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Analysis of the Time Fractional-Order Coupled Burgers Equations with Non-Singular Kernel Operators

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Abstract: In this article, we have investigated the fractional-order Burgers equation via Natural decomposition method with nonsingular kernel derivatives. The two types of fractional derivatives are used in the article of Caputo–Fabrizio and Atangana–Baleanu derivative. We employed Natural transform on fractional-order Burgers equation followed by inverse Natural transform, to achieve the result of the equations. To validate the method, we have considered a two examples and compared with the exact results.

Keywords: Caputo–Fabrizio and Atangana–Baleanu operators; Adomian decomposition method; Natural transform; Burgers equations

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1. Introduction

Fractional calculus is a developing area in many areas of science. Scholars are paying attention to fractional differential equations as they are applied to model various implementations such as heat conduction, viscoelasticity, dynamical systems, biology, and so on [1–3]. Because of its significance in various fields, numerous methods for studying the computational and exact results of fractional differential equations have been developed. Other than the modelling, convergence and divergence of the results are also equally importants. A appropriate definition is required for a fractional generalisation of a physical system. Many fractional derivative definitions introduce in the last few centuries. Caputo, Riemann–Liouville, Fabrizio, Atangana–Baleanu, Grunwald–Letnikov, and Riesz fractional derivatives are some famous definitions in the literature. We refer to [4,5] and the references therein for additional information. The kernel of the Riemann–Liouville and Caputo fractional derivatives is unique. Atangana–Baleanu and Caputo–Fabrizio have recently created two non-singular kernel fractional derivative definitions. Numerous techniques for analyzing fractional differential equations for accuracy and dependability are being investigated. Some of the famous numerical and analytical techniques such as fractional differential transform method [6,7], variational iteration method [8], homotopy analysis transform method [9], homotopy perturbation transform method [10], q-homotopy analysis transform method [11], residual power series method [12], operational matrix method [13], Natural decomposition transform technique [14] and Adam Bashforth's Moulton technique [15].

The aim of this article is to implement the Natural decomposition method to the coupled Burgers equations.

The two dimensional non-linear fractional partial differential equation is

$$\begin{aligned} &\frac{\partial^{\alpha} u}{\partial \tau^{\alpha}} + u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \zeta} = \frac{1}{Re}\left[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \zeta^2}\right], \\ &\frac{\partial^{\beta} v}{\partial \tau^{\beta}} + u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \zeta} = \frac{1}{Re}\left[\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \zeta^2}\right], \ \ 0 < \alpha, \beta \leq 1. \end{aligned} \tag{1}$$

The Burgers model of turbulence is a very significant fluid dynamic model. Several researchers have considered studying this theory and model of shock waves to achieve theoretical knowledge of a physical flow class and to analyze different approximate techniques. The unique function of Equation (1) is that it is the straightforward the competition's numerical formulation among viscous diffusion and non-linear advection. It represents the most basic forms of the dissipation term $v\frac{\partial u}{\partial \zeta}$ and the non-linear advection term $u\frac{\partial u}{\partial \xi}$, where $V = [u, v]$ and Re is the Reynolds number used to simulate the physical phenomenon of waves motions and thus determine the solution's behavior. Cole [16] investigated the mathematical properties of Equation (1). Non-linear phenomena play an important roles in physics and applied mathematics. The signification of achieving the actual or approximated results of partial differential equations in mathematics and physics is in terms of seeking new techniques, this is still a hot topic for achieving new actual or approximated results [17–20]. Different methods for obtaining various actual results of many physicals model described applying non-linear partial differential equations have been proposed for this purpose. Bateman [21] developed a well-known model and discovered its steady results, which are descriptive of many flow of viscous. Burgers [16] later suggested it as one of a class model defining mathematical problems of turbulence. Hopf [22] and Cole [23] described it in the context of gas dynamics. They also showed independently that the Burgers equation can be achieved the actual solution for any initial condition. Numerical solutions to Burgers' one-dimensional problem have been studied by Benton and Platzman [24]. There's no doubt that the non-linear convection terms and the viscosity term simplify the Navier–Stokes equation [25].

The aim of this article is to apply natural decomposition method to solve fractional-order coupled Burgers equations. Rawashdeh and Maitama [26] introduce natural decomposition method for a class of non-linear partial differential equations. Natural decomposition method do not require prescribed assumptions, linearization, discretization or perturbation and prevent any roundoff errors. Recently, natural decomposition method applied to fractional-order Fisher's equation [27]. The paper is organized as follows. Section 2 discusses briefly the fundamental definitions of singular and nonsingular fractional calculus definitions, natural transforms, and fractional derivatives. In Section 3, we introduced the NTDM for solving fractional-order Burgers equations with non-singular definitions. We discussed the uniqueness and convergence of the results in Section 4. In Section 5, two examples of fractional-order coupled Burgers equations given to validate the present techniques. In Section 6, brief conclusions of this article are represented.

2. Basic Preliminaries

There are several fractional derivative definitions available in the literature; for more information, see [28–30]. For the benefit of the readers, we have provided definitions of Riemann–Liouville, Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives in this section.

Definition 1. *The fractional Riemann–Liouville integral operator of a function $f \in C_v, v \geq -1$ is defined as [31]*

$$\begin{aligned} I^{\alpha} f(\omega) =& \frac{1}{\Gamma(\alpha)} \int_0^{\omega} (\omega - \zeta)^{\alpha - 1} f(\zeta) d\zeta, \ \ \alpha > 0, \ \ \omega > 0. \\ \text{and} \ \ I^0 f(\omega) =& f(\omega) \end{aligned} \tag{2}$$

Definition 2. *The Caputo of fractional derivative $f(\omega)$ is defined as [31]*

$$D_{\omega}^{\alpha} f(\omega) = I^{m-\alpha} D^{m} f(\omega) = \frac{1}{m-\alpha} \int_{\omega}^{0} (\omega - \zeta)^{m-\alpha-1} f^{m}(\zeta) d\zeta \tag{3}$$

for $m-1 < \alpha \leq m,\ \ m \in N,\ \ \omega > 0, f \in C_{v}^{m}, v \geq -1$.

Definition 3. *The fractional Caputo–Fabrizio derivative of $f(\omega)$ is defined by [31]*

$$D_{\omega}^{\alpha} f(\omega) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{\omega} \exp\left(\frac{-\alpha(\omega-\zeta)}{1-\alpha}\right) D(f(\zeta)) d\zeta \tag{4}$$

where $0 < \alpha < 1$ and $B(\alpha)$ is a normalization function, where $B(0) = B(1) = 1$.

Definition 4. *The fractional Atangana–Baleanu Caputo derivative of $f(\omega)$ is expressed as [31]*

$$D_{\omega}^{\alpha} f(\omega) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{\omega} E_{\alpha}\left(\frac{-\alpha(\omega-\zeta)}{1-\alpha}\right) D(f(\zeta)) d\zeta \tag{5}$$

where $0 < \alpha < 1$. Normalization function is $B(\alpha)$ and the Mittag–Leffler function is $E_{\alpha}(z) = \sum_{m=0}^{\infty} \frac{z^{m}}{\Gamma(\alpha^{m}+1)}$.

Definition 5. *The Natural transformation of $u(\tau)$ is given by*

$$\mathbb{N}(u(\tau)) = \mathcal{U}(s,\mu) = \int_{-\infty}^{\infty} e^{-s\tau} u(\tau) d\tau,\ \ s \in (-\infty, \infty). \tag{6}$$

For $\tau \in (0, \infty)$, Natural transformation of $u(\tau)$ is expressed as

$$\mathbb{N}(u(\tau)H(\tau)) = \mathbb{N}^{+} = \mathcal{U}^{+}(s,\mu) = \int_{-\infty}^{\infty} e^{-s\tau} u(\tau) d\tau,\ \ s \in (0, \infty). \tag{7}$$

where $H(\tau)$ is the Heaviside function.

Definition 6. *The inverse Natural transformation of $\mathcal{U}(s,\mu)$ is defined by*

$$\mathbb{N}^{-1}[\mathcal{U}(s,\mu)] = u(\tau),\ \ \forall \tau \geq 0 \tag{8}$$

Lemma 1. *If Natural transformation of linearity property of $u_1(\tau)$ is $u_1(s,\mu)$ and $u_2(\tau)$ is $u_2(s,\mu)$, then*

$$\mathbb{N}[c_1 u_1(\tau) + c_2 u_2(\tau)] = c_1 \mathbb{N}[u_1(\tau)] + c_2 \mathbb{N}[u_2(\tau)] = c_1 u_1(s,\mu) + c_2 u_2(s,\mu), \tag{9}$$

where c_1 and c_2 are constants.

Lemma 2. *(inverse property) If inverse Natural transformation of $\mathcal{U}_1(s,\mu)$ and $\mathcal{U}_2(s,\mu)$ are $u_1(\tau)$ and $u_2(\tau)$ respectively then*

$$\mathbb{N}^{-1}[c_1 \mathcal{U}_1(s,\mu) + c_2 \mathcal{U}_2(s,\mu)] = c_1 \mathbb{N}^{-1}[\mathcal{U}_1(s,\mu)] + c_2 \mathbb{N}^{-1}[\mathcal{U}_2(s,\mu)] = c_1 u_1(\tau) + c_2 u_2(\tau), \tag{10}$$

where c_1 and c_2 are constants.

Definition 7. *The Caputo operator of Natural transformation of $D_{\tau}^{\alpha} u(\tau)$ is defined as [31]*

$$\mathbb{N}[D_{\tau}^{\alpha}] = \left(\frac{s}{\mu}\right)^{\alpha} \left(\mathbb{N}[u(\tau)] - \left(\frac{1}{s}\right) u(0)\right) \tag{11}$$

Definition 8. *Natural transformation of $D_\tau^\alpha u(\tau)$ by means of Caputo–Fabrizio is defined as [31]*

$$\mathbb{N}[D_\tau^\alpha] = \frac{1}{1-\alpha+\alpha(\frac{\mu}{s})}\left(\mathbb{N}[u(\tau)] - \left(\frac{1}{s}\right)u(0)\right) \tag{12}$$

Definition 9. *Natural transformation of $D_\tau^\alpha u(\tau)$ by means of Atangana–Baleanu Caputo derivative is expressed as [31]*

$$\mathbb{N}[D_\tau^\alpha] = \frac{M[\alpha]}{1-\alpha+\alpha(\frac{\mu}{s})^\alpha}\left(\mathbb{N}[u(\tau)] - \left(\frac{1}{s}\right)u(0)\right) \tag{13}$$

3. Methodology

In this section, we introduce a general numerical methodology for the following equation based on Natural transform.

$$D_\tau^\alpha u(\xi,\tau) = \mathcal{L}(u(\xi,\tau)) + \mathcal{N}(u(\xi,\tau)) + h(\xi,\tau), \tag{14}$$

with the initial condition

$$u(\xi,0) = \phi(\xi), \tag{15}$$

where $\mathcal{L}$, $\mathcal{N}$ and $h(\xi,\tau)$ are linear, non-linear and source terms respectively.

3.1. Case I $(NTDM_{CF})$

By using Natural transformation of Equation (14), with the help of Caputo–Fabrizio fractional derivative we achieve,

$$\frac{1}{p(\alpha,\mu,s)}\left(\mathbb{N}[u(\xi,\tau)] - \frac{\phi(\xi)}{s}\right) = \mathbb{N}[M(\xi,\tau)], \tag{16}$$

where

$$p(\alpha,\mu,s) = 1-\alpha+\alpha(\frac{\mu}{s}). \tag{17}$$

By applying inverse Natural transformation, we can write Equation (16) as,

$$u(\xi,\tau) = \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + p(\alpha,\mu,s)\mathbb{N}[M(\xi,\tau)]\right), \tag{18}$$

$\mathcal{N}(u(\xi,\tau))$ can be decomposed into

$$\mathcal{N}(u(\xi,\tau)) = \sum_{i=0}^{\infty} A_t, \tag{19}$$

where A_τ is the Adomian polynomials. We suppose that Equation (14) has the numerical expansion

$$u(\xi,\tau) = \sum_{i=0}^{\infty} u_i(\xi,\tau). \tag{20}$$

By putting Equations (19) and (20) into (18), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(\xi,\tau) =& \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + p(\alpha,\mu,s)\mathbb{N}[h(\xi,\tau)]\right)\\ &+ \mathbb{N}^{-1}\left(p(\alpha,\mu,s)\mathbb{N}\left[\sum_{i=0}^{\infty}\mathcal{L}(u_i(\xi,\tau)) + A_\tau\right]\right)\end{aligned} \tag{21}$$

From (21), we get

$$\begin{aligned} u_0^{CF}(\xi,\tau) =& \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + p(\alpha,\mu,s)\mathbb{N}[h(\xi,\tau)]\right), \\ u_1^{CF}(\xi,\tau) =& \mathbb{N}^{-1}(p(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_0(\xi,\tau)) + A_0]), \\ & \vdots \\ u_{l+1}^{CF}(\xi,\tau) =& \mathbb{N}^{-1}(p(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_l(\xi,\tau)) + A_l]), \quad l = 1,2,3,\cdots \end{aligned} \tag{22}$$

By putting (22) into (20), we get the $NTDM_{CF}$ solution of (14) as

$$u^{CF}(\xi,\tau) = u_0^{CF}(\xi,\tau) + u_1^{CF}(\xi,\tau) + u_2^{CF}(\xi,\tau) + \cdots \tag{23}$$

3.2. Case I $(NTDM_{ABC})$

By applying Natural transformation of Equation (14), with the help of Atangana–Baleanu derivative we achieve,

$$\frac{1}{q(\alpha,\mu,s)}\left(\mathbb{N}[u(\xi,\tau)] - \frac{\phi(\xi)}{s}\right) = \mathbb{N}[M(\xi,\tau)], \tag{24}$$

where

$$q(\alpha,\mu,s) = \frac{1-\alpha+\alpha(\frac{\mu}{s})^{\alpha}}{B(\alpha)}. \tag{25}$$

By using inverse Natural transformation (8), we can write (24) as,

$$u(\xi,\tau) = \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + q(\alpha,\mu,s)\mathbb{N}[M(\xi,\tau)]\right), \tag{26}$$

$\mathcal{N}(u(\xi,\tau))$ can be decomposed into

$$\mathcal{N}(u(\xi,\tau)) = \sum_{i=0}^{\infty} A_\tau, \tag{27}$$

where A_τ is the Adomian polynomials [32,33]. We suppose that, the Equation (14) has the numerical expansion

$$u(\xi,\tau) = \sum_{i=0}^{\infty} u_i(\xi,\tau). \tag{28}$$

By putting Equations (27) and (28) into (26), we achieve

$$\begin{aligned} \sum_{i=0}^{\infty} u_i(\xi,\tau) =& \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + q(\alpha,\mu,s)\mathbb{N}[h(\xi,\tau)]\right) \\ &+ \mathbb{N}^{-1}\left(q(\alpha,\mu,s)\mathbb{N}\left[\sum_{i=0}^{\infty}\mathcal{L}(u_i(\xi,\tau)) + A_\tau\right]\right) \end{aligned} \tag{29}$$

From (21), we get

$$\begin{aligned} u_0^{ABC}(\xi,\tau) =& \mathbb{N}^{-1}\left(\frac{\phi(\xi)}{s} + q(\alpha,\mu,s)\mathbb{N}[h(\xi,\tau)]\right), \\ u_1^{ABC}(\xi,\tau) =& \mathbb{N}^{-1}(q(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_0(\xi,\tau)) + A_0]), \\ & \vdots \\ u_{l+1}^{ABC}(\xi,\tau) =& \mathbb{N}^{-1}(q(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_l(\xi,\tau)) + A_l]), \quad l = 1,2,3,\cdots \end{aligned} \tag{30}$$

By putting (30) into (28), we get the $NTDM_{CF}$ solution of (14) as

$$u^{ABC}(\xi,\tau) = u_0^{ABC}(\xi,\tau) + u_1^{ABC}(\xi,\tau) + u_2^{ABC}(\xi,\tau) + \cdots \tag{31}$$

4. Convergence Analysis

In this section, we discuss convergence and uniqueness of the $NTDM_{CF}$ and $NTDM_{ABC}$.

Theorem 1. *The $NTDM_{CF}$ result of (14) is unique when $0 < (\delta_1 + \delta_2)(1 - \alpha + \alpha\tau) < 1$.*
Proof Let $F = (C[J], ||.||)$ be the Banach space with the norm $||\phi(\tau)|| = max_{\tau\in J}|\phi(\tau)|, \forall$ continuous function on J. Let $G : F \to F$ is a non-linear mapping, where

$$u_{l+1}^C = u_0^C + \mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_l(\zeta,\tau)) + \mathcal{N}(u_l(\zeta,\tau))]], \;\; l \geq 0.$$

Suppose that $|\mathcal{L}(u) - \mathcal{L}(u^*)| < \delta_1|u - u^*|$ and $|\mathcal{N}(u) - \mathcal{N}(u^*)| < \delta_2|u - u^*|$, where δ_1 and δ_2 are Lipschitz constants and $u := u(\zeta,\tau)$ and $u^* := u^*(\zeta,\tau)$ are are two different function values.

$$\begin{aligned}
||Gu - Gu^*|| &\leq max_{t\in J}|\mathbb{N}^{-1}\Big[p(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u) - \mathcal{L}(u^*)] \\
&+ p(\alpha,\mu,s)\mathbb{N}[\mathcal{N}(u) - \mathcal{N}(u^*)]|\Big] \\
&\leq max_{\tau\in J}\Big[\delta_1\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[|u - u^*|]] \\
&+ \delta_2\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[|u - u^*|]]\Big] \\
&\leq max_{t\in J}(\delta_1 + \delta_2)\Big[\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}|u - u^*|]\Big] \\
&\leq (\delta_1 + \delta_2)\Big[\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}||u - u^*||]\Big] \\
&= (\delta_1 + \delta_2)(1 - \alpha + \alpha\tau)||u - u^*||
\end{aligned} \tag{32}$$

G is contraction as $0 < (\delta_1 + \delta_2)(1 - \alpha + \alpha\tau) < 1$. The result of (14) is unique from Banach fixed point theorem.

Theorem 2. *The $NTDM_{ABC}$ result of (14) is unique when $0 < (\delta_1 + \delta_2)(1 - \alpha + \alpha\frac{\tau^{mu}}{\Gamma(mu+1)}) < 1$.*
Proof: Let $F = (C[J], ||.||)$ be a Banach space with the norm $||\phi(\tau)|| = max_{\tau\in J}|\phi(\tau)|, \forall$ continuous function on J. Let $G : F \to F$ be a non-linear mapping, where

$$u_{l+1}^C = u_0^C + \mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u_l(\zeta,\tau)) + \mathcal{N}(u_l(\zeta,\tau))]], \;\; l \geq 0.$$

Suppose that $|\mathcal{L}(u) - \mathcal{L}(u^*)| < \delta_1|u - u^*|$ and $|\mathcal{N}(u) - \mathcal{N}(u^*)| < \delta_2|u - u^*|$, where δ_1 and δ_2 are Lipschitz constants and $u := u(\zeta,\tau)$ and $u^* := u^*(\zeta,\tau)$ are two different function values.

$$\begin{aligned}
||Gu - Gu^*|| &\leq max_{t\in J}|\mathbb{N}^{-1}\Big[q(\alpha,\mu,s)\mathbb{N}[\mathcal{L}(u) - \mathcal{L}(u^*)] \\
&+ q(\alpha,\mu,s)\mathbb{N}[\mathcal{N}(u) - \mathcal{N}(u^*)]|\Big] \\
&\leq max_{t\in J}\Big[\delta_1\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}[|u - u^*|]] \\
&+ \delta_2\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}[|u - u^*|]]\Big] \\
&\leq max_{t\in J}(\delta_1 + \delta_2)\Big[\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}|u - u^*|]\Big] \\
&\leq (\delta_1 + \delta_2)\Big[\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}||u - u^*||]\Big] \\
&= (\delta_1 + \delta_2)(1 - \alpha + \alpha\frac{\tau^{\alpha}}{\Gamma\alpha + 1})||u - u^*||
\end{aligned} \tag{33}$$

G is a contraction as $0 < (\delta_1 + \delta_2)(1 - \alpha + \alpha\frac{\tau^{\alpha}}{\Gamma\alpha+1}) < 1$. The result of (14) is unique from Banach fixed point theorem.

Theorem 3. *The $NTDM_{CF}$ result of (14) is convergent.*
Proof: Let $u_m = \sum_{r=0}^{m} u_r(\zeta,\tau)$. To prove that u_m is a Cauchy sequence in F, consider,

$$\begin{aligned}
||u_m - u_n|| &= max_{\tau\in J}|\sum_{r=n+1}^{m} u_r|, \;\; n = 1,2,3,\cdots \\
&\leq max_{\tau\in J}\left|\mathbb{N}^{-1}\left[p(\alpha,\mu,s)\mathbb{N}\left[\sum_{r=n+1}^{m}(\mathcal{L}(u_{r-1})+\mathcal{N}(u_{r-1}))\right]\right]\right| \\
&= max_{\tau\in J}\left|\mathbb{N}^{-1}\left[p(\alpha,\mu,s)\mathbb{N}\left[\sum_{r=n+1}^{m-1}(\mathcal{L}(u_{r})+\mathcal{N}(u_{r}))\right]\right]\right| \\
&\leq max_{\tau\in J}|\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[(\mathcal{L}(u_{m-1})-\mathcal{L}(u_{n-1})+\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1}))]]| \\
&\leq \delta_1 max_{\tau\in J}|\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[(\mathcal{L}(u_{m-1})-\mathcal{L}(u_{n-1}))]]| \\
&+ \delta_2 max_{\tau\in J}|\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[(\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1}))]]| \\
&= (\delta_1+\delta_2)(1-\alpha+\alpha\tau)||u_{m-1}-u_{n-1}||
\end{aligned} \tag{34}$$

Let $m = n+1$, then

$$||u_{n+1}-u_n|| \leq \delta||u_n - u_{n-1}|| \leq \delta^2||u_{n-1}u_{n-2}|| \leq \cdots \leq \delta^n||u_1-u_0||, \tag{35}$$

where $\delta = (\delta_1+\delta_2)(1-\alpha+\alpha\tau)$. Similarly, we have

$$\begin{aligned}
||u_m - u_n|| &\leq ||u_{n+1}-u_n|| + ||u_{n+2}u_{n+1}|| + \cdots + ||u_m - u_{m-1}||, \\
&(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})||u_1 - u_0|| \\
&\leq \delta^n\left(\frac{1-\delta^{m-n}}{1-\delta}\right)||u_1||,
\end{aligned} \tag{36}$$

As $0 < \delta < 1$, we get $1-\delta^{m-n} < 1$. Therefore,

$$||u_m - u_n|| \leq \frac{\delta^n}{1-\delta} max_{\tau\in J}||u_1||. \tag{37}$$

Since $||u_1|| < \infty$, $||u_m - u_n|| \to 0$ when $n \to \infty$. Hence u_m is a Cauchy sequence in F, therefore the series u_m is convergent.

Theorem 4. *The $NTDM_{ABC}$ result of (14) is convergent.*
Proof: Let $u_m = \sum_{r=0}^{m} u_r(\zeta,\tau)$. To prove that u_m is a Cauchy sequence in F, consider,

$$\begin{aligned}
||u_m - u_n|| &= max_{\tau\in J}|\sum_{r=n+1}^{m} u_r|, \;\; n = 1,2,3,\cdots \\
&\leq max_{\tau\in J}\left|\mathbb{N}^{-1}\left[q(\alpha,\mu,s)\mathbb{N}\left[\sum_{r=n+1}^{m}(\mathcal{L}(u_{r-1})+\mathcal{N}(u_{r-1}))\right]\right]\right| \\
&= max_{\tau\in J}\left|\mathbb{N}^{-1}\left[q(\alpha,\mu,s)\mathbb{N}\left[\sum_{r=n+1}^{m-1}(\mathcal{L}(u_{r})+\mathcal{N}(u_{r}))\right]\right]\right| \\
&\leq max_{\tau\in J}|\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}[(\mathcal{L}(u_{m-1})-\mathcal{L}(u_{n-1})+\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1}))]]| \\
&\leq \delta_1 max_{\tau\in J}|\mathbb{N}^{-1}[q(\alpha,\mu,s)\mathbb{N}[(\mathcal{L}(u_{m-1})-\mathcal{L}(u_{n-1}))]]| \\
&+ \delta_2 max_{\tau\in J}|\mathbb{N}^{-1}[p(\alpha,\mu,s)\mathbb{N}[(\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1}))]]| \\
&= (\delta_1+\delta_2)(1-\alpha+\alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)})||u_{m-1}-u_{n-1}||
\end{aligned} \tag{38}$$

Let $m = n+1$, then

$$||u_{n+1} - u_n|| \leq \delta||u_n - u_{n-1}|| \leq \delta^2||u_{n-1}u_{n-2}|| \leq \cdots \leq \delta^n||u_1 - u_0||, \tag{39}$$

where $\delta = (\delta_1 + \delta_2)(1 - \alpha + \alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)})$. Similarly, we have

$$\begin{aligned} ||u_m - u_n|| &\leq ||u_{n+1} - u_n|| + ||u_{n+2}u_{n+1}|| + \cdots + ||u_m - u_{m-1}||, \\ &(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})||u_1 - u_0|| \\ &\leq \delta^n\left(\frac{1-\delta^{m-n}}{1-\delta}\right)||u_1||, \end{aligned} \tag{40}$$

As $0 < \delta < 1$, we get $1 - \delta^{m-n} < 1$. Therefore,

$$||u_m - u_n|| \leq \frac{\delta^n}{1-\delta} max_{t\in J}||u_1||. \tag{41}$$

Since $||u_1|| < \infty$, $||u_m - u_n|| \to 0$ when $n \to \infty$. Hence u_m is a Cauchy sequence in F, therefore the series u_m is convergent.

5. Numerical Examples

This section includes the numerical results for a few problems of Burgers equation. We have chosen these equations as the closed form results are available and also well known techniques applied to analyze the results in the literature.

Example 1. *Consider the fractional-order system of Burgers equations*

$$\begin{aligned} D_\tau^\alpha u + u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \zeta} &= \frac{1}{Re}\left[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \zeta^2}\right], \\ D_\tau^\alpha v + u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \zeta} &= \frac{1}{Re}\left[\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \zeta^2}\right], \ 0 < \alpha, \beta \leq 1, \end{aligned} \tag{42}$$

with initial conditions

$$\begin{aligned} u(\xi,\zeta,0) &= \frac{3}{4} - \frac{1}{4(1+\exp((Re/32) - 4\xi + 4\zeta))}, \\ v(\xi,\zeta,0) &= \frac{3}{4} + \frac{1}{4(1+\exp((Re/32) - 4\xi + 4\zeta))}. \end{aligned} \tag{43}$$

where Re is the Reynolds number. Now using the Natural transform (42), we get

$$\begin{aligned} \mathbb{N}[D_\tau^\alpha u(\xi,\zeta,\tau)] &= -\mathbb{N}\left\{u\frac{\partial u}{\partial \xi}\right\} - \mathbb{N}\left\{v\frac{\partial u}{\partial \xi}\right\} + \frac{1}{Re}\left[\mathbb{N}\left\{\frac{\partial^2 u}{\partial \xi^2}\right\} + \mathbb{N}\left\{\frac{\partial^2 u}{\partial \zeta^2}\right\}\right], \\ \mathbb{N}[D_\tau^\beta v(\xi,\zeta,\tau)] &= -\mathbb{N}\left\{u\frac{\partial v}{\partial \xi}\right\} - \mathbb{N}\left\{v\frac{\partial v}{\partial \xi}\right\} + \frac{1}{Re}\left[\mathbb{N}\left\{\frac{\partial^2 v}{\partial \xi^2}\right\} + \mathbb{N}\left\{\frac{\partial^2 v}{\partial \zeta^2}\right\}\right]. \end{aligned} \tag{44}$$

Define the non-linear operator as

$$\begin{aligned} \frac{1}{s^\alpha}\mathbb{N}[u(\xi,\zeta,\tau)] - s^{2-\alpha}u(\xi,\zeta,0) &= \mathbb{N}\left[-u\frac{\partial u}{\partial \xi} - v\frac{\partial u}{\partial \zeta} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \zeta^2}\right)\right], \\ \frac{1}{s^\beta}\mathbb{N}[v(\xi,\zeta,\tau)] - s^{2-\alpha}\mu(\xi,\zeta,0) &= \mathbb{N}\left[-u\frac{\partial v}{\partial \xi} - v\frac{\partial v}{\partial \zeta} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \zeta^2}\right)\right]. \end{aligned} \tag{45}$$

By the above equation, we get

$$\begin{aligned}\mathbb{N}[u(\xi,\zeta,\tau)] &= s^2\left[\frac{3}{4}-\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\right]+\frac{\alpha(s-\alpha(s-\alpha))}{s^2}\left[-u\frac{\partial u}{\partial\xi}-v\frac{\partial u}{\partial\zeta}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\zeta^2}\right)\right],\\ \mathbb{N}[v(\xi,\zeta,\tau)] &= s^2\left[\frac{3}{4}+\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\right]+\frac{\alpha(s-\beta(s-\alpha))}{s^2}\left[-u\frac{\partial v}{\partial\xi}-v\frac{\partial v}{\partial\zeta}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\zeta^2}\right)\right].\end{aligned} \tag{46}$$

Apply inverse Natural transformation on Equation (46) and then it reduces to

$$\begin{aligned}u(\xi,\zeta,\tau) &= \left[\frac{3}{4}-\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\right]\\ &\quad+\mathbb{N}^{-1}\left[\frac{\alpha(s-\alpha(s-\alpha))}{s^2}\mathbb{N}\left\{-u\frac{\partial u}{\partial\xi}-v\frac{\partial u}{\partial\zeta}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\zeta^2}\right)\right\}\right],\\ v(\xi,\zeta,\tau) &= \left[\frac{3}{4}+\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\right]\\ &\quad+\mathbb{N}^{-1}\left[\frac{\alpha(s-\beta(s-\alpha))}{s^2}\mathbb{N}\left\{-u\frac{\partial v}{\partial\xi}-v\frac{\partial v}{\partial\zeta}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\zeta^2}\right)\right\}\right].\end{aligned} \tag{47}$$

Now We Implement NDM_{CF}

Suppose that the infinite series results of the unknown functions $u(\xi,\zeta,\tau)$ and $v(\xi,\zeta,\tau)$ are respectively as follows

$$u(\xi,\zeta,\tau)=\sum_{l=0}^{\infty}u_l(\xi,\zeta,\tau)\ \ and\ \ v(\xi,\zeta,\tau)=\sum_{l=0}^{\infty}v_l(\xi,\zeta,\tau) \tag{48}$$

Note that $uu_\xi=\sum_{l=0}^{\infty}A_l$, $uv_\xi=\sum_{l=0}^{\infty}B_l$, $vu_\zeta=\sum_{l=0}^{\infty}C_l$ and $vv_\zeta=\sum_{l=0}^{\infty}D_l$ are the Adomian polynomials and they signify the non-linear terms. Applying the these terms, Equation (47) can be written as

$$\begin{aligned}\sum_{l=0}^{\infty}u_{l+1}(\xi,\zeta,\tau) &= \frac{3}{4}-\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\\ &\quad+\mathbb{N}^{-1}\left[\frac{\alpha(s-\alpha(s-\alpha))}{s^2}\mathbb{N}\left\{-\sum_{l=0}^{\infty}A_l-\sum_{l=0}^{\infty}B_l+\frac{1}{Re}\left(\sum_{l=0}^{\infty}u_{l\xi\xi}+\sum_{l=0}^{\infty}u_{l\zeta\zeta}\right)\right\}\right],\\ \sum_{l=0}^{\infty}v_{l+1}(\xi,\zeta,\tau) &= \frac{3}{4}+\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}\\ &\quad+\mathbb{N}^{-1}\left[\frac{\alpha(s-\beta(s-\alpha))}{s^2}\mathbb{N}\left\{-\sum_{l=0}^{\infty}C_l-\sum_{l=0}^{\infty}D_l+\frac{1}{Re}\left(\sum_{l=0}^{\infty}v_{l\xi\xi}+\sum_{l=0}^{\infty}v_{l\zeta\zeta}\right)\right\}\right].\end{aligned} \tag{49}$$

By both sides comparing of Equation (49), we can easily achieve the recursive relation as shown below

$$\begin{aligned}u_0(\xi,\zeta,\tau) &= \frac{3}{4}-\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))},\\ v_0(\xi,\zeta,\tau) &= \frac{3}{4}+\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}.\end{aligned}$$

$$u_1(\xi,\zeta,\tau) = -\frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)(\alpha(\tau-1)+1)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2},$$
$$v_1(\xi,\zeta,\tau) = \frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)(\beta(\tau-1)+1)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2}. \tag{50}$$

$$\begin{aligned} u_2(\xi,\zeta,\tau) &= -\frac{\tau(1-\alpha)+\frac{\alpha\tau^2}{2}}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\left(-\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) \\ &+\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right)+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right), \\ v_2(\xi,\zeta,\tau) &= \frac{\tau(1-\beta)+\frac{\beta\tau^2}{2}}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4\Gamma(\alpha+\beta+1)}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\Big((-1+\exp((Re/8)(\xi-\zeta)) \\ &+\exp((Re/4)(\xi-\zeta)))\Big)\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right)+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right). \end{aligned} \tag{51}$$

The remaining components of u_l and $v_l (l \geq 3)$ of NDM solution can be smoothly achieved. Consequently, we calculate the series solution as

$$\begin{aligned} u(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) = u_0(\xi,\zeta,\tau)+u_1(\xi,\zeta,\tau)+u_2(\xi,\zeta,\tau)+\cdots, \\ u(\xi,\zeta,\tau) &= \frac{3}{4}-\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}-\frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)(\alpha(\tau-1)+1)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2} \\ &-\frac{\tau(1-\alpha)+\frac{\alpha\tau^2}{2}}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\left(-\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) \\ &+\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right)+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)+\cdots. \end{aligned} \tag{52}$$

$$\begin{aligned} v(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) = v_0(\xi,\zeta,\tau)+v_1(\xi,\zeta,\tau)+v_2(\xi,\zeta,\tau)+\cdots, \\ v(\xi,\zeta,\tau) &= \frac{3}{4}+\frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}+\frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)(\beta(\tau-1)+1)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2} \\ &\frac{\tau(1-\beta)+\frac{\beta\tau^2}{2}}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\Big((-1+\exp((Re/8)(\xi-\zeta)) \\ &+\exp((Re/4)(\xi-\zeta)))\Big)\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right)+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)+\cdots. \end{aligned} \tag{53}$$

Now We Implement NDM_{ABC}

Suppose that the infinite series results of the unknown functions $u(\xi,\zeta,\tau)$ and $v(\xi,\zeta,\tau)$ are respectively as follows

$$u(\xi,\zeta,\tau) = \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) \;\; and \;\; v(\xi,\zeta,\tau) = \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) \tag{54}$$

Note that $uu_\xi = \sum_{l=0}^{\infty} A_l$, $uv_\xi = \sum_{l=0}^{\infty} B_l$, $vu_\zeta = \sum_{l=0}^{\infty} C_l$ *and* $vv_\zeta = \sum_{l=0}^{\infty} D_l$ *are the Adomian polynomials and they signify the non-linear terms. Applying the these terms, Equation (47) can be written as*

$$\begin{aligned}
\sum_{l=0}^{\infty} u_{l+1}(\xi,\zeta,\tau) &= \frac{3}{4} - \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))} \\
&\quad + \mathbb{N}^{-1}\left[\frac{\mu^\alpha(s^\alpha+\alpha(\mu^\alpha-s^\alpha))}{s^{2\alpha}}\mathbb{N}\left\{-\sum_{l=0}^{\infty} A_l - \sum_{l=0}^{\infty} B_l + \frac{1}{Re}\left(\sum_{l=0}^{\infty} u_{l\xi\xi} + \sum_{l=0}^{\infty} u_{l\zeta\zeta}\right)\right\}\right], \\
\sum_{l=0}^{\infty} v_{l+1}(\xi,\zeta,\tau) &= \frac{3}{4} + \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))} \\
&\quad + \mathbb{N}^{-1}\left[\frac{\mu^\beta(s^\beta+\beta(\mu^\beta-s^\beta))}{s^{2\beta}}\mathbb{N}\left\{-\sum_{l=0}^{\infty} C_l - \sum_{l=0}^{\infty} D_l + \frac{1}{Re}\left(\sum_{l=0}^{\infty} v_{l\xi\xi} + \sum_{l=0}^{\infty} v_{l\zeta\zeta}\right)\right\}\right].
\end{aligned} \tag{55}$$

By comparing both sides of Equation (55), we can easily achieve the recursive relation as shown below

$$\begin{aligned}
u_0(\xi,\zeta,\tau) &= \frac{3}{4} - \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}, \\
v_0(\xi,\zeta,\tau) &= \frac{3}{4} + \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))}.
\end{aligned}$$

$$\begin{aligned}
u_1(\xi,\zeta,\tau) &= -\frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\left(1-\alpha+\frac{\alpha\tau^\alpha}{\Gamma(\alpha+1)}\right)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2}, \\
v_1(\xi,\zeta,\tau) &= \frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\left(1-\beta+\frac{\beta\tau^\beta}{\Gamma(\beta+1)}\right)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2}.
\end{aligned} \tag{56}$$

$$\begin{aligned}
u_2(\xi,\zeta,\tau) &= -\frac{\left(\tau-\alpha\tau+\frac{\alpha\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right)}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\left(-\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) \\
&\quad + \left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) + \exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right), \\
v_2(\xi,\zeta,\tau) &= \frac{\left(\tau-\beta\tau+\frac{\beta\tau^{\beta+1}}{\Gamma(\beta+2)}\right)}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\Big((-1+\exp((Re/8)(\xi-\zeta)) \\
&\quad + \exp((Re/4)(\xi-\zeta)))\Big)\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) + \exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right).
\end{aligned} \tag{57}$$

Continuing in the same procedure, the remaining components of u_l *and* $v_l (l \geq 3)$ *of Elzaki decomposition method solution can be smoothly obtained. Consequently, we determine the series solution as*

$$
\begin{aligned}
u(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) = u_0(\xi,\zeta,\tau) + u_1(\xi,\zeta,\tau) + u_2(\xi,\zeta,\tau) + \cdots, \\
u(\xi,\zeta,\tau) &= \frac{3}{4} - \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))} - \frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\left(1-\alpha+\frac{\alpha\tau^{\alpha}}{\Gamma(\alpha+1)}\right)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2} \\
&\quad - \frac{\left(\tau-\alpha\tau+\frac{\alpha\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right)}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\left(-\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) \\
&\quad + \left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) + \exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right) + \cdots.
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
v(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) = v_0(\xi,\zeta,\tau) + v_1(\xi,\zeta,\tau) + v_2(\xi,\zeta,\tau) + \cdots, \\
v(\xi,\zeta,\tau) &= \frac{3}{4} + \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta))} + \frac{Re\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\left(1-\beta+\frac{\beta\tau^{\beta}}{\Gamma(\beta+1)}\right)}{128(1+\exp((\frac{Re}{8})(\xi-\zeta)))^2} \\
&\quad \frac{\left(\tau-\beta\tau+\frac{\beta\tau^{\beta+1}}{\Gamma(\beta+2)}\right)}{4096(\exp((\frac{Re}{8})(\xi-\zeta)))^4}\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)Re^2\Big((-1+\exp((Re/8)(\xi-\zeta)) \\
&\quad + \exp((Re/4)(\xi-\zeta)))\Big)\left(-1+\exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right)\right) + \exp\left(\left(\frac{Re}{8}\right)(\xi-\zeta)\right) + \cdots.
\end{aligned} \tag{59}
$$

The exact solution for Equation (42) at $\alpha = \beta = 1$ is

$$
\begin{aligned}
u(\xi,\zeta,\tau) &= \frac{3}{4} - \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta-\tau))}, \\
v(\xi,\zeta,\tau) &= \frac{3}{4} + \frac{1}{4(1+\exp((Re/32)-4\xi+4\zeta-\tau))}.
\end{aligned} \tag{60}
$$

Example 2. *Consider the fractional-order system of Burgers equations*

$$
\begin{aligned}
D_{\tau}^{\alpha}u &= 2u\frac{\partial u}{\partial\xi} + 2v\frac{\partial u}{\partial\zeta} + \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\zeta^2}, \\
D_{\tau}^{\alpha}v &= 2u\frac{\partial v}{\partial\xi} + 2v\frac{\partial v}{\partial\zeta} + \frac{\partial^2 v}{\partial\xi^2} + \frac{\partial^2 v}{\partial\zeta^2}, \quad 0 < \alpha,\beta \le 1,
\end{aligned} \tag{61}
$$

with initial conditions

$$
\begin{aligned}
u(\xi,\zeta,0) &= 1 - \tanh(-\xi+2\zeta+1), \\
v(\xi,\zeta,0) &= 1 - 2\tanh(-\xi+2\zeta+1).
\end{aligned} \tag{62}
$$

Now by applying Natural transform on Equation (61), we get

$$
\begin{aligned}
\mathbb{N}[D_{\tau}^{\alpha}u(\xi,\zeta,\tau)] &= \mathbb{N}\left[2u\frac{\partial u}{\partial\xi}\right] + \mathbb{N}\left[2v\frac{\partial u}{\partial\xi}\right] + \mathbb{N}\left[\frac{\partial^2 u}{\partial\xi^2}\right] + \mathbb{N}\left[\frac{\partial^2 u}{\partial\zeta^2}\right], \\
\mathbb{N}[D_{\tau}^{\beta}v(\xi,\zeta,\tau)] &= \mathbb{N}\left[2u\frac{\partial v}{\partial\xi}\right] + \mathbb{N}\left[2v\frac{\partial v}{\partial\xi}\right] + \mathbb{N}\left[\frac{\partial^2 v}{\partial\xi^2}\right] + \mathbb{N}\left[\frac{\partial^2 v}{\partial\zeta^2}\right].
\end{aligned} \tag{63}
$$

Define the non-linear operator as

$$\begin{aligned}\frac{1}{s^{\alpha}}\mathbb{N}[u(\xi,\zeta,\tau)]-s^{2-\alpha}u(\xi,\zeta,0)&=\mathbb{N}\left[2u\frac{\partial u}{\partial\xi}+2v\frac{\partial u}{\partial\zeta}+\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\zeta^2}\right],\\ \frac{1}{s^{\beta}}\mathbb{N}[v(\xi,\zeta,\tau)]-s^{2-\beta}u(\xi,\zeta,0)&=\mathbb{N}\left[2u\frac{\partial v}{\partial\xi}+2v\frac{\partial v}{\partial\zeta}+\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\zeta^2}\right].\end{aligned}\tag{64}$$

On simplification, the above equation reduces to

$$\begin{aligned}\mathbb{N}[u(\xi,\zeta,\tau)]&=s^2\left[1-\tanh(-\xi+2\zeta+1)\right]+\frac{\mu(s-\alpha(s-\mu))}{s^2}\left[2u\frac{\partial u}{\partial\xi}+2v\frac{\partial u}{\partial\zeta}+\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\zeta^2}\right],\\ \mathbb{N}[v(\xi,\zeta,\tau)]&=s^2\left[1+2\tanh(-\xi+2\zeta+1)\right]+\frac{\mu(s-\beta(s-\mu))}{s^2}\left[2u\frac{\partial v}{\partial\xi}+2v\frac{\partial v}{\partial\zeta}+\frac{\partial^2 v}{\partial\xi^2}+\frac{\partial^2 v}{\partial\zeta^2}\right].\end{aligned}\tag{65}$$

Applying inverse NT on Equation (65), we have

$$\begin{aligned}u(\xi,\zeta,\tau)&=1-\tanh(-\xi+2\zeta+1)+\mathbb{N}^{-1}\left[\frac{\mu(s-\alpha(s-\mu))}{s^2}\mathbb{N}\left\{2uu_{\xi}+2vu_{y}+u_{\xi\xi}+u_{\zeta\zeta}\right\}\right],\\ v(\xi,\zeta,\tau)&=1+2\tanh(-\xi+2\zeta+1)+\mathbb{N}^{-1}\left[\frac{\mu(s-\beta(s-\mu))}{s^2}\mathbb{N}\left\{2uv_{\xi}+2vv_{y}+v_{\xi\xi}+v_{\zeta\zeta}\right\}\right].\end{aligned}\tag{66}$$

Now We Apply NDM_{CF}

Suppose that the infinite series results of the unknown functions $u(\xi,\zeta,\tau)$ and $v(\xi,\zeta,\tau)$ are respectively as follows

$$u(\xi,\zeta,\tau)=\sum_{l=0}^{\infty}u_l(\xi,\zeta,\tau)\ \ and\ \ v(\xi,\zeta,\tau)=\sum_{l=0}^{\infty}v_l(\xi,\zeta,\tau)\tag{67}$$

Note that $uu_{\xi}=\sum_{l=0}^{\infty}A_l$, $uv_{\xi}=\sum_{l=0}^{\infty}B_l$, $vu_{\zeta}=\sum_{l=0}^{\infty}C_l$ and $vv_{\zeta}=\sum_{l=0}^{\infty}D_l$ are the Adomian polynomials and they signify the non-linear terms. Applying these terms, Equation (66) can be written as

$$\begin{aligned}\sum_{l=0}^{\infty}u_l(\xi,\zeta,\tau)&=1-\tanh(-\xi+2\zeta+1)+\mathbb{N}^{-1}\left[\frac{\mu(s-\alpha(s-\mu))}{s^2}\mathbb{N}\left\{2\sum_{l=0}^{\infty}A_l+2\sum_{l=0}^{\infty}B_l+\sum_{l=0}^{\infty}u_{l\xi\xi}+\sum_{l=0}^{\infty}u_{l\zeta\zeta}\right\}\right],\\ \sum_{l=0}^{\infty}v_l(\xi,\zeta,\tau)&=1+2\tanh(-\xi+2\zeta+1)+\mathbb{N}^{-1}\left[\frac{\mu(s-\beta(s-\mu))}{s^2}\mathbb{N}\left\{2\sum_{l=0}^{\infty}C_l+2\sum_{l=0}^{\infty}D_l+\sum_{l=0}^{\infty}v_{l\xi\xi}+\sum_{l=0}^{\infty}v_{l\zeta\zeta}\right\}\right].\end{aligned}\tag{68}$$

By comparing both sides of Equation (68), they can be written as follows

$$\begin{aligned}u_0(\xi,\zeta,\tau)&=1-\tanh(-\xi+2\zeta+1),\\ v_0(\xi,\zeta,\tau)&=1+2\tanh(-\xi+2\zeta+1).\\ u_1(\xi,\zeta,\tau)&=-2sech(-\xi+2\zeta+1)(\alpha(\tau-1)+1),\\ v_1(\xi,\zeta,\tau)&=4sech(-\xi+2\zeta+1)(\beta(\tau-1)+1).\end{aligned}$$

$$\begin{aligned}u_2(\xi,\zeta,\tau)&=8sech^2(-\xi+2\zeta+1)(-2sech^2(-\xi+2\zeta+1)+(2sech^2(-\xi+2\zeta+1)\\ &\quad+\tanh(-\xi+2\zeta+1)))\tau(1-\alpha)+\frac{\alpha\tau^2}{2},\\ v_2(\xi,\zeta,\tau)&=-8sech^2(-\xi+2\zeta+1)(-sech^2(-\xi+2\zeta+1)+(2sech^2(-\xi+2\zeta+1)\\ &\quad+2\tanh(-\xi+2\zeta+1)))\tau(1-\beta)+\frac{\beta\tau^2}{2}.\end{aligned}$$

the remaining components of u_l and $v_l (l \geq 3)$ of Natural decomposition method result can be smoothly achieved. Consequently, we calculated the series form result as

$$\begin{aligned}
u(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) = u_0(\xi,\zeta,\tau) + u_1(\xi,\zeta,\tau) + u_2(\xi,\zeta,\tau) + \cdots, \\
u(\xi,\zeta,\tau) &= 1 - \tanh(-\xi+2\zeta+1) - 2sech(-\xi+2\zeta+1)(\alpha(\tau-1)+1) \\
&\quad + 8sech^2(-\xi+2\zeta+1)(-2sech^2(-\xi+2\zeta+1) + (2sech^2(-\xi+2\zeta+1) \\
&\quad + \tanh(-\xi+2\zeta+1)))\tau(1-\alpha) + \frac{\alpha\tau^2}{2} + \cdots, \\
v(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) = v_0(\xi,\zeta,\tau) + v_1(\xi,\zeta,\tau) + v_2(\xi,\zeta,\tau) + \cdots, \\
v(\xi,\zeta,\tau) &= 1 + 2\tanh(-\xi+2\zeta+1) + 4sech(-\xi+2\zeta+1)(\beta(\tau-1)+1) \\
&\quad - 8sech^2(-\xi+2\zeta+1)(-sech^2(-\xi+2\zeta+1) + (2sech^2(-\xi+2\zeta+1) \\
&\quad + 2\tanh(-\xi+2\zeta+1)))\tau(1-\beta) + \frac{\beta\tau^2}{2} + \cdots.
\end{aligned} \tag{69}$$

Now We Apply NDM_{ABC}

Suppose that the infinite series results of the unknown function $u(\xi,\zeta,\tau)$ and $v(\xi,\zeta,\tau)$ are respectively as follows

$$u(\xi,\zeta,\tau) = \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) \;\; and \;\; v(\xi,\zeta,\tau) = \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) \tag{70}$$

Note that $uu_\xi = \sum_{l=0}^{\infty} A_l$, $uv_\xi = \sum_{l=0}^{\infty} B_l$, $vu_\zeta = \sum_{l=0}^{\infty} C_l$ and $vv_\zeta = \sum_{l=0}^{\infty} D_l$ are the Adomian polynomials and they signify the non-linear terms. Applying the these terms, Equation (66) can be written as

$$\begin{aligned}
\sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) &= \left[1 - \tanh(-\xi+2\zeta+1)\right] \\
&\quad + \mathbb{N}^{-1}\left[\frac{\mu^\alpha(s^\alpha + \alpha(\mu^\alpha - s^\alpha))}{s^{2\alpha}}\mathbb{N}\left\{2\sum_{l=0}^{\infty} A_l + 2\sum_{l=0}^{\infty} B_l + \sum_{l=0}^{\infty} u_{l\xi\xi} + \sum_{l=0}^{\infty} u_{l\zeta\zeta}\right\}\right], \\
\sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) &= \left[1 + 2\tanh(-\xi+2\zeta+1)\right] \\
&\quad + \mathbb{N}^{-1}\left[\frac{\mu^\beta(s^\beta + \beta(\mu^\beta - s^\beta))}{s^{2\beta}}\mathbb{N}\left\{2\sum_{l=0}^{\infty} C_l + 2\sum_{l=0}^{\infty} D_l + \sum_{l=0}^{\infty} v_{l\xi\xi} + \sum_{l=0}^{\infty} v_{l\zeta\zeta}\right\}\right].
\end{aligned} \tag{71}$$

By comparing both sides of Equation (71), we can write as follows

$$\begin{aligned}
u_0(\xi,\zeta,\tau) &= 1 - \tanh(-\xi+2\zeta+1), \\
v_0(\xi,\zeta,\tau) &= 1 + 2\tanh(-\xi+2\zeta+1). \\
u_1(\xi,\zeta,\tau) &= -2sech(-\xi+2\zeta+1)\left(1 - \alpha + \frac{\alpha\tau^\alpha}{\Gamma(\alpha+1)}\right), \\
v_1(\xi,\zeta,\tau) &= 4sech(-\xi+2\zeta+1)\left(1 - \beta + \frac{\alpha\tau^\beta}{\Gamma(\beta+1)}\right).
\end{aligned}$$

$$
\begin{aligned}
u_2(\xi,\zeta,\tau) &= 8sech^2(-\xi+2\zeta+1)(-2sech^2(-\xi+2\zeta+1)\\
&\quad+(2sech^2(-\xi+2\zeta+1)+\tanh(-\xi+2\zeta+1)))\left(\tau-\alpha\tau+\frac{\alpha\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right),\\
v_2(\xi,\zeta,\tau) &= -8sech^2(-\xi+2\zeta+1)(-sech^2(-\xi+2\zeta+1)\\
&\quad+(2sech^2(-\xi+2\zeta+1)+2\tanh(-\xi+2\zeta+1)))\left(\tau-\beta\tau+\frac{\beta\tau^{\beta+1}}{\Gamma(\beta+2)}\right).
\end{aligned}
$$

the remaining components of u_l and $v_l (l \geq 3)$ of Natural decomposition method (NDM) result can be smoothly achieved. Consequently, we calculated the series form result as

$$
\begin{aligned}
u(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} u_l(\xi,\zeta,\tau) = u_0(\xi,\zeta,\tau)+u_1(\xi,\zeta,\tau)+u_2(\xi,\zeta,\tau)+\cdots,\\
u(\xi,\zeta,\tau) &= 1-\tanh(-\xi+2\zeta+1)-2sech(-\xi+2\zeta+1)\left(1-\alpha+\frac{\alpha\tau^{\alpha}}{\Gamma(\alpha+1)}\right)\\
&\quad+8sech^2(-\xi+2\zeta+1)(-2sech^2(-\xi+2\zeta+1)+(2sech^2(-\xi+2\zeta+1)\\
&\quad+\tanh(-\xi+2\zeta+1)))\left(\tau-\alpha\tau+\frac{\alpha\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right)+\cdots,\\
v(\xi,\zeta,\tau) &= \sum_{l=0}^{\infty} v_l(\xi,\zeta,\tau) = v_0(\xi,\zeta,\tau)+v_1(\xi,\zeta,\tau)+v_2(\xi,\zeta,\tau)+\cdots,\\
v(\xi,\zeta,\tau) &= 1+2\tanh(-\xi+2\zeta+1)+4sech(-\xi+2\zeta+1)\left(1-\beta+\frac{\beta\tau^{\beta}}{\Gamma(\beta+1)}\right)\\
&\quad-8sech^2(-\xi+2\zeta+1)(-sech^2(-\xi+2\zeta+1)+(2sech^2(-\xi+2\zeta+1)\\
&\quad+2\tanh(-\xi+2\zeta+1)))\left(\tau-\beta\tau+\frac{\beta\tau^{\beta+1}}{\Gamma(\beta+2)}\right)+\cdots.
\end{aligned}
\tag{72}
$$

The exact result for Equation (61) is given by

$$
\begin{aligned}
u(\xi,\zeta,\tau) &= 1-\tanh(-\xi+2\zeta+2\tau+1),\\
v(\xi,\zeta,\tau) &= 1+2\tanh(-\xi+2\zeta+2\tau+1).
\end{aligned}
\tag{73}
$$

Numerical Results and Discussion

In this study, we have successfully applied two novel methods to investigate the numerical solution of fractional coupled Burgers equations. Find numerical data for the system of Burgers equations at any order for different values of space and time variables with Maple 13. In Tables 1 and 2, we perform numerical simulations for various Brownian motions with different ξ and τ values for the system in problem 1. The numerical comparison of variational iteration method, Natural decomposition method in terms of absolute error for Equation (42) is presented in Tables 3 and 4. Tables 5 and 6 show the results of a numerical study for the coupled system considered in problem 2. Analogously, in Table 7, we compare the solution to Equation (61) obtained by variational iteration method, Natural decomposition method. Based on the data in the tables above, we can conclude that the results achieved by the Natural decomposition method are more reliable. The behavior of the Natural decomposition method result from $u(\xi,\zeta,\tau)$ for problem 1 is represented in Figure 1, and the nature of the actual result and different fractional-order of α are provided in Figure 2, respectively. In Figure 3 show that the different fractional-order at $\alpha = 1, 0.8, 0.6$ and 0.4. In the same way, the achieved result $v(\xi,\zeta,\tau)$ for Equation (61) can be seen in Figure 4. Figures 5 and 6 are the response of acquired results for problem 1 with different standard motion and Brownian motions $\beta = 0.8$ and 0.6. The behavior of the Natural decomposition method result from $u(\xi,\zeta,\tau)$ for problem 2 is represented in Figure 7, and the nature of the actual result and different fractional-order of α are provided in Figure 8, respectively. In Figure 9 show that the different fractional-order at $\alpha = 1, 0.8, 0.6$ and 0.4. In the same way, the achieved result $v(\xi,\zeta,\tau)$ for Equation (61) can be seen in Figure 10.

Figures 11 and 12 are the response of acquired results for problem 1 with different standard motion and Brownian motion $\beta = 0.8$ and 0.6.

Table 1. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ solutions of $v(\xi, \zeta, \tau)$ for Example 1 at different fractional-order of α and β with $l = 4$ for different ξ and τ when $\zeta = 1$.

ξ	t	$\alpha = \beta = 0.6$	$\alpha = \beta = 0.75$	$\alpha = \beta = 0.9$	$\alpha = \beta = 1(Approx)$	$\alpha = \beta = 1(Exact)$
	0.2	0.760060	0.760035	0.760025	0.760021	0.760021
	0.4	0.760148	0.760081	0.760050	0.760039	0.760040
0.2	0.6	0.760284	0.760158	0.760095	0.760071	0.760074
	0.8	0.760473	0.760278	0.760167	0.760123	0.760138
	1	0.760720	0.760449	0.760278	0.760205	0.760258
	0.2	0.760728	0.760430	0.760304	0.760258	0.760258
	0.4	0.761784	0.760978	0.760610	0.760477	0.760482
0.4	0.6	0.763420	0.761915	0.76149	0.760861	0.760898
	0.8	0.765698	0.763357	0.762026	0.761493	0.761673
	1	0.768663	0.765415	0.763362	0.762482	0.763108
	0.2	0.768374	0.765097	0.763645	0.763104	0.763108
	0.4	0.769658	0.761211	0.767177	0.765676	0.765744
0.6	0.6	0.796654	0.781353	0.763235	0.760073	0.760522
	0.8	0.819877	0.796626	0.782885	0.767192	0.768965
	1	0.849767	0.818099	0.797332	0.788131	0.793241
	0.2	0.821594	0.807747	0.807683	0.793103	0.793241
	0.4	0.848508	0.839515	0.833718	0.804635	0.805676
0.8	0.6	0.857888	0.871744	0.867444	0.844863	0.847161
	0.8	0.844810	0.899765	0.897914	0.884778	0.886000
	1	0.805199	0.918764	0.919533	0.925065	0.912839
	0.2	0.950038	0.944350	0.941824	0.923764	0.922839
	0.4	0.915159	0.958238	0.973309	0.958357	0.954325
1	0.6	0.791894	0.916308	0.977100	0.978048	0.976759
	0.8	0.569588	0.795564	0.931177	0.964956	0.991035
	1	0.238108	0.571732	0.807906	0.894048	0.909478

Table 2. Comparative study between varational iteration method (VIM) [34], NDM_{CF} and NDM_{ABC} for the numerical result $u(\xi, \zeta, \tau)$ of Example 1 at $Re = 100$, $\zeta = 1$, $\alpha = 1$, and $\beta = 1$ for $l = 4$.

τ	ξ	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert u_{Exact} - u_{HPTM} \rvert$	$\lvert u_{Exact} - u_{EDM} \rvert$
	0.1	3.2300×10^{-8}	5.4246×10^{-10}	5.4246×10^{-10}
	0.2	6.9000×10^{-8}	3.4568×10^{-10}	3.4568×10^{-10}
0.1	0.3	1.6200×10^{-7}	2.2468×10^{-9}	2.2468×10^{-9}
	0.4	5.9700×10^{-7}	6.3267×10^{-9}	6.3267×10^{-9}
	0.5	1.8660×10^{-6}	2.1326×10^{-8}	2.1326×10^{-8}
	0.1	2.4400×10^{-7}	4.7421×10^{-9}	4.7421×10^{-9}
	0.2	8.3100×10^{-7}	3.1235×10^{-8}	3.1235×10^{-8}
0.2	0.3	2.8500×10^{-6}	4.5682×10^{-8}	4.5682×10^{-8}
	0.4	9.7940×10^{-6}	3.5223×10^{-7}	3.5223×10^{-7}
	0.5	3.2012×10^{-7}	2.9315×10^{-7}	2.9315×10^{-7}
	0.1	2.2981×10^{-6}	3.2245×10^{-7}	3.2245×10^{-7}
	0.2	5.4602×10^{-6}	4.2659×10^{-7}	4.2659×10^{-7}
0.3	0.3	2.5432×10^{-5}	1.5348×10^{-6}	1.5348×10^{-6}
	0.4	6.4229×10^{-4}	8.2374×10^{-6}	8.2374×10^{-6}
	0.5	2.8364×10^{-4}	4.1975×10^{-5}	4.1975×10^{-5}
	0.1	5.5428×10^{-5}	2.1351×10^{-6}	2.1351×10^{-6}
	0.2	2.4133×10^{-5}	2.6276×10^{-6}	2.6276×10^{-6}
0.4	0.3	6.3743×10^{-5}	2.2334×10^{-5}	2.2334×10^{-5}
	0.4	2.9070×10^{-4}	1.2035×10^{-5}	1.2035×10^{-5}
	0.5	6.9763×10^{-4}	2.2145×10^{-4}	2.2145×10^{-4}
	0.1	2.2529×10^{-5}	2.3223×10^{-6}	2.3223×10^{-6}
	0.2	4.9868×10^{-5}	3.2721×10^{-5}	3.2721×10^{-5}
0.5	0.3	4.1932×10^{-4}	3.0767×10^{-5}	3.0767×10^{-5}
	0.4	5.5568×10^{-4}	2.3742×10^{-4}	2.3742×10^{-4}
	0.5	2.4350×10^{-3}	1.3223×10^{-3}	1.3223×10^{-3}

Table 3. Comparative study between VIM [34], $NTDM_{CF}$ and $NTDM_{ABC}$ for the approximate solution $v(\xi, \zeta, \tau)$ of Example 1 at $Re = 100$, $\zeta = 1$, $\alpha = 1$, and $\beta = 1$ for $l = 4$.

τ	ξ	$\lvert v_{Exact} - v_{VIM} \rvert$	$\lvert v_{Exact} - v_{NTDM_{CF}} \rvert$	$\lvert v_{Exact} - v_{NTDM_{ABC}} \rvert$
0.1	0.1	9.0202×10^{-9}	1.3770×10^{-11}	8.6253×10^{-11}
	0.2	5.4060×10^{-9}	4.8036×10^{-10}	3.1054×10^{-10}
	0.3	2.7960×10^{-8}	1.6734×10^{-9}	1.1992×10^{-9}
	0.4	6.5902×10^{-8}	5.8013×10^{-9}	5.5827×10^{-9}
	0.5	3.9762×10^{-7}	1.9773×10^{-8}	3.6150×10^{-8}
0.2	0.1	3.3610×10^{-8}	2.3548×10^{-9}	2.8894×10^{-9}
	0.2	9.1810×10^{-8}	8.2143×10^{-8}	1.0171×10^{-8}
	0.3	1.9482×10^{-7}	2.8611×10^{-8}	3.6538×10^{-8}
	0.4	9.8750×10^{-7}	9.2053×10^{-7}	1.4010×10^{-7}
	0.5	4.2127×10^{-6}	3.3727×10^{-7}	6.3855×10^{-7}
0.3	0.1	2.3872×10^{-7}	1.2779×10^{-8}	2.3189×10^{-8}
	0.2	5.3523×10^{-7}	4.4573×10^{-8}	8.1227×10^{-8}
	0.3	2.5565×10^{-6}	1.5522×10^{-7}	2.8704×10^{-7}
	0.4	6.3787×10^{-6}	5.3749×10^{-6}	1.0445×10^{-6}
	0.5	2.8365×10^{-5}	1.8244×10^{-6}	4.1512×10^{-6}
0.4	0.1	5.2272×10^{-6}	4.3423×10^{-7}	1.0363×10^{-7}
	0.2	2.6215×10^{-6}	1.5145×10^{-7}	3.6229×10^{-7}
	0.3	6.3642×10^{-6}	5.2723×10^{-6}	1.2717×10^{-6}
	0.4	2.9203×10^{-5}	1.8244×10^{-6}	4.5250×10^{-6}
	0.5	6.9958×10^{-5}	6.1751×10^{-5}	1.6814×10^{-5}
0.5	0.1	2.2532×10^{-6}	1.1434×10^{-7}	3.3630×10^{-7}
	0.2	4.8935×10^{-6}	3.9879×10^{-6}	1.1745×10^{-6}
	0.3	2.4921×10^{-5}	1.3880×10^{-6}	4.1074×10^{-6}
	0.4	5.8486×10^{-5}	4.7977×10^{-5}	1.4434×10^{-5}
	0.5	1.6542×10^{-4}	1.6182×10^{-5}	5.1527×10^{-5}

Table 4. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ solutions of $u(\xi, \zeta, \tau)$ for Example 2 at different fractional-order of α and β with $l = 4$ for different ξ and τ when $\zeta = 1$.

ξ	τ	$\alpha = \beta = 0.6$	$\alpha = \beta = 0.75$	$\alpha = \beta = 0.9$	$\alpha = \beta = 1(Approx)$	$\alpha = \beta = 1(Exact)$
0.2	0.2	0.007531	0.003305	0.003056	0.003355	0.003318
	0.4	0.037374	0.010555	0.003583	0.002299	0.001492
	0.6	0.107634	0.037076	0.011908	0.005587	0.000671
	0.8	0.228832	0.095214	0.035864	0.017981	0.000301
	1	0.409637	0.198269	0.086536	0.047619	0.000135
0.4	0.2	0.010890	0.004952	0.004581	0.005015	0.004945
	0.4	0.054113	0.015998	0.005662	0.003639	0.002225
	0.6	0.156343	0.056189	0.019107	0.009354	0.001000
	0.8	0.333035	0.144112	0.057207	0.029946	0.000450
	1	0.596925	0.299764	0.137217	0.078479	0.000202
0.6	0.2	0.015625	0.007446	0.006882	0.007503	0.007368
	0.4	0.077841	0.024529	0.009141	0.005878	0.003318
	0.6	0.225856	0.086120	0.031439	0.016138	0.001492
	0.8	0.482298	0.220396	0.093435	0.051355	0.000671
	1	0.865819	0.457619	0.222442	0.132987	0.000301
0.8	0.2	0.022393	0.011281	0.010372	0.011236	0.010973
	0.4	0.112097	0.038380	0.015138	0.009711	0.004945
	0.6	0.326798	0.134623	0.053250	0.028657	0.002225
	0.8	0.699653	0.343269	0.156889	0.090621	0.001000
	1	1.257990	0.710643	0.370200	0.231727	0.000450
1	0.2	0.032749	0.017333	0.015694	0.016848	0.016325
	0.4	0.165420	0.062144	0.025783	0.016410	0.007368
	0.6	0.483746	0.217538	0.092962	0.052080	0.003318
	0.8	1.036890	0.551429	0.271315	0.163669	0.001492
	1	1.865360	1.136210	0.633917	0.413380	0.000670

Table 5. The exact, $NTDM_{CF}$ and $NTDM_{ABC}$ solutions of $v(\xi, \zeta, \tau)$ for Example 2 at different fractional-order of α and β with $l = 4$ for different ξ and τ when $\zeta = 1$.

ξ	τ	$\alpha = \beta = 0.6$	$\alpha = \beta = 0.75$	$\alpha = \beta = 0.9$	$\alpha = \beta = 1(Approx)$	$\alpha = \beta = 1(Exact)$
	0.2	3.010560	2.998480	2.994809	2.993571	2.993365
	0.4	3.060490	3.019625	3.003995	2.999902	2.997016
0.2	0.6	3.142602	3.063333	3.024227	3.011611	2.998659
	0.8	3.256140	3.135462	3.063606	3.036047	2.999397
	1	3.400174	3.239966	3.129121	3.080563	2.999729
	0.2	3.015831	2.997739	2.992253	2.990409	2.990109
	0.4	3.090282	3.029146	3.005834	2.999757	2.995550
0.4	0.6	3.212598	3.093978	3.035646	3.016907	2.998000
	0.8	3.381655	3.200917	3.093647	3.052668	2.999101
	1	3.596061	3.355824	3.190150	3.117848	2.999596
	0.2	3.023772	2.996638	2.988438	2.985692	2.985263
	0.4	3.134729	3.043180	3.008425	2.999423	2.993365
0.6	0.6	3.316769	3.139066	3.052084	3.024316	2.997016
	0.8	3.568204	3.297118	3.136975	3.076147	2.998659
	1	3.886965	3.525992	3.278240	3.170692	2.999397
	0.2	3.035766	2.995000	2.982740	2.978658	2.978054
	0.4	3.200960	3.063729	3.011961	2.998675	2.990109
0.8	0.6	3.471437	3.204907	3.075310	3.034312	2.995550
	0.8	3.844671	3.437381	3.198374	3.108342	2.997999
	1	4.317582	3.773870	3.403204	3.243539	2.999101
	0.2	3.053892	2.992555	2.974226	2.968170	2.967350
	0.4	3.299308	3.093483	3.016524	2.997036	2.985263
1	0.6	3.699990	3.299910	3.107172	3.046991	2.993365
	0.8	4.252157	3.639306	3.283009	3.150375	2.997015
	1	4.951244	4.130244	3.575765	3.339528	2.998659

Table 6. Comparative study between VIM, $NTDM_{CF}$ and $NTDM_{ABC}$ for the approximate solution $u(\xi, \zeta, \tau)$ of Example 2 at $Re = 100$, $\zeta = 0.3$, $\alpha = 1$, and $\beta = 1$ for $n = 4$.

τ	ξ	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert u_{Exact} - u_{NTDM_{CF}} \rvert$	$\lvert u_{Exact} - u_{NTDM_{ABC}} \rvert$
	0.1	0.00388488	0.000040388	0.0000403885
	0.2	0.02058640	0.000643080	0.0006430800
0.1	0.3	0.08300040	0.003204730	0.0032047300
	0.4	0.19216900	0.009896630	0.0098966300
	0.5	0.35630400	0.023490500	0.0234905000
	0.1	0.00340343	0.000037786	0.0000377862
	0.2	0.02549380	0.000619573	0.0006195730
0.2	0.3	0.09466810	0.003156030	0.0031560300
	0.4	0.21124200	0.009911520	0.0099115200
	0.5	0.38165100	0.023838100	0.0238381000
	0.1	0.00290734	0.000030901	0.0000309013
	0.2	0.02730130	0.000536002	0.0005360020
0.3	0.3	0.09244390	0.002840270	0.0028402700
	0.4	0.19342500	0.009181990	0.0091819900
	0.5	0.33037500	0.022574100	0.0225741000
	0.1	0.00271456	0.000018180	0.0000181808
	0.2	0.02207170	0.000368845	0.0003688450
0.4	0.3	0.06204090	0.002146160	0.0021461600
	0.4	0.10393800	0.007381420	0.0073814200
	0.5	0.13365400	0.018959600	0.0189596000
	0.1	0.00334970	1.950740×10^{-6}	1.9507400×10^{-6}
	0.2	0.00408341	0.000093131	0.0000931316
0.5	0.3	0.01675610	0.000951695	0.0009516950
	0.4	0.10582700	0.004143110	0.0041431100
	0.5	0.30400000	0.012150100	0.0121501000

Table 7. Comparative study between VIM [34], $NTDM_{CF}$ and $NTDM_{ABC}$ for the approximate solution $v(\xi, \zeta, \tau)$ of Example 2 at $Re = 100$, $\zeta = 0.3$, $\alpha = 1$, and $\beta = 1$ for $l = 4$.

τ	ξ	$\lvert v_{Exact} - v_{VIM} \rvert$	$\lvert v_{Exact} - v_{NTDM_{CF}} \rvert$	$\lvert v_{Exact} - v_{NTDM_{ABC}} \rvert$
0.1	0.1	0.000409157	0.000080777	0.000080777
	0.2	0.00179639	0.00128616	0.00128616
	0.3	0.00575008	0.00640947	0.00640947
	0.4	0.0154079	0.0197933	0.0197933
	0.5	0.0351083	0.046981	0.046981
0.2	0.1	0.000358137	0.0000755724	0.0000755724
	0.2	0.000618968	0.00123915	0.00123915
	0.3	0.00097818	0.00631207	0.00631207
	0.4	0.00333882	0.019823	0.019823
	0.5	0.0109793	0.0476761	0.0476761
0.3	0.1	0.000169508	0.0000618027	0.0000618027
	0.2	0.00190188	0.001072	0.001072
	0.3	0.0086713	0.00568053	0.00568053
	0.4	0.0207692	0.018364	0.018364
	0.5	0.0372769	0.0451482	0.0451482
0.4	0.1	0.000255194	0.0000363616	0.0000363616
	0.2	0.00656152	0.00073769	0.00073769
	0.3	0.0259301	0.00429232	0.00429232
	0.4	0.0634979	0.0147628	0.0147628
	0.5	0.122694	0.0379192	0.0379192
0.5	0.1	0.00104781	3.90148×10^{-6}	3.90148×10^{-6}
	0.2	0.0143686	0.000186263	0.000186263
	0.3	0.0541981	0.00190339	0.00190339
	0.4	0.132966	0.00828621	0.00828621
	0.5	0.261277	0.0243001	0.0243001

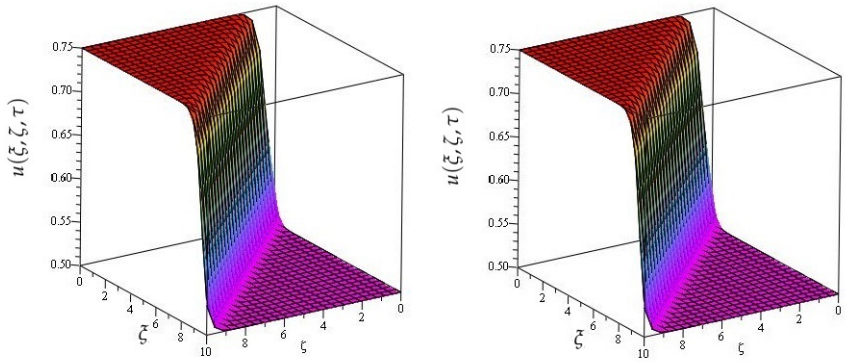

Figure 1. The exact and numerical solutions of $u(\xi, \zeta, \tau)$ of Example 1 at $\alpha = 1$.

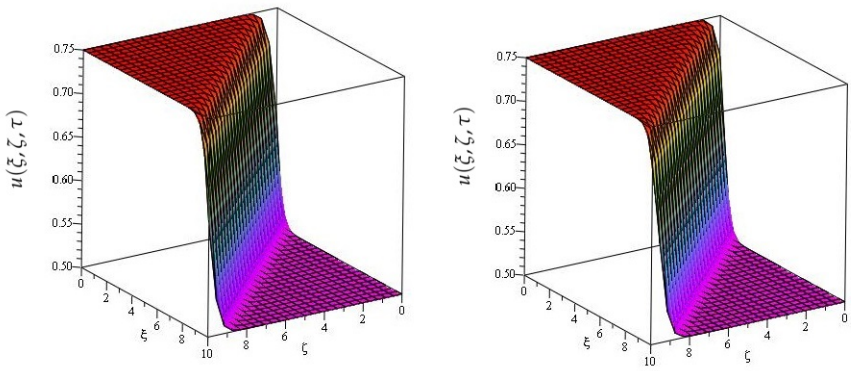

Figure 2. The numerical solution graph of $u(\xi, \zeta, \tau)$ of Example 1 at $\alpha = 0.8$ and 0.6.

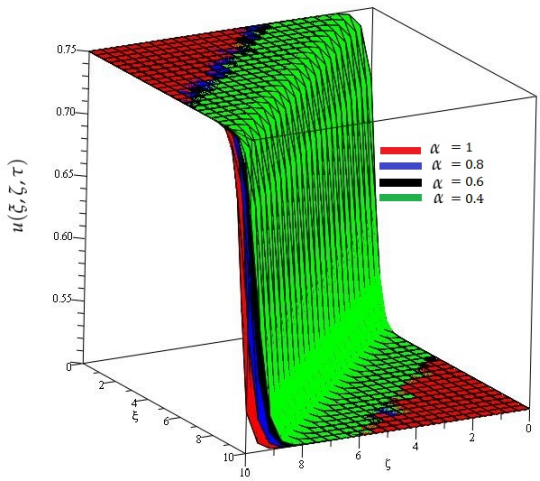

Figure 3. The numerical solution graph of $u(\xi, \zeta, \tau)$ of Example 1 at different value of α.

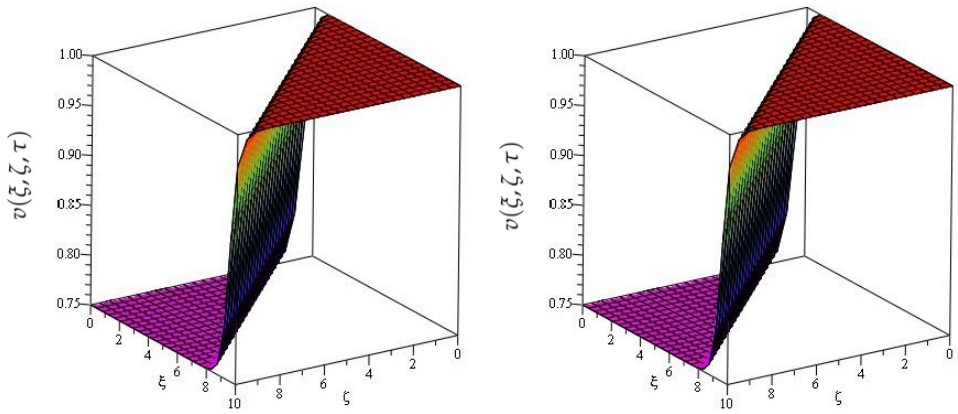

Figure 4. The exact and numerical solutions of $v(\xi, \zeta, \tau)$ of Example 1 at $\beta = 1$.

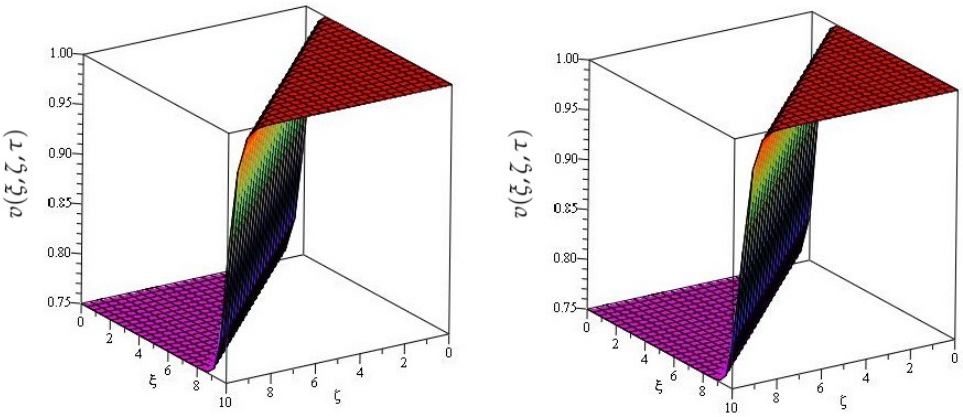

Figure 5. The numerical solution graph of $v(\xi, \zeta, \tau)$ of Example 1 at $\beta = 0.8$ and 0.6.

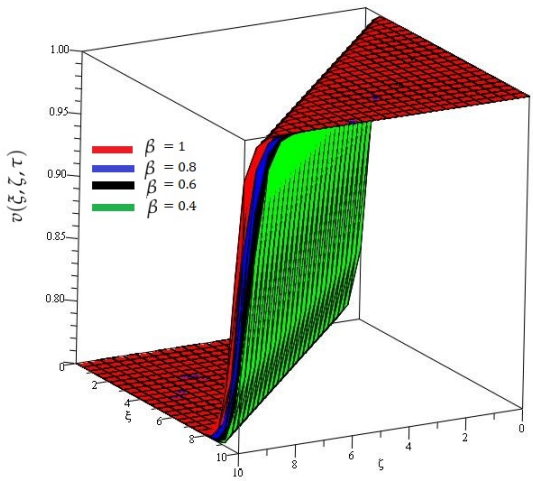

Figure 6. The numerical solution graph of $v(\xi, \zeta, \tau)$ of Example 1 at different value of β.

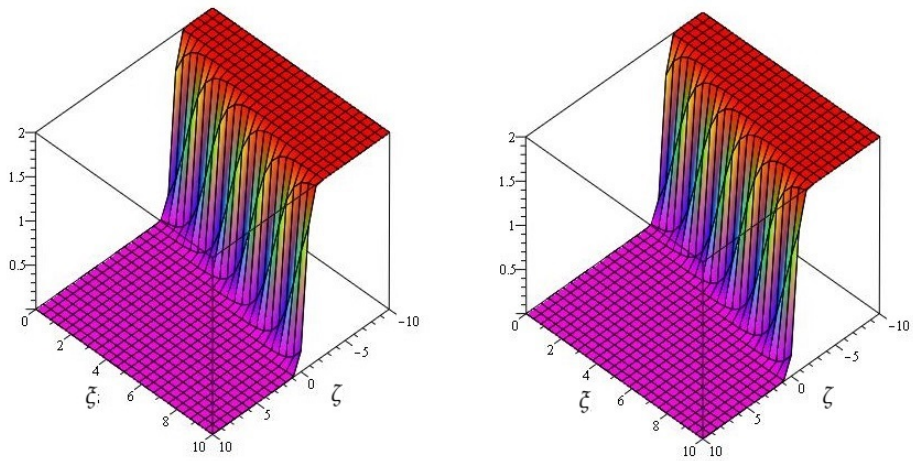

Figure 7. The exact and numerical solutions of $u(\xi, \zeta, \tau)$ of Example 2 at $\alpha = 1$.

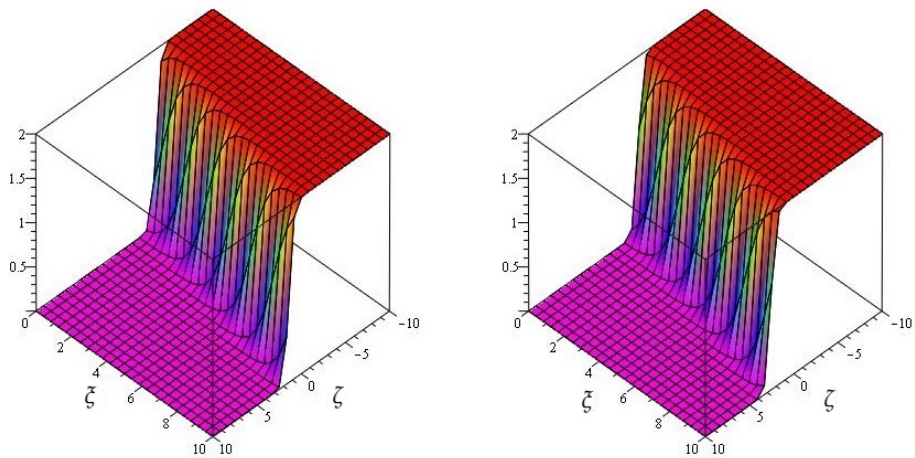

Figure 8. The numerical solution graph of $u(\xi, \zeta, \tau)$ of Example 2 at $\alpha = 0.8$ and 0.6.

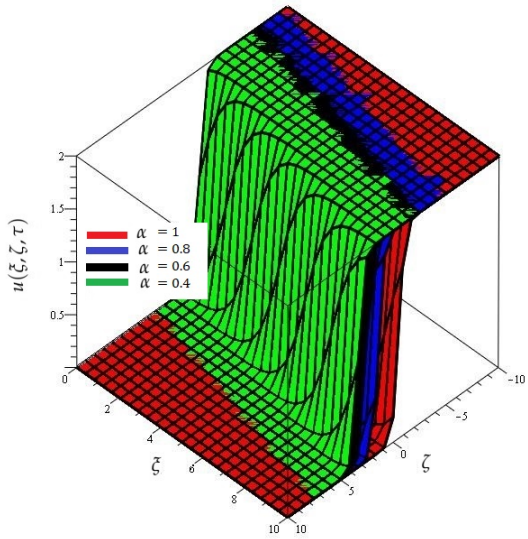

Figure 9. The numerical solution graph of $u(\xi, \zeta, \tau)$ of Example 2 at different value of α.

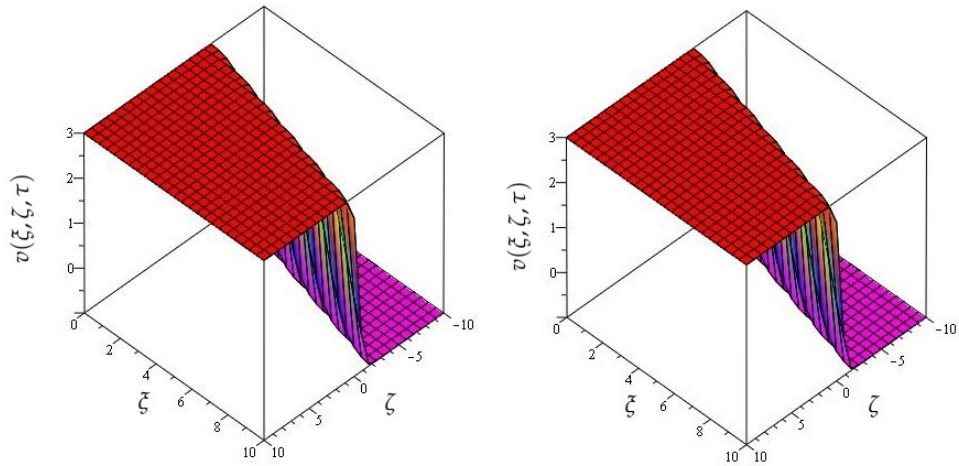

Figure 10. The exact and numerical solutions of $v(\xi, \zeta, \tau)$ of Example 2 at $\beta = 1$.

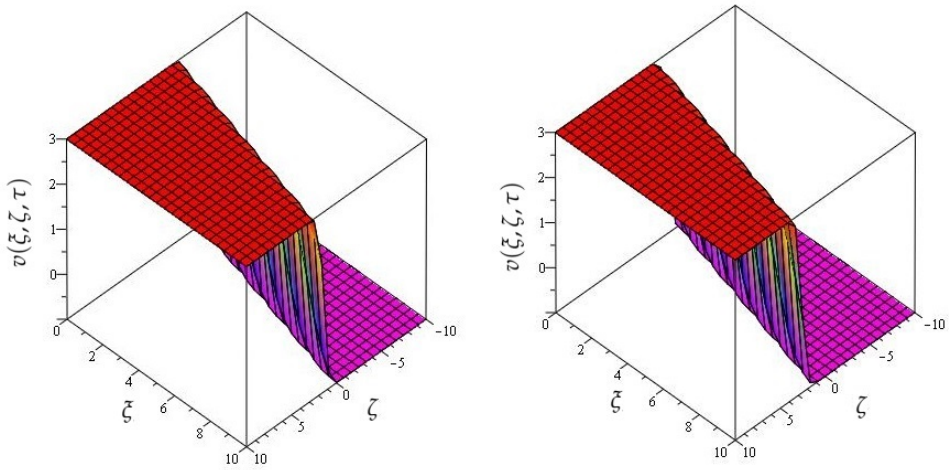

Figure 11. The numerical solution graph of $v(\xi, \zeta, \tau)$ of Example 2 at $\beta = 0.8$ and 0.6.

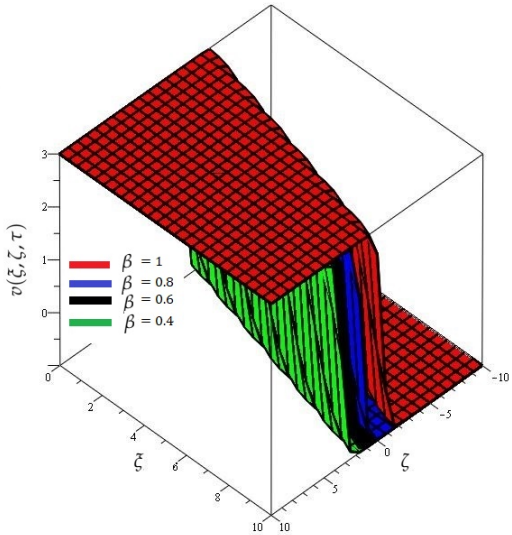

Figure 12. The numerical solution graph of $v(\xi, \zeta, \tau)$ of Example 2 at different value of β.

6. Conclusions

In the present article, Natural decomposition method is applied for the solution of coupled systems of fractional Burger equations. The graphical and tabular representations of the derived results have been done. The solutions are obtained for fractional systems which are closely related to their actual solutions. The suggested technique provides a series result in a form of recurrence relation with high accuracy and minimal calculations. Numerous computational results are compared with well-known numerical techniques and the exact results when $\alpha = \beta = 1$. These representation of the obtained results have clearly confirmed the higher accuracy of the suggested methods. The convergence of fractional solutions to integer order solution have been shown. The less calculations and higher accuracy are the valuable themes of the present methods. The researchers are then modified it to solve other systems with fractional partial differential equations.

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