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On Golden Lorentzian Manifolds Equipped with Generalized Symmetric Metric Connection

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Abstract: This research deals with the generalized symmetric metric U-connection defined on golden Lorentzian manifolds. We also derive sharp geometric inequalities that involve generalized normalized δ-Casorati curvatures for submanifolds of golden Lorentzian manifolds equipped with generalized symmetric metric U-connection.

Keywords: Riemannian manifolds; golden structure; scalar curvature

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1. Background

Golden ratios have been investigated by different researchers for many centuries. Taking inspiration from golden mean, the golden structure introduced by [1] as a polynomial structure [2,3] came into picture, and the structure polynomial was written as $\varphi^2 = \varphi + I$, φ being $(1,1)$ tensor field. In 2007, the authors in [4] investigated invariant submanifolds isometrically immersed in golden Riemannian manifolds, highlighting new ideas. In the same ambient manifold [5], work was carried out on an induced structure, producing some new studies. Recently, Bahadir and Sirajuddin et al. [6] studied slant submanifolds in golden Riemannian manifolds, developing different useful results. The study of golden structures was also carried out on semi-Riemannian manifolds by M. Ozkan [7]. One can refer to [8–11], etc., and the references therein for recent developments in golden differential geometry.

Another important development in the submanifold theory was the introduction of δ-invariants [12] (also known as Chen's invariants). With the help of these new Riemannian invariants, B. Y. Chen not only established a relation in the form of an optimal inequality but also defined and studied a new concept known as ideal immersion. This investigation of Chen's invariants have been extensively used by many researchers in different ambient spaces ([13–15], etc.).

It is known that Gauss curvature might vanish for intuitively curved looking surfaces. That is why F. Casorati [16] introduced another notion that is known today as the Casorati curvature. In 2007, Decu et al. [17] studied optimal inequalities and produced the notions of normalized Casorati curvatures (see also [18]). The same authors obtained an extension of the above results and produced the notion of generalized normalized δ-Casorati curvatures [19]. Later on, these research techniques were used for establishing optimal inequalities by many researchers in different ambient spaces ([20–22], etc.).

The present study deals with the generalized symmetric metric U-connection on golden Lorentzian manifolds. We also study the lower bounds for submanifolds immersed

in golden Lorentzian manifolds equipped with generalized symmetric metric U-connection. Moreover, submanifolds for which equality holds are also discussed.

2. Preliminaries

Consider the Riemannian manifolds (N^m, g) and (M^n, g) such that N isometrically immerses in M. Represent the shape operator of N by $S_\xi, \xi \in \Gamma(T^\perp N)$. Further, on M, assume that $\overline{\nabla}$ stands for the Levi-Civita connection and $\nabla^\perp$ for connection in a normal bundle. Furthermore, for all $Y_1, Y_2 \in \Gamma(TN)$ and $\xi \in \Gamma(T^\perp N)$, we write [23]

$$\overline{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h(Y_1, Y_2),$$

$$\overline{\nabla}_{Y_1} \xi = -S_\xi Y_1 + \nabla^\perp_{Y_1} \xi,$$

and

$$g(S_\xi Y_1, Y_2) = g(h(Y_1, Y_2), \xi).$$

One can also recall the Gauss equation as [23]

$$\begin{aligned} \overline{R}(Y_1, Y_2, Y_3, Y_4) &= R(Y_1, Y_2, Y_3, Y_4) - g(h(Y_1, Y_4), h(Y_2, Y_3)) \\ &\quad + g(h(Y_1, Y_3), h(Y_2, Y_4)), \qquad \forall Y_i \in \Gamma(TN), i = 1, 2, 3, 4. \end{aligned}$$

Let us fix $\{E_1, \ldots, E_m, E_{m+1}, \ldots, E_n\}$ for a local orthonormal frame in M^n. Define the scalar curvature τ as follows

$$\tau = \sum_{1 \le i < j \le m} R(E_i, E_j, E_j, E_i).$$

Set

$$\rho = \frac{2}{(m^2 - m)} \tau$$

as the normalized scalar curvature and

$$\mathcal{H} = \sum_{i=1}^{m} \frac{1}{m} h(E_i, E_i)$$

as the mean curvature. For our convenience, put $h^r_{ij} = g(h(E_i, E_j), E_r)$. In this way, $\forall r \in \{m+1, \ldots, n\}$, we can write

$$||\mathcal{H}||^2 = \frac{1}{m^2} \sum_{r=m+1}^{n} \left(\sum_{i=1}^{m} h^r_{ii} \right)^2$$

and

$$||h||^2 = \sum_{r=m+1}^{n} \sum_{i,j=1}^{m} \left(h^r_{ij}\right)^2.$$

One recalls the *Casorati curvature* for the submanifold N as

$$\mathcal{C} = \frac{1}{n} ||h||^2. \tag{1}$$

Let us represent any linear subspace of TN with the help of $\mathcal{A}$ and fix its dimension equal to t such that $t \ge 2$, spanned by $\{E_1, \ldots, E_t\}$. Then, for a t-plane section $\mathcal{A}$, one can define the scalar curvature by

$$\tau(\mathcal{A}) = \sum_{1 \le i < j \le t} R(E_i, E_j, E_j, E_i),$$

$$\mathcal{C}(\mathcal{A}) = \frac{1}{t}\sum_{r=m+1}^{n}\sum_{i,j=1}^{t}\left(h_{ij}^{r}\right)^{2}.$$

Let $\mathcal{L}$ be a hyperplane of T_pN. Then, Decu et al. [17] introduced the normalized δ-Casorati curvatures as

$$[\delta_c(m-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{m+1}{2m(m-1)}\inf\{\mathcal{C}(\mathcal{L})\},$$

$$[\widehat{\delta}_c(m-1)]_p = 2\mathcal{C}_p - \frac{2m-1}{2m}\sup\{\mathcal{C}(\mathcal{L})\}.$$

If $r \neq m(m-1)$ is a positive real number, then according to [19] the generalized normalized δ-Casorati curvatures

$$\begin{aligned}[\delta_c(r;m-1)]_p &= r\mathcal{C}_p \\ &+ \frac{(m^2-m-r)(m+r)(m-1)}{rm}\inf\{\mathcal{C}(\mathcal{L})\},\end{aligned}$$

if $(m^2-m) > r > 0$ and

$$\begin{aligned}[\widehat{\delta}_c(r;m-1)]_p &= r\mathcal{C}_p \\ &+ \frac{(m^2-m-r)(m+r)(m-1)}{rm}\sup\{\mathcal{C}(\mathcal{L})\}\end{aligned}$$

provided $r > (m^2-m)$.

We refer to [24] for further definitions and formulas.

2.1. Golden Riemannian Manifolds

We identify a Riemannian manifold with the help of (M^n, g) and of a $(1,1)$-tensor field with $\mathtt{G}$ satisfying [4,6,10,11].

$$K(Y) = Y^n + a_nY^{n-1} + \ldots + a_2Y + a_1I = 0,$$

where I stands for identity transformation, and at $p \in M$ (for $Y = \mathtt{G}$), $\mathtt{G}^{n-1}(p), \mathtt{G}^{n-2}(p), \ldots, \mathtt{G}(p), I$ are linearly independent. Then $K(Y)$ represents structure polynomial.

Further, M is equipped with a golden structure if [6].

$$\varphi^2 = \varphi + I.$$

In the above equation, I denotes the identity transformation, and φ stands for $(1,1)$ tensor field. When (M,g) is equipped with the golden structure φ, M becomes golden Riemannian manifold if the following relations holds good for φ-compatible metric

$$g(\varphi Y_1, Y_2) = g(Y_1, \varphi Y_2), \qquad \forall Y_1, Y_2 \in \Gamma(TM),$$

and

$$g(\varphi Y_1, \varphi Y_2) = g(\varphi^2 Y_1, Y_2) = g(\varphi Y_1, Y_2) + g(Y_1, Y_2).$$

The Riemannian curvature tensor R of the locally golden space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$ is written according to [11] as

$$\begin{aligned} R(X,Y)Z = \quad & \tfrac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10}[g(Y,Z)X - g(X,Z)Y] \\ & + \tfrac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10}[g(\varphi Y,Z)X - g(\varphi X,Z)Y \\ & + g(Y,Z)\varphi X - g(X,Z)\varphi Y] \\ & + \tfrac{c_1+c_2}{5}[g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y]. \end{aligned} \tag{2}$$

We define the golden Lorentzian manifold.

Definition 1. *Let us consider a semi-Riemannian manifold (M^n, g), where g has the signature $(-,+,+,...,+(n-1\ times))$. Then M stands for golden Lorentzian manifold if it is endowed with a golden structure φ and g is φ-compatible.*

Example 1. *Let $\mathbb{R}^3_1$ represent the semi-Euclidean space and consider the signature of g as $(-,+,+)$. If Φ stands for a $(1,1)$ tensor field, then it is easy to show that if*

$$\Phi(s_1,s_2,s_3) = \frac{1}{2}(s_1+\sqrt{5}s_2, s_2+\sqrt{5}s_1, 2\psi s_3),$$

for any vector field $(s_1,s_2,s_3) \in \mathbb{R}^3_1$, where $\psi = \frac{1+\sqrt{5}}{2}$ is the golden mean, then

$$\Phi^2 = \phi + I,$$

and hence, Φ is a golden structure on $\mathbb{R}^3_1$. Moreover, g may also be verified to be Φ-compatible. Thus, $(\mathbb{R}^3_1, g, \Phi)$ becomes a golden Lorentzian manifold.

Example 2. *For any semi-Euclidean space R^5_1 and $(1,1)$ tensor field Φ defined on R^5_1 as*

$$\Phi(s_1,s_2,s_3,s_4,s_5) = (\psi s_1, (1-\psi)s_2, \psi s_3, (1-\psi)s_4, \psi s_5),$$

where ψ is golden mean. We can see that $\Phi^2 = \Phi + I$. Moreover, g is Φ-compatible, where g is used for the metric tensor with signature $(-,+,+,+,+)$ on R^5_1. Hence, (R^5_1, g, Φ) represents a golden Lorentzian manifold.

According to the study in [8], we have

Theorem 1. *On any golden Lorentzian manifold (M, g, φ), the golden structure φ is integrable if and only if*

$$\nabla\varphi = 0.$$

where ∇ denotes the Levi-Civita connection associated with g.

The structure φ has been considered an integrable golden structure in this work.

2.2. Generalized Symmetric Metric (g.s.m.) Connection in a Golden Lorentzian Manifold

Let (M, g, φ) denote a golden Lorentzian manifold. We write its associated torsion tensor $\overline{T}$ by

$$\overline{T}(Y_1,Y_2) = -\alpha\{u(Y_1)Y_2 - u(Y_2)Y_1\} - \beta\{u(Y_1)\varphi Y_2 - u(Y_2)\varphi Y_1\}, \tag{3}$$

where Y_i belong to $\Gamma(TM)$ for $i = 1, 2$, and α, β represent smooth functions on the manifold M. In the present scenario, $\overline{\nabla}$ is known as generalized symmetric connection. One can also write the relation

$$u(Y_1) = g(U, Y_1),$$

where U is a unitary (spacelike or timelike) vector field on M, and u stands for a 1-form. In addition to this, if

$$\overline{\nabla} g = 0,$$

then $\overline{\nabla}$ represents a generalized metric connection. Otherwise, it is said to be a non-metric connection. Considering ∇ as a Levi-Civita connection, one notices that

$$\overline{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + H(Y_1, Y_2), \forall Y_1, Y_2 \in \Gamma(TM). \tag{4}$$

Suppose that H presents any $(1,2)$ tensor and let $\overline{T}$ be a torsion tensor of the generalized symmetric metric (g.s.m.) connection $\overline{\nabla}$. Then one obtains

$$H(Y_1, Y_2) = \frac{1}{2}[\overline{T}(Y_1, Y_2) + T^{'}(Y_2, Y_1) + T^{'}(Y_1, Y_2)], \forall Y_1, Y_2 \in \Gamma(TM), \tag{5}$$

and

$$g(\overline{T}(Y_3, Y_1), Y_2) = g(T^{'}(Y_1, Y_2), Y_3), \forall Y_1, Y_2, Y_3 \in \Gamma(TM). \tag{6}$$

This way Equations (3) and (6) produce the following

$$T^{'}(Y_1, Y_2) = \beta\{u(Y_1)\phi Y_2 - g(\phi Y_1, Y_2)U\} - \alpha\{g(Y_1, Y_2)U - u(Y_1)Y_2\}, \forall Y_1, Y_2 \in \Gamma(TM). \tag{7}$$

Using Equations (3), (5) and (7), we obtain

$$H(Y_1, Y_2) = \beta\{u(Y_2)\phi Y_1 - g(\phi Y_1, Y_2)U\} - \alpha\{g(Y_1, Y_2)U - u(Y_2)Y_1\}, \forall Y_1, Y_2 \in \Gamma(TM).$$

So, one may write

Theorem 2. *If (M, g, φ) is a golden Lorentzian manifold and $\overline{\nabla}$ denotes the g.s.m. connection of (α, β)-type, then we have*

$$\overline{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \alpha\kappa(Y_1, Y_2) + \beta\kappa(\phi Y_1, Y_2), \forall Y_1, Y_2 \in \Gamma(TM) \tag{8}$$

where $\kappa(Y_1, Y_2) = -g(Y_1, Y_2)U + u(Y_2)Y_1$.

Corollary 1. *(i) In view of Equation (8), when*

- $(\alpha, \beta) = (1, 0)$, *one obtains a semi-symmetric metric connection;*
- $(\alpha, \beta) = (0, 1)$, *one obtains a quarter symmetric metric connection.*

(ii) Again using Equation (8), the g.s.m. connection will be

- α *semi symmetric metric connection with* $\beta = 0$;
- β *quarter symmetric metric connection provided* $\alpha = 0$.

Definition 2. *Let (M, g, φ) be golden Lorentzian manifold equipped with a golden structure φ and $\overline{\nabla}$ be the g.s.m. connection of (α, β)-type. If the associated vector field U is parallel, then $\overline{\nabla}$ is called the generalized symmetric metric U-connection on M.*

Now, when the associated vector field U in Equation (3) satisfies the following relation

$$g(U, U) = u(U) = \varepsilon, \tag{9}$$

where $\varepsilon = \pm 1$. Taking into use Theorem 2 and the fact that U is parallel w.r.t. $\overline{\nabla}$, we obtain

$$\overline{\nabla}_{Y_1} U = 0 \Leftrightarrow \nabla_{Y_1} U = -\alpha\kappa(Y_1, U) - \beta\kappa(\varphi Y_1, U), \forall Y_1 \in \Gamma(TM). \tag{10}$$

Using Equations (8)–(10), we have

$$(\overline{\nabla}_{Y_1}u)Y_2 = (\nabla_{Y_1}u)Y_2 - \alpha u(\kappa(Y_1,Y_2)) - \beta u(\kappa(\varphi Y_1,Y_2)), \forall Y_1,Y_2 \in \Gamma(TM).$$

Furthermore, $(\overline{\nabla}_{Y_1}u)Y_2 = 0$, and thus,

$$(\nabla_{Y_1}u)Y_2 = \alpha u(\kappa(Y_1,Y_2)) + \beta u(\kappa(\varphi Y_1,Y_2)), \forall Y_1,Y_2 \in \Gamma(TM). \tag{11}$$

Proposition 1. *For any golden Lorentzian manifold (M^n, g, φ) with generalized symmetric metric U-connection $\overline{\nabla}$, we have*

$$\begin{aligned} R(Y_1,Y_2)U &= \beta^2[u(\varphi Y_2)\varphi Y_1 - u(\varphi Y_1)\varphi Y_2] \\ &+ \alpha\beta[-u(Y_1)\varphi Y_2 + u(Y_2)\varphi Y_1 + u(\varphi Y_2)Y_1 - u(\varphi Y_1)Y_2] \\ &- \alpha^2[u(Y_1)Y_2 - u(Y_2)Y_1], \end{aligned} \tag{12}$$

$$\begin{aligned} R(U,Y_1)Y_2 &= -\alpha^2\kappa(Y_1,Y_2)\beta^2[u(\varphi Y_2)\varphi Y_1 + u(\varphi Y_1)\varphi Y_2] \\ &+ \alpha\beta[-u(\varphi Y_2)Y_1 - \kappa(\varphi Y_1,Y_2) + g(Y_1,Y_2)\varphi U], \end{aligned} \tag{13}$$

$$\begin{aligned} u(R(Y_1,Y_2)Y_3) &= -\beta^2[u(\varphi Y_2)g(\varphi Y_1,Y_3) - u(\varphi Y_1)g(\varphi Y_2,Y_3)] \\ &-\alpha\beta[u(Y_2)g(\varphi Y_1,Y_3) - u(Y_1)g(\varphi Y_2,Y_3) \\ &+u(\varphi Y_2)g(Y_1,Y_3) - u(\varphi Y_1)g(Y_2,Y_3)] \\ &+\alpha^2[u(Y_1)g(Y_2,Y_3) - u(Y_2)g(Y_1,Y_3)], \end{aligned} \tag{14}$$

where R is the Curvature tensor.

Proof. Using Equations (9)–(11), we have Equation (12). With the aid of Equation (12), we can easily find the Equations (13) and (14). □

Let M be a golden Lorentzian manifold with generalized symmetric metric U-connection $\overline{\nabla}$. Let $\overline{R}$ denote the curvature tensor w.r.t. $\overline{\nabla}$. Then, for all $Y_1, Y_2, Y_3 \in \Gamma(TM)$, we write

$$\overline{R}(Y_1,Y_2)Y_3 = \overline{\nabla}_{Y_1}\overline{\nabla}_{Y_2}Y_3 - \overline{\nabla}_{Y_2}\overline{\nabla}_{Y_1}Y_3 - \overline{\nabla}_{[Y_1,Y_2]}Y_3. \tag{15}$$

Due to Equations (8) and (15), we reach

$$\begin{aligned} \overline{R}(Y_1,Y_2)Y_3 &= R(Y_1,Y_2)Y_3 + \alpha^2\{g(Y_1,Y_3)Y_2 - g(Y_2,Y_3)Y_1\} \\ &+\beta^2\{g(\varphi Y_1,Y_3)\varphi Y_2 - g(\varphi Y_2,Y_3)\varphi Y_1\} \\ &+\alpha\beta\{g(\varphi Y_1,Y_3)Y_2 - g(\varphi Y_2,Y_3)Y_1 \\ &+g(Y_1,Y_3)\varphi Y_2 - g(Y_2,Y_3)\varphi Y_1\}. \end{aligned}$$

Considering an orthonormal frame field on M and contracting over Y, one can write the Ricci tensor identified by $\overline{S}$ as

$$\begin{aligned} \overline{S}(Y_2,Y_3) &= S(Y_2,Y_3) + \alpha^2(1-n\varepsilon)g(Y_2,Y_3) \\ &+\beta^2\{g(\varphi Y_2,\varphi Y_3) - g(\varphi Y_2,Y_3)trace\varphi\} \\ &+\alpha\beta\{(2-n\varepsilon)g(\varphi Y_2,Y_3) - g(Y_2,Y_3)trace\varphi\}, \end{aligned}$$

and the Ricci operator by

$$\begin{aligned} \overline{Q}Y_2 &= QY_2 + \alpha^2(1-n\varepsilon)Y_2 + \beta^2\{\varphi Y_2 + Y - \varphi Y_2 trace\varphi\} \\ &+\alpha\beta\{(2-n\varepsilon)\varphi Y_2 - Y_2 trace\varphi\}. \end{aligned} \tag{16}$$

Let us identify with R the curvature tensor and denote by S the Ricci tensor w.r.t. Levi-Civita connection. Then, the contraction of Equation (16) along Y helps us to write the scalar curvature with respect to $\overline{\nabla}$, as follows

$$\begin{aligned}\overline{\tau} &= \tau + \alpha^2(\varepsilon - n)n + \beta^2\{trace\varphi + n\varepsilon - (trace\varphi)^2\} \\ &\quad + \alpha\beta(2 - 2n\varepsilon)trace\varphi. \end{aligned} \tag{17}$$

in the above relations, Q represents Ricci operator, and τ stands for scalar curvature associated to Levi-Civita connection.

Theorem 3. *For any golden Lorentzian manifold equipped with generalized symmetric metric U-connection, we have the following characterizations in Table 1.*

Table 1. Characterizations of any golden Lorentzian manifold equipped with generalized symmetric metric U-connection.

Connection type	Ricci tensor
α semi-symmetric	$S(Y_2, Y_3) - \alpha^2(n\varepsilon - 1)g(Y_2, Y_3)$
β quarter symmetric	$S(Y_2, Y_3) + \beta^2\{g(\varphi Y_2, \varphi Y_3) - g(\varphi Y_2, Y_3)trace\varphi\}$
Semi-symmetric	$S(Y_2, Y_3) - (n\varepsilon - 1)g(Y_2, Y_3)$
Quarter symmetric	$S(Y_2, Y_3) + \{g(\varphi Y_2, \varphi Y_3) - g(\varphi Y_2, Y_3)trace\varphi\}$
Connection type	Scalar curvature
α semi-symmetric	$\tau - \alpha^2(n - \varepsilon)n$
β quarter symmetric	$\tau + \beta^2\{trace\varphi + n\varepsilon - (trace\varphi)^2\}$
Semi-symmetric	$\tau - n(n - \varepsilon)$
Quarter symmetric	$\tau + \{trace\varphi + n\varepsilon - (trace\varphi)^2\}$

Using Equations (3) and (17), we have

$$\begin{aligned}\overline{\tau} &= \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10}(n - \varepsilon)n \\ &\quad + \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(2n\varepsilon - 2)trace\varphi] \\ &\quad + \frac{c_1 + c_2 - 5\beta^2}{5}[(trace\varphi)^2 - trace\varphi - n\varepsilon].\end{aligned}$$

Using Equation (23) in Theorem 3, we have the following theorem.

Theorem 4. *For any locally golden product Lorentzian manifold equipped with generalized symmetric metric U-connection, we have the following characterizations in Table 2.*

Table 2. Characterizations of any locally golden product Lorentzian manifold equipped with generalized symmetric metric U-connection.

Connection type	Scalar curvature
α semi-symmetric	$(A - \alpha^2)(n - \varepsilon)n + B(2n\varepsilon - 2)trace\varphi + C((trace\varphi)^2 - trace\varphi - n\varepsilon)$
β quarter symmetric	$A(n - \varepsilon)n + B(2n\varepsilon - 2)trace\varphi + (C - \beta^2)((trace\varphi)^2 - trace\varphi - n\varepsilon)$
Semi-symmetric	$(A - 1)(n - \varepsilon)n + B(2n\varepsilon - 2)trace\varphi + C((trace\varphi)^2 - trace\varphi - n\varepsilon)$
Quarter symmetric	$A(n - \varepsilon)n + B(2n\varepsilon - 2)trace\varphi + (C - 1)((trace\varphi)^2 - trace\varphi - n\varepsilon)$

where $A = \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2}{10}$, $B = \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{10}$ and $C = \frac{c_1 + c_2}{5}$.

3. Inequalities for Golden Lorentzian Manifolds Equipped with Generalized Symmetric Metric U-Connection

From now on, let M^n stand for locally golden product Lorentzian manifold with g.s.m. U-connection.

Theorem 5. *For submanifold N^m of M^n, we have the following inequalities*

(i) for $\delta_c(r;m-1)$,

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(r;m-1)] \\ & +\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon) \\ & +\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & +\frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned} \tag{18}$$

where the real number r satisfies $(m^2-m)>r>0$;

(ii) for $\widehat{\delta}_c(r;m-1)$,

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(r;m-1)] \\ & +\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon) \\ & +\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & +\frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned} \tag{19}$$

where $(m^2-m)<r$.

In addition, the relations in Equations (18) and (19) become equalities if in an orthonormal frame $\{E_1,\ldots,$
$E_m,E_{m+1},\ldots,E_n\}$, the shape operator can be represented as follows:

$$S_{m+1}=\begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{bm}{r}(m-1) \end{pmatrix}, S_{m+2}=\cdots=S_n=0. \tag{20}$$

Proof. (i) For the locally golden product Lorentzian manifold M equipped with g.s.m. U-connection, considering the orthonormal frame $\{E_1,\ldots,E_m,E_{m+1},\ldots,E_n\}$ and using the Gauss equation and the relations in Equations (1) and (17), one achieves

$$\begin{aligned}2\overline{\tau}(p) &= \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon]+n^2||\mathcal{H}||^2-n\mathcal{C}.\end{aligned}$$

We write a quadratic polynomial $\mathcal{T}$ as follows

$$\begin{aligned}\mathcal{T} &= r\mathcal{C}+\frac{(n-1)(n^2-n-r)(n+r)}{rn}\mathcal{C}(\mathcal{J})-2\overline{\tau}(p)\\ &+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon],\end{aligned}$$

where $\mathcal{J}$ stands for a hyperplane of T_pN. Now, consider the situation where, without loss of generality, $\{E_1,\ldots,E_{m-1}\}$ spans $\mathcal{J}$. In this way, we get

$$\begin{aligned}\mathcal{T} &= \frac{r}{m}\sum_{\alpha=m+1}^{n}\sum_{i,j=1}^{m}(h_{ij}^{\alpha})^2+(\frac{1}{m}+\frac{1}{r})(m^2-m-r)\sum_{\alpha=m+1}^{n}\sum_{i,j=1}^{m-1}(h_{ij}^{\alpha})^2\\ &-2\overline{\tau}(p)+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon].\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{T} &= (\frac{r}{m}+1)\sum_{\alpha=m+1}^{n}\sum_{i,j=1}^{m}(h_{ij}^{\alpha})^2+(\frac{1}{m}+\frac{1}{r})(m^2-m-r)\sum_{\alpha=m+1}^{n}\sum_{i,j=1}^{m-1}(h_{ij}^{\alpha})^2\\ &-\sum_{\alpha=m+1}^{n}(\sum_{i=1}^{m}h_{ii}^{\alpha})^2.\end{aligned}$$

This leads us to the following

$$\begin{aligned}\mathcal{T} &= \sum_{\alpha=m+1}^{n}\sum_{i=1}^{m-1}\Big[\Big\{\frac{m^2}{r}+\frac{(r-1)}{r}m-2\Big\}(h_{ii}^{\alpha})^2+\frac{2r+2m}{m}(h_{im}^{\alpha})^2\Big]\\ &+\sum_{\alpha=m+1}^{n}\Big[\frac{2r+2m}{r}(m-1)\sum_{i<j=1}^{m-1}(h_{ij}^{\alpha})^2-2\sum_{i<j=1}^{m}h_{ii}^{\alpha}h_{jj}^{\alpha}\\ &+\frac{r}{m}(h_{mm}^{\alpha})^2\Big].\end{aligned}\tag{21}$$

With the help of Equation (21), the critical points

$$h^c=(h_{11}^{m+1},h_{12}^{m+1},\ldots,h_{mm}^{m+1},\ldots,h_{11}^{n},\ldots,h_{mm}^{n})$$

are the solutions of (for $1 \leq i, j \leq m-1$)

$$\begin{aligned} \frac{\partial \mathcal{T}}{\partial h^{\alpha}_{ii}} &= 2[\frac{(r+m)}{r}(m-1)h^{\alpha}_{ii} - \sum_{l=1}^{m} h^{\alpha}_{ll}] = 0, \\ \frac{\partial \mathcal{T}}{\partial h^{\alpha}_{mm}} &= \frac{2r}{m}h^{\alpha}_{mm} - 2\sum_{l=1}^{m-1} h^{\alpha}_{ll} = 0, \\ \frac{\partial \mathcal{T}}{\partial h^{\alpha}_{ij}} &= \frac{(m+r)}{r}(4m-4)h^{\alpha}_{ij} = 0, \\ \frac{\partial \mathcal{T}}{\partial h^{\alpha}_{im}} &= \frac{4r+4m}{m}h^{\alpha}_{im} = 0, \end{aligned} \tag{22}$$

$\forall \alpha \in \{m+1, \ldots, n\}$ and $i \neq j$.
When i is not equal to j, Equation (22) produces $h^{\alpha}_{ij} = 0$ for every solution of h^c, and the first two sets of equations in Equation (22) have the determinant equal to zero (submanifolds that are not totally geodesic have solutions). Apart from this, one can express the Hessian matrix by

$$H(\mathcal{T}) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} \frac{2(r+m)(m-1)}{r} - 2 & -2 & \ldots & -2 & -2 \\ -2 & \frac{2(m-1)(r+m)}{r} - 2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & \frac{2(r+m)(m-1)}{r} - 2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2r}{m} \end{pmatrix},$$

and 0 stands for the null matrices of corresponding sizes. We also write

$$H_2 = \mathrm{diag}\Big(\frac{2}{r}(r+m)(m-1), \frac{2}{r}(m-1)(r+m), \ldots, \frac{2}{r}(r+m)(m-1)\Big),$$

$$H_3 = \mathrm{diag}\Big(\frac{2(r+m)}{m}, \frac{2(r+m)}{m}, \ldots, \frac{2(r+m)}{m}\Big).$$

In this way, we find the following eigenvalues for $H(\mathcal{T})$

$$\lambda_{11} = 0,\ \lambda_{22} = \frac{2(m^3 - m^2 + r^2)}{rm},\ \lambda_{33} = \cdots = \lambda_{mm} = \frac{2(m-1)(r+m)}{r},$$

$$\lambda_{ij} = \frac{2}{r}(m-1)(r+m),\ \lambda_{in} = \frac{2(r+m)}{m},$$

where in we have assumed that $i \neq j$.
This concludes $\mathcal{T}$ to be parabolic and confirms its approach to a minimum $\mathcal{T}(h^c)$ at any solution h^c of Equation (22). Now, Equations (21) and (22) lead us to $\mathcal{T}(h^c) = 0$, establishing $\mathcal{T} \geq 0$ and providing the following

$$\begin{aligned} 2\overline{\tau}(p) &\leq r\mathcal{C} + \frac{(m-1)(m^2-m-r)(m+r)}{rm}\mathcal{C}(\mathcal{J}) \\ &+ \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10}(m-\varepsilon)m \\ &+ \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(2m\varepsilon-2)trace\varphi] \\ &+ \frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2 - trace\varphi - m\varepsilon], \end{aligned}$$

where by, we get

$$\begin{aligned} \rho &\leq \mathcal{C}\frac{r}{(m^2-m)} + \frac{(r+m)}{rm^2}(m^2-m-r)\mathcal{C}(\mathcal{J}) \\ &+ \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10(m-1)}(m-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ &+ \frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2 - trace\varphi - m\varepsilon], \end{aligned}$$

where $\mathcal{J}$ stands for any tangent hyperplane of T_pN such that Equation (18) holds in view of the above equation. In addition, $\forall i, j \in \{1, \dots m\}$ with $i \neq j$, Equation (18) holds for equality if

$$h^{\alpha}_{ij} = 0, \tag{23}$$

and

$$h^{\alpha}_{mm} = \frac{(m-1)}{r}mh^{\alpha}_{11} = \frac{(m-1)}{r}mh^{\alpha}_{22} \cdots = \frac{(m-1)}{r}mh^{\alpha}_{m-1m-1}, \quad m+1 \leq \alpha \leq n. \tag{24}$$

Therefore, by virtue of Equations (23) and (24), one concludes that with trivial normal connection, N is invariantly quasi-umbilical in M^n if and only if Equation (18) holds for equality, in such a way that S appears as Equation (20) for some local orthonormal frames.

(ii) Equation (19) can also be established in a similar fashion.

□

Next, we write the following result.

Corollary 2. *For any submanifold N^m immersed in a locally golden product Lorentzian manifold M^n endowed with a generalized symmetric metric U-connection, the following inequalities hold*

(i) for $\delta_c(m-1)$;

$$\begin{aligned} \rho &\leq \delta_c(m-1) \\ &+ \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10(m-1)}(m-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ &+ \frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2 - trace\varphi - m\varepsilon], \end{aligned} \tag{25}$$

where $0 \leq r \leq (m^2-m)$;

(ii) for $\widehat{\delta}_c(m-1)$*;*

$$\begin{aligned}\rho \;\leq\; & \widehat{\delta}_c(m-1)\\ & +\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon)\\ & +\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi]\\ & +\frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned} \tag{26}$$

where $(m^2-m)<r$.

In addition, Equations (25) and (26) hold for equality if for an orthonormal frame $\{E_1,\ldots,E_m,E_{m+1},\ldots,E_n\}$*, operator S can be represented as follows*

$$S_{m+1}=\begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0\\ 0 & b & 0 & \ldots & 0 & 0\\ 0 & 0 & b & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & b & 0\\ 0 & 0 & 0 & \ldots & 0 & 2b \end{pmatrix}, \qquad S_{m+2}=\cdots=S_n=0;$$

and

$$S_{m+1}=\begin{pmatrix} 2b & 0 & 0 & \ldots & 0 & 0\\ 0 & 2b & 0 & \ldots & 0 & 0\\ 0 & 0 & 2b & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 2b & 0\\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \qquad S_{m+2}=\cdots=S_n=0.$$

Now, we will give some results for the theorem 5

4. Some Consequences

Corollary 3. *For a Riemannian manifold* N^m *isometrically immersed in* M^n*, we have the following:*

(i) for $\delta_c(r;m-1)$*, for every* $r\in\{0,...,m(m-1)\}$*:*

(a) M^n *is equipped with a semi-symmetric metric U-connection*

$$\begin{aligned}\rho \;\leq\; & \frac{1}{(m^2-m)}[\delta_c(r;m-1)]\\ & +\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon)\\ & +\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi]\\ & +\frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(b) M^n *is equipped with* β *quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(r;m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(c) M^n *is equipped with semi-symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(r;m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(d) M^n *is equipped with quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(r;m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon].\end{aligned}$$

(ii) *for* $\widehat{\delta}_c(r;m-1)$, *for every* $r > m(m-1)$*:*

(a) M^n *is equipped with* α *semi-symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(r;m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(b) M^n *is equipped with* β *quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(r;m-1)]\\ &+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon)\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(c) M^n *is equipped with semi-symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(r;m-1)]\\ &+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10}{10(m-1)}(m-\varepsilon)\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(d) M^n *is equipped with quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(r;m-1)]\\ &+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon)\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi]\\ &+\frac{c_1+c_2-5}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon].\end{aligned}$$

Moreover, the relations in the above results become equalities if in some orthonormal frame $\{E_1,\ldots,E_m,E_{m+1},\ldots,E_n\}$*, the operator S reduces to:*

$$S_{m+1}=\begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0\\ 0 & b & 0 & \ldots & 0 & 0\\ 0 & 0 & b & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & b & 0\\ 0 & 0 & 0 & \ldots & 0 & \frac{b(m^2-m)}{r}\end{pmatrix},\qquad S_{m+2}=\cdots=S_n=0. \tag{27}$$

Corollary 4. *When* N^m *represents a Riemannian manifold isometrically immersed in a golden Lorentzian manifold* M^n *equipped with a g.s.m. U-connection, we have the following relations*

(i) for $\delta_c(m-1)$*, for every* $r\in\{0,\ldots(m^2-m)\}$*:*

(a) M^n *is equipped with* α *semi-symmetric metric* U*-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(b) M^n *is equipped with* β *quarter symmetric metric* U*-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(c) M^n *is equipped with semi-symmetric metric* U*-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(d) M^n *is equipped with quarter symmetric metric* U*-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\delta_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon].\end{aligned}$$

(ii) *for* $\widehat{\delta}_c(m-1)$*, for every* $(m^2-m) < r$.

(a) M^n *is equipped with* α *semi-symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(b) M^n *is equipped with* β *quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5\beta^2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(c) M^n *is equipped with semi-symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(m-1)] \\ & + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon],\end{aligned}$$

(d) M^n *is equipped with quarter symmetric metric U-connection*

$$\begin{aligned}\rho \leq\ & \frac{1}{(m^2-m)}[\widehat{\delta}_c(m-1)] \\ & + + \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10(m-1)}(m-\varepsilon) \\ & + \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10(m^2-m)}[(2m\varepsilon-2)trace\varphi] \\ & + \frac{c_1+c_2-5}{5(m^2-m)}[(trace\varphi)^2-trace\varphi-m\varepsilon].\end{aligned}$$

Equalities hold for all relations in the above results if in some orthonormal frame $\{E_1, \ldots, E_m, E_{m+1}, \ldots, E_n\}$, *the shape operators take the following form*

$$S_{m+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2b \end{pmatrix}, \qquad S_{m+2} = \cdots = S_n = 0 \tag{28}$$

and

$$S_{m+1} = \begin{pmatrix} 2b & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2b & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \qquad S_{m+2} = \cdots = S_n = 0. \tag{29}$$

Remark 1. *The proofs of all the corollaries are similar to that of Theorem 5.*

Remark 2. *For a Lorentzian manifold M equipped with generalized symmetric metric U-connection, we can define structures of different types [4]:*

- *silver type* $\sigma_{2,1} = 1 + \sqrt{2}$ *if* $p = 2$ *and* $q = 1$;
- *subtle type* $\sigma_{4,1} = 2 + \sqrt{5}$ *if* $p = 4$ *and* $q = 1$;
- *copper type* $\sigma_{1,2} = 2$ *with* $p = 1$ *and* $q = 2$;
- *bronze type* $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ *with* $p = 3$ *and* $q = 1$;
- *nickel type* $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ *if* $p = 1$ *and* $q = 3$, *etc.*

Using the above structures, one can establish inequalities similar to Theorem 5.

5. Discussion

The present study deals with generalized symmetric metric U-connection on golden Lorentzian manifolds. We also study lower bounds for submanifolds immersed in golden Lorentzian manifolds equipped with generalized symmetric metric U-connection. Moreover, submanifolds for which equality holds are also discussed.

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