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On an Extension of a Hardy–Hilbert-Type Inequality with Multi-Parameters

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Abstract: Making use of weight coefficients as well as real/complex analytic methods, an extension of a Hardy–Hilbert-type inequality with a best possible constant factor and multiparameters is established. Equivalent forms, reverses, operator expression with the norm, and a few particular cases are also considered.

Keywords: Hardy–Hilbert-type inequality; weight coefficient; equivalent form; reverse; operator

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1. Introduction

In this paper, we generalize the classical Hardy–Hilbert inequality, which can be stated as follows: assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} > 0,$$

$||g||_q > 0$. We have the following Hardy–Hilbert integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$.

If $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$,

$$||a||_p = \left\{ \sum_{m=1}^\infty a_m^p \right\}^{\frac{1}{p}} > 0,$$

$||b||_q > 0$, then we have the following Hardy–Hilbert inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1–12]). Assuming that $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N} = \{1, 2, \cdots\})$,

$$U_m := \sum_{i=1}^{m} \mu_i, V_n := \sum_{j=1}^{n} \upsilon_j \ (m, n \in \mathbf{N}), \tag{3}$$

we have the following inequality (cf. [1], Theorem 321, replacing $\mu_m^{1/q} a_m$ (resp. $\upsilon_n^{1/p} b_n$) by a_m (resp. b_n)):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i = \upsilon_j = 1$ $(i, j \in \mathbf{N})$, (4) reduces to (2). Inequality (4) is known as Hardy–Hilbert-type inequality.

Note. The authors of [1] did not prove that the constant factor in (4) is the best possible.

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [13] provided an extension of (1) for $p = q = 2$. Improving upon the method of [13], Yang [6] presented the following best possible extensions of (1) and (2):

If $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{ \int_0^\infty \phi(x) |f(x)|^p dx \}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{5}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ keeps a finite value and $k_\lambda(x, y) x^{\lambda_1 - 1}$ $(k_\lambda(x, y) y^{\lambda_2 - 1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \{ \sum_{n=1}^{\infty} \phi(n) |a_n|^p \}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, it follows that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_\lambda(m, n) a_m b_n < k(\lambda_1) ||a||_{p,\phi} ||b||_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for

$$\lambda = 1, \ k_1(x, y) = \frac{1}{x + y}, \ \lambda_1 = \frac{1}{q}, \ \lambda_2 = \frac{1}{p},$$

inequality (5) reduces to (1), while (6) reduces to (2).

For $s \in \mathbf{N}, 0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x, y) = \frac{1}{\prod_{k=1}^{s} (x^{\lambda/s} + c_k y^{\lambda/s})} \ (0 < c_1 < \cdots < c_s).$$

Then, by (6), we derive that (cf. [14])

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(m^{\lambda/s}+c_k n^{\lambda/s})} < k_s(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{7}$$

where the constant factor

$$k_s(\lambda_1) = \frac{\pi s}{\lambda \sin(\frac{\pi s \lambda_1}{\lambda})}\sum_{k=1}^{s} c_k^{\frac{s\lambda_1}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j - c_k}\ (\in \mathbf{R}_+) \tag{8}$$

is the best possible.

Some other kinds of results, such as Hilbert-type integral inequalities, half-discrete Hilbert-type inequalities, and multidimensional Hilbert-type inequalities are provided in [15–42].

In the present paper, making use of weight coefficients as well as real/complex analytic methods, a Hardy–Hilbert-type inequality with a best possible constant factor and multiparameters is established (for $p > 1$). This inequality constitutes an extension of (4) and (7). Equivalent forms, reverses (two cases of $0 < p < 1$ and $p < 0$), operator expression with the norm, and a few particular cases are also considered.

2. Some Lemmas

In this section we prove the inequalities of the weight functions, which are used to prove the main results. In the sequel, we assume for the multiparameters that $p \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$,

$$\frac{1}{p}+\frac{1}{q} = 1,\ 0 < \lambda_1,\ \lambda_2 \leq 1,\ \lambda_1 + \lambda_2 = \lambda,\ 0 < c_1 \leq \cdots \leq c_s\ (s \in \mathbf{N}),$$

$k_s(\lambda_1)$ is indicated by (8), $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N})$, U_m and V_n are defined by (3), $a_m, b_n \geq 0$ $(m, n \in \mathbf{N})$,

$$||a||_{p,\Phi_\lambda} = \left(\sum_{m=1}^{\infty}\Phi_\lambda(m)a_m^p\right)^{\frac{1}{p}} \in \mathbf{R}_+,\ ||b||_{q,\Psi_\lambda} = \left(\sum_{n=1}^{\infty}\Psi_\lambda(n)b_n^q\right)^{\frac{1}{q}} \in \mathbf{R}_+,$$

where we define

$$\Phi_\lambda(m) := \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\ \Psi_\lambda(n) := \frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}}\ (m, n \in \mathbf{N}).$$

Lemma 1. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$,

$$z_k \in \mathbf{C} \backslash \{z | Re(z) \geq 0,\ Im(z) = 0\}\ \ (k = 1, 2, \cdots, n)$$

are different points, the function $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* z_i $(i = 1, 2, \cdots, n)$, *and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, then for* $\alpha \in \mathbf{R}$, *we have*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}}\sum_{k=1}^{n} Re(s)[f(z)z^{\alpha-1}, z_k], \tag{9}$$

where

$$0 < Im(\ln z) = \arg z < 2\pi.$$

In particular, if z_k $(k = 1, \cdots, n)$ *are all poles of order 1, setting*

$$\varphi_k(z) = (z - z_k)f(z)\ (\varphi_k(z_k) \neq 0),$$

then

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{\pi}{\sin \pi\alpha}\sum_{k=1}^{n}(-z_k)^{\alpha-1}\varphi_k(z_k). \tag{10}$$

Proof. By [43] (p. 118), we obtain (9). We have that

$$\begin{aligned} 1-e^{2\pi\alpha i} &= 1-\cos 2\pi\alpha - i\sin 2\pi\alpha \\ &= -2i\sin\pi\alpha(\cos\pi\alpha + i\sin\pi\alpha) = -2ie^{i\pi\alpha}\sin\pi\alpha. \end{aligned}$$

In particular, since

$$f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1}),$$

it is obvious that

$$Re(s)[f(z)z^{\alpha-1}, -a_k] = {z_k}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then, by (9), we obtain (10).
This completes the proof of the lemma. □

Example 1. *For $s \in \mathbf{N}$, $\varepsilon > 0$, we set*

$$k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(x^{\lambda/s} + c_k y^{\lambda/s})},$$

and $\widetilde{c}_k = c_k + (k-1)\varepsilon$ $(k = 1,\cdots,s)$. By (10), we get that

$$\begin{aligned} \widetilde{k}_s(\lambda_1) \quad &: = \int_0^\infty \prod_{k=1}^{s}\frac{1}{t^{\lambda/s}+\widetilde{c}_k}t^{\lambda_1-1}dt \\ &= \frac{s}{\lambda}\int_0^\infty \prod_{k=1}^{s}\frac{1}{u+\widetilde{c}_k}u^{\frac{s\lambda_1}{\lambda}-1}du \\ &= \frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^{s}\widetilde{c}_k^{\frac{s\lambda_1}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{\widetilde{c}_j-\widetilde{c}_k} \in \mathbf{R}_+. \end{aligned}$$

Since we have

$$\begin{aligned} 0 \quad &< \quad \widetilde{k}_s(\lambda_1) = \frac{s}{\lambda}\int_0^\infty \prod_{k=1}^{s}\frac{1}{u+\widetilde{c}_k}u^{\frac{s\lambda_1}{\lambda}-1}du \\ &\leq \frac{s}{\lambda}\int_0^\infty \frac{1}{(u+c_1)^s}u^{\frac{s\lambda_1}{\lambda}-1}du \\ &= \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}}\int_0^\infty \frac{1}{(v+1)^s}v^{\frac{s\lambda_1}{\lambda}-1}dv \\ &= \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}}B(\frac{s\lambda_1}{\lambda},\frac{s\lambda_2}{\lambda}) \in \mathbf{R}_+, \end{aligned}$$

it follows that

$$\begin{aligned} k_s(\lambda_1) &= \lim_{\varepsilon\to 0^+}\widetilde{k}_s(\lambda_1) \\ &= \frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\lambda_1}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k} \in \mathbf{R}_+. \end{aligned}$$

In particular, for $s = 1$, we obtain

$$k_1(\lambda_1) = \frac{1}{\lambda}\int_0^\infty \frac{u^{(\lambda_1/\lambda)-1}}{u+c_1}du = \frac{\pi}{\lambda c_1^{\lambda_2/\lambda}\sin(\frac{\pi\lambda_1}{\lambda})}; \tag{11}$$

for $c_s = \cdots = c_1$, we derive that

$$k(\lambda_1) := \int_0^\infty \frac{t^{\lambda_1-1}}{(t^{\lambda/s}+c_1)^s}dt = \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}}B(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}). \tag{12}$$

Lemma 2. *Define the following weight coefficients:*

$$\begin{aligned}\omega_s(\lambda_2, m) \quad &: = \sum_{n=1}^\infty \frac{1}{\prod_{k=1}^s (U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{\lambda_1} v_n}{V_n^{1-\lambda_2}}, m \in \mathbf{N}, &(13)\\ \varpi_s(\lambda_1, n) \quad &: = \sum_{m=1}^\infty \frac{1}{\prod_{k=1}^s (U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{V_n^{\lambda_2} \mu_m}{U_m^{1-\lambda_1}}, n \in \mathbf{N}. &(14)\end{aligned}$$

Then, we have the following inequalities:

$$\begin{aligned}\omega_s(\lambda_2, m) \quad &< \quad k_s(\lambda_1)\ (0 < \lambda_2 \le 1, \lambda_1 > 0; m \in \mathbf{N}), &(15)\\ \varpi_s(\lambda_1, n) \quad &< \quad k_s(\lambda_1)\ (0 < \lambda_1 \le 1, \lambda_2 > 0; n \in \mathbf{N}). &(16)\end{aligned}$$

Proof. We set

$$\mu(t) := \mu_m, t \in (m-1, m]\ (m \in \mathbf{N});\ v(t) := v_n, t \in (n-1, n]\ (n \in \mathbf{N}),$$

$$U(x) := \int_0^x \mu(t)dt\ (x \ge 0), V(y) := \int_0^y v(t)dt\ (y \ge 0).$$

Then by (3), it follows that

$$U(m) = U_m, V(n) = V_n\ (m, n \in \mathbf{N}).$$

For $x \in (m-1, m]$,

$$U'(x) = \mu(x) = \mu_m\ (m \in \mathbf{N});$$

for $y \in (n-1, n]$,

$$V'(y) = v(y) = v_n\ (n \in \mathbf{N}).$$

Since $V(y)$ is strictly increasing in $(n-1, n]$, $\frac{\lambda}{s} > 0$ and $1 - \lambda_2 \ge 0$, in view of the decreasing property, we obtain that

$$\begin{aligned}\omega_s(\lambda_2, m) \quad &= \quad \sum_{n=1}^\infty \int_{n-1}^n \frac{1}{\prod_{k=1}^s (U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}}V'(y)dy\\ &< \quad \sum_{n=1}^\infty \int_{n-1}^n \frac{1}{\prod_{k=1}^s (U_m^{\lambda/s} + c_k V^{\lambda/s}(y))}\frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(y)}V'(y)dy.\end{aligned}$$

Setting

$$t = \left(\frac{U_m}{V(y)}\right)^{\lambda/s},$$

we obtain

$$V'(y)dy = -\frac{s}{\lambda}U_m t^{-\frac{s}{\lambda}-1}dt$$

and

$$\begin{aligned}\omega_s(\lambda_2, m) &< \frac{-s}{\lambda}\sum_{n=1}^{\infty}\int_{(\frac{U_m}{V(n-1)})^{\lambda/s}}^{(\frac{U_m}{V(n)})^{\lambda/s}} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1}dt \\ &= \frac{s}{\lambda}\int_{(\frac{U_m}{V(\infty)})^{\lambda/s}}^{\infty} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1}dt \\ &\le \frac{s}{\lambda}\int_{0}^{\infty} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1}dt = k_s(\lambda_1).\end{aligned}$$

Since $U(x)$ is strictly increasing in $(m-1, m]$, $\frac{\lambda}{s} > 0$ and $0 < \lambda_1 \le 1$, similarly, we have

$$\begin{aligned}\varpi_s(\lambda_1, n) &= \sum_{m=1}^{\infty}\int_{m-1}^{m} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{V_n^{\lambda_2} U'(x)}{U_m^{1-\lambda_1}} dx \\ &< \sum_{m=1}^{\infty}\int_{m-1}^{m} \frac{1}{\prod_{k=1}^{s}(U^{\lambda/s}(x) + c_k V_n^{\lambda/s})} \frac{V_n^{\lambda_2} U'(x)}{U^{1-\lambda_1}(x)} dx \ (t = (\frac{U(x)}{V_n})^{\lambda/s}) \\ &= \frac{s}{\lambda}\sum_{m=1}^{\infty}\int_{(\frac{U(m-1)}{V(n)})^{\lambda/s}}^{(\frac{U(m)}{V(n)})^{\lambda/s}} \frac{t^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^{s}(t+c_k)} dt \\ &= \frac{s}{\lambda}\int_{0}^{(\frac{U(\infty)}{V(n)})^{\lambda/s}} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1}dt \le k_s(\lambda_1).\end{aligned}$$

Hence, we deduce (15) and (16).

This completes the proof of the lemma. □

Lemma 3. *If $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ $(m \in \{m_0, m_0+1, \cdots\})$, $\upsilon_n \ge \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, $U(\infty) = V(\infty) = \infty$, then*
(i) for $m, n \in \mathbf{N}$, we have

$$k_s(\lambda_1)(1-\theta(\lambda_2, m)) < \omega_s(\lambda_2, m) \ (0 < \lambda_2 \le 1, \lambda_1 > 0), \tag{17}$$
$$k_s(\lambda_1)(1-\vartheta(\lambda_1, n)) < \varpi_s(\lambda_1, n) \ (0 < \lambda_1 \le 1, \lambda_2 > 0), \tag{18}$$

where

$$\theta(\lambda_2, m) = O\left(\frac{1}{U_m^{\lambda_2}}\right) \in (0,1), \vartheta(\lambda_1, n) = O\left(\frac{1}{V_n^{\lambda_1}}\right) \in (0,1);$$

(ii) for any $a > 0$, we have

$$\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+a}} = \frac{1}{a}\left(\frac{1}{U_{m_0}^{a}} + aO(1)\right), \tag{19}$$
$$\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+a}} = \frac{1}{a}\left(\frac{1}{V_{n_0}^{a}} + a\widetilde{O}(1)\right). \tag{20}$$

Proof. Since $v_n \geq v_{n+1}$ $(n \geq n_0), 1-\lambda_2 \geq 0$ and $V(\infty) = \infty$, we have

$$\begin{aligned}
\omega_s(\lambda_2, m) &\geq \sum_{n=n_0}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} v_{n+1} \\
&> \sum_{n=n_0}^{\infty} \int_n^{n+1} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V^{\lambda/s}(y))} \frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(y)} V'(y) dy \\
&= \frac{-s}{\lambda} \sum_{n=n_0}^{\infty} \int_{(\frac{U_m}{V(n)})^{\lambda/s}}^{(\frac{U_m}{V(n+1)})^{\lambda/s}} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1} dt \\
&= \frac{s}{\lambda} \int_{(\frac{U_m}{V(\infty)})^{\lambda/s}}^{(\frac{U_m}{V_{n_0}})^{\lambda/s}} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1} dt \\
&= \frac{s}{\lambda} \int_0^{(\frac{U_m}{V_{n_0}})^{\lambda/s}} \frac{t^{\frac{s\lambda_1}{\lambda}-1}}{\prod_{k=1}^{s}(t+c_k)} dt = k_s(\lambda_1)(1-\theta(\lambda_2, m)),
\end{aligned}$$

where

$$\theta(\lambda_2, m) := \frac{s}{\lambda k_s(\lambda_1)} \int_{(\frac{U_m}{V_{n_0}})^{\lambda/s}}^{\infty} \frac{1}{\prod_{k=1}^{s}(t+c_k)} t^{\frac{s\lambda_1}{\lambda}-1} dt \in (0,1).$$

We obtain

$$\begin{aligned}
0 &< \theta(\lambda_2, m) \leq \frac{s}{\lambda k_s(\lambda_1)} \int_{(\frac{U_m}{V_{n_0}})^{\lambda/s}}^{\infty} \frac{1}{t^s} t^{\frac{s\lambda_1}{\lambda}-1} dt \\
&= \frac{s}{\lambda k_s(\lambda_1)} \int_{(\frac{U_m}{V_{n_0}})^{\lambda/s}}^{\infty} t^{\frac{-s\lambda_2}{\lambda}-1} dt = \frac{1}{\lambda_2 k_s(\lambda_1)} (\frac{V_{n_0}}{U_m})^{\lambda_2},
\end{aligned}$$

and then

$$\theta(\lambda_2, m) = O\left(\frac{1}{U_m^{\lambda_2}}\right).$$

Hence, we deduce (17). Similarly, we obtain (18).
For $a > 0$, we have that

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+a}} &= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^{\infty} \frac{\mu_m}{U_m^{1+a}} \\
&= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U_m^{1+a}} dx \\
&< \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U^{1+a}(x)} dx \\
&= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}} + \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+a}(x)} = \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}} + \frac{1}{aU_{m_0}^a} \\
&= \frac{1}{a}\left(\frac{1}{U_{m_0}^a} + a\sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+a}}\right),
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+a}} \geq \sum_{m=m_0}^{\infty} \frac{\mu_{m+1}}{U_m^{1+a}} = \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x)}{U_m^{1+a}} dx \\
&> \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x)dx}{U^{1+a}(x)} = \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+a}(x)} = \frac{1}{aU_{m_0}^a}.
\end{aligned}$$

Hence, we derive (19). Similarly, we also get (20).
This completes the proof of the lemma. □

3. Main Results and Operator Expressions

In this section, by using Lemma 3, we obtain Theorems 1 and 2.

Theorem 1. *For $p > 1$, we have the following equivalent inequalities:*

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}} + c_k V_n^{\frac{\lambda}{s}})} < k_s(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \tag{21}$$

$$J := \left\{\sum_{n=1}^{\infty}\frac{v_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}} + c_k V_n^{\frac{\lambda}{s}})}\right]^p\right\}^{\frac{1}{p}} < k_s(\lambda_1)||a||_{p,\Phi_\lambda}. \tag{22}$$

Proof. By Hölder's inequality with weight (cf. [44]), we have

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right]^p \\
&= \left[\sum_{m=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}} + c_k V_n^{\frac{\lambda}{s}})}\left(\frac{U_m^{(1-\lambda_1)/q} v_n^{1/p} a_m}{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}\right)\left(\frac{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}{U_m^{(1-\lambda_1)/q} v_n^{1/p}}\right)\right]^p \\
&\le \sum_{m=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\left(\frac{U_m^{(1-\lambda_1)p/q} v_n}{V_n^{1-\lambda_2}\mu_m^{p/q}} a_m^p\right) \\
&\quad\times\left[\sum_{m=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{V_n^{(1-\lambda_2)(q-1)}\mu_m}{U_m^{1-\lambda_1} v_n^{q-1}}\right]^{p-1} \\
&= \frac{(\varpi_s(\lambda_1, n))^{p-1}}{V_n^{p\lambda_2 - 1} v_n}\sum_{m=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p.
\end{aligned} \tag{23}$$

In view of (16), we obtain that

$$\begin{aligned}
J &\le (k_s(\lambda_1))^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p\right]^{\frac{1}{p}} \\
&= (k_s(\lambda_1))^{\frac{1}{q}}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p\right]^{\frac{1}{p}} \\
&= (k_s(\lambda_1))^{\frac{1}{q}}\left[\sum_{m=1}^{\infty}\omega_s(\lambda_2, m)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{24}$$

Then, by (15), we have (22).
By Hölder's inequality (cf. [44]), we obtain that

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\left[\frac{v_n^{1/p}}{V_n^{\frac{1}{p}-\lambda_2}}\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right]\left(\frac{V_n^{\frac{1}{p}-\lambda_2}}{v_n^{1/p}} b_n\right) \\
&\le J||b||_{q,\Psi_\lambda}.
\end{aligned} \tag{25}$$

Then, by (22), we derive (21). On the other hand, assuming that (21) is valid, we set

$$b_n := \frac{v_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right]^{p-1}, n \in \mathbf{N}.$$

Then, we get that $J^p = ||b||_{q,\Psi_\lambda}^q$.

If $J = 0$, then (22) is trivially valid; if $J = \infty$, then by (24) and (15), this is impossible. Suppose that $0 < J < \infty$. By (21), it follows that

$$||b||_{q,\Psi_\lambda}^q = J^p = I < k_s(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \quad (26)$$

$$||b||_{q,\Psi_\lambda}^{q-1} = J < k_s(\lambda_1)||a||_{p,\Phi_\lambda}, \quad (27)$$

and then (22) follows, which is equivalent to (21).

This completes the proof of the theorem. □

Theorem 2. *If $p > 1$, $m_0, n_0 \in \mathbf{N}$, $\mu_m \geq \mu_{m+1}$ $(m \in \{m_0, m_0+1, \cdots\})$, $\upsilon_n \geq \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, $U(\infty) = V(\infty) = \infty$, then the constant factor $k_s(\lambda_1)$ in (21) and (22) is the best possible.*

Proof. For $\varepsilon \in (0, p\lambda_1)$, we set

$$\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \ (\in (0,1)), \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} \ (> 0),$$

and

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - \varepsilon - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n. \quad (28)$$

Then, by (19) and (20), we have

$$\begin{aligned} ||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\Psi_\lambda} &= \left(\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon \widetilde{O}(1)\right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \widetilde{I} \ :&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \\ &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{\mu_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{V_n^{\widetilde{\lambda}_2}}{U_m^{1-\widetilde{\lambda}_1}}\right] \frac{\upsilon_n}{V_n^{\varepsilon+1}} \\ &= \sum_{n=1}^{\infty} \varpi_s(\widetilde{\lambda}_1, n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \geq k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}(1 - \vartheta(\widetilde{\lambda}_1, n))\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\ &= k_s(\widetilde{\lambda}_1)\left(\sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{\varepsilon+1}} - \sum_{n=1}^{\infty} O(\frac{\upsilon_n}{V_n^{\frac{\varepsilon}{q}+\lambda_1+1}})\right) \\ &= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon(\widetilde{O}(1) - O(1))\right]. \end{aligned}$$

If there exists a positive constant $K \leq k_s(\lambda_1)$, such that (21) is valid when we replace $k_s(\lambda_1)$ by K, then in particular, we have

$$\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\Psi_\lambda},$$

namely

$$k_s(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon(\widetilde{O}(1)-O(1))\right]$$
$$< \quad K\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k_s(\lambda_1) \leq K$ $(\varepsilon \to 0^+)$. Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (21).

The constant factor $k_s(\lambda_1)$ in (22) is still the best possible. Otherwise, we would reach a contradiction by (25) that the constant factor in (21) is not the best possible.

This completes the proof of the theorem. □

For $p > 1$,

$$\Psi_\lambda^{1-p}(n) = \frac{v_n}{V_n^{1-p\lambda_2}},$$

we define the following normed spaces:

$$\begin{aligned} l_{p,\Phi_\lambda} \quad &: \quad = \{a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\Phi_\lambda} < \infty\}, \\ l_{q,\Psi_\lambda} \quad &: \quad = \{b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\Psi_\lambda} < \infty\}, \\ l_{p,\Psi_\lambda^{1-p}} \quad &: \quad = \{c = \{c_n\}_{n=1}^{\infty}; ||c||_{p,\Psi_\lambda^{1-p}} < \infty\}. \end{aligned}$$

Assuming that $a = \{a_m\}_{m=1}^{\infty} \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}, n \in \mathbf{N},$$

we can rewrite (22) as:

$$||c||_{p,\Psi_\lambda^{1-p}} < k_s(\lambda_1)||a||_{p,\Phi_\lambda} < \infty,$$

namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 1. *Define a Hilbert-type operator* $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ *as follows: For any* $a = \{a_m\}_{m=1}^{\infty} \in l_{p,\Phi_\lambda}$*, there exists a unique representation* $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$*. Define the formal inner product of* Ta *and* $b = \{b_n\}_{n=1}^{\infty} \in l_{q,\Psi_\lambda}$ *as follows:*

$$(Ta, b) := \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right] b_n. \tag{29}$$

We can express the above results in operator forms as:

$$(Ta, b) \quad < \quad k_s(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \tag{30}$$
$$||Ta||_{p,\Psi_\lambda^{1-p}} \quad < \quad k_s(\lambda_1)||a||_{p,\Phi_\lambda}. \tag{31}$$

Define the norm of the operator T as follows:

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{||Ta||_{p,\Psi_\lambda^{1-p}}}{||a||_{p,\Phi_\lambda}}.$$

Then, by (31), we get that $||T|| \leq k_s(\lambda_1)$. Since the constant factor in (31) is the best possible, we have $||T|| = k_s(\lambda_1)$.

4. Some Reverses

In the following, we also set

$$\begin{aligned}\widetilde{\Phi}_{\lambda}(m) &= (1-\theta(\lambda_2,m))\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\\ \widetilde{\Psi}_{\lambda}(n) &= (1-\vartheta(\lambda_1,n))\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}}\ (m,n\in\mathbf{N}).\end{aligned}$$

For $0<p<1$ or $p<0$, we still use the formal symbols $||a||_{p,\Phi_\lambda}$, $||b||_{q,\Psi_\lambda}$, $||a||_{p,\widetilde{\Phi}_\lambda}$ and $||b||_{q,\widetilde{\Psi}_\lambda}$.

Theorem 3. *If* $0<p<1$, $m_0,n_0\in\mathbf{N}$, $\mu_m\geq\mu_{m+1}$ $(m\in\{m_0,m_0+1,\cdots\})$, $\upsilon_n\geq\upsilon_{n+1}$ $(n\in\{n_0,n_0+1,\cdots\})$, $U(\infty)=V(\infty)=\infty$, *then we have the following equivalent inequalities with the best possible constant factor* $k_s(\lambda_1)$*:*

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}}+c_kV_n^{\frac{\lambda}{s}})} > k_s(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}||b||_{q,\Psi_\lambda}, \tag{32}$$

$$J = \left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}}+c_kV_n^{\frac{\lambda}{s}})}\right]^p\right\}^{\frac{1}{p}} > k_s(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}. \tag{33}$$

Proof. By the reverse Hölder inequality (cf. [44]), we derive the reverses of (23–25). Then, by (17), we obtain (33). By (33) and the reverse of (25), we have (32). On the other hand, assuming that (32) is valid, we set b_n as in Theorem 1. Then, we get that $J^p=||b||_{q,\Psi_\lambda}^q$. If $J=\infty$, then (33) is trivially valid; if $J=0$, then by the reverse of (24) and (17), this is impossible.

Suppose that $0<J<\infty$. By (32), it follows that

$$||b||_{q,\Psi_\lambda}^q = J^p = I > k_s(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}||b||_{q,\Psi_\lambda}, \tag{34}$$

$$||b||_{q,\Psi_\lambda}^{q-1} = J > k_s(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}, \tag{35}$$

and then (33) follows, which is equivalent to (32).

For $\varepsilon\in(0,p\lambda_1)$, we set $\widetilde{\lambda}_1,\widetilde{\lambda}_2,\widetilde{a}_m$ and $\widetilde{b}_n$ as in (28). Then by (19), (20) and (16), we find

$$
\begin{aligned}
||a||_{p,\widetilde{\Phi}_\lambda}||b||_{q,\Psi_\lambda} &= \left(\sum_{m=1}^{\infty}(1-\theta(\lambda_2,m))\frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}}-\sum_{m=1}^{\infty}O\left(\frac{\mu_m}{U_m^{1+\lambda_2+\frac{\varepsilon}{q}}}\right)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon(O(1)-O_1(1))\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}, \\
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\prod_{k=1}^{s}(U_m^{\lambda/s}+c_kV_n^{\lambda/s})} \\
&= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{\mu_m}{\prod_{k=1}^{s}(U_m^{\lambda/s}+c_kV_n^{\lambda/s})}\frac{V_n^{\widetilde{\lambda}_2}}{U_m^{1-\widetilde{\lambda}_1}}\right]\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^{\infty}\varpi_s(\widetilde{\lambda}_1,n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \le k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right).
\end{aligned}
$$

If there exists a positive constant $K \ge k_s(\lambda_1)$, such that (32) is valid when we replace $k_s(\lambda_1)$ by K, then in particular, we have

$$\varepsilon\widetilde{I} > \varepsilon K||\widetilde{a}||_{p,\widetilde{\Phi}_\lambda}||\widetilde{b}||_{q,\Psi_\lambda},$$

namely,

$$
\begin{aligned}
& k_s(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right) \\
> \; & K\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon(O(1)-O_1(1))\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.
\end{aligned}
$$

It follows that $k_s(\lambda_1) \ge K$ $(\varepsilon \to 0^+)$. Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (32). The constant factor $k_s(\lambda_1)$ in (33) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (25) that the constant factor in (32) is not the best possible.

This completes the proof of the theorem. □

Theorem 4. *If $p < 0, m_0, n_0 \in \mathbf{N}, \mu_m \ge \mu_{m+1}$ $(m \in \{m_0, m_0+1, \cdots\})$, $\upsilon_n \ge \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, $U(\infty) = V(\infty) = \infty$, then we have the following equivalent inequalities with the best possible constant factor $k_s(\lambda_1)$:*

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}}+c_kV_n^{\frac{\lambda}{s}})} > k_s(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\widetilde{\Psi}_\lambda}, \tag{36}$$

$$
\begin{aligned}
J_1 &:= \left\{\sum_{n=1}^{\infty}\frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1,n))^{p-1}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}}+c_kV_n^{\frac{\lambda}{s}})}\right]^p\right\}^{\frac{1}{p}} \\
&> k_s(\lambda_1)||a||_{p,\Phi_\lambda}.
\end{aligned} \tag{37}
$$

Proof. By the reverse Hölder inequality with weight (cf. [44]), since $p < 0$, by (18), we have

$$
\begin{aligned}
& \left[\sum_{m=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right]^p \\
= & \left[\sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\frac{\lambda}{s}} + c_k V_n^{\frac{\lambda}{s}})} \left(\frac{U_m^{(1-\lambda_1)/q} v_n^{1/p} a_m}{V_n^{(1-\lambda_2)/p} \mu_m^{1/q}}\right)\left(\frac{V_n^{(1-\lambda_2)/p} \mu_m^{1/q}}{U_m^{(1-\lambda_1)/q} v_n^{1/p}}\right)\right]^p \\
\leq & \sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{(1-\lambda_1)p/q} v_n}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p \\
& \times \left[\sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{V_n^{(1-\lambda_2)(q-1)} \mu_m}{U_m^{1-\lambda_1} v_n^{q-1}}\right]^{p-1} \\
= & \frac{(\varpi_s(\lambda_1, n))^{p-1}}{V_n^{p\lambda_2 - 1} v_n} \sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \\
\leq & (k_s(\lambda_1))^{p-1} \frac{(1 - \vartheta(\lambda_1, n))^{p-1}}{V_n^{p\lambda_2 - 1} v_n} \\
& \times \sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p, \\
J_1 \geq & (k_s(\lambda_1))^{\frac{1}{q}} \left\{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}} \\
= & (k_s(\lambda_1))^{\frac{1}{q}} \left\{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \frac{U_m^{(1-\lambda_1)(p-1)} v_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}} \\
= & (k_s(\lambda_1))^{\frac{1}{q}} \left\{\sum_{m=1}^{\infty} \omega_s(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}}.
\end{aligned} \tag{38}
$$

Then by (15), we obtain (37).

By the reverse Hölder inequality (cf. [44]), we have

$$
\begin{aligned}
I = & \sum_{n=1}^{\infty} \frac{V_n^{\lambda_2 - \frac{1}{p}} v_n^{1/p}}{(1 - \vartheta(\lambda_1, n))^{1/q}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right] \\
& \times \left[(1 - \vartheta(\lambda_1, n))^{\frac{1}{q}} \frac{V_n^{\frac{1}{p} - \lambda_2}}{v_n^{1/p}} b_n\right] \geq J_1 ||b||_{q, \widetilde{\Psi}_\lambda}.
\end{aligned} \tag{39}
$$

Then, by (37), we deduce (36). On the other hand, assuming that (36) is valid, we set b_n as follows:

$$
b_n := \frac{V_n^{p\lambda_2 - 1} v_n}{(1 - \vartheta(\lambda_1, n))^{p-1}} \left[\sum_{m=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\right]^{p-1}, \; n \in \mathbf{N}.
$$

Then, we obtain that $J_1^p = ||b||_{q, \widetilde{\Psi}_\lambda}^q$. If $J_1 = \infty$, then (37) is trivially valid; if $J_1 = 0$, then by (15) and (38), this is impossible. Suppose that $0 < J_1 < \infty$. By (36), it follows that

$$
\begin{aligned}
||b||_{q, \widetilde{\Psi}_\lambda}^q &= J_1^p = I > k_s(\lambda_1) ||a||_{p, \Phi_\lambda} ||b||_{q, \widetilde{\Psi}_\lambda}, \\
||b||_{q, \widetilde{\Psi}_\lambda}^{q-1} &= J_1 > k_s(\lambda_1) ||a||_{p, \Phi_\lambda},
\end{aligned}
$$

and then (37) follows, which is equivalent to (36).

For $\varepsilon \in (0, |p|\lambda_1)$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}(>0), \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}(\in (0,1))$, and

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1 - \varepsilon}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n.$$

Then, by (17), (19) and (20), we have

$$\begin{aligned}
||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\widetilde{\Psi}_\lambda} &= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(1-\vartheta(\lambda_1,n))\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right]^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}} - \sum_{n=1}^{\infty}O\left(\frac{\upsilon_n}{V_n^{1+\lambda_1+\varepsilon}}\right)\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon(\widetilde{O}(1) - O_1(1))\right]^{\frac{1}{q}}, \\
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})} \\
&= \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{\upsilon_n}{\prod_{k=1}^{s}(U_m^{\lambda/s} + c_k V_n^{\lambda/s})}\frac{U_m^{\widetilde{\lambda}_1}}{V_n^{1-\widetilde{\lambda}_2}}\right]\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&= \sum_{m=1}^{\infty}\omega_s(\widetilde{\lambda}_2, m)\frac{\mu_m}{U_m^{1+\varepsilon}} \le k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right).
\end{aligned}$$

If there exists a positive constant $K \ge k_s(\lambda_1)$, such that (36) is valid when we replace $k_s(\lambda_1)$ by K, then in particular, we have

$$\varepsilon\widetilde{I} > \varepsilon K||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\widetilde{\Psi}_\lambda},$$

namely,

$$\begin{aligned}
&k_s(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right) \\
&> K\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon(\widetilde{O}(1) - O_1(1))\right]^{\frac{1}{q}}.
\end{aligned}$$

It follows that $k_s(\lambda_1) \ge K(\varepsilon \to 0^+)$. Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (36).

The constant factor $k_s(\lambda_1)$ in (37) is still the best possible. Otherwise, we would reach a contradiction by (39) that the constant factor in (36) is not the best possible.

This completes the proof of the theorem. □

Remark 1. *(i) For $\mu_i = \upsilon_j = 1 (i, j \in \mathbf{N})$, (21) reduces to (7).*
(ii) For

$$s = \lambda = c_1 = 1,\ \lambda_1 = \frac{1}{q},\ \lambda_2 = \frac{1}{p},$$

(21) reduces to (4); for $s = \lambda = c_1 = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$, (21) reduces to the dual form of (4) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{U_m^{p-2}}{\mu_m^{p-1}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{V_n^{q-2}}{\upsilon_n^{q-1}}b_n^q\right)^{\frac{1}{q}}. \quad (40)$$

(iii) For $p = q = 2$, both (4) and (40) reduce to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m + V_n} < \pi\left(\sum_{m=1}^{\infty}\frac{a_m^2}{\mu_m}\sum_{n=1}^{\infty}\frac{b_n^2}{\upsilon_n}\right)^{\frac{1}{2}}. \quad (41)$$

5. Conclusions

In the present paper, making use of weight coefficients as well as real/complex analytic methods, a Hardy–Hilbert-type inequality with a best possible constant factor and multiparameters and the equivalent forms are established in Theorems 1 and 2. Reverses, operator expression with the norm, and a few particular cases are also considered in Theorems 3 and 4, Definition 1, and Remark 1. The lemmas and theorems provide an extensive account of this type of inequality.

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