18th century, how remote from the applications pure mathematics of that type were.

Ref. [49] mentions that Borel exhibited the results of his book "*Leçon sur la théorie des fonction*" between 1896–1897 at the Ecole Normale Supérieure in Paris, where [50] was his student, adding to the definition of measure the notion of "numerably additive", extending the ordinary length of an interval to open sets, based on the property: every open set is the countable and disjoint union of open intervals, which he called measurable sets; but he did not study its properties. Lebesgue rigorously analysed those properties, obtaining a special collection of "measurable sets", which he called a σ -algebra. The new notion introduced by Borel is the ideal framework in which Lebesgue developed its integral. Lebesgue delved into the Riemann integral, finding its limitations. In response, the integral of Lebesgue in 1901 emerged, broader than Riemann's, whose development is sustained on the notion of "measure", as in the ancient Greeks. In 1904, Lebesgue defined *measurable functions* as those that allow the development of a much broader and more satisfactory theory of integration than that of Riemann. The path that Riemann, Darboux, and Lebesgue led in the construction of a deeper and more rigorous calculus established the necessary and sufficient conditions for integrability, not only in the Riemann sense, for bounded functions, but also for a significant generalization of the Riemann integral.

The sequential overcoming of these difficulties with the Cauchy and Riemann integrals encouraged the search for a more powerful concept of integral, which Jordan, Borel, and Baire began, and culminated in Lebesgue's definition establishing a solid, strong and structured theory of integration, the latter, based on the idea of changing the partitions of the domain of a function by partitions in the range. Denjoy in 1912, and Perron in 1914 built integrals that managed to integrate any derived function, however, these integrals

turned out to be similar. Today, it is named Denjoy-Perron integral. In 1957, Kurzweil, and two years later, Henstock, each defined integral that solve the derivative inversion problem; years later, they resulted equivalent, and it is currently known as the Henstock-Kurzweil integral. Table 4 shows the structure of the GEC3.

Table 4. GEC3 Integral Calculus Foundation.

Components	Description
Problem situations	- Lack of the formalization of “new calculus” due to problems of rigor and theoretical foundation; - Reconsider the integrability conditions of the function over a given interval; - Problems representing series related to integration. Existence of a bounded derivative which is not Riemann-integrable (limitations to the Riemann integral). - Need to establish necessary and sufficient conditions for the integrability of any function; - Notation of the integral and the extremes of integration; - Need for a good definition of function, giving rise to improper integrals of the second kind; - Difficulties with Cauchy and Riemann integrals; - Formalisation of infinitesimal calculus (differential and integral).
Languages	Analytical, algebraic, geometric.
Rules (definitions, propositions, procedures)	- Degeometrization of 18th century analysis. Expansion of the calculus of variations centred on analytical dynamics in terms of extreme principles (minimum operation or virtual rapidity). - Formalisation of the concept of function by Dirichlet; - Incorporation in the notation of the extremes of integration; - Rigorous definition of the integral by Cauchy and properties; - Precise meaning of the integral of a function defined in an interval that must be bounded and closed; - In-depth review of Riemann’s notion of the integral; - Development and formalisation of the concepts of indefinite and defined integrals and the extensions (improper integrals); - Formalisation of improper integrals of the first and second kinds; - Definition of measurable functions; - The teleological metaphysics of rational mechanics is annulled with the works of Euler, Lagrange, and Laplace.
Arguments	Conception of calculus not as a system for studying curves and geometric objects, but as a tool for analysing functions composed of variables and constants. Work with multivariate functions by analysing their orthogonal trajectories and the continuum mechanics concept of integrals based on rigor and precision, reaching generalisation. Necessary and sufficient conditions are established for integrability, not only for limited functions. Definitive separation of calculus of geometry. Definitive separation of the integral from differential calculus; Cauchy demonstrated the inverse relationship of the derivative and the integral, through the fundamental theorem of calculus in its first historical version. The integral ceases to be a tool to become a new concept with its specific problems and methods. Rescue of the Leibnitzian legacy: mathematics can be understood as reasoning from the manipulation of symbols, regardless of metaphysical concerns. The usefulness of symbols for mathematical calculations is defended. The concept of “measurable sets” that names a σ-algebra is introduced, and the Lebesgue integral emerged.

Table 4. Cont.

Components	Description
Relations	Foundation of Integral Calculus in three periods: - Geometric, when problems and conceptions of geometry predominated, and led to the “degeometrization” of eighteenth-century analysis; - Analytical or “algebraic”, which began in 1740 with Euler and was developed at the end of the century with Lagrange; - Classic analysis, which began at the beginning of the 19th century with Cauchy’s writings; - “Eulerian calculation”, which was detached from the “Leibnitzian calculation”; - Development and formalisation of the concept of indefinite and defined integrals and their extensions (improper integrals); - Formalisation of the improper integral by DeMorgan with convergent series; - Study of simple and multivariate functions, instead of curves and surfaces; - Positioning of an analytical definition of definite integrals for continuous functions, by means of concepts of limit, function, and convergence; - Definition of measurable functions; - Required and appropriate conditions are established for the integrability of any function.
	Formalisation of: - Partial differential equations; - The calculus of variations; - Analytical mechanics; - The algebraic character of differential and integral calculus; - Riemann, Darboux and Lebesgue built a deeper and more rigorous calculus, established the necessary and sufficient conditions for integrability; - New integrals: Denjoy–Perron and Henstock–Kurzweil

Source: own creation.

In OSA the appearance of the secondary object is treated as a global mention of one or several configurations of primary objects, which is describe by the shared effect and produced by the processes associated with five different dualities [5]. The unitary-systemic duality allows considering a unitary configuration, for example, GEC1-3, Tables 2–4; or the set formed by the three configurations, as a systemic entity, Figure 6, given that, when a new topic is studied, what is done is a systemic presentation of the topic, the socio-epistemic configurations are studied and the practices that these configurations allow. However, when a new theme is initiated, the previously studied configuration and the practices that it enables are considered as a whole, as something known and, consequently, formed by unitary (elementary) entities. These same objects have to be considered systemically in order to be learned [5].

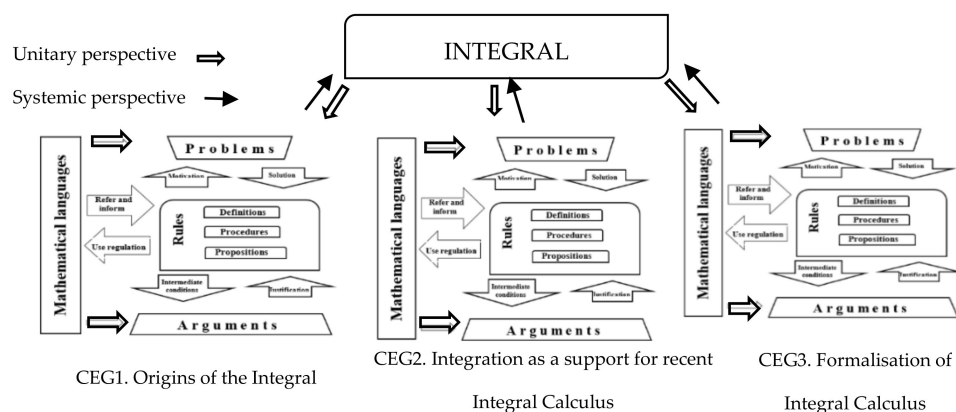


Figure 6. The integral object as a global reference.

Hence, Figure 6 facilitates managing the complexity of the integral by identifying three large (unitary) meanings, which allow the teacher to use them to present partial meanings for the integral, e.g., Barrow's rule for calculating definite integrals, calculating the area under a curve, properties of the integral, methods of integration, among others, which only from a well-structured articulation will it be possible to understand the global context in which integral calculus is applied in extra-mathematical situations. This allows to duplicate the considered object, to identify its illustration and the represented object as different entities; allows the teacher to show that the different configurations (GEC1-3) are partial displays and explanations of the emergent object, emerging from mathematical practices useful to present and formalize specific situations such as: types of integral, extensions to the concept of integral, applications of improper integrals in complex analysis, to name a few. The ostensive-non-ostensive duality approves to reflect that the symbolised object is an ideal object dissimilar from its material representations, while the extensional-intentional duality leads to think, in general, that object as a general "something", the integral calculus, which achieves objectivity by considering the personal-institutional duality. The grouping of these dichotomies produces the appearance of a global reference not only for the integral but also for integral calculus, on which it is possible to carry out certain actions in order to improve the pedagogical practices implemented. In this process, the interaction and intersubjectivity of the subjects who construct and reconstruct their representations are fundamental to enable quality teaching and learning, which is fundamental in higher education.

5.2. Results Related to Experimentation with University Students

After analysing the curriculum of the subject and the textbooks proposed in the bibliography, the Calculus 2 syllabus was analysed, seeking to identify partial meanings for the integral. This classification was compared with the meanings identified in the three global epistemic configurations proposed in this study. It was found that the integral appeared through representations, different definitions, propositions, procedures, and arguments in the following order: first, indefinite integrals are presented; then, definite integrals, integration methods, calculations of areas between curves, and improper integrals. In each of these situations there are some applications which end up being more exercises than problems that allow modelling. In the syllabuses analysed, students are expected to master integration techniques and understand the integral basically as an operator, but they should not seek to develop mathematical competences to apply the integral to solve problems in different contexts.

Based on this situation, the programmes were adjusted by designing and implementing a sequence of activities aimed at presenting a representative sample of partial meanings for the integral connected to each other, which would allow mathematical competences to be developed in different contexts. In particular, a balance was sought between the conceptual development of the basic ideas of integral calculus with the appropriate handling of its algorithms, thus offering a global meaning for the integral, where it is identified as a systemic entity, allowing students to develop specific skills such as abstracting, representing, conceptualising, generalising and synthesising; in other words, developing competences in the use of the integral when solving a variety of problems proposed in different contexts. For reasons of space, it is omitted here, as it is explained in detail in [8]. The implementation provided evidence that students connected the interpretation of the integral as an operator with the generalisation of the sum of infinitely small sums (as a continuous sum) and with the notion of function. They also made additional mathematical connections, applying the integral to other contexts: continuity equations present in physics problems, calculation of the quantity of motion, energy, total rate of change of a moving mobile. They recognised the integral as a useful tool for modelling problems in other sciences involving continuously varying quantities, as it facilitates the interpretation of different phenomena observed when performing experimental operations. They calculated continuously varying areas, volumes, velocities, resistances. They accurately applied the integral operator

iteratively when working with functions of two and three variables, which allowed them to understand phenomena that require numerical determination, whether to calculate areas of flat surfaces with a double integral, volumes of bodies with a double or triple integral, area of surfaces with a surface integral, centres of gravity and moments of inertia, among others. From their productions, there was evidence that the students in these groups managed to use a representative sample of the global meaning of the integral that allowed them to solve a variety of problems in different contexts. I was also found that, during the following year, the students in the focus groups and now taking Vector Calculus proved for themselves that when applying line integrals in a vector field, this coincides with the line integral of a scalar field. Achievements that the teachers in charge highlighted in these students, aspects referenced in [8] and other extensions in [9].

6. Final Considerations

6.1. In Relation to the Complexity of the Integral

In this work we have exposed that emerges a secondary object, called an integral, that plays the global reference role of all the primary object configurations that have allowed us to model the complexity of the integral. This global reference in the mathematical activity takes the form of a specific configuration of primary objects. Therefore, what can be done with this second-level object is determined by this configuration of primary (first level) objects. In OSA, the entity that assumes the role of global reference is seen as simultaneously single and multiple, since metaphorically, it is interpreted as a multiplicity of options opening up from associated primary objects in different configurations.

Table 1 shows the characterisation of the complexity of the integral, considering the three established periods. We consider them useful in solving problems in intra and extra mathematical contexts that involve application of the different meanings for the integral. For this reason, we consider it pertinent to know that, although a single meaning is intended for this object, there is an epistemic complexity that requires an articulation of partial (primary) meanings, and that only from a well-structured articulation will be possible to understand the global context in which the integral calculus is applied. Hence, we share [51] position when he indicates that this rethinking leads us to assume that mathematical knowledge is not an objective replica of a single reality external to the subject, but rather a personal and social construction of meanings, the result of a historical evolution, a cultural process in permanent development, located in a specific context. We consider that, in this process, the interaction and intersubjectivity of the subjects who build and reconstruct their representations are essential to enable quality teaching and learning, fundamental in higher education. Hence, one of the contributions of this work is the characterisation of the complexity of the integral object through partial meanings.

This scope leads us to study the complexity of the concept, since the current trend is to consider that mathematics should be applied to extra mathematical contexts (which entails reflection on the complexity of the mathematical objects taught). By sharing this articulation of meanings for the integral with the students group, we demonstrated development of advanced mathematical thinking skills such as: abstract, visualize, estimate, justify, reason under hypothesis, categorize, conjecturing, generalize, synthesize, define, significant advances in demonstrate and formalize, which enables them to know how, that is, for an illustrated doing that implies: enlightened action and performance, transversal use of knowledge, design of appropriate ways to formulate and solve problems, not only in intra and extra-school contexts of their mathematical knowledge, but also expanding their areas of proximal development, by assuming cognitive and volitional challenges and “risks” in their subsequent professional work.

Some Suggestions for Teaching the Integral

The complex look applied to the mathematical object allows to deepen into the connection process between its partial meanings. The complexity, structured in terms of a set of epistemic configurations, specifies which components are to be connected. We agree

with [52] on the concept of integrality, considered from the different stages of its historical, development opens a vast range of possibilities to apply it to problem situations in different areas of knowledge; that ratifies the importance of our classification (GEC1, GEC2, GEC3). Such range leads us to study the complexity of the concept, given that the tendency is to consider that mathematics should be applied to extra-mathematical contexts, without realizing that the process involves reflection on the complexity of the mathematical objects.

It is about going from a naive and optimistic point of view, which presupposes that the student will easily carry out the transfer of the mathematical knowledge generated in a single context to other new and different contexts. Another more prudent point of view is that, although it is considered that the possibility of creative transfer can occur, we assume that without a work on a representative sample of the complexity of the mathematical object that is to be taught, involving the articulation and connection of the components of this complexity, the students will hardly be able apply the mathematical object to different contexts.

The above thoughts allow us to share the proposal in [7] that a strategy to ensure students' competence in the use of integral for problem-solving consists of designing sequences of tasks aimed at presenting different partial meanings of the integral connected to each other. We implemented this strategy with three groups of students, where we evidenced development of advanced mathematical thinking skills, mentioned in [8]. Therefore, this work's second contribution was to bring forward briefly a teaching and learning experience of the integral oriented to present a representative sample of partial meanings of the well-connected integral, which allowed us to find evidence of the development of students' competence in use of the integral to solve problems in different contexts.

6.2. Limitations of the Study

There were some limitations in the development of the research, among the most important of which are the following:

- It was quite difficult to access primary bibliographic sources; therefore, resorting to them was onerous;
- Proposing a paradigm shift in the curricular structure that the faculties had in place was not easy. Persuading them to allow the project to be developed by making adjustments to the established curriculum structure, applying it, and looking at its benefits, advantages and difficulties, was hard work;
- Systematising the information collected, which covered more than 20 centuries, was a time-consuming task, requiring almost exclusive work and dedication of time for more than three years, in order to be able to organise and present the work exposed here.

6.3. The Prospective of the Research

Studies such as the present one, which historiographically traces the evolution of a mathematical concept that, for those who study it, is considered difficult to learn, allows the teacher to: reflect on the complexity of the object to be taught; reorient the instructional process with a view to achieving changes in teaching paradigms traditionally centred on a formal mechanistic approach, which does little to enable students to develop mathematical skills; and recognise that the difficulties linked to the epistemic complexity of mathematical objects are often the origin of multiple errors, difficulties and obstacles that students face when they fail to find connections and articulations of the concepts studied with the everyday problem situations they face, preventing them from being mathematically competent. It can also be interpreted as a model for other researchers interested in improving their pedagogical practices by studying the intrinsic complexity of other mathematical objects that, due to their epistemological nature, are difficult for students to learn.

How to use the results of this study: This work has identified the epistemic complexity of the integral mathematical object, and its evolution and articulation until it became Integral Calculus. Metaphorically, it presents a thorough and detailed examination of the

way in which this branch of mathematics was constituted which, step by step, progressed, giving answers to different everyday situations in each historical period in which it was developed, considering its state and the factors that intervened for its progress. In spite of responding to some problematic situations, other unsolved situations remained open, which led to the emergence of other more elaborated concepts that made it possible to respond satisfactorily to different situations and at the same time to receive an adequate foundation that consolidated Integral Calculus as another branch of mathematics. Awareness of this complexity allows teachers to identify different meanings for the integral which, when articulated and well connected when planning their classes, allows them to select specific problem situations that enable students to understand different meanings for the integral, to give them meaning, and to know how and when they can use them to find solutions to everyday situations specific to their professional work; in other words, to develop mathematical competences. Following the model proposed here for the integral can be a guide for developing other similar studies for different mathematical objectives.

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