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Equivalence of Certain Iteration Processes Obtained by Two New Classes of Operators

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Abstract: The aim of this paper is two fold: the first is to define two new classes of mappings and show the existence and iterative approximation of their fixed points; the second is to show that the Ishikawa, Mann, and Krasnoselskij iteration methods defined for such classes of mappings are equivalent. An application of the main results to solve split feasibility and variational inequality problems are also given.

Keywords: fixed point; Ishikawa iteration; Mann iteration; Krasnoselskij iteration

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1. Introduction and Preliminaries

In certain cases, such as solving a system of nonlinear functional equations, optimization problems, variational inequality problems, split feasibility problems, and equilibrium point problems, the transformation of the given problem into a fixed-point problem of a certain operator, requires an appropriate space that acts as the domain of a corresponding operator and contains the solution set of the problem. In most of the cases, when finding an analytic solution of the corresponding fixed-point problem is not possible, an approximation of the solution of a particular fixed-point problem is obtained via fixed-point iteration methods. For details on the subject, we refer the reader to [1] and the references therein. For different classes of mappings, fixed-point iteration methods may behave differently. A fixed-point iteration method may be convergent for one class of mappings; it might not be suitable for others.

To decide whether an iteration method is useful for the approximation of the solution of the given problem, it is of paramount importance to answer the following questions:

(i) Does it converge to the fixed point of an operator?
(ii) Is it equivalent to some other iteration methods?

Before we address the above questions, let us recall the following concepts:

Let K be a nonempty convex subset of a normed space $(X, \|\cdot\|)$ and $T : K \to K$. We denote the set $\{x \in K : Tx = x\}$ of fixed points of T by $Fix\,(T)$.

Define $T^0 = I$ (the identity map on K) and $T^n = T^{n-1} \circ T$, called the n^{th} iterate of T for $n \geq 1$. Let $x_0 \in K$ be an initial guess that approximates the solution of a functional equation $Tx = x$ in X.

A sequence $\{x_n\}_{n=0}^{\infty}$ in X is called a Picard iteration associated with T if:

$$x_n = T^n x_0, \quad n = 1, 2, \dots \tag{1}$$

Let $\lambda \in [0,1]$. A sequence $\{x_n\}_{n=0}^{\infty}$ given by:

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \ \ n = 0,1,2,\ldots \tag{2}$$

is called the Krasnoselskij iteration sequence.

Note that the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ sequence given by (2) is exactly the Picard iteration corresponding to an averaged operator:

$$T_\lambda = (1-\lambda)I + \lambda T. \tag{3}$$

Moreover, for $\lambda = 1$, the Krasnoselskij iteration method reduces to the Picard iteration method. Furthermore, $Fix(T) = Fix(T_\lambda)$, for all $\lambda \in (0,1]$.

The Mann iteration [2] method associated with T is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ \ n = 0,1,2,\ldots \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies certain appropriate conditions. Note that the Mann iteration $\{x_n\}_{n=0}^{\infty}$ sequence given by (4) is exactly the Krasnoselskij iteration method with varying step sizes.

The Ishikawa iteration [3] associated with T, was first employed to establish the strong convergence of a sequence to a fixed point of a Lipschitzian and pseudo-contractive self-map on a convex compact subset of a Hilbert space.

It is defined as follows:

$$\begin{aligned} u_0 &\in K. \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T v_n, \\ v_n &= (1-\beta_n)u_n + \beta_n T u_n, \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ are appropriate sequences of parameters.

In the last three decades, both the Mann and Ishikawa iteration methods have been successfully used by several authors to approximate fixed points of various class of operators in Banach spaces.

In [4], the following conjecture was given: if the Mann iteration sequence associated with a certain mapping T converges to its fixed point, then so does the Ishikawa iteration sequence associated with T.

In a series of papers [5–9], a positive answer to the above conjecture was given. The following is the key result in [4].

Theorem 1 ([4])**.** *Let K be a nonempty convex subset of a normed space $(X, \|\cdot\|)$ and $T : K \to K$ satisfy the following inequality:*

$$\|Tx - Ty\| \le c \max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\right\}, \tag{6}$$

for all $x, y \in K$, $0 \le c < 1$. Suppose that T possesses a fixed point x^* in K. Then, the Picard iteration and the certain Mann and Ishikawa iteration associated with T converge strongly to x^*.

The mapping T satisfying (6) is known as a quasi-contraction mapping.

We now pose the following

Question

Let $(X, \|\cdot\|)$ be Banach space and $T : X \to X$ satisfy a certain contractive condition such that $Fix(T) \neq \emptyset$. Does there exist $\lambda \in (0,1]$ such that the following statement holds?

If the Mann iteration method associated with T_λ converges to the fixed point, then so does the Ishikawa iteration associated with T_λ.

In 1966, Browder and Petryshyn [10] introduced the concept of the asymptotic regularity in connection with the study of fixed points of nonexpansive mappings. As a matter of fact, the same property was used in 1955 by Krasnoselskij [11] to prove that if

K is a compact convex subset of a uniformly convex Banach space and $T : X \to X$ is a nonexpansive mapping, then for any $x_0 \in K$, the sequence:

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),\ n \geq 0, \tag{7}$$

converges to the fixed point of T.

In proving this result, Krasnoselskij used the fact that if T is nonexpansive, which, in general, is not asymptotically regular, then the averaged mapping $T_{\frac{1}{2}}$ in (7) is asymptotically regular.

Therefore, an averaged operator T_λ enriches the class of nonexpansive mappings with respect to the asymptotic regularity. This fact suggests that one could enrich the classes of contractive mappings in metrical fixed-point theory by imposing a certain contractive condition on T_λ instead of T itself.

In this way, the following mapping classes were introduced and studied: in enriched contractions and enriched ϕ contractions [12], enriched Kannan contractions [13], enriched Chatterjea mappings [14], enriched nonexpansive mappings in Hilbert spaces [15], enriched multivalued contractions [16], enriched cyclic contractions [17], etc.

Following the authors of [12], a mapping $T : X \to X$ is called an enriched contraction or (b, θ)-enriched contraction if there exist two constants, $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that for all $x, y \in X$,

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\|. \tag{8}$$

As shown in [12], many well-known contractive conditions from the literature imply the (b, θ)-enriched contraction. It was proven that any enriched contraction mapping defined on a Banach space has a unique fixed point, which can be approximated by means of the Krasnoselskij iterative scheme.

The aim of this paper is to enrich the quasi-contraction (6) and the weak contraction [18] mappings on a Banach space and to answer the above question, which exactly support the conjecture given in [4].

2. Two New Classes of Operators on a Normed Space

We introduce the following.

Definition 1. *Let $(X, \|\cdot\|)$ be a normed space. A mapping $T : X \to X$ is said to be an enriched quasi-contraction if there exist two constants, $b \in [0, \infty)$ and $c \in [0, 1)$, such that for all $x, y \in X$,*

$$\begin{aligned} \|b(x-y) + Tx - Ty\| \leq c\max\{&\|(b+1)(x-y)\|, \|x - Tx\|, \\ &\|y - Ty\|, \|b(x-y) + x - Ty\|, \\ &\|b(y-x) + y - Tx\|\}. \end{aligned} \tag{9}$$

To highlight an involvement of constants b and c in (9), we shall also call T a (b, c)-enriched quasi-contraction.

Example 1. *Any quasi-contraction mapping T with contraction constant c is a $(0, c)$-enriched quasi-contraction.*

We now give an example of an enriched quasi-contraction, which is not a quasi-contraction.

Example 2. *Let $X = [0, 1]$ be endowed with the usual norm and $T : X \to X$ be defined by $Tx = 1 - x$, for all $x \in [0, 1]$. Then, T is not a quasi-contraction, but T is an enriched quasi-contraction.*

Indeed, if T is a quasi-contraction then, by (6), there exists $c \in [0, 1)$ such that for all $x, y \in [0, 1]$, we have:

$$|x-y| \leq c\max\{|x-y|, |2x-1|, |2y-1|, |y+x-1|, |x+y-1|\},$$

which upon taking $x = 0$ and $y = 1$ gives $1 \leq c < 1$, a contradiction.

On the other hand, for $b = 1$, T satisfies the inequality (9) for all $x, y \in [0,1]$.

Example 3. *Let (Y, μ) be a finite measure space. The classical Lebesgue space $X = L^2(Y, \mu)$ is defined as the collection of all Borel measurable functions $f : Y \to \mathbb{R}$ such that $\int_Y |f(y)|^2 d\mu(y) < \infty$. We know that the space X equipped with the norm $\|f\|_X = \left(\int_Y |f|^2 d\mu \right)^{\frac{1}{2}}$ is a Banach space. Define the mapping $T : L^2(Y, \mu) \to CB(L^2(Y, \mu))$ by:*

$$Tf = g - 2f,$$

where $g(y) = 1$, $\forall\, y \in Y$. Clearly, $g \in L^2(Y, \mu)$ as $\mu(Y) < \infty$.

Note that T is a $(2, 0.5)$-enriched quasi-contraction mapping, but not a quasi-contraction. Indeed, if T were a quasi-contraction, then, by (6), there exists $c \in [0,1)$ such that for all $f, h \in L^2(Y, \mu)$, we have:

$$\|-2f + 2h\|_X \leq c \max\{\|f - h\|_X, \|3f - h\|_X, \|3h - g\|_X, \|f - g + 2h\|_X, \|h - g + 2f\|_X\}.$$

which upon taking $f(y) = 0$ and $h(y) = 1$, for all $y \in Y$, gives $1 \leq c$, a contradiction.

We need the following technical notations.

Definition 2 ([19])**.** *Let T be a self-mapping on a normed space $(X, \|\cdot\|)$. For $A \subset X$, let $\Lambda[A] = \sup\{||x - y|| : x, y \in A\}$, and for each $x \in X$, let:*

$$O(T, x, n) = \{x, Tx, \ldots, T^n x\},\ n = 1, 2, 3, \ldots$$
$$O(T, x, \infty) = \{x, Tx, T^2 x, \ldots\}.$$

A normed space $(X, \|\cdot\|)$ is said to be a T-orbital Banach space if every Cauchy sequence contained in $O(T, x, \infty)$ for some $x \in X$ converges in X.

Before stating the main result, we first prove two lemmas for the class of enriched quasi-contraction mappings.

Lemma 1. *Let T be a (b, c)-enriched quasi-contraction on a normed space $(X, \|\cdot\|)$ and n be any positive integer. Then, there exists $\lambda \in (0,1]$ such that for each $x \in X$ and for all positive integers s and t in $\{1, 2, \ldots, n\}$, we have:*

$$\left\|T_\lambda^s x - T_\lambda^t x\right\| \leq c\Lambda[O(T_\lambda, x, n)]. \tag{10}$$

Proof. Let us denote $\lambda = \frac{1}{b+1}$. Clearly, $0 < \lambda < 1$. Note that, for any $x, y \in X$, (9) becomes:

$$\begin{aligned} \left\| \left(\frac{1}{\lambda} - 1\right)(x - y) + Tx - Ty \right\| \leq c \max \Bigg\{ & \frac{1}{\lambda}\|x - y\|, \|x - Tx\|, \|y - Ty\|, \\ & \left\| \left(\frac{1}{\lambda} - 1\right)(x - y) + x - Ty \right\|, \\ & \left\| \left(\frac{1}{\lambda} - 1\right)(y - x) + y - Tx \right\| \Bigg\} \end{aligned}$$

that is,

$$\frac{1}{\lambda}\|(1-\lambda)(x-y)+\lambda Tx-\lambda Ty\| \le c\max\left\{\frac{1}{\lambda}\|x-y\|, \|x-Tx\|, \|y-Ty\|,\right.$$
$$\frac{1}{\lambda}\|(1-\lambda)(x-y)+\lambda x-\lambda Ty\|,$$
$$\left.\frac{1}{\lambda}\|(1-\lambda)(y-x)+\lambda y-\lambda Tx\|\right\},$$

which can be written in an equivalent form as:

$$\|T_\lambda x - T_\lambda y\| \le c\max\{\|x-y\|, \|x-T_\lambda x\|, \|y-T_\lambda y\|, \|x-T_\lambda y\|, \|y-T_\lambda x\|\}. \tag{11}$$

Let $x \in X$ be arbitrary and n a fixed positive integer. By using (11), we have:

$$\begin{aligned}\left\|T_\lambda^s x - T_\lambda^t x\right\| &= \left\|T_\lambda T_\lambda^{s-1}x - T_\lambda T_\lambda^{t-1}x\right\| \\ &\le c\max\left\{\left\|T_\lambda^{s-1}x - T_\lambda^{t-1}x\right\|, \left\|T_\lambda^{s-1}x - T_\lambda^{s}x\right\|, \left\|T_\lambda^{t-1}x - T_\lambda^{t}x\right\|,\right. \\ &\left.\left\|T_\lambda^{s-1}x - T_\lambda^{t}x\right\|, \left\|T_\lambda^{s}x - T_\lambda^{t-1}x\right\|\right\}.\end{aligned}$$

This implies that:

$$\left\|T_\lambda^s x - T_\lambda^t x\right\| \le c\Lambda[O(T_\lambda, x, n)].$$

□

Remark 1. *It follows from Lemma 1 that if T is a (b,c)-enriched quasi-contraction and $x \in X$, then for any positive integer n, there exists a positive integer $k \le n$, such that:*

$$\left\|x - T_\lambda^k x\right\| = \Lambda[O(T_\lambda, x, n)].$$

Lemma 2. *If T is a (b,c)-enriched quasi-contraction on a normed space $(X, \|\cdot\|)$, then there exists $\lambda \in (0,1]$ such that:*

$$\Lambda[O(T_\lambda, x, \infty)] \le \frac{1}{1-c}\|x - T_\lambda x\| \tag{12}$$

holds for all $x \in X$.

Proof. Take $\lambda = \frac{1}{b+1}$. Let $x \in X$ be arbitrary. Since:

$$\Lambda[O(T_\lambda, x, 1)] \le \Lambda[O(T_\lambda, x, 2)] \le, \ldots.$$

Note that,

$$\Lambda[O(T_\lambda, x, \infty)] = \sup\{\Lambda[O(T_\lambda, x, n)] : n \in \mathbb{N}\}.$$

Then, (12) follows, if we show that:

$$\Lambda[O(T_\lambda, x, n)] \le \frac{1}{1-c}\|x - T_\lambda x\|, \ \forall n \in \mathbb{N}.$$

Let n be any positive integer. By Remark 1, there exists $T_\lambda^k \in O(T_\lambda, x, n)(1 \le k \le n)$ such that:

$$\left\|x - T_\lambda^k x\right\| = \Lambda[O(T_\lambda, x, n)].$$

Using Lemma 1 and the triangle inequality, we have:

$$\begin{aligned}\left\|x - T_\lambda^k x\right\| &\leq \|x - T_\lambda x\| + \left\|T_\lambda x - T_\lambda^k x\right\| \\ &\leq \|x - T_\lambda x\| + c\Lambda[O(T_\lambda, x, n)] \\ &= \|x - T_\lambda x\| + c\left\|x - T_\lambda^k x\right\|.\end{aligned}$$

Therefore,

$$\Lambda[O(T_\lambda, x, n)] = \left\|x - T_\lambda^k x\right\| \leq \frac{1}{1-c}\|x - T_\lambda x\|.$$

Since n is arbitrary, the proof is complete. □

We are now in a position to prove the following result.

Theorem 2. *Let T be a (b, c)-enriched quasi-contraction on normed space $(X, \|\cdot\|)$. Then, T has unique fixed point $x^* \in X$. Moreover, for $\lambda = \frac{1}{b+1}$, the iterative algorithm $\{x_n\}_{n=0}^\infty$ given by:*

$$x_n = (1-\lambda)x_{n-1} + \lambda T x_{n-1}, \quad n \geq 1, \tag{13}$$

converges to x^* for any $x_0 \in X$ provided that X is a T_λ-orbital Banach space.

Proof. Following a similar argument in the proof of Lemma 1 for $\lambda = \frac{1}{b+1}$, we have:

$$\|T_\lambda x - T_\lambda y\| \leq c \max\{\|x - y\|, \|x - T_\lambda x\|, \|y - T_\lambda y\|, \|x - T_\lambda y\|, \|y - T_\lambda x\|\}. \tag{14}$$

In view of (1), the Krasnoselskij iterative process $\{x_n\}_{n=1}^\infty$ defined by (13) is exactly the Picard iteration associated with T_λ, that is,

$$x_n = T_\lambda x_{n-1} = T_\lambda^n x_0, \quad n \geq 0. \tag{15}$$

We now show that the sequence of iterates $\{x_n\}_{n=1}^\infty$ defined by (15) is a Cauchy sequence. Let n and m $(n < m)$ be any positive integers. By Lemma 1, we obtain:

$$\begin{aligned}\|x_n - x_m\| &= \|T_\lambda^n x_0 - T_\lambda^m x_0\| \\ &= \left\|T_\lambda T_\lambda^{n-1} x_0 - T_\lambda^{m-n+1} T_\lambda^{n-1} x_0\right\| \\ &= \left\|T_\lambda x_{n-1} - T_\lambda^{m-n+1} x_{n-1}\right\| \\ &\leq c\Lambda[O(T_\lambda, x_{n-1}, m-n+1)].\end{aligned}$$

By Remark 1, there exists an integer p, $1 \leq p \leq m - n + 1$ such that the following holds:

$$\Lambda[O(T_\lambda, x_{n-1}, m-n+1)] = \left\|x_{n-1} - x_{n+p-1}\right\|.$$

It follows from Lemma 1 that:

$$\begin{aligned}\left\|x_{n-1} - x_{n+p-1}\right\| &= ||T_\lambda x_{n-2} - T_\lambda^{p+1} x_{n-2}|| \\ &\leq c\Lambda[O(T_\lambda, x_{n-2}, p+1)]\end{aligned}$$

this implies that:

$$\left\|x_{n-1} - x_{n+p-1}\right\| \leq c\Lambda[O(T_\lambda, x_{n-2}, m-n+2)].$$

Therefore, we have:

$$\|x_n - x_m\| \leq c\Lambda[O(T_\lambda, x_{n-1}, m-n+1)] \leq c^2\Lambda[O(T_\lambda, x_{n-2}, m-n+2)].$$

Continuing, we obtain that:

$$\|x_n - x_m\| \le c\Lambda[O(T_\lambda, x_{n-1}, m-n+1)] \le \ldots \le c^n \Lambda[O(T_\lambda, x_0, m)].$$

From Lemma 2, we obtain:

$$\|x_n - x_m\| \le \frac{c^n}{1-c}\|x_0 - T_\lambda x_0\|.$$

Upon taking the limit as n tends to infinity, we have that $\{x_n\}$ is a Cauchy sequence. Since X is a T_λ-orbital Banach space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Note that,

$$\begin{aligned}
\|x^* - T_\lambda x^*\| &\le \|x^* - x_{n+1}\| + \|x_{n+1} - T_\lambda x^*\| \\
&= \|x^* - x_{n+1}\| + \|T_\lambda x_n - T_\lambda x^*\| \\
&\le \|x^* - x_{n+1}\| + c\max\big\{\|x_n - x^*\|, \|x_n - x_{n+1}\|, \\
&\quad \|x^* - T_\lambda x^*\|, \|x_n - T_\lambda x^*\|, \|x_{n+1} - x^*\|\big\} \\
&\le \|x^* - x_{n+1}\| + c\big\{\|x_n - x^*\| + \|x_n - x_{n+1}\| \\
&\quad + \|x^* - T_\lambda x^*\| + \|x_{n+1} - x^*\|\big\}.
\end{aligned}$$

Hence:

$$\|x^* - T_\lambda x^*\| \le \frac{1}{1-c}\Big\{(1+c)\|x_{n+1} - x^*\| + c\|x_n - x^*\| + c\|x_n - x_{n+1}\|\Big\}.$$

As $\lim_{n\to\infty} x_n = x^*$, we have $\|x^* - T_\lambda x^*\| = 0$, that is x^* is the fixed point of T_λ. The uniqueness follows from (14). □

If we take $b = 0$ in Theorem 2, we obtain Theorem 1 of [19] in the setting of normed spaces.

Corollary 1 ([19])**.** *Let T be a quasi-contraction mapping on a normed space $(X, \|\cdot\|)$. Then, T has a unique fixed point, provided that X is a T-orbital Banach space.*

Now, we prove the following fixed-point theorem for a (b, c)-enriched quasi-contraction in a Banach space.

Corollary 2. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a (b, c)-enriched quasi-contraction. Then, T has a unique fixed point.*

Proof. Following arguments similar to those in the proof Theorem 2, the result follows. □

By Corollary 2, we obtain the following corollaries.

Corollary 3 ([12])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, θ)-enriched contraction, that is an operator satisfying:*

$$||b(x-y) + Tx - Ty|| \le \theta||x-y||,\ \forall\, x, y \in X \tag{16}$$

with $b \in [0, \infty)$ and $\theta \in [0, b+1)$. Then, T has a unique fixed point.

Proof. Take $\lambda = \frac{1}{b+1}$. Obviously, $0 < \lambda < 1$, and the (b, θ)-enriched contraction condition (16) becomes:

$$||(\frac{1}{\lambda} - 1)(x-y) + Tx - Ty|| \le \theta||x-y||,\ \forall\, x, y \in X,$$

which can be written in an equivalent form as:

$$||T_\lambda x - T_\lambda y|| \leq d||x - y||,\ \forall\, x, y \in X. \tag{17}$$

where we denote $d = \lambda\theta$. Since $\theta \in (0, b+1)$, it follows that $c \in [0, 1)$, and therefore, by (17) T_λ is a d-contraction. It follows from [20] that T_λ satisfies Condition (17) and also satisfies Condition (11), since for the value of $\lambda = \frac{1}{b+1}$, the inequality (11) is the same as condition (9). This suggests that T is an enriched quasi-contraction. Corollary 2 leads to the conclusion. □

Corollary 4 ([13]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, a)-enriched Kannan contraction, that is an operator satisfying:*

$$||b(x - y) + Tx - Ty|| \leq a\{||x - Tx|| + ||y - Ty||\},\ \forall\, x, y \in X \tag{18}$$

with $b \in [0, \infty)$ and $a \in [0, 1/2)$. Then, T has a unique fixed point.

Proof. Take $\lambda = \frac{1}{b+1}$. Obviously, $0 < \lambda < 1$, and the (b, a)-enriched Kannan contraction condition (18) becomes:

$$||(\frac{1}{\lambda} - 1)(x - y) + Tx - Ty|| \leq a\{||x - Tx|| + ||y - Ty||\},\ \forall\, x, y \in X,$$

which can be written in an equivalent form as:

$$||T_\lambda x - T_\lambda y|| \leq a\{||x - T_\lambda x|| + ||y - T_\lambda y||\},\ \forall\, x, y \in X. \tag{19}$$

Therefore, by (19), T_λ is a Kannan contraction. It follows from [20] that T_λ satisfies Condition (19) and also satisfies Condition (11), since for the value of $\lambda = \frac{1}{b+1}$, the inequality (11) is the same as Condition (A). This suggests that T is an enriched quasi-contraction. Corollary 2 leads to the conclusion. □

Corollary 5 ([14]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, k)-enriched Chatterjea contraction, that is an operator satisfying:*

$$||b(x-y) + Tx - Ty|| \leq k\{||(b+1)(x-y) + y - Ty|| + ||(b+1)(y-x) + x - Tx||\}, \tag{20}$$

for all $x, y \in X$, with $b \in [0, \infty)$ and $k \in [0, 1/2)$. Then, T has a unique fixed point.

Proof. Take $\lambda = \frac{1}{b+1}$. Obviously, $0 < \lambda < 1$, and the (b, k)-enriched Chatterjea contraction condition (20) can be written in an equivalent form as:

$$||T_\lambda x - T_\lambda y|| \leq k\{||x - T_\lambda y|| + ||y - T_\lambda x||\},\ \forall\, x, y \in X. \tag{21}$$

Therefore, by (21), T_λ is a Chatterjea contraction. It follows from [20] that T_λ satisfies Condition (21) and also satisfies Condition (11), since for the value of $\lambda = \frac{1}{b+1}$, the inequality (11) is the same as Condition (A). This suggests that T is an enriched quasi-contraction. Corollary 2 leads to the conclusion. □

Corollary 6 ([21]). *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be an (b, a, k)-enriched Ćirić–Reich–Rus contraction, that is an operator satisfying:*

$$||b(x - y) + Tx - Ty|| \leq a||x - y|| + k\{||x - Tx|| + ||y - Ty||\},\ \forall\, x, y \in X \tag{22}$$

with $a, b \in [0, \infty)$ and $k \in [0, 1/2)$ satisfying $a + 2k < 1$. Then, T has a unique fixed point.

Proof. Take $\lambda = \frac{1}{b+1}$. Then, the (b, a, k)-enriched Ćirić–Reich–Rus contraction condition (22) can be written in an equivalent form as:

$$||T_\lambda x - T_\lambda y|| \leq \lambda k||x - y|| + k\{||x - T_\lambda x|| + ||y - T_\lambda y||\}, \tag{23}$$

for all $x, y \in X$. It follows from [20] that T_λ satisfies Condition (23) and also satisfies Condition (11), since for the value of $\lambda = \frac{1}{b+1}$, the inequality (11) is the same as condition (9). This suggests that T is an enriched quasi-contraction. Corollary 2 leads to the conclusion. □

Corollary 7 ([22])**.** *Let $(X, \|\cdot\|)$ be a Banach space, m, n, p real numbers with $p \in [0, 1)$, $m, n \in [0, 1/2)$, and $T : X \to X$ a Zamfirescu operator, such that for each couple of different points $x, y \in X$, at least one of the following conditions is satisfied:*

1. $||Tx - Ty|| \leq p||x - y||$,
2. $||Tx - Ty|| \leq m\{||x - Tx|| + ||y - Ty||\}$,
3. $||Tx - Ty|| \leq n\{||x - Ty|| + ||y - Tx||\}$.

Then, T has a unique fixed point.

Proof. It follows from [20] that an operator T satisfying the contractive conditions in Corollary 7 is a $(0, c)$-enriched quasi-contraction, for some $c \in [0, 1)$. Corollary 2 leads to the conclusion. □

Now, we introduce the enriched weak contraction mapping as follows:

Definition 3. *Let $(X, \|\cdot\|)$ be a normed space and $T : X \to X$. If there exist $b \in [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$, and:*

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\big(\|x - y\| - \phi(\|x - y\|)\big), \tag{24}$$

holds for all $x, y \in X$, then the mapping T is said to be a (b, ϕ)-enriched weak contraction.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (b, ϕ) weak enriched contraction. Then, T has a unique fixed point in X, provided that:*

1. *ϕ is continuous and nondecreasing;*
2. $\lim_{t\to\infty} \phi(t) = \infty$.

Proof. Let us denote $\lambda = \frac{1}{b+1}$. By the (b, ϕ) weak enriched contraction condition (24), we have:

$$\begin{aligned} & ||(\frac{1}{\lambda} - 1)(x - y) + Tx - Ty|| \\ \leq \; & \frac{1}{\lambda}(\|x - y\| - \phi(\|x - y\|), \quad \forall\, x, y \in X, \end{aligned}$$

which can be written in an equivalent form as follows:

$$\|T_\lambda x - T_\lambda y\| \leq \|x - y\| - \phi(\|x - y\|), \quad \forall\, x, y \in X. \tag{25}$$

Let $x_0 \in X$. Define the Krasnoselskij iteration (2) with the help of T. From (25), we have:

$$\begin{aligned} \|T_\lambda x_n - T_\lambda x_{n+1}\| &= \|x_{n+1} - x_{n+2}\| \\ &\leq \|x_n - x_{n+1}\| - \phi(\|x_n - x_{n+1}\|). \end{aligned}$$

Set $\delta_n = \|x_n - x_{n+1}\|$. Then, we have:

$$\delta_{n+1} \leq \delta_n - \phi(\delta_n) \leq \delta_n. \tag{26}$$

Therefore, $\{\delta_n\}$ is a non-negative nonincreasing sequence, and hence possesses a limit $\delta \geq 0$. Suppose that $\delta > 0$. Since ϕ is nondecreasing, $\phi(\delta_n) \geq \phi(\delta) > 0$. By (26), we have $\delta_{n+1} \leq \delta_n - \phi(\delta)$. Thus, $\delta_{N+m} \leq \delta_m - N\phi(\delta)$, a contradiction for N large enough. Therefore, $\delta = 0$.

Fix $\epsilon > 0$ and choose N so that $\|x_N - x_{N+1}\| \leq \min\{\epsilon/2, \phi(\epsilon/2)\}$. We show that T_λ is a self-map of the closed ball $B(x_N, \epsilon)$. Let $x \in B(x_N, \epsilon)$.

CASE 1. Then, $\|x - x_N\| \leq \frac{\epsilon}{2}$ gives:

$$\begin{aligned}\|T_\lambda x - x_N\| &\leq \|T_\lambda x - T_\lambda x_N\| + \|T_\lambda x_N - x_N\| \\ &\leq \|x - x_N\| - \phi(\|x - x_N\|) + \|x_{N+1} - x_N\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

CASE 2. If $\epsilon/2 < \|x - x_N\| \leq \epsilon$, then $\phi(\|x - x_N\|) \geq \phi(\epsilon/2)$. Therefore:

$$\begin{aligned}\|T_\lambda x - x_N\| &\leq \|x - x_N\| - \phi(\|x - x_N\|) + \|x_{N+1} - x_N\| \\ &\leq \|x - x_N\| - \phi(\epsilon/2) + \phi(\epsilon/2) \\ &= \|x - x_N\| \leq \epsilon.\end{aligned}$$

Since T_λ is a self-map of $B(x_N, \epsilon)$, it follows that each $x_n \in B(x_N, \epsilon)$ for $n > N$. Since ϵ is arbitrary, $\{x_n\}$ is a Cauchy sequence and, hence, convergent, whose limit is a fixed point of T_λ by the continuity of T_λ. The uniqueness is clear from (25). □

3. Equivalence between Iteration Methods

In this section, we show that the Mann and Ishikawa iteration methods are equivalent with respect to approximating fixed points of:

(i) (b, θ)-enriched contractions;
(ii) (b, c)-enriched quasi-contractions;
(iii) (b, ϕ)-enriched weak contractions.

The Mann iteration associated with T_λ, starting from $x_0 \in K$, is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_\lambda x_n, \tag{27}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfies certain appropriate conditions.

The Ishikawa iteration associated with T_λ, starting from $u_0 \in K$, is the sequence $\{u_n\}_{n=1}^{\infty}$ defined by:

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n T_\lambda v_n, \\ v_n &= (1 - \beta_n)u_n + \beta_n T_\lambda u_n, \quad n \geq 0, \end{aligned} \tag{28}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfy certain appropriate conditions.

Let us recall the following lemma from [23].

Lemma 3. *Let $\{a_n\}_n$ be a non-negative sequence that satisfies the inequality:*

$$a_{n+1} \leq (1 - \omega_n)a_n + \sigma_n, \tag{29}$$

where $\omega_n \in (0, 1)$ for each $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \omega_n = \infty$, $\sigma_n = \epsilon_n \omega_n$, and $\lim_{n\to\infty} \epsilon_n = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Theorem 4. *Let K be a nonempty closed convex subset of a normed space $(X, \|\cdot\|)$ and $T : K \to K$ a (b, θ)-enriched contraction. Suppose that T has a unique fixed point $x^* \in K$. Let $x_0 = u_0 \in K$.*

Define $\{x_n\}$ and $\{u_n\}$ by (27) and (28), respectively, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

1. $0 \leq \alpha_n, \beta_n \leq 1,\ \forall\, n \geq 0$;
2. $\lim \alpha_n = \lim \beta_n = 0$;
3. $\sum \alpha_n = \infty$.

Then, there exists $\lambda \in (0,1]$ such that the following are equivalent:

(i) The Mann iteration associated with T_λ (27) converges strongly to x^*;
(ii) The Ishikawa iteration associated with T_λ (28) converges strongly to x^*.

Proof. Note that (ii) implies (i), which is obvious by setting $\beta_n = 0$ in (28).

Take $\lambda = \frac{1}{b+1}$. Clearly, $0 < \lambda < 1$. In this case, (8) becomes:

$$\left\| \left(\frac{1}{\lambda} - 1\right)(x-y) + Tx - Ty \right\| \leq \theta \|x-y\|, \quad \forall\, x, y \in X,$$

which can be written in an equivalent form as:

$$\|T_\lambda x - T_\lambda y\| \leq d\|x-y\|, \tag{30}$$

where $\theta\lambda = d$. As $\theta \in [0, b+1)$, we have $d \in (0,1)$. Now,

$$\|x_{n+1} - u_{n+1}\| = \|(1-\alpha_n)(x_n - u_n) + \alpha_n(T_\lambda x_n - T_\lambda v_n)\|.$$

Using (30), we have:

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|T_\lambda x_n - T_\lambda v_n\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + d\alpha_n\|v_n - x_n\|.\end{aligned}$$

Using the value of v_n from (28) in the above inequality, we obtain:

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|x_n - u_n\| + d\alpha_n\|[(1-\beta_n)u_n + \beta_n T_\lambda u_n] - x_n\| \\ &= (1-\alpha_n)\|x_n - u_n\| + d\alpha_n\|(1-\beta_n)(u_n - x_n) + \beta_n(T_\lambda u_n - x_n)\| \\ &\leq (1-\alpha_n)\|x_n - u_n\| + d\alpha_n(1-\beta_n)\|u_n - x_n\| + d\alpha_n\beta_n\|T_\lambda u_n - x_n\| \\ &\leq [1 - \alpha_n(1 - d(1-\beta_n)]\|x_n - u_n\| + \alpha_n\beta_n\|T_\lambda u_n - x_n\|,\end{aligned}$$

which implies that:

$$\|x_{n+1} - u_{n+1}\| \leq [1 - \alpha_n(1 - d(1-\beta_n)]\|x_n - u_n\| + \alpha_n\beta_n\|T_\lambda u_n - x_n\|. \tag{31}$$

We now claim that $\{\|T_\lambda u_n - x_n\|\}_n$ is bounded. It suffices to show that $\{\|u_n\|\}_n$ is bounded. Note that,

$$\begin{aligned}\|u_{n+1}\| &= \|(1-\alpha_n)u_n + \alpha_n T_\lambda v_n\| \\ &\leq (1-\alpha_n)\|u_n\| + \alpha_n\|T_\lambda v_n\|.\end{aligned}$$

It follows from (30) and simple induction that:

$$\begin{aligned}\|u_{n+1}\| &\leq (1-\alpha_n)\|u_n\| + d\alpha_n\|v_n\| \\ &= (1-\alpha_n)\|u_n\| + d\alpha_n\|(1-\beta_n)u_n + \beta_n T_\lambda u_n\| \\ &\leq (1-\alpha_n)\|u_n\| + d\alpha_n(1-\beta_n)\|u_n\| + d\alpha_n\beta_n\|T_\lambda u_n\| \\ &\leq (1-\alpha_n)\|u_n\| + d\alpha_n(1-\beta_n)\|u_n\| + d\alpha_n\beta_n\|u_n\| \\ &= \|u_n\| \leq \ldots \leq \|u_0\|.\end{aligned}$$

This implies that $\|T_\lambda u_n - x_n\| \leq \vartheta$, $\forall\, n \geq 0$. Then, (31) becomes,

$$\|x_{n+1} - u_{n+1}\| \leq [1 - \alpha_n(1 - d(1-\beta_n)]\|x_n - u_n\| + \vartheta\alpha_n\beta_n.$$

An inequality (29) of Lemma 3 is satisfied if we take $a_n := \|x_n - u_n\|$, $\omega_n := \alpha_n(1 - d(1-\beta_n) \in (0,1)$, and $\sigma_n := \vartheta\alpha_n\beta_n$, for each $n \in \mathbb{N}$, in the above inequality. Therefore,

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{32}$$

Since (i) is true, using (32), we obtain that:

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \|x_n - u_n\|,$$

which implies that $\lim_{n\to\infty}\|u_n - x^*\| = 0$. □

Theorem 5. *Let M be a nonempty closed convex subset of a normed space $(X, \|.\|)$ and $T : M \to M$ a (b,c)-enriched quasi-contraction on M. Suppose that T has a unique fixed point $x^* \in M$. Then, the Krasnoselskij, Mann, and Ishikawa iterations associated with T_λ converge strongly to x^*, where $\lambda = \frac{1}{b+1}$.*

Proof. Following arguments similar to those given in the proof of Theorem 2, we have:

$$\|T_\lambda x - T_\lambda y\| \leq c\max\{\|x-y\|, \|x - T_\lambda x\|, \|y - T_\lambda y\|, \|x - T_\lambda y\|, \|y - T_\lambda x\|\}.$$

That is T_λ is a quasi-contraction. It follows from Theorem 2 that the Krasnoselskij iterative process $\{x_n\}_{n=0}^\infty$ defined by (13) converges strongly to x^*.

In [4], it was shown that the Mann iteration for T_λ satisfying (6) with a sequence $\{\alpha_n\}$ in $(0,1)$, which is bounded away from zero, converges strongly to the unique fixed point of T_λ and the Ishikawa method associated with T_λ with each $\alpha_n > 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$ converges strongly to x^*. □

Theorem 6. *Let X be a Banach space, K a closed convex subset of X and $T : K \to K$ a (b,ϕ) weak enriched contraction. Then, the Mann iteration (27) associated with T_λ with (i) $0 \leq \alpha_n \leq 1$ and (ii) $\sum\alpha_n = \infty$ converges to the unique fixed point x^* of T, where $\lambda = \frac{1}{b+1}$.*

Proof. From Theorem 3, T has a unique fixed point. Call it x^*. Using (27), we have:

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T_\lambda x_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_\lambda x_n - T_\lambda x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n[\|x_n - x^*\| - \phi(\|x_n - x^*\|)] \\ &\leq \|x_n - x^*\| - \alpha_n\phi(\|x_n - x^*\|).\end{aligned}$$

This implies that:

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|. \tag{33}$$

Therefore, $\{\|x_n - x^*\|\}$ is a non-negative nonincreasing sequence, which converges to a limit $\gamma \geq 0$. Suppose $\gamma > 0$.

For notational convenience, define $\lambda_n = \|x_n - x^*\|$. Then, $\lambda_n \geq \gamma$. For any fixed integer N, it follows from (33) that:

$$\sum_{n=N}^{\infty}\alpha_n\phi(\gamma) \leq \sum_{n=N}^{\infty}\alpha_n\phi(\lambda_n) \leq \sum_{n=N}^{\infty}(\lambda_n - \lambda_{n+1}) \leq \lambda_N,$$

a contradiction to (ii). Therefore, $\gamma = 0$. □

Theorem 7. *Let X be a Banach space, K a closed convex subset of X, and $T : K \to K$ a (b, ϕ) weak enriched contraction. Then, the Ishikawa iteration associated with T_λ with (i) $0 \leq \alpha_n, \beta_n \leq 1$ and (ii) $\sum \alpha_n \beta_n = \infty$, converges to the unique fixed point x^* of T, where $\lambda = \frac{1}{b+1}$.*

Proof. The proof of Theorem 7 is similar to Theorem 6 and is omitted. □

4. Applications to Split Feasibility and Variational Inequality Problems

Variational inequality theory is an important tool in economics, engineering mechanics, mathematical programming, transportation, and other fields. Many numerical methods have been constructed to solve variational inequalities and optimization problems. The aim of this section is to present generic convergence theorems for Krasnoselskij-type algorithms that solve variational inequality problems and split feasibility problems, respectively.

4.1. Solving Variational Inequality Problems

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C \subset H$ be closed and convex. A mapping $S : H \to H$ is called monotone if:

$$\langle Sx - Sy, x - y \rangle \geq 0, \ \forall\, x, y \in H.$$

The variational inequality problem with respect to S and C, denoted by $VIP(S, C)$, is to find $x^* \in C$ such that:

$$\langle Sx^*, x - x^* \rangle \geq 0, \ \forall\, x \in H.$$

It is well known (see for example [24]) that if $\gamma > 0$, then $x^* \in C$ is a solution of $VIP(S, C)$ if and only if x^* is a solution of the fixed-point problem:

$$x = P_C(I - \gamma G)x,$$

where P_C is the nearest point projection onto C.

In [24], it was proven, amongst many others results, that if $I - \gamma G$ and $P_C(I - \gamma G)$ are averaged nonexpansive mappings, then, under some additional assumptions, the iterative algorithm $\{x_n\}_{n=0}^{\infty}$ defined by:

$$x_{n+1} = P_C(I - \gamma G)x_n, \ n \geq 0,$$

converges weakly to a solution of $VIP(S, C)$, if such solutions exist.

Our alternative is to consider $VIP(S, C)$ for enriched quasi-contraction mappings, which are in general discontinuous mappings, instead of nonexpansive mappings, which are always continuous. In this case, we shall have $VIP(S, C)$ with a unique solution, as shown by the next theorem. Moreover, the considered algorithm (34) will converge strongly to the solution of $VIP(S, C)$.

Theorem 8. *Assume that for $\gamma > 0$, $P_C(I - \gamma G)$ is a (b, c)-enriched quasi-contraction mapping. Then, there exists $\lambda \in (0, 1]$ such that the iterative algorithm $\{x_n\}_{n=0}^{\infty}$ defined by:*

$$x_{n+1} = (1 - \lambda)x_n + \lambda P_C(I - \gamma G)x_n, \ n \geq 0, \tag{34}$$

converges strongly to the unique solution x^* of $VIP(S, C)$, for any $x_0 \in C$.

Proof. Since C is closed, we take $X := C$ and $T := P_C(I - \gamma G)$ and apply Corollary 2. □

4.2. Solving Split Feasibility Problems

The split feasibility problem (SFP), introduced by Censor and Elfving in 1994 [25], is:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{35}$$

where C and Q are closed convex subsets of the Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

If we assume that the SFP (35) is consistent, that is it has a solution and denote by W the solution set of (35), then (see [26]) $x^* \in C$ is a solution of (35) if and only if it is a solution of the fixed-point problem.

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x,$$

where P_C and P_Q are the nearest point projections onto C and Q, respectively, $\gamma > 0$, and A^* is the adjoint operator of A. It was shown in [24] that if δ is the spectral radius of A^*A and $\gamma \in (0, 2/\delta)$, then the operator:

$$T = P_C\big(I - \gamma A^*(I - P_Q)A\big)$$

is averaged and nonexpansive and the so-called CQ algorithm:

$$x_{n+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_n, \;\; n \geq 0,$$

converges weakly to a solution of the SFP.

In the case of averaged nonexpansive mappings, the problem of turning the weak convergence above into the strong convergence has received a great deal of research work. This usually consists of considering additional assumptions; see [26] for a recent survey on Halpern-type algorithms.

We propose here an alternative to all those approaches, by considering enriched quasi-contraction mappings, which are in general discontinuous mappings, instead of nonexpansive mappings, which are always continuous. In this case, we have a SFP with a unique solution, as shown by the next theorem, while the considered algorithm (36) will converge strongly.

Theorem 9. *Assume that the SFP (35) is consistent, $\gamma \in (0, 2/\delta)$, and $P_C\big(I - \gamma A^*(I - P_Q)A\big)$ is a (b, c)-enriched quasi-contraction mapping. Then, there exists $\lambda \in (0, 1]$ such that the iterative algorithm $\{x_n\}_{n=0}^{\infty}$ defined by:*

$$x_{n+1} = (1 - \lambda)x_n + \lambda P_C\big(I - \gamma A^*(I - P_Q)A\big)x_n, \; n \geq 0, \tag{36}$$

converges strongly to the unique solution x^* of the SFP (35) for any $x_0 \in C$.

Proof. Since C is closed, we take $X := C$ and $T := P_C(I - \gamma G)$ and apply Corollary 2. □

5. Conclusions

(1) In this paper, we first introduced a large class of contractive mappings, called enriched quasi-contractions, that includes the usual quasi-contraction mappings, enriched contractions, enriched Kannan mappings, enriched Chatterjea mappings, Zamfirescu mappings, and enriched Ćirić–Reich–Rus mappings;
(2) We studied the set of fixed points and constructed an algorithm of the Krasnoselskij-type in order to approximate fixed points of enriched quasi-contraction mappings for which we have proven the strong convergence theorem;
(3) We then extended the weak contractions to the larger class of enriched weak contractions and constructed the corresponding algorithm of the Krasnoselskij-type in order to approximate fixed points of enriched quasi-contraction mappings for which we proved the strong convergence theorem;
(4) We showed that the Ishikawa, Mann, and Krasnoselskij iteration methods defined with the help of enriched quasi-contractions and enriched weak contraction mappings are equivalent;
(5) As applications of our main results, we presented two Krasnoselskij-projection-type algorithms to solve split feasibility problems and variational inequality problems

in the class of enriched quasi-mappings, thus improving the existence and weak convergence results for split feasibility problems and variational inequality problems in [24] to existence and uniqueness, as well as to strong convergence theorems.

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