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A Survey on Function Spaces of John–Nirenberg Type

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Abstract: In this systematic review, the authors give a survey on the recent developments of both the John–Nirenberg space JN_p and the space BMO as well as their vanishing subspaces such as VMO, XMO, CMO, VJN_p, and CJN_p on $\mathbb{R}^n$ or a given cube $Q_0 \subset \mathbb{R}^n$ with finite side length. In addition, some related open questions are also presented.

Keywords: Euclidean space; cube; congruent cube; BMO; JN_p; (localized) John–Nirenberg–Campanato space; Riesz–Morrey space; vanishing John–Nirenberg space; duality; commutator

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1. Introduction

In this article, a *cube* Q means that it has finite side length and all its sides parallel to the coordinate axes, but Q is not necessarily open or closed. Moreover, we always let $\mathcal{X}$ be $\mathbb{R}^n$ or a given cube of $\mathbb{R}^n$. Recall that the *Lebesgue space* $L^q(\mathcal{X})$ with $q \in [1,\infty]$ is defined to be the set of all measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{L^q(\mathcal{X})} := \begin{cases} \left[\int_{\mathcal{X}} |f(x)|^q \, dx\right]^{\frac{1}{q}} & \text{when} \quad q \in [1,\infty), \\ \operatorname*{ess\,sup}_{x\in\mathcal{X}} |f(x)| & \text{when} \quad q = \infty \end{cases}$$

is finite. In what follows, we use $\mathbf{1}_E$ to denote the *characteristic function* of a set $E \subset \mathbb{R}^n$, and for any given $q \in [1,\infty)$, $L^q_{\rm loc}(\mathcal{X})$ to denote the set of all measurable functions f on $\mathcal{X}$ such that $f\mathbf{1}_E \in L^q(\mathcal{X})$ for any bounded measurable set $E \subset \mathcal{X}$.

It is well known that $L^p(\mathcal{X})$ with $p \in [1,\infty]$ plays a leading role in the modern analysis of mathematics. In particular, when $p \in (1,\infty)$, the space $L^p(\mathcal{X})$ enjoys some elegant properties, such as the reflexivity and the separability, which no longer hold true in $L^\infty(\mathcal{X})$. Thus, many studies related to $L^p(\mathcal{X})$ need some modifications when $p = \infty$: for instance, the boundedness of Calderón–Zygmund operators. Recall that the Calderón–Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for any given $p \in (1,\infty)$, but not bounded on $L^\infty(\mathbb{R}^n)$. Indeed, T maps $L^\infty(\mathbb{R}^n)$ into the space $\mathrm{BMO}\,(\mathbb{R}^n)$ which was introduced by John and Nirenberg [1] in 1961 to study the functions of *bounded mean oscillation*; here and thereafter,

$$\mathrm{BMO}\,(\mathcal{X}) := \left\{ f \in L^1_{\rm loc}(\mathcal{X}) : \ \|f\|_{\mathrm{BMO}\,(\mathcal{X})} := \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left|f(x) - f_Q\right| dx < \infty \right\}$$

with

$$f_Q := \fint_Q f(y)\,dy := \frac{1}{|Q|}\int_Q f(y)\,dy$$

and the supremum taken over all cubes Q of $\mathcal{X}$. This implies that $\mathrm{BMO}\,(\mathcal{X})$ is a fine substitute of $L^\infty(\mathcal{X})$. Furthermore, it should be mentioned that, in the sense modulo constants, $\mathrm{BMO}\,(\mathcal{X})$ is a Banach space, but, for simplicity, we regard $f \in \mathrm{BMO}\,(\mathcal{X})$ as a function rather than an equivalent class $f + \mathbb{C} := \{f + c : \ c \in \mathbb{C}\}$ if there exists no confusion. Moreover, the space $\mathrm{BMO}\,(\mathcal{X})$ and its numerous variants as well as their vanishing subspaces have attracted a lot of attention since 1961. For instance, Fefferman and Stein [2] proved that the dual space of the Hardy space $H^1(\mathbb{R}^n)$ is $\mathrm{BMO}\,(\mathbb{R}^n)$; Coifman et al. [3] showed an equivalent characterization of the boundedness of Calderón–Zygmund commutators via $\mathrm{BMO}\,(\mathbb{R}^n)$; Coifman and Weiss [4,5] introduced the space of homogeneous type and studied the Hardy space and the BMO space in this context; Sarason [6] obtained the equivalent characterization of $\mathrm{VMO}\,(\mathbb{R}^n)$, the closure in $\mathrm{BMO}\,(\mathbb{R}^n)$ of uniformly continuous functions, and used it to study stationary stochastic processes satisfying the strong mixing condition and the algebra $H^\infty + C$; Uchiyama [7] established an equivalent characterization of the compactness of Calderón–Zygmund commutators via $\mathrm{CMO}\,(\mathbb{R}^n)$ which is defined to be the closure in $\mathrm{BMO}\,(\mathbb{R}^n)$ of infinitely differentiable functions on $\mathbb{R}^n$ with compact support; Nakai and Yabuta [8] studied pointwise multipliers for functions on $\mathbb{R}^n$ of bounded mean oscillation; and Iwaniec [9] used the compactness theorem in Uchiyama [7] to study linear complex Beltrami equations and the $L^p(\mathbb{C})$ theory of quasiregular mappings. All these classical results have wide generalizations as well as applications and have inspired a myriad of further studies in recent years: see, for instance, the References [10–13] for their applications in singular integral operators as well as their commutators, the References [14–19] for their applications in pointwise multipliers, the References [20–22] for their applications in partial differential equations, and the References [23–28] for more variants and properties of $\mathrm{BMO}\,(\mathbb{R}^n)$. In particular, we refer the reader to Chang and Sadosky [29] for an instructive survey on functions of bounded mean oscillation and also Chang et al. [25] for BMO spaces on the Lipschitz domain of $\mathbb{R}^n$.

Naturally, $\mathrm{BMO}\,(\mathcal{X})$ extends $L^\infty(\mathcal{X})$, in the sense that $L^\infty(\mathcal{X}) \subsetneqq \mathrm{BMO}\,(\mathcal{X})$ and, moreover, $\|\cdot\|_{\mathrm{BMO}\,(\mathcal{X})} \le 2\|\cdot\|_{L^\infty(\mathcal{X})}$. Similarly, such extension exists for any $L^p(\mathcal{X})$ with $p \in (1,\infty)$. Indeed, John and Nirenberg [1] also introduced a generalized version of the BMO condition which was subsequently used to define the so-called John–Nirenberg space $JN_p(Q_0)$ with exponent $p \in (1,\infty)$ and Q_0 being any given cube of $\mathbb{R}^n$. Recall that for any given $p \in (1,\infty)$ and any given cube Q_0 of $\mathbb{R}^n$, the *John–Nirenberg space* $JN_p(Q_0)$ is defined to be the set of all $f \in L^1(Q_0)$ such that

$$\|f\|_{JN_p(Q_0)} := \sup\left[\sum_i |Q_i|\left\{ ⨍_{Q_i} \left|f(x) - f_{Q_i}\right| dx\right\}^p\right]^{\frac{1}{p}} < \infty, \tag{1}$$

where the supremum is taken over all collections of *interior pairwise disjoint* cubes $\{Q_i\}_i$ of Q_0. It is easy to see that the limit of $JN_p(Q_0)$ when $p \to \infty$ is just $\mathrm{BMO}\,(Q_0)$ (see also Corollary 2 below). Moreover, the John–Nirenberg space is closely related to the Lebesgue space $L^p(Q_0)$ and the weak Lebesgue space $L^{p,\infty}(Q_0)$ which is defined in Definition 1 below. Precisely, let $p \in (1,\infty)$. On the one hand, the inequality obtained in ([1], Lemma 3) (see also Theorem 2 below) implies that $JN_p(Q_0) \subset L^{p,\infty}(Q_0)$; additionally, by ([30], Example 3.5), we further know that $JN_p(Q_0) \subsetneqq L^{p,\infty}(Q_0)$. On the other hand, it is obvious that $L^p(Q_0) \subset JN_p(Q_0)$ with $\|\cdot\|_{JN_p(Q_0)} \le 2\|\cdot\|_{L^p(Q_0)}$, but the striking nontriviality was shown very recently by Dafni et al. ([31], Proposition 3.2 and Corollary 4.2), who say that $L^p(Q_0) \subsetneqq JN_p(Q_0)$. Combining these facts, we conclude that

$$L^p(Q_0) \subsetneqq JN_p(Q_0) \subsetneqq L^{p,\infty}(Q_0). \tag{2}$$

Therefore, John–Nirenberg spaces are new spaces between Lebesgue spaces and weak Lebesgue spaces, which motivates us to study the properties of JN_p. Furthermore, various John–Nirenberg-type spaces have also attracted a lot of attention in recent years (see, for instance, [31–37] for the Euclidean space case and [30,38–40] for the metric measure space case).

It should be mentioned that the mean oscillation truly makes a difference in both BMO and JN_p; for instance,

(i) Via the characterization of distribution functions, we know that BMO is closely related to the space $L_{\exp}$ whose definition (see (6) below) is similar to an equivalent expression of BMO but with $f - f_Q$ replaced by f (see Proposition 3 below);

(ii) There exists an interesting observation presented by Riesz [41], which says that in (1), if we replace $f - f_{Q_i}$ by f, then $JN_p(Q_0)$ turns to be $L^p(Q_0)$. Moreover, this conclusion also holds true when Q_0 is replaced by $\mathbb{R}^n$ (see Proposition 28 below).

The main purpose of this article is to give a survey on some recent developments of both the John–Nirenberg space JN_p and the space BMO, including their several generalized (or related) spaces and some vanishing subspaces. We begin in Section 2 by recalling some definitions and basic properties of BMO and JN_p. Section 3 summarizes some recent developments of the John–Nirenberg–Campanato space, the localized John–Nirenberg–Campanato space, and the special John–Nirenberg–Campanato space via congruent cubes. Section 4 focuses on the Riesz-type space, which differs from the John–Nirenberg space in subtracting integral means, and its congruent counterpart. In Section 5, we pay attention to some vanishing subspaces of the aforementioned John–Nirenberg-type spaces, such as VMO, XMO, CMO, VJN_p, and CJN_p on $\mathbb{R}^n$ or any given cube Q_0 of $\mathbb{R}^n$. In addition, several related open questions are also summarized in this survey.

More precisely, the remainder of this survey is organized as follows.

Section 2 is split into two subsections. In Section 2.1, via recalling the definitions of distribution functions and some related function spaces (including the weak Lebesgue space, the Morrey space, and the space $L_{\exp}$), we present the relation

$$L^\infty(Q_0) \subsetneqq \mathrm{BMO}\,(Q_0) \subsetneqq L_{\exp}(Q_0)$$

in Proposition 2 below, which is a counterpart of (2) above, and also show two equivalent Orlicz-type norms on $\mathrm{BMO}\,(\mathbb{R}^n)$ in Proposition 3 below; moreover, the corresponding results for the localized BMO space are also obtained in Corollary 1 below. Section 2.2 is devoted to some significant results of JN_p, including the famous John–Nirenberg inequality (see Theorem 2 below), and the accurate relations of JN_p and L^p as well as $L^{p,\infty}$ (see Remark 2 below). Furthermore, some recent progress of JN_p is also briefly listed at the end of this subsection.

Section 3 is split into three subsections. In Section 3.1, we first recall the notions of the John–Nirenberg–Campanato space (for short, JNC space), the corresponding Hardy-type space, and their basic properties, which include the limit results and the relations with other classical spaces. Then we review the dual theorem between these two spaces and the independence over the second sub-index of JNC spaces and Hardy-type spaces. Section 3.2 is devoted to the localized counterpart of Section 3.1. The aim of Section 3.3 is the summary of the special JNC space defined via congruent cubes (for short, congruent JNC space), including their basic properties corresponding to those in Section 3.1. Furthermore, some applications about the boundedness of operators on congruent spaces are mentioned as well.

In Section 4, via subtracting integral means in the JNC space, we first give the definition of the Riesz-type space appearing in [37] and then present some basic facts about this space in Section 4.1. Moreover, the predual space (namely, the block-type space) and the corresponding dual theorem of the Riesz-type space are also displayed in this subsection. Section 4.2 is devoted to the congruent counterpart of the Riesz-type space and the boundedness of some important operators.

Section 5 is split into three subsections. Section 5.1 is devoted to several vanishing subspaces of $\mathrm{BMO}\,(\mathbb{R}^n)$, including $\mathrm{VMO}\,(\mathbb{R}^n)$, $\mathrm{CMO}\,(\mathbb{R}^n)$, $\mathrm{MMO}\,(\mathbb{R}^n)$, $\mathrm{XMO}\,(\mathbb{R}^n)$, and $\mathrm{X_1MO}\,(\mathbb{R}^n)$. We first recall their definitions and then review their (except $\mathrm{MMO}\,(\mathbb{R}^n)$) mean oscillation characterizations, respectively, in Theorems 11–13 below. Meanwhile, an open question on the corresponding equivalent characterization of $\mathrm{MMO}\,(\mathbb{R}^n)$ is also listed in Question 11 below. Then, we further review the compactness theorems of the Calderón–Zygmund commutators $[b, T]$, where b belongs to the vanishing subspaces $\mathrm{CMO}\,(\mathbb{R}^n)$ as well as $\mathrm{XMO}\,(\mathbb{R}^n)$, and propose an open question on $[b, T]$ with

$b \in \mathrm{XMO}(\mathbb{R}^n)$. Moreover, the characterizations via Riesz transforms of $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{VMO}(\mathbb{R}^n)$, and $\mathrm{CMO}(\mathbb{R}^n)$, as well as the localized results of these vanishing subspaces, are presented. Furthermore, some open questions are listed in this subsection. Section 5.2 devotes to the vanishing subspaces of JNC spaces. We first recall the definition of the vanishing JNC space on cubes in Definition 17 and then review its equivalent characterization as well as its dual result, respectively, in Theorems 19 and 20. Moreover, for the case of $\mathbb{R}^n$, we review the corresponding results for $VJN_p(\mathbb{R}^n)$ and $CJN_p(\mathbb{R}^n)$, which are, respectively, counterparts of $\mathrm{VMO}(\mathbb{R}^n)$ and $\mathrm{CMO}(\mathbb{R}^n)$ (see Theorems 21 and 22 below). As before, some open questions are also listed at the end of this subsection. Section 5.3 is devoted to the congruent counterpart of Section 5.2, and some similar conclusions are listed in this subsection; meanwhile, some open questions on the JNC space have affirmative answers in the congruent setting (see Proposition 32 below).

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$. We always denote by C and $\widetilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. Moreover, we use $C_{(\gamma,\, \beta,\, \ldots)}$ to denote a positive constant depending on the indicated parameters γ, $\beta, \ldots$ Constants with subscripts, such as C_0 and A_1, do not change in different occurrences. Moreover, the symbol $f \lesssim g$ represents that $f \le Cg$ for some positive constant C. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. For any $p \in [1, \infty]$, let p' be its *conjugate index*, that is, p' satisfies $1/p + 1/p' = 1$. We use $\mathbf{1}_E$ to denote the *characteristic function* of a set $E \subset \mathbb{R}^n$, $|E|$ to denote the *Lebesgue measure* when $E \subset \mathbb{R}^n$ is measurable, and $\mathbf{0}$ to denote the *origin* of $\mathbb{R}^n$. For any function f on $\mathbb{R}^n$, let $\mathrm{supp}\,(f) := \{x \in \mathbb{R}^n : f(x) \neq 0\}$. Let $\mathbb{X}$ be a normed linear space. We use $(\mathbb{X})^*$ to denote its dual space.

2. BMO and JN_p

It is well known that the space BMO has played an important role in harmonic analysis, partial differential equations, and other mathematical fields since it was introduced by John and Nirenberg in their celebrated article [1]. However, in the same article [1], another mysterious space appeared as well, which is now called the John–Nirenberg space JN_p. Indeed, BMO can be viewed as the limit space of JN_p as $p \to \infty$ (see Proposition 6 and Corollary 2 below with $\alpha := 0$). To establish the relations of BMO and JN_p, and also to summarize some recent works of John–Nirenberg-type spaces, we first recall some basic properties of BMO and JN_p in this section.

This section is devoted to some well-known results of $\mathrm{BMO}(\mathcal{X})$ and $JN_p(\mathcal{X})$, respectively, in Sections 2.1 and 2.2. In addition, it is trivial to find that all the results in Section 2.1 also hold true with the cube Q_0 replaced by the ball B_0 of $\mathbb{R}^n$.

2.1. *(Localized) BMO and* $L_{\exp}$

This subsection is devoted to several equivalent norms of the spaces BMO and localized BMO. To this end, we begin with the *distribution function*

$$\mathfrak{D}(f;\mathcal{X})(t) := |\{x \in \mathcal{X} : |f(x)| > t\}|, \tag{3}$$

where $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $t \in (0, \infty)$. Recall that the distribution function is closely related to the following weak Lebesgue space.

Definition 1. *Let $p \in (0, \infty)$. The* weak Lebesgue space $L^{p,\infty}(\mathcal{X})$ *is defined by setting*

$$L^{p,\infty}(\mathcal{X}) := \left\{ f \text{ is measurable on } \mathcal{X} : \|f\|_{L^{p,\infty}(\mathcal{X})} < \infty \right\},$$

where, for any measurable function f on $\mathcal{X}$,

$$\|f\|_{L^{p,\infty}(\mathcal{X})} := \sup_{t \in (0,\infty)} \left[t |\{x \in \mathcal{X} : |f(x)| > t\}|^{\frac{1}{p}} \right].$$

Moreover, the distribution function also features $\mathrm{BMO}\,(\mathcal{X})$, which is exactly the famous result obtained by John and Nirenberg ([1], Lemma 1′): there exist positive constants C_1 and C_2, depending only on the dimension n, such that, for any given $f \in \mathrm{BMO}\,(\mathcal{X})$, any given cube $Q \subset \mathcal{X}$, and any $t \in (0, \infty)$,

$$\left|\left\{x \in Q : |f(x) - f_Q| > t\right\}\right| \le C_1 e^{-\frac{C_2}{\|f\|_{\mathrm{BMO}\,(\mathcal{X})}}t}|Q|. \tag{4}$$

The main tool used in the proof of (4) is the following well-known *Calderón–Zygmund decomposition* (see, for instance, [42], p. 34, Theorem 2.11, and also [43], p. 150, Lemma 1).

Theorem 1. *For a given function f which is integrable and non-negative on $\mathcal{X}$, and a given positive number λ, there exists a sequence $\{Q_j\}_j$ of disjoint dyadic cubes of $\mathcal{X}$ such that*

(i) $f(x) \le \lambda$ *for almost every* $x \in \mathcal{X} \setminus \bigcup_j Q_j$;

(ii) $|\bigcup_j Q_j| \le \frac{1}{\lambda}\|f\|_{L^1(\mathcal{X})}$;

(iii) $\lambda < \fint_{Q_j} f(x)\,dx \le 2^n\lambda$.

As an application of (4), we find that for any given $q \in (1, \infty)$, $f \in \mathrm{BMO}\,(\mathbb{R}^n)$ if and only if $f \in L^1_{\mathrm{loc}}\,(\mathbb{R}^n)$ and

$$\|f\|_{\mathrm{BMO}_q(\mathbb{R}^n)} := \sup_{\text{cube } Q \subset \mathbb{R}^n}\left[\fint_Q \left|f(x) - f_Q\right|^q dx\right]^{\frac{1}{q}} < \infty.$$

Meanwhile, $\|\cdot\|_{\mathrm{BMO}\,(\mathbb{R}^n)} \sim \|\cdot\|_{\mathrm{BMO}_q(\mathbb{R}^n)}$ (see, for instance, [42], p. 125, Corollary 6.12).

Recently, Bényi et al. [44] gave a comprehensive approach for the boundedness of weighted commutators via a new equivalent Orlicz-type norm

$$\|f\|_{\mathcal{BMO}(\mathcal{X})} := \sup_{\text{cube } Q \subset \mathcal{X}} \|f - f_Q\|_{L_{\exp}(Q)}. \tag{5}$$

This equivalence is proved in Proposition 3 below. Here and thereafter, for any given cube Q of $\mathbb{R}^n$ and any measurable function g, the *locally normalized Orlicz norm* $\|g\|_{L_{\exp}(Q)}$ is defined by setting

$$\|g\|_{L_{\exp}(Q)} := \inf\left\{\lambda \in (0, \infty) : \fint_Q \left[e^{\frac{|g(x)|}{\lambda}} - 1\right] dx \le 1\right\}. \tag{6}$$

Moreover, for any given cube Q of $\mathbb{R}^n$, the *space* $L_{\exp}(Q)$ is defined by setting

$$L_{\exp}(Q) := \left\{f \text{ is measurable on } Q : \exists\, \lambda \in (0, \infty) \text{ such that } \fint_Q e^{\frac{|f(x)|}{\lambda}}\,dx < \infty\right\}.$$

The space $L_{\exp}(Q)$ was studied in the interpolation of operators (see, for instance, [45], p. 243), and it is closely related to the space $\mathrm{BMO}\,(Q)$ (see Proposition 3 below).

On the Orlicz function in (6), we have the following properties.

Lemma 1. *For any $t \in [0, \infty)$, let $\Phi(t) := e^t - 1$. Then,*

(i) Φ *is of* lower type 1, *namely for any $s \in (0, 1)$ and $t \in (0, \infty)$,*

$$\Phi(st) \le s\Phi(t);$$

(ii) Φ *is of* critical lower type 1, *namely there exists no $p \in (1, \infty)$, such that for any $s \in (0, 1)$ and $t \in (0, \infty)$,*

$$\Phi(st) \le Cs^p\Phi(t)$$

holds true for some constant $C \in [1, \infty)$ independent of s and t.

Proof. We first show (i). For any $s \in (0,1)$ and $t \in (0,\infty)$, let

$$h(s,t) := \Phi(st) - s\Phi(t) = e^{st} - 1 - s(e^t - 1).$$

Then,

$$\frac{\partial}{\partial t} h(s,t) = se^{st} - se^t = s(e^{st} - e^t).$$

From this and $s \in (0,1)$, we deduce that for any $t \in (0,\infty)$, $\frac{\partial}{\partial t} h(s,t) < 0$, and hence $h(s,t) \le h(s,0) = 0$, which shows that Φ is of lower type 1 and hence completes the proof of (i).

Next, we show that Φ is of critical lower type 1. Suppose that there exist a $p \in (1,\infty)$ and a constant $C \in [1,\infty)$, such that for any $s \in (0,1)$ and $t \in (0,\infty)$, $\Phi(st) \le Cs^p\Phi(t)$, namely

$$e^{st} - 1 \le Cs^p(e^t - 1). \tag{7}$$

From $p \in (1,\infty)$ and the L'Hospital rule, we deduce that

$$\lim_{s\to 0^+} \frac{\Phi(st)}{s^p\Phi(t)} = \lim_{s\to 0^+} \frac{e^{st} - 1}{s^p(e^t - 1)} = \lim_{s\to 0^+} \frac{te^{st}}{ps^{p-1}(e^t - 1)} = \infty,$$

which contradicts (7), and hence Φ is of critical lower type 1. Here and thereafter, $s \to 0^+$ means $s \in (0,1)$ and $s \to 0$. This finishes the proof of (ii) and hence of Lemma 1. □

Before showing the equivalent Orlicz-type norms of $\mathrm{BMO}\,(\mathcal{X})$, we first prove the following equivalent characterizations of $\mathrm{BMO}\,(\mathcal{X})$. These characterizations might be well known. However, to the best of our knowledge, we did not find a complete proof. For the convenience of the reader, we present the details here.

Proposition 1. *The following three statements are mutually equivalent:*

(i) $f \in \mathrm{BMO}\,(\mathcal{X})$;

(ii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and there exist positive constants* C_3 *and* C_4*, such that for any cube* $Q \subset \mathcal{X}$ *and any* $t \in (0,\infty)$*,*

$$\left|\left\{x \in Q : |f(x) - f_Q| > t\right\}\right| \le C_3 e^{-C_4 t}|Q|;$$

(iii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and there exists a* $\lambda \in (0,\infty)$*, such that*

$$\sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q e^{\frac{|f(x)-f_Q|}{\lambda}}\,dx < \infty.$$

Proof. We prove this proposition via showing (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

First, the implication (i) $\Longrightarrow$ (ii) was proved by John and Nirenberg in [1], Lemma 1′ (see (4) above).

Next, we show the implication (ii) $\Longrightarrow$ (iii). Suppose that f satisfies (ii). Then, there exist positive constants C_3 and C_4, such that for any cube $Q \subset \mathcal{X}$ and any $t \in (0,\infty)$,

$$\left|\left\{x \in Q : |f(x) - f_Q| > t\right\}\right| \le C_3 e^{-C_4 t}|Q|$$

and hence

$$\begin{aligned}&\fint_Q e^{\frac{C_4}{2}|f(x)-f_Q|}\,dx \\ &\quad = \frac{1}{|Q|}\int_0^\infty \left|\left\{x \in Q : e^{\frac{C_4}{2}|f(x)-f_Q|} > t\right\}\right| dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|Q|}\left(\int_0^1 + \int_1^\infty\right)\left|\left\{x \in Q:\ e^{\frac{C_4}{2}|f(x)-f_Q|} > t\right\}\right| dt \\
&\le 1 + \frac{1}{|Q|}\int_1^\infty \left|\left\{x \in Q:\ |f(x)-f_Q| > 2C_4^{-1}\log t\right\}\right| dt \\
&\le 1 + \frac{1}{|Q|}\int_1^\infty C_3 e^{-C_4 2C_4^{-1}\log t}|Q|\, dt \\
&= 1 + C_3\int_1^\infty t^{-2}\, dt = 1 + C_3,
\end{aligned} \tag{8}
$$

which implies that f satisfies (iii). This shows the implication (ii) $\Longrightarrow$ (iii).

Finally, we show the implication (iii) $\Longrightarrow$ (i). Suppose that f satisfies (iii). Then, there exists a $\lambda \in (0,\infty)$, such that

$$\sup_{Q\subset\mathcal{X}} \fint_Q e^{\frac{|f(x)-f_Q|}{\lambda}}\, dx < \infty.$$

From this and the basic inequality $x \le e^x - 1$ for any $x \in \mathbb{R}$, we deduce that

$$\sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left|f(x)-f_Q\right| dx \le \lambda \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx < \infty,$$

which implies that f satisfies (i), and hence the implication (iii) $\Longrightarrow$ (i) holds true. This finishes the proof of Proposition 1. □

In what follows, for any normed space $\mathbb{Y}(\mathcal{X})$, equipped with the norm $\|\cdot\|_{\mathbb{Y}(\mathcal{X})}$, whose elements are measurable functions on $\mathcal{X}$, let

$$\mathbb{Y}(\mathcal{X})/\mathbb{C} := \left\{f \text{ is measurable on } \mathcal{X}:\ \|f\|_{\mathbb{Y}(\mathcal{X})/\mathbb{C}} := \inf_{c\in\mathbb{C}} \|f+c\|_{\mathbb{Y}(\mathcal{X})} < \infty\right\}.$$

Proposition 2. *Let Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$[L^\infty(Q_0)/\mathbb{C}] \subsetneqq \mathrm{BMO}\,(Q_0) \subsetneqq \left[L_{\exp}(Q_0)/\mathbb{C}\right].$$

Proof. Indeed, on the one hand, from

$$\fint_Q \left|f(x)-f_Q\right| dx \le 2\fint_Q \left|f(x)+c\right| dx \le 2\|f+c\|_{L^\infty(Q_0)}$$

for any $c \in \mathbb{C}$, we deduce that $[L^\infty(Q_0)/\mathbb{C}] \subset \mathrm{BMO}\,(Q_0)$. Moreover, let $g(\cdot) := \log|\cdot - c_0|$, where c_0 is the center of Q_0. Then, $g \in \mathrm{BMO}\,(Q_0) \setminus [L^\infty(Q_0)/\mathbb{C}]$ (see [46], Example 3.1.3, for this fact).

On the other hand, by Proposition 1(iii), we easily find that $\mathrm{BMO}\,(Q_0) \subset [L_{\exp}(Q_0)/\mathbb{C}]$. Moreover, without loss of generality, we may assume that $Q_0 := (-1,1)$ and let

$$g(x) := \begin{cases} -\log(-x), & x \in (-1,0), \\ 0, & x = 0, \\ \log(x), & x \in (0,1). \end{cases}$$

We claim that $g \in [L_{\exp}(Q_0)/\mathbb{C}] \setminus \mathrm{BMO}\,(Q_0)$. Indeed, for any $\epsilon \in (0,1)$, let $I_\epsilon := (-\epsilon,\epsilon)$. Then,

$$\fint_{I_\epsilon} \left|g(x) - g_{I_\epsilon}\right| dx = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \left|\log|x|\right| dx = -\frac{1}{\epsilon}\int_0^\epsilon \log(x)\, dx = 1 - \log(\epsilon) \to \infty$$

as $\epsilon \to 0^+$, which implies that $g \notin \mathrm{BMO}(Q_0)$. However,

$$\int_{Q_0} e^{\frac{1}{2}|g(x)|}\,dx = 2\int_0^1 e^{-\frac{1}{2}\log(x)}\,dx = 2\int_0^1 x^{-\frac{1}{2}}\,dx = 4 < \infty,$$

which implies that $g \in L_{\exp}(Q_0)$. Therefore, $\mathrm{BMO}(Q_0) \subsetneqq [L_{\exp}(Q_0)/\mathbb{C}]$, which completes the proof of Proposition 2. □

Now, we show that the two Orlicz-type norms, (5) and

$$\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} := \inf\left\{\lambda \in (0,\infty) : \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx \le 1\right\}$$

for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, are equivalent norms of $\mathrm{BMO}(\mathcal{X})$.

Proposition 3. *The following three statements are mutually equivalent:*
(i) $f \in \mathrm{BMO}(\mathcal{X})$;
(ii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $\|f\|_{\mathcal{BMO}(\mathcal{X})} < \infty$;
(iii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} < \infty$.

Moreover, $\|\cdot\|_{\mathrm{BMO}(\mathcal{X})} \sim \|\cdot\|_{\mathcal{BMO}(\mathcal{X})} \sim \|\cdot\|_{\widetilde{L_{\exp}}(\mathcal{X})}$.

Proof. To prove this proposition, we only need to prove that for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$,

$$\|f\|_{\mathrm{BMO}(\mathcal{X})} \sim \|f\|_{\mathcal{BMO}(\mathcal{X})} \sim \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}.$$

We first show that for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, $\|f\|_{\mathrm{BMO}(\mathcal{X})} \le \|f\|_{\mathcal{BMO}(\mathcal{X})}$ and $\|f\|_{\mathrm{BMO}(\mathcal{X})} \le \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}$. To this end, let $f \in L^1_{\mathrm{loc}}(\mathcal{X})$. For any cube $Q \subset \mathcal{X}$ and any $\lambda \in (0,\infty)$, by $t \le e^t - 1$ for any $t \in (0,\infty)$, we have

$$\fint_Q \left|f(x) - f_Q\right| dx \le \lambda \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx \le \lambda,$$

which implies that

$$\fint_Q \left|f(x) - f_Q\right| dx \le \|f - f_Q\|_{L_{\exp}(Q)}$$

and hence

$$\|f\|_{\mathrm{BMO}(\mathcal{X})} \le \|f\|_{\mathcal{BMO}(\mathcal{X})}.$$

Moreover, to show $\|f\|_{\mathrm{BMO}(\mathcal{X})} \le \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}$, it suffices to assume that $f \in \widetilde{L_{\exp}}(\mathcal{X})$; otherwise, $\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} = \infty$, and hence the desired inequality automatically holds true. Then, by $t \le e^t - 1$ for any $t \in (0,\infty)$, we conclude that for any $n \in \mathbb{N}$ and any cube $Q \subset \mathcal{X}$,

$$\fint_Q \frac{|f(x) - f_Q|}{\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} + \frac{1}{n}}\,dx \le \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} + \frac{1}{n}}} - 1\right] dx. \tag{9}$$

From the definition of $\|\cdot\|_{\widetilde{L_{\exp}}(\mathcal{X})}$, we deduce that for any $n \in \mathbb{N}$, there exists a

$$\lambda_n \in \left(\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}, \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} + \frac{1}{n}\right)$$

such that

$$\sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda_n}} - 1\right] dx \le 1.$$

By this, (9), and the monotonicity of $e^{(\cdot)} - 1$, we conclude that, for any $n \in \mathbb{N}$ and any cube $Q \subset \mathcal{X}$,

$$\fint_Q \frac{|f(x) - f_Q|}{\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} + \frac{1}{n}} \, dx \le 1$$

and hence

$$\fint_Q |f(x) - f_Q| \, dx \le \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} + \frac{1}{n}.$$

Letting $n \to \infty$, we then obtain

$$\|f\|_{\mathrm{BMO}\,(\mathcal{X})} = \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q |f(x) - f_Q| \, dx \le \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}.$$

To summarize, we have, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$,

$$\|f\|_{\mathrm{BMO}\,(\mathcal{X})} \le \|f\|_{\mathcal{BMO}(\mathcal{X})} \text{ and } \|f\|_{\mathrm{BMO}\,(\mathcal{X})} \le \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})}. \tag{10}$$

Next, we show that the reverse inequalities hold true for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, respectively. In fact, we may assume that $f \in \mathrm{BMO}\,(\mathcal{X})$ because, otherwise, the desired inequalities automatically hold true. Now, let $f \in \mathrm{BMO}\,(\mathcal{X})$. Then, for any cube $Q \subset \mathcal{X}$ and any $\lambda \in (C_2^{-1}\|f\|_{\mathrm{BMO}\,(\mathcal{X})}, \infty)$, by (4) and the calculation of (8), we obtain

$$\begin{aligned}
&\fint_Q e^{\frac{|f(x)-f_Q|}{\lambda}} \, dx \\
&\quad \le 1 + \frac{1}{|Q|} \int_1^\infty \left|\left\{x \in Q : \ |f(x) - f_Q| > \lambda \log t\right\}\right| dt \\
&\quad \le 1 + \frac{1}{|Q|} \int_1^\infty C_1 e^{-\frac{C_2}{\|f\|_{\mathrm{BMO}\,(\mathcal{X})}} \lambda \log t} |Q| \, dt \\
&\quad = 1 + C_1 \int_1^\infty t^{-\frac{C_2\lambda}{\|f\|_{\mathrm{BMO}\,(\mathcal{X})}}} \, dt = 1 + C_1
\end{aligned}$$

and hence

$$\fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx \le C_1,$$

where $C_1 \in (1, \infty)$ is as in (4). From this and Lemma 1(i) with s replaced by $1/C_1$, we deduce that

$$\fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda C_1}} - 1\right] dx \le \frac{1}{C_1} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx \le 1. \tag{11}$$

On the one hand, by (11) and

$$\frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})} < \lambda C_1 < \infty,$$

we conclude that

$$\|f - f_Q\|_{L_{\exp}(Q)} = \inf\left\{\widetilde{\lambda} > 0 : \ \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\widetilde{\lambda}}} - 1\right] dx \le 1\right\}$$

$$\leq \frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})}$$

and hence

$$\|f\|_{\mathcal{BMO}(\mathcal{X})} = \sup_{\text{cube } Q\subset\mathcal{X}} \|f - f_Q\|_{L_{\exp}(Q)} \leq \frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})}. \tag{12}$$

On the other hand, by (11), we conclude that

$$\sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda C_1}} - 1\right] dx \leq 1.$$

From this and

$$\frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})} < \lambda C_1 < \infty,$$

we deduce that

$$\begin{aligned}\|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} &= \inf\left\{\lambda \in (0,\infty) : \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_Q|}{\lambda}} - 1\right] dx \leq 1\right\}\\ &\leq \frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})}.\end{aligned}$$

Combining this with (12), we have, for any $f \in \mathrm{BMO}\,(\mathcal{X})$,

$$\|f\|_{\mathcal{BMO}(\mathcal{X})} \leq \frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})} \text{ and } \|f\|_{\widetilde{L_{\exp}}(\mathcal{X})} \leq \frac{C_1}{C_2}\|f\|_{\mathrm{BMO}\,(\mathcal{X})}.$$

This, together with (10), then finishes the proof of Proposition 3. □

Remark 1. *There exists another norm on $L_{\exp}(Q_0)$, defined by the distribution functions as follows. Let f be a measurable function on Q_0. The* decreasing rearrangement *f^* of f is defined by setting, for any $u \in [0,\infty)$,*

$$f^*(u) := \inf\{t \in (0,\infty) : |\{x \in Q_0 : |f(x)| > t\}| \leq u\}.$$

Moreover, for any $v \in (0,\infty)$, let

$$f^{**}(v) := \frac{1}{v}\int_0^v f^*(u)\,du.$$

Then, $f \in L_{\exp}(Q_0)$ if and only if f is measurable on Q_0 and

$$\|f\|_{L^*_{\exp}(Q_0)} := \sup_{v\in(0,|Q_0|]} \frac{f^{**}(v)}{1+\log(\frac{|Q_0|}{v})} < \infty.$$

Meanwhile, $\|\cdot\|_{L^*_{\exp}(Q_0)}$ is a norm of $L_{\exp}(Q_0)$ (see [45], p. 246, Theorem 6.4, for more details). Furthermore, from [45] (p. 7, Corollary 1.9), we deduce that $\|\cdot\|_{L^*_{\exp}(Q_0)}$ and $\|\cdot\|_{L_{\exp}(Q_0)}$ are equivalent. Notice that f^* and f^{**} are fundamental tools in the theory of Lorentz spaces *(see [47], p. 48, for more details).*

Recently, Izuki et al. [48] obtained both the John–Nirenberg inequality and the equivalent characterization of $\mathrm{BMO}\,(\mathbb{R}^n)$ on the ball Banach function space which contains Morrey spaces, (weighted, mixed-norm, variable) Lebesgue spaces, and Orlicz-slice spaces as special cases (see [48], Definition 2.8, and also [49], for the related definitions). Precisely, let X be a ball Banach function space satisfying the additional assumption that the Hardy–

Littlewood maximal operator M is bounded on X' (the associated space of X; see [48], Definition 2.9, for its definition), and for any $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|b\|_{\mathrm{BMO}_X} := \sup_B \frac{1}{\|\mathbf{1}_B\|_X} \left\| |b - b_B| \mathbf{1}_B \right\|_{X'},$$

where the supremum is taken over all balls B of $\mathbb{R}^n$. It is obvious that $\|\cdot\|_{\mathrm{BMO}_{L^1(\mathbb{R}^n)}} = \|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Moreover, in [48] (Theorem 1.2), Izuki et al. showed that under the above assumption of X, $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\|b\|_{\mathrm{BMO}_X} < \infty$; meanwhile,

$$\|\cdot\|_{\mathrm{BMO}_X} \sim \|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Furthermore, the John–Nirenberg inequality on X was also obtained in [48] (Theorem 3.1), which shows that there exists some positive constant $\widetilde{C}$, such that for any ball $B \subset \mathbb{R}^n$ and any $\tau \in [0, \infty)$,

$$\left\| \mathbf{1}_{\{x \in B:\ |b(x) - b_B| > \tau 2^{n+2} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\}} \right\|_X \le \widetilde{C} 2^{-\frac{\tau}{1 + 2^{n+4} \|M\|_{X' \to X'}}} \|\mathbf{1}_B\|_X,$$

where $\|M\|_{X' \to X'}$ denotes the *operator norm* of M on X'. Later, these results were applied in [49] to establish the compactness characterization of commutators on ball Banach function spaces.

Now, we come to the localized counterpart. The local space $\mathrm{BMO}(\mathbb{R}^n)$, denoted by $\mathrm{bmo}(\mathbb{R}^n)$, was originally introduced by Goldberg [50]. In the same article, Goldberg also introduced the localized Campanato space $\Lambda_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, \infty)$, which proves the dual space of the localized Hardy space. Later, Jonsson et al. [51] constructed the localized Hardy space and the localized Campanato space on the subset of $\mathbb{R}^n$; Chang [52] studied the localized Campanato space on bounded Lipschitz domains; Chang et al. [20] studied the localized Hardy space and its dual space on smooth domains as well as their applications to boundary value problems; and Dafni and Liflyand [53] characterized the localized Hardy space in the sense of Goldberg, respectively, by means of the localized Hilbert transform and localized molecules. In what follows, for any cube Q of $\mathbb{R}^n$, we use $\ell(Q)$ to denote its side length, and let $\ell(\mathbb{R}^n) := \infty$. Recall that

$$\mathrm{bmo}(\mathcal{X}) := \left\{ f \in L^1_{\mathrm{loc}}(\mathcal{X}) :\ \|f\|_{\mathrm{bmo}(\mathcal{X})} < \infty \right\},$$

where

$$\|f\|_{\mathrm{bmo}(\mathcal{X})} := \sup_Q ⨍_Q \left| f(x) - f_{Q,c_0} \right| dx$$

with

$$f_{Q,c_0} := \begin{cases} f_Q & \text{if } \ell(Q) \in (0, c_0), \\ 0 & \text{if } \ell(Q) \in [c_0, \ell(\mathcal{X})) \end{cases} \tag{13}$$

for some given $c_0 \in (0, \ell(\mathcal{X}))$, and the supremum taken over all cubes Q of $\mathcal{X}$. Furthermore, a well-known fact is that $\mathrm{bmo}(\mathcal{X})$ is independent of the choice of c_0 (see, for instance, [54], Lemma 6.1).

Proposition 4. *Let $\mathcal{X}$ be $\mathbb{R}^n$ or a cube Q_0 of $\mathbb{R}^n$. Then,*

$$[L^\infty(\mathcal{X})/\mathbb{C}] \subset [\mathrm{bmo}(\mathcal{X})/\mathbb{C}] \subset \mathrm{BMO}(\mathcal{X}) \tag{14}$$

and

$$\|\cdot\|_{\mathrm{BMO}(\mathcal{X})} \le 2 \inf_{c \in \mathbb{C}} \|\cdot + c\|_{\mathrm{bmo}(\mathcal{X})} \le 4 \inf_{c \in \mathbb{C}} \|\cdot + c\|_{L^\infty(\mathcal{X})}. \tag{15}$$

Moreover,

$$[L^\infty(\mathbb{R}^n)/\mathbb{C}] \subsetneqq [\mathrm{bmo}\,(\mathbb{R}^n)/\mathbb{C}] \subsetneqq \mathrm{BMO}\,(\mathbb{R}^n) \tag{16}$$

and, for any cube Q_0 of $\mathbb{R}^n$,

$$[L^\infty(Q_0)/\mathbb{C}] \subsetneqq [\mathrm{bmo}\,(Q_0)/\mathbb{C}] = \mathrm{BMO}\,(Q_0) \subsetneqq \left[L_{\exp}(Q_0)/\mathbb{C}\right] \tag{17}$$

with

$$\|\cdot\|_{\mathrm{BMO}\,(Q_0)} \le 2\inf_{c\in\mathbb{C}} \|\cdot + c\|_{\mathrm{bmo}\,(Q_0)} \le 4\|\cdot\|_{\mathrm{BMO}\,(Q_0)}.$$

Proof. First, we prove (15). To this end, let $f \in L^1_{\mathrm{loc}}(\mathcal{X})$. Then, for any $c \in \mathbb{C}$ and any cube Q of $\mathcal{X}$,

$$\begin{aligned} \fint_Q \left|f(x) - f_Q\right| dx &= \fint_Q \left|[f(x) + c] - (f + c)_Q\right| dx \\ &\le 2 \fint_Q \left|f(x) + c\right| dx \le 2\|f + c\|_{L^\infty(Q)}. \end{aligned}$$

From this and the definitions of $\|\cdot\|_{\mathrm{BMO}\,(\mathcal{X})}$ and $\|\cdot\|_{\mathrm{bmo}\,(\mathcal{X})}$, it follows that (15) holds true, which further implies (14).

We now show (16). Indeed, let

$$g_1(x) := \begin{cases} \log(|x|) & \text{if } x \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \\ 0 & \text{if } x = \mathbf{0}. \end{cases}$$

From [46] (Example 3.1.3), we deduce that $g_1 \in \mathrm{BMO}\,(\mathbb{R}^n)$. However, $g_1 \notin \mathrm{bmo}\,(\mathbb{R}^n)$ because, for any $M > \max\{c_0, 1\}$, by the sphere coordinate changing method, we have

$$\fint_{B(\mathbf{0},M)} \left|\log(|x|)\right| dx \sim \log(M),$$

which tends to infinity as $M \to \infty$. Thus, $g_1 \in \mathrm{BMO}\,(\mathbb{R}^n) \setminus [\mathrm{bmo}\,(\mathbb{R}^n)/\mathbb{C}]$, and hence we have $[\mathrm{bmo}\,(\mathbb{R}^n)/\mathbb{C}] \subsetneqq \mathrm{BMO}\,(\mathbb{R}^n)$. Moreover, define

$$g_2(x) := \begin{cases} \log(|x|) & \text{if } |x| \in (0,1), \\ 0 & \text{if } |x| \in \{0\} \bigcup [1,\infty). \end{cases}$$

Notice that $g_2 \notin L^\infty(\mathbb{R}^n)$ and $g_2 = \max\{g_1, 0\} \in \mathrm{BMO}\,(\mathbb{R}^n)$. Then, for any cube $Q \subset \mathbb{R}^n$, if $\ell(Q) \in (0, c_0)$, then

$$\fint_Q \left|g_2(x) - (g_2)_Q\right| dx \le \|g_2\|_{\mathrm{BMO}\,(\mathbb{R}^n)};$$

if $\ell(Q) \in [c_0, \infty)$, then

$$\fint_Q \left|g_2(x)\right| dx \le \fint_{B(\mathbf{0},1)} \log(|x|)\, dx \sim \|g_2\|_{L^1(\mathbb{R}^n)} \sim 1.$$

To summarize, $\|g_2\|_{\mathrm{bmo}\,(\mathbb{R}^n)} \lesssim 1 + \|g_2\|_{\mathrm{BMO}\,(\mathbb{R}^n)}$, which implies that $g_2 \in \mathrm{bmo}\,(\mathbb{R}^n)$ and hence $L^\infty(\mathbb{R}^n) \subsetneqq \mathrm{bmo}\,(\mathbb{R}^n)$. This shows (16).

We next prove (17). By the above example g_2, we conclude that $L^\infty(Q_0) \subsetneqq \mathrm{bmo}\,(Q_0)$. Meanwhile, $\mathrm{BMO}\,(Q_0) \subsetneqq [L_{\exp}(Q_0)/\mathbb{C}]$ was obtained in Proposition 2. Moreover, for any given $f \in \mathrm{BMO}\,(Q_0)$, we have $f \in L^1(Q_0)$ and hence

$$\inf_{c\in\mathbb{C}} \|f - c\|_{\mathrm{bmo}\,(Q_0)}$$

$$= \begin{cases} \fint_Q \left|f(x) - f_Q\right| dx \le \|f\|_{\mathrm{BMO}\,(Q_0)} & \text{if } \ell(Q) \in (0, c_0), \\ \inf_{c\in\mathbb{C}} \fint_Q |f(x) - c|\, dx \le 2\|f\|_{\mathrm{BMO}\,(Q_0)} & \text{if } \ell(Q) \in [c_0, \ell(Q_0)), \end{cases}$$
$$\le 2\|f\|_{\mathrm{BMO}\,(Q_0)}.$$

Combining this with the observations that $[\mathrm{bmo}\,(Q_0)/\mathbb{C}] \subset \mathrm{BMO}\,(Q_0)$ and that, for any $c \in \mathbb{C}$,

$$\|f\|_{\mathrm{BMO}\,(Q_0)} = \|f + c\|_{\mathrm{BMO}\,(Q_0)} \le 2\|f + c\|_{\mathrm{bmo}\,(Q_0)},$$

we find that $[\mathrm{bmo}\,(Q_0)/\mathbb{C}] = \mathrm{BMO}\,(Q_0)$ and

$$\|f\|_{\mathrm{BMO}\,(Q_0)} \le 2 \inf_{c\in\mathbb{C}} \|f + c\|_{\mathrm{bmo}\,(Q_0)} \le 4\|f\|_{\mathrm{BMO}\,(Q_0)}.$$

To summarize, we obtain (17). This finishes the proof of Proposition 4. □

Let $f \in L^1_{\mathrm{loc}}(\mathcal{X})$. Similar to Proposition 3, let

$$\|f\|_{\mathrm{bmo}_1(\mathcal{X})} := \sup_{\text{cube } Q\subset\mathcal{X}} \left\|f - f_{Q,c_0}\right\|_{L_{\exp}(Q)} \tag{18}$$

and

$$\|f\|_{\mathrm{bmo}_2(\mathcal{X})} := \inf\left\{\lambda \in (0,\infty) :\ \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left[e^{\frac{|f(x)-f_{Q,c_0}|}{\lambda}} - 1\right] dx \le 1\right\}, \tag{19}$$

where $c_0 \in (0, \ell(\mathcal{X}))$, and f_{Q,c_0} is as in (13). To show that they are equivalent norms of $\mathrm{bmo}\,(\mathcal{X})$, we first establish the following John–Nirenberg inequality for $\mathrm{bmo}\,(\mathcal{X})$, namely Proposition 5 below. In what follows, for any given cube Q of $\mathbb{R}^n$, $(a_1, \ldots, a_n)$ denotes the *left and lower vertex* of Q, which means that for any $(x_1, \ldots, x_n) \in Q$, $x_i \ge a_i$ for any $i \in \{1, \ldots, n\}$. Recall that for any given cube Q of $\mathbb{R}^n$, the *dyadic system* $\mathscr{D}_Q$ of Q is defined by setting

$$\mathscr{D}_Q := \bigcup_{j=0}^{\infty} \mathscr{D}_Q^{(j)}, \tag{20}$$

where, for any $j \in \{0, 1, \ldots\}$, $\mathscr{D}_Q^{(j)}$ denotes the set of all $(x_1, \ldots, x_n) \in Q$, such that for any $i \in \{1, \ldots, n\}$, either

$$x_i \in \left[a_i + k_i 2^{-j}\ell(Q), a_i + (k_i + 1)2^{-j}\ell(Q)\right)$$

for some $k_i \in \{0, 1, \ldots, 2^j - 2\}$ or

$$x_i \in \left[a_i + (1 - 2^{-j})\ell(Q), a_i + \ell(Q)\right].$$

Proposition 5. *Let $f \in \mathrm{bmo}\,(\mathcal{X})$ and $c_0 \in (0, \ell(\mathcal{X}))$. Then, there exist positive constants C_5 and C_6, such that for any given cube $Q \subset \mathcal{X}$ and any $t \in (0,\infty)$,*

$$\left|\left\{x \in Q :\ |f(x) - f_{Q,c_0}| > t\right\}\right| \le C_5 e^{-\frac{C_6}{\|f\|_{\mathrm{bmo}\,(\mathcal{X})}} t} |Q|. \tag{21}$$

Proof. Indeed, this proof is a slight modification of the proof of [1] (Lemma 1) or [42] (Theorem 6.11). We give some details here, again for the sake of completeness.

Let $f \in \mathrm{bmo}\,(\mathcal{X})$. Then, from Proposition 4, we deduce that $f \in \mathrm{BMO}\,(\mathcal{X})$ with $\|f\|_{\mathrm{BMO}\,(\mathcal{X})} \le 2\|f\|_{\mathrm{bmo}\,(\mathcal{X})}$, which further implies that for any cube $Q \subset \mathcal{X}$ with $\ell(Q) < c_0$ and any $t \in (0, \infty)$,

$$\mathfrak{D}\Big(f - f_{Q,c_0}; Q\Big)(t) = \mathfrak{D}\Big(f - f_Q; Q\Big)(t) \le C_1 e^{-\frac{C_2}{\|f\|_{\mathrm{BMO}\,(\mathcal{X})}}t}|Q|$$
$$\le C_1 e^{-\frac{C_2}{2\|f\|_{\mathrm{bmo}\,(\mathcal{X})}}t}|Q|,$$

where C_1 and C_2 are as in (4), and the distribution function $\mathfrak{D}$ is defined as in (3). Therefore, to show (21), it remains to prove that for any given cube Q with $\ell(Q) \ge c_0$, and any $t \in (0, \infty)$,

$$\Big|\{x \in Q : |f(x)| > t\}\Big| \le C_5 e^{-\frac{C_6}{\|f\|_{\mathrm{bmo}\,(\mathcal{X})}}t}|Q|.$$

Notice that, in this case, there exists a unique $m_0 \in \mathbb{Z}_+$ such that $2^{-(m_0+1)}\ell(Q) < c_0 \le 2^{-m_0}\ell(Q)$. Moreover, since inequality (21) is not altered when we multiply both f and t by the same constant, without loss of generality, we may assume that $\|f\|_{\mathrm{bmo}\,(\mathcal{X})} = 1$. Let Q_0 be any given dyadic subcube of Q with level m_0, namely $Q_0 \in \mathscr{D}_Q^{(m_0)}$. Then, by $c_0 \le 2^{-m_0}\ell(Q) = \ell(Q_0)$ and the definition of $\|f\|_{\mathrm{bmo}\,(\mathcal{X})}$, we have

$$\fint_{Q_0} |f(x)|\,dx \le \|f\|_{\mathrm{bmo}\,(\mathcal{X})} = 1. \tag{22}$$

From the Calderón–Zygmund decomposition (namely Theorem 1) of f with height $\lambda := 2$, we deduce that there exists a family $\{Q_{1,j}\}_j \subset \mathscr{D}_{Q_0}^{(1)}$, such that for any j,

$$2 < \fint_{Q_{1,j}} |f(x)|\,dx \le 2^{n+1}$$

and $|f(x)| \le 2$ when $x \in Q \setminus \bigcup_j Q_{1,j}$. By this and (22), we conclude that

$$\sum_j |Q_{1,j}| \le \frac{1}{2}\sum_j \int_{Q_{1,j}} |f(x)|\,dx \le \frac{1}{2}\int_{Q_0} |f(x)|\,dx \le \frac{1}{2}|Q_0|$$

and, for any j,

$$\Big|f_{Q_{1,j}}\Big| \le \left|\fint_{Q_{1,j}} f(x)\,dx\right| \le 2^{n+1}.$$

Moreover, for any j, from the Calderón–Zygmund decomposition of $f - f_{Q_{1,j}}$ with height 2, we deduce that there exists a family $\{Q_{1,j,k}\}_k \subset \mathscr{D}_{Q_{1,j}}^{(1)}$, such that for any k,

$$2 < \fint_{Q_{1,j,k}} |f(x) - f_{Q_{1,j}}|\,dx \le 2^{n+1}$$

and $|f(x) - f_{Q_{1,j}}| \le 2$ when $x \in Q \setminus \bigcup_k Q_{1,j,k}$. Meanwhile, by the construction of $\{Q_{1,j}\}_j$, we know that $\ell(Q_{1,j}) = \frac{1}{2}\ell(Q_0) = 2^{-(m_0+1)}\ell(Q)$, which, combined with the facts $\|f\|_{\mathrm{bmo}\,(\mathcal{X})} = 1$ and $2^{-(m_0+1)}\ell(Q) < c_0$, further implies that

$$\fint_{Q_{1,j}} \Big|f(x) - f_{Q_{1,j}}\Big|\,dx \le \|f\|_{\mathrm{bmo}\,(\mathcal{X})} = 1.$$

Thus, we obtain, for any j,

$$\sum_k \left|Q_{1,j,k}\right| \le \frac{1}{2} \sum_j \int_{Q_{1,j,k}} |f(x) - f_{Q_{1,j}}|\,dx$$
$$\le \frac{1}{2} \int_{Q_{1,j}} |f(x) - f_{Q_{1,j}}|\,dx \le \frac{1}{2}|Q_{1,j}|$$

and, for any k,

$$\left|f_{Q_{1,j,k}} - f_{Q_{1,j}}\right| \le \fint_{Q_{1,j,k}} |f(x) - f_{Q_{1,j}}|\,dx \le 2^{n+1}.$$

Rewrite $\bigcup_{j,k}\{Q_{1,j,k}\} =: \bigcup_j \{Q_{2,j}\}$. Then, we have

$$\sum_j \left|Q_{2,j}\right| \le \frac{1}{2} \sum_j \left|Q_{1,j}\right| \le \frac{1}{4}|Q_0|$$

and, for any $x \in Q \setminus \bigcup_j Q_{2,j}$,

$$|f(x)| \le \left|f(x) - f_{Q_{1,j}}\right| + \left|f_{Q_{1,j}}\right| \le 2 + 2^{n+1} \le 2 \cdot 2^{n+1}.$$

Repeating this process, then, for any $T \in \mathbb{N}$, we obtain a family $\{Q_{T,j}\}_j \subset \mathscr{D}_{Q_0}$ of disjoint dyadic cubes, such that

$$\sum_j \left|Q_{T,j}\right| \le 2^{-T}|Q_0|$$

and, for any $x \in Q_0 \setminus \bigcup_j Q_{T,j}$,

$$|f(x)| \le T2^{n+1}.$$

Notice that, for any $t \in [2^{n+1}, \infty)$, there exists a unique $T \in \mathbb{N}$, such that $T2^{n+1} \le t < (T+1)2^{n+1} \le T2^{n+2}$. Therefore, we obtain

$$\begin{aligned} |\{x \in Q_0 :\ |f(x)| > t\}| &\le \sum_j \left|Q_{T,j}\right| \le 2^{-T}|Q_0| \\ &= e^{-T\log 2}|Q_0| \le e^{-C_6 t}|Q_0|, \end{aligned} \tag{23}$$

where $C_6 := 2^{-(n+2)} \log 2$. Furthermore, observe that if $t \in (0, 2^{n+1})$, then $C_6 t < 2^{-1} \log 2$ and hence

$$|\{x \in Q_0 :\ |f(x)| > t\}| \le |Q_0| \le e^{2^{-1}\log 2 - C_6 t}|Q_0| = C_5 e^{-C_6 t}|Q_0|,$$

where $C_5 := \sqrt{2}$. By this, (23), and the arbitrariness of $Q_0 \in \mathscr{D}_Q^{(m_0)}$, we conclude that for any $t \in (0, \infty)$,

$$\begin{aligned} |\{x \in Q :\ |f(x)| > t\}| &= \sum_{Q_0 \in \mathscr{D}_Q^{(m_0)}} |\{x \in Q_0 :\ |f(x)| > t\}| \\ &\le C_5 e^{-C_6 t} \sum_{Q_0 \in \mathscr{D}_Q^{(m_0)}} |Q_0| = C_5 e^{-C_6 t}|Q| \end{aligned}$$

and hence (21) holds true. This finishes the proof of Proposition 5. □

As a corollary of Proposition 5, we have the following result: namely, $\|\cdot\|_{\mathrm{bmo}_1(\mathcal{X})}$ in (18) and $\|\cdot\|_{\mathrm{bmo}_2(\mathcal{X})}$ in (19) are equivalent norms of $\mathrm{bmo}\,(\mathcal{X})$. The proof of Corollary 1 is just a repetition of the proof of Proposition 3 with (4) replaced by (21); we omit the details here.

Corollary 1. *The following three statements are mutually equivalent:*

(i) $f \in \mathrm{bmo}\,(\mathcal{X})$;

(ii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $\|f\|_{\mathrm{bmo}_1(\mathcal{X})} < \infty$;

(iii) $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $\|f\|_{\mathrm{bmo}_2(\mathcal{X})} < \infty$.

Moreover, $\|\cdot\|_{\mathrm{bmo}\,(\mathcal{X})} \sim \|\cdot\|_{\mathrm{bmo}_1(\mathcal{X})} \sim \|\cdot\|_{\mathrm{bmo}_2(\mathcal{X})}$.

2.2. John–Nirenberg Space JN_p

Although there exist many fruitful studies of the space BMO in recent years, as was mentioned before, the structure of JN_p is largely a mystery, and there still exist many unsolved problems on JN_p. The first well-known property of JN_p is the following *John–Nirenberg inequality* obtained in [1] (Lemma 3), which says that $JN_p(Q_0)$ is embedded into the weak Lebesgue space $L^{p,\infty}(Q_0)$ (see Definition 1).

Theorem 2 (John–Nirenberg)**.** *Let $p \in (1,\infty)$ and Q_0 be a given cube of $\mathbb{R}^n$. If $f \in JN_p(Q_0)$, then $f - f_{Q_0} \in L^{p,\infty}(Q_0)$, and there exists a positive constant $C_{(n,p)}$, depending only on n and p, but independent of f, such that*

$$\left\|f - f_{Q_0}\right\|_{L^{p,\infty}(Q_0)} \le C_{(n,p)}\|f\|_{JN_p(Q_0)}.$$

It should be mentioned that the proof of Theorem 2 relies on the Calderón–Zygmund decomposition (namely Theorem 1) as well. Moreover, as an application of Theorem 2, Dafni et al. recently showed in [31] (Proposition 5.1) that for any given $p \in (1,\infty)$ and $q \in [1,p)$, $f \in JN_p(Q_0)$ if and only if $f \in L^1(Q_0)$ and

$$\|f\|_{JN_{p,q}(Q_0)} := \sup\left[\sum_i |Q_i|\left(\fint_{Q_i}\left|f(x) - f_{Q_i}\right|^q dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} < \infty,$$

where the supremum is taken in the same way as in (1); meanwhile, $\|\cdot\|_{JN_p(Q_0)} \sim \|\cdot\|_{JN_{p,q}(Q_0)}$. Furthermore, in [31] (Proposition 5.1), Dafni et al. also showed that for any given $p \in (1,\infty)$ and $q \in [p,\infty)$, the spaces $JN_{p,q}(Q_0)$ and $L^q(Q_0)$ coincide as sets.

Remark 2.

(i) *As a counterpart of Proposition 2, for any given $p \in (1,\infty)$ and any given cube Q_0 of $\mathbb{R}^n$, we have*
$$L^p(Q_0) \subsetneqq JN_p(Q_0) \subsetneqq L^{p,\infty}(Q_0).$$
Indeed, $L^p(Q_0) \subset JN_p(Q_0)$ is obvious from their definitions; $JN_p(Q_0) \subset L^{p,\infty}(Q_0)$ is just Theorem 2; $JN_p(Q_0) \subsetneqq L^{p,\infty}(Q_0)$ was shown in [30] (Example 3.5); and the desired function is just $x^{-1/p}$ on $[0,2]$. However, the fact $L^p(Q_0) \subsetneqq JN_p(Q_0)$ is extremely non-trivial and was obtained in [31] (Proposition 3.2 and Corollary 4.2) via constructing a nice fractal function based on skillful dyadic techniques. Moreover, in [31] (Theorem 1.1 and Remark 2.4), Dafni et al. showed that for any given $p \in (1,\infty)$ and any given interval $I_0 \subset \mathbb{R}$, no matter whether bounded or not, monotone functions are in $JN_p(I_0)$ if and only if they are also in $L^p(I_0)$. Thus, $JN_p(\mathcal{X})$ may be very "close" to $L^p(\mathcal{X})$ for any given $p \in (1,\infty)$.

(ii) *$JN_1(Q_0)$ coincides with $L^1(Q_0)$. To be precise, let Q_0 be any given cube of $\mathbb{R}^n$, and*
$$JN_1(Q_0) := \left\{f \in L^1(Q_0) : \|f\|_{JN_1(Q_0)} < \infty\right\},$$
where $\|f\|_{JN_1(Q_0)}$ is defined as in (1) with p replaced by 1. Then, we claim that $JN_1(Q_0) = [L^1(Q_0)/\mathbb{C}]$ with equivalent norms. Indeed, for any $f \in JN_1(Q_0)$, by the definition of $\|f\|_{JN_1(Q_0)}$, we have
$$\|f\|_{JN_1(Q_0)} \ge \left\|f - f_{Q_0}\right\|_{L^1(Q_0)} \ge \inf_{c\in\mathbb{C}}\|f + c\|_{L^1(Q_0)} =: \|f\|_{L^1(Q_0)/\mathbb{C}}.$$

Conversely, for any given $f \in L^1(Q_0)$ and any $c \in \mathbb{C}$, we have

$$\begin{aligned}\|f\|_{JN_1(Q_0)} &= \sup \sum_i \int_{Q_i} \left|f(x) - f_{Q_i}\right| dx \\ &\le 2 \sup \sum_i \int_{Q_i} \left|f(x) + c\right| dx \\ &\le 2\|f + c\|_{L^1(Q_0)},\end{aligned}$$

which implies that $\|f\|_{JN_1(Q_0)} \le \|f\|_{L^1(Q_0)/\mathbb{C}}$ and hence the above claim holds true. Moreover, the relation between $JN_1(\mathbb{R})$ and $L^1(\mathbb{R})$ was studied in [33] (Proposition 2).

(iii) *Garsia and Rodemich in [55] (Theorem 7.4) showed that for any given $p \in (1,\infty)$, $f \in L^{p,\infty}(Q_0)$ if and only if $f \in L^1(Q_0)$ and*

$$\|f\|_{\mathrm{GaRo}_p(Q_0)} := \sup \frac{1}{(\sum_i |Q_i|)^{1/p'}} \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} \left|f(x) - f(y)\right| dx\, dy < \infty,$$

where the supremum is taken in the same way as in (1); meanwhile,

$$\|\cdot\|_{L^{p,\infty}(Q_0)} \sim \|\cdot\|_{\mathrm{GaRo}_p(Q_0)};$$

(see also [35], Theorem 5(ii), for this equivalence). Moreover, in [35] (Theorem 5(i)), Milman showed that $\|\cdot\|_{\mathrm{GaRo}_p(Q_0)} \le 2\|\cdot\|_{JN_p(Q_0)}$.

Recall that the predual space of $\mathrm{BMO}(\mathcal{X})$ is the Hardy space $H^1(\mathcal{X})$ (see, for instance, [5], Theorem B). Similar to this duality, Dafni et al. [31] also obtained the predual space of $JN_p(Q_0)$ for any given $p \in (1,\infty)$, which is denoted by the Hardy kind space $HK_{p'}(Q_0)$, here and thereafter $1/p + 1/p' = 1$. Later, these properties, including equivalent norms and duality, were further studied on several John–Nirenberg-type spaces, such as John–Nirenberg–Campanato spaces, localized John–Nirenberg–Campanato spaces, congruent John–Nirenberg–Campanato spaces (see Section 3 for more details), and Riesz-type spaces (see Section 4 for more details).

Finally, let us briefly recall some other related studies concerning the John–Nirenberg space JN_p, which will not be stated in detail in this survey, although all of them are quite instructive:

- Stampacchia [56] introduced the space $N^{(p,\lambda)}$, which coincides with $JN_{(p,1,0)_\alpha}(Q_0)$ in Definitions 3 if we write $\lambda = p\alpha$ with $p \in (1,\infty)$ and $\alpha \in (-\infty,\infty)$, and applied them to the context of interpolation of operators.
- Campanato [57] also used the John–Nirenberg spaces to study the interpolation of operators.
- In the context of doubling metric spaces, JN_p and median-type JN_p were studied, respectively, by Aalto et al. in [30] and Myyryläinen in [58].
- Hurri-Syrjänen et al. [34] established a local-to-global result for the space $JN_p(\Omega)$ on an open subset Ω of $\mathbb{R}^n$. More precisely, it was proved that the norm $\|\cdot\|_{JN_p(\Omega)}$ is dominated by its local version $\|\cdot\|_{JN_{p,\tau}(\Omega)}$ modulus constants; here, $\tau \in [1,\infty)$; for any open subset Ω of $\mathbb{R}^n$, the related "norm" $\|\cdot\|_{JN_p(\Omega)}$ is defined in the same way as $\|\cdot\|_{JN_p(Q_0)}$ in (1) with Q_0 replaced by Ω; and $\|\cdot\|_{JN_{p,\tau}(\Omega)}$ is defined in the same way as $\|\cdot\|_{JN_p(\Omega)}$ with an additional requirement $\tau Q \subset \Omega$ for all chosen cubes Q in the definition of $\|\cdot\|_{JN_p(\Omega)}$.
- Marola and Saari [40] studied the corresponding results of Hurri-Syrjänen et al. [34] on metric measure spaces and obtained the equivalence between the local and the global JN_p norms. Moreover, in both articles [34,40], a global John–Nirenberg inequality for $JN_p(\Omega)$ was established.

- Berkovits et al. [32] applied the dyadic variant of $JN_p(Q_0)$ in the study of self-improving properties of some Poincaré-type inequalities. Later, the dyadic $JN_p(Q_0)$ was further studied by Kinnunen and Myyryläinen in [59].
- A. Brudnyi and Y. Brudnyi [60] introduced a class of function spaces $V_\kappa([0,1]^n)$ which coincides with $JN_{(p,q,s)_\alpha}([0,1]^n)$, defined below for suitable range of indices (see [61], Proposition 2.9, for more details). Very recently, Domínguez and Milman [62] further introduced and studied sparse Brudnyi and John–Nirenberg spaces.
- Blasco and Espinoza-Villalva [33] computed the concrete value of $\|\mathbf{1}_A\|_{JN_p(\mathbb{R})}$ for any given $p\in[1,\infty]$ and any measurable set $A\subset\mathbb{R}$ of positive and finite Lebesgue measure, where $JN_\infty(\mathbb{R}):=\mathrm{BMO}\,(\mathbb{R})$.
- The $JN_p(Q_0)$-type norm $\|\cdot\|_{\mathrm{GaRo}_p(Q_0)}$ in Remark 2(iii) was further generalized and studied in Astashkin and Milman [63] via the Strömberg–Jawerth–Torchinsky local maximal operator.

3. John–Nirenberg–Campanato Space

The main target of this section is to summarize the main results of John–Nirenberg–Campanato spaces, localized John–Nirenberg–Campanato spaces, and congruent John–Nirenberg–Campanato spaces obtained, respectively, in [36,61,64]. Moreover, at the end of each part, we list some open questions which are still unsolved so far. Now, we first recall some definitions of some basic function spaces.

- For any $s\in\mathbb{Z}_+$ (the set of all non-negative integers), let $\mathcal{P}_s(Q)$ denote the set of all polynomials of degree not greater than s on the cube Q, and $P_Q^{(s)}(f)$ denote the unique polynomial of degree not greater than s, such that
$$\int_Q\left[f(x)-P_Q^{(s)}(f)(x)\right]x^\gamma\,dx=0,\quad\forall\,|\gamma|\le s,\tag{24}$$
where $\gamma:=(\gamma_1,\ldots,\gamma_n)\in\mathbb{Z}_+^n:=(\mathbb{Z}_+)^n$, $|\gamma|:=\gamma_1+\cdots+\gamma_n$, and $x^\gamma:=x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ for any $x:=(x_1,\ldots,x_n)\in\mathbb{R}^n$.
- Let $q\in[1,\infty]$ and Q_0 be a given cube of $\mathbb{R}^n$. For any measurable function f, let
$$\|f\|_{L^q(Q_0,|Q_0|^{-1}dx)}:=\left[\fint_{Q_0}|f(x)|^q\,dx\right]^{\frac{1}{q}}.$$
- Let $q\in(1,\infty)$, $s\in\mathbb{Z}_+$, and Q_0 be a given cube of $\mathbb{R}^n$. The *space* $L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)$ is defined by setting
$$L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0):=\left\{f\in L^q(Q_0):\ \|f\|_{L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)}<\infty\right\},$$
where
$$\|f\|_{L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)}:=\inf_{m\in\mathcal{P}_s(Q_0)}\|f+m\|_{L^q(Q_0,|Q_0|^{-1}dx)}.$$
- For any given $v\in[1,\infty]$ and $s\in\mathbb{Z}_+$, and any measurable subset $E\subset\mathbb{R}^n$, let
$$L_s^v(E):=\left\{f\in L^v(E):\ \int_E f(x)x^\gamma\,dx=0,\ \forall\,\gamma\in\mathbb{Z}_+^n,\ |\gamma|\le s\right\}.$$

Let Q be any given cube of $\mathbb{R}^n$. It is well known that $P_Q^{(0)}(f)=f_Q$, and for any $s\in\mathbb{Z}_+$, there exists a constant $C_{(s)}\in[1,\infty)$, independent of f and Q, such that
$$\left|P_Q^{(s)}(f)(x)\right|\le C_{(s)}\fint_Q|f(x)|\,dx,\quad\forall\,x\in Q.\tag{25}$$

Indeed, let $\{\varphi_Q^{(\gamma)}:\ \gamma \in \mathbb{Z}_+^n,\ |\gamma| \le s\}$ denote the Gram–Schmidt orthonormalization of $\{x^\gamma:\ \gamma \in \mathbb{Z}_+^n,\ |\gamma| \le s\}$ on the cube Q with respect to the weight $1/|Q|$, namely for any γ, ν, $\mu \in \mathbb{Z}_+^n$ with $|\gamma| \le s$, $|\nu| \le s$, and $|\mu| \le s$, $\varphi_Q^{(\gamma)} \in \mathcal{P}_s(Q)$ and

$$\langle \varphi_Q^{(\nu)}, \varphi_Q^{(\mu)} \rangle := \frac{1}{|Q|} \int_Q \varphi_Q^{(\nu)}(x) \varphi_Q^{(\mu)}(x)\, dx = \begin{cases} 1, & \nu = \mu, \\ 0, & \nu \neq \mu. \end{cases}$$

Then,

$$P_Q^{(s)}(f)(x) := \sum_{\{\gamma \in \mathbb{Z}_+^n:\ |\gamma| \le s\}} \langle \varphi_Q^{(\gamma)}, f \rangle \varphi_Q^{(\gamma)}(x), \quad \forall\, x \in Q,$$

and we can choose $C_{(s)} := \sum_{\{\gamma \in \mathbb{Z}_+^n:\ |\gamma| \le s\}} \|\varphi_Q^{(\gamma)}\|_{L^\infty(Q)}^2$ satisfying (25) (see [65], p. 83, and [66], p. 54, Lemma 4.1, for more details).

3.1. John–Nirenberg–Campanato Spaces

In this subsection, we first recall the definitions of Campanato spaces, John–Nirenberg–Campanato spaces (for short, JNC spaces), and Hardy-type spaces, respectively, in Definitions 2, 3, and 6 below. Moreover, we review some properties of JNC spaces and Hardy-type spaces, including their limit spaces (Proposition 6 and Corollary 2 below), relations with the Lebesgue space (Propositions 7 and 8 below), the dual result (Theorem 3 below), the monotonicity over the first sub-index (Proposition 9 below), the John–Nirenberg-type inequality (Theorem 4 below), and the equivalence over the second sub-index (Propositions 10 and 11 below).

A general dual result for Hardy spaces was given by Coifman and Weiss [5] who proved that for any given $p \in (0,1]$ and $q \in [1,\infty]$, and s being a non-negative integer not smaller than $n(\frac{1}{p}-1)$, the dual space of the Hardy space $H^p(\mathbb{R}^n)$ is the Campanato space $C_{\frac{1}{p}-1,q,s}(\mathbb{R}^n)$, which was introduced by Campanato [67] and coincides with $\mathrm{BMO}(\mathbb{R}^n)$ when $p = 1$.

Definition 2. *Let $\alpha \in [0,\infty)$, $q \in [1,\infty)$, and $s \in \mathbb{Z}_+$.*

(i) *The* Campanato space *$C_{\alpha,q,s}(\mathcal{X})$ is defined by setting*

$$C_{\alpha,q,s}(\mathcal{X}) := \left\{ f \in L^q_{\mathrm{loc}}(\mathcal{X}) :\ \|f\|_{C_{\alpha,q,s}(\mathcal{X})} < \infty \right\},$$

where

$$\|f\|_{C_{\alpha,q,s}(\mathcal{X})} := \sup |Q|^{-\alpha} \left[\fint_Q \left| f - P_Q^{(s)}(f) \right|^q \right]^{\frac{1}{q}}$$

and the supremum is taken over all cubes Q of $\mathcal{X}$. In addition, the "norm" $\|\cdot\|_{C_{\alpha,q,s}(\mathcal{X})}$ of polynomials is zero, and for simplicity, the space $C_{\alpha,q,s}(\mathcal{X})$ is regarded as the quotient space $C_{\alpha,q,s}(\mathcal{X})/\mathcal{P}_s(\mathcal{X})$.

(ii) *The* dual space *$(C_{\alpha,q,s}(\mathcal{X}))^*$ of $C_{\alpha,q,s}(\mathcal{X})$ is defined to be the set of all continuous linear functionals on $C_{\alpha,q,s}(\mathcal{X})$ equipped with the weak-$*$ topology.*

In what follows, for any $\ell \in (0,\infty)$, $Q(\mathbf{0},\ell)$ denotes the cube centered at the origin $\mathbf{0}$ with side length ℓ.

Remark 3. *Let $0 < q \le p \le \infty$. The* Morrey space *$M_q^p(\mathbb{R}^n)$, introduced by Morrey in [68], is defined by setting*

$$M_q^p(\mathbb{R}^n) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) :\ \|f\|_{M_q^p(\mathbb{R}^n)} < \infty \right\},$$

where, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{M^p_q(\mathbb{R}^n)} := \sup_{\text{cube } Q\subset\mathbb{R}^n} |Q|^{\frac{1}{p}}\left[\fint_Q |f(y)|^q\,dy\right]^{\frac{1}{q}}.$$

From Campanato ([67], Theorem 6.II), it follows that for any given $q \in [1,\infty)$ *and* $\alpha \in [-\frac{1}{q}, 0)$, *and any* $f \in C_{q,\alpha,0}(\mathcal{X})$,

$$\|f\|_{C_{q,\alpha,0}(\mathcal{X})} \sim \left\|f - \sigma(f)\right\|_{M_q^{-1/\alpha}(\mathcal{X})}, \tag{26}$$

where the positive equivalence constants are independent of f, *and*

$$\sigma(f) := \begin{cases} \lim\limits_{\ell\to\infty} \dfrac{1}{|Q(\mathbf{0},\ell)|}\displaystyle\int_{Q(\mathbf{0},\ell)} f(x)\,dx & \text{if } \mathcal{X} = \mathbb{R}^n, \\ \dfrac{1}{|Q_0|}\displaystyle\int_{Q_0} f(x)\,dx & \text{if } \mathcal{X} = Q_0; \end{cases}$$

see also Nakai [16], Theorem 2.1 and Corollary 2.3, for this conclusion on spaces of homogeneous type. In addition, a surprising result says that in the definition of supremum $\|\cdot\|_{M^p_q(\mathbb{R}^n)}$, *if "cubes" were changed into "measurable sets", then the Morrey norm* $\|\cdot\|_{M^p_q(\mathbb{R}^n)}$ *becomes an equivalent norm of the weak Lebesgue space (see Definition 1). To be precise, for any given* $0 < q < p < \infty$, $f \in L^{p,\infty}(\mathbb{R}^n)$ *if and only if* $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\widetilde{M^p_q}(\mathbb{R}^n)} := \sup_{A\subset\mathbb{R}^n,\,|A|\in(0,\infty)} |A|^{\frac{1}{p}}\left[\fint_A |f(y)|^q\,dy\right]^{\frac{1}{q}} < \infty;$$

moreover,

$$\|\cdot\|_{L^{p,\infty}(\mathbb{R}^n)} \le \|\cdot\|_{\widetilde{M^p_q}(\mathbb{R}^n)} \le \left(\frac{p}{p-q}\right)^{\frac{1}{q}} \|\cdot\|_{L^{p,\infty}(\mathbb{R}^n)};$$

see, for instance, [69], p. 485, Lemma 2.8. Another interesting JN_p*-type equivalent norm of the weak Lebesgue space was presented in Remark 2(iii).*

Inspired by the relation between BMO and the Campanato space, as well as the relation between BMO and JN_p, Tao et al. [61] introduced a Campanato-type space $JN_{(p,q,s)_\alpha}(\mathcal{X})$ in the spirit of the John–Nirenberg space $JN_p(Q_0)$, which contains $JN_p(Q_0)$ as a special case. This John–Nirenberg–Campanato space is defined not only on any cube Q_0 but also on the whole space $\mathbb{R}^n$.

Definition 3. *Let* $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, *and* $\alpha \in \mathbb{R}$.

(i) *The* John–Nirenberg–Campanato space *(for short, JNC* space*)* $JN_{(p,q,s)_\alpha}(\mathcal{X})$ *is defined by setting*

$$JN_{(p,q,s)_\alpha}(\mathcal{X}) := \left\{f \in L^q_{\mathrm{loc}}(\mathcal{X}) :\ \|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})} < \infty\right\},$$

where

$$\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})} := \sup\left\{\sum_i |Q_i|\left[|Q_i|^{-\alpha}\left\{\fint_{Q_i}\left|f(x) - P^{(s)}_{Q_i}(f)(x)\right|^q dx\right\}^{\frac{1}{q}}\right]^p\right\}^{\frac{1}{p}},$$

$P^{(s)}_{Q_i}(f)$ *for any* i *is as in (24) with* Q *replaced by* Q_i, *and the supremum is taken over all collections of interior pairwise disjoint cubes* $\{Q_i\}_i$ *of* $\mathcal{X}$. *Furthermore, the "norm"* $\|\cdot\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}$

of polynomials is zero, and for simplicity, the space $JN_{(p,q,s)_\alpha}(\mathcal{X})$ is regarded as the quotient space $JN_{(p,q,s)_\alpha}(\mathcal{X})/\mathcal{P}_s(\mathcal{X})$.

(ii) *The* dual space $(JN_{(p,q,s)_\alpha}(\mathcal{X}))^*$ *of $JN_{(p,q,s)_\alpha}(\mathcal{X})$ is defined to be the set of all continuous linear functionals on $JN_{(p,q,s)_\alpha}(\mathcal{X})$ equipped with the weak-$*$ topology.*

Remark 4. *In [61], the JNC space was introduced only for any given $\alpha \in [0,\infty)$ to study its relation with the Campanato space in Definition 2, and for any given $p \in (1,\infty)$ due to Remark 2(ii). However, many results in [61] also hold true when $\alpha \in \mathbb{R}$ and $p = 1$, just with some slight modifications of their proofs. Thus, in this survey, we introduce the JNC space for any given $\alpha \in \mathbb{R}$ and $p \in [1,\infty)$ and naturally extend some related results with some identical proofs omitted.*

The following proposition, which is just [61] (Proposition 2.6), means that the classical Campanato space serves as a limit space of $JN_{(p,q,s)_\alpha}(\mathcal{X})$, similar to the Lebesgue spaces $L^\infty(\mathcal{X})$ and $L^p(\mathcal{X})$ when $p \to \infty$.

Proposition 6. *Let $\alpha \in [0,\infty)$, $q \in [1,\infty)$, and $s \in \mathbb{Z}_+$. Then,*

$$\lim_{p\to\infty} JN_{(p,q,s)_\alpha}(\mathcal{X}) = C_{\alpha,q,s}(\mathcal{X})$$

in the following sense: for any $f \in \bigcup_{r\in[1,\infty)} \bigcap_{p\in[r,\infty)} JN_{(p,q,s)_\alpha}(\mathcal{X})$,

$$\lim_{p\to\infty} \|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})} = \|f\|_{C_{\alpha,q,s}(\mathcal{X})}.$$

In Proposition 6, if we take $\mathcal{X} = Q_0$, we then have the following corollary, which is just [61] (Corollary 2.8).

Corollary 2. *Let $q \in [1,\infty)$, $\alpha \in [0,\infty)$, $s \in \mathbb{Z}_+$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$C_{\alpha,q,s}(Q_0) = \left\{ f \in \bigcap_{p\in[1,\infty)} JN_{(p,q,s)_\alpha}(Q_0) : \lim_{p\to\infty} \|f\|_{JN_{(p,q,s)_\alpha}(Q_0)} < \infty \right\}$$

and for any $f \in C_{\alpha,q,s}(Q_0)$,

$$\|f\|_{C_{\alpha,q,s}(Q_0)} = \lim_{p\to\infty} \|f\|_{JN_{(p,q,s)_\alpha}(Q_0)}.$$

Remark 5.

(i) *Let $p \in (1,\infty)$ and Q_0 be a given cube of $\mathbb{R}^n$. It is easy to show that*

$$\mathrm{BMO}\,(Q_0) \subset JN_p(Q_0).$$

However, we claim that

$$\mathrm{BMO}\,(\mathbb{R}^n) \not\subset JN_p(\mathbb{R}^n).$$

Indeed, for the simplicity of the presentation, without loss of generality, we may show this claim only in $\mathbb{R}$. Let $g(x) := \log(|x|)$ for any $x \in \mathbb{R} \setminus \{0\}$, and $g(0) := 0$. Then, $g \in \mathrm{BMO}\,(\mathbb{R})$ due to [46] (Example 3.1.3), and hence it suffices to prove that $g \notin JN_p(\mathbb{R})$ for any given $p \in (1,\infty)$. To do this, let $I_t := (0,t)$ for any $t \in (0,\infty)$. Then, by some simple calculations, we obtain

$$g_{I_t} = \fint_{I_t} g(x)\,dx = \frac{1}{t}\int_0^t \log(x)\,dx = \log(t) - 1$$

and hence

$$\left|\left\{ x \in I_t : \left|g(x) - g_{I_t}\right| > \frac{1}{2} \right\}\right|$$

$$= \left|\left\{x \in (0,t) : \left|\log(x) - [\log(t) - 1]\right| > \frac{1}{2}\right\}\right|$$
$$\geq t - te^{-\frac{1}{2}} = t\left(1 - e^{-\frac{1}{2}}\right) \to \infty$$

as $t \to \infty$. However, the John–Nirenberg inequality of $JN_p(I_t)$ in Theorem 2 implies that for any $t \in (0,\infty)$,

$$\left|\left\{x \in I_t : \left|g(x) - g_{I_t}\right| > \frac{1}{2}\right\}\right| \lesssim \left[\frac{\|g\|_{JN_p(I_t)}}{\frac{1}{2}}\right]^p \lesssim \|g\|^p_{JN_p(\mathbb{R})}$$

with the implicit positive constants depending only on p. Thus, $g \notin JN_p(\mathbb{R})$, and hence the above claim holds true.

(ii) *The predual counterpart of Corollary 2 is still unclear so far (see Question 2 below for more details).*

Obviously, $JN_{(p,q,0)_0}(Q_0)$ is just $JN_{p,q}(Q_0)$. From this and [31] (Proposition 5.1), we deduce that when $p \in (1,\infty)$ and $q \in [1,p)$, $JN_{(p,q,0)_0}(Q_0)$ coincides with $JN_p(Q_0)$ in the sense of equivalent norms, and when $p \in (1,\infty)$ and $q \in [p,\infty)$, $JN_{(p,q,0)_0}(Q_0)$ and $L^q(Q_0)$ coincide as sets. Moreover, by adding a particular weight of $|Q_0|$, the authors of this article showed that the aforementioned coincidence (as sets) can be modified into equivalent norms (see Proposition 7 below, which is just [61], Proposition 2.5). In what follows, for any given positive constant A and any given function space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$, we write $A\mathbb{X} := \{Af : f \in \mathbb{X}\}$ with its norm defined by setting, for any $Af \in A\mathbb{X}$, $\|Af\|_{A\mathbb{X}} := A\|f\|_{\mathbb{X}}$.

Proposition 7. *Let $p \in [1,\infty)$, $q \in [p,\infty)$, $s \in \mathbb{Z}_+$, $\alpha = 0$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$\left[|Q_0|^{-\frac{1}{p}} JN_{(p,q,s)_\alpha}(Q_0)\right] = \left[L^q(Q_0, |Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)\right]$$

with equivalent norms, namely

$$\|f\|_{L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)} \leq |Q_0|^{-\frac{1}{p}} \|f\|_{JN_{(p,q,s)_0}(Q_0)}$$
$$\leq 2^{p-\frac{p}{q}}\left[1 + C_{(s)}\right]^{\frac{p}{q}} \|f\|_{L^q(Q_0,|Q_0|^{-1}dx)/\mathcal{P}_s(Q_0)},$$

where $C_{(s)}$ is as in (25).

It is a very interesting open question to find a counterpart of Proposition 7 when $\alpha \in \mathbb{R} \setminus \{0\}$ (see Question 1 below for more details).

Now, we review the predual of the John–Nirenberg–Campanato space via introducing atoms, polymers, and Hardy-type spaces in order, which coincide with the same notation as in [31] when $u \in (1,\infty)$, $v \in (u,\infty]$, and $\alpha = 0 = s$ (see [61], Remarks 3.4 and 3.8, for more details). In particular, when $\alpha = 0$, the $(u,v,s)_0$-atom below is just the classic atom of the Hardy space (see [61], Remark 3.2).

Definition 4. *Let $u, v \in [1,\infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. A function a is called a $(u,v,s)_\alpha$-atom on a cube Q if*

(i) $\operatorname{supp}(a) := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset Q$;

(ii) $\|a\|_{L^v(Q)} \leq |Q|^{\frac{1}{v}-\frac{1}{u}-\alpha}$;

(iii) $\int_Q a(x)x^\gamma\, dx = 0$ *for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \leq s$.*

In what follows, for any $u \in [1,\infty]$, let u' denote its *conjugate index*, namely $1/u + 1/u' = 1$, and for any $\{\lambda_j\}_j \subset \mathbb{C}$, let

$$\left\|\{\lambda_j\}_j\right\|_{\ell^u} := \begin{cases} \left(\sum_j |\lambda_j|^u\right)^{\frac{1}{u}} & \text{when } u \in [1,\infty), \\ \sup_j |\lambda_j| & \text{when } u = \infty. \end{cases} \tag{27}$$

Definition 5. *Let $u, v \in [1,\infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. The* space of $(u,v,s)_\alpha$-polymers*, denoted by $\widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})$, is defined to be the set of all $g \in (JN_{(u',v',s)_\alpha}(\mathcal{X}))^*$ satisfying that there exist $(u,v,s)_\alpha$-atoms $\{a_j\}_j$ supported, respectively, in interior pairwise disjoint cubes $\{Q_j\}_j$ of $\mathcal{X}$, and $\{\lambda_j\}_j \subset \mathbb{C}$ with $|\lambda_j|^u < \infty$, such that*

$$g = \sum_j \lambda_j a_j$$

*in $(JN_{(u',v',s)_\alpha}(\mathcal{X}))^*$. Moreover, any $g \in \widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})$ is called a $(u,v,s)_\alpha$-*polymer *with its norm $\|g\|_{\widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})}$ defined by setting*

$$\|g\|_{\widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})} := \inf\left\|\{\lambda_j\}_j\right\|_{\ell^u},$$

where the infimum is taken over all decompositions of g as above.

Definition 6. *Let $u, v \in [1,\infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. The* Hardy-type space $HK_{(u,v,s)_\alpha}(\mathcal{X})$ *is defined by setting*

$$HK_{(u,v,s)_\alpha}(\mathcal{X}) := \Bigg\{ g \in (JN_{(u',v',s)_\alpha}(\mathcal{X}))^* : \; g = \sum_i g_i \text{ in } (JN_{(u',v',s)_\alpha}(\mathcal{X}))^*, \\ \{g_i\}_i \subset \widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X}), \text{ and } \sum_i \left\|g_i\right\|_{\widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})} < \infty \Bigg\}$$

and for any $g \in HK_{(u,v,s)_\alpha}(\mathcal{X})$, let

$$\|g\|_{HK_{(u,v,s)_\alpha}(\mathcal{X})} := \inf \sum_i \|g_i\|_{\widetilde{HK}_{(u,v,s)_\alpha}(\mathcal{X})},$$

where the infimum is taken over all decompositions of g as above. Moreover, the finite atomic Hardy-type space $HK^{\mathrm{fin}}_{(u,v,s)_\alpha}(\mathcal{X})$ *is defined to be the set of all finite summations $\sum_{m=1}^M \lambda_m a_m$, where $M \in \mathbb{N}$, $\{\lambda_m\}_{m=1}^M \subset \mathbb{C}$, and $\{a_m\}_{m=1}^M$ are $(u,v,s)_\alpha$-atoms.*

The significant dual relation between $JN_{(p,q,s)_\alpha}(\mathcal{X})$ and $HK_{(p',q',s)_\alpha}(\mathcal{X})$ reads as follows, which is just [61] (Theorem 3.9) with $\alpha \in [0,\infty)$ replaced by $\alpha \in \mathbb{R}$ (this makes sense because the crucial lemma ([61], Lemma 3.12) still holds true with the corresponding replacement).

Theorem 3. *Let $p, q \in (1,\infty)$, $1/p = 1/p' = 1 = 1/q + 1/q'$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then, $(HK_{(p',q',s)_\alpha}(\mathcal{X}))^* = JN_{(p,q,s)_\alpha}(\mathcal{X})$ in the following sense:*

(i) *If $f \in JN_{(p,q,s)_\alpha}(\mathcal{X})$, then f induces a linear functional $\mathcal{L}_f$ on $HK_{(p',q',s)_\alpha}(\mathcal{X})$ and*

$$\|\mathcal{L}_f\|_{(HK_{(p',q',s)_\alpha}(\mathcal{X}))^*} \le C\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})},$$

where C is a positive constant independent of f.

(ii) *If $\mathcal{L} \in (HK_{(p',q',s)_\alpha}(\mathcal{X}))^*$, then there exists an $f \in JN_{(p,q,s)_\alpha}(\mathcal{X})$, such that for any $g \in HK^{\mathrm{fin}}_{(p',q',s)_\alpha}(\mathcal{X})$,*

$$\mathcal{L}(g) = \int_{\mathcal{X}} f(x)g(x)\,dx,$$

and

$$\|\mathcal{L}\|_{(HK_{(p',q',s)_\alpha}(\mathcal{X}))^*} \sim \|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}$$

with the positive equivalence constants independent of f.

When $\mathcal{X} := Q_0$, $\alpha = 0 = s$, and $q \in [1, p)$, by [61] (Remark 3.10 and Proposition 10), we know that Theorem 3 in this case coincides with [31] (Theorem 6.6). As an application of Theorem 3, the authors obtained the following atomic characterization of $L^{q'}_s(Q_0)$ for any given $q' \in (1, \infty)$ and $s \in \mathbb{Z}_+$, which is just [61] (Corollary 3.13).

Proposition 8. *Let $p \in (1, \infty)$, $q \in [p, \infty)$, $1/p = 1/p' = 1 = 1/q + 1/q'$, $s \in \mathbb{Z}_+$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$L^{q'}_s(Q_0, |Q_0|^{q'-1}dx) = |Q_0|^{\frac{1}{p}} HK_{(p',q',s)_0}(Q_0)$$

with equivalent norms.

From Theorem 2 and [47] (p. 14, Exercise 1.1.11), we deduce that for any $1 < p_1 < p_2 < \infty$,

$$JN_{p_2}(Q_0) \subset L^{p_2,\infty}(Q_0) \subset L^{p_1}(Q_0) \subset JN_{p_1}(Q_0).$$

Moreover, it is easy to show the following monotonicity over the first sub-index of both $JN_{(p,q,s)_\alpha}(Q_0)$ and $HK_{(u,v,s)_\alpha}(Q_0)$.

Proposition 9. *Let $s \in \mathbb{Z}_+$ and Q_0 be a given cube of $\mathbb{R}^n$.*

(i) *Let $1 < u_1 < u_2 < \infty$. If $v \in (1, \infty)$ and $\alpha \in \mathbb{R}$, or $v = \infty$ and $\alpha \in [0, \infty)$, then*

$$HK_{(u_2,v,s)_\alpha}(Q_0) \subset HK_{(u_1,v,s)_\alpha}(Q_0)$$

and

$$\|\cdot\|_{HK_{(u_1,v,s)_\alpha}(Q_0)} \leq |Q_0|^{\frac{1}{u_1} - \frac{1}{u_2}} \|\cdot\|_{HK_{(u_2,v,s)_\alpha}(Q_0)}.$$

(ii) *Let $1 < p_1 < p_2 < \infty$. If $q \in (1, \infty)$ and $\alpha \in \mathbb{R}$, or $q = 1$ and $\alpha \in [0, \infty)$, then*

$$JN_{(p_2,q,s)_\alpha}(Q_0) \subset JN_{(p_1,q,s)_\alpha}(Q_0)$$

and there exists some positive constant C, such that

$$\|\cdot\|_{JN_{(p_1,q,s)_\alpha}(Q_0)} \leq C|Q_0|^{\frac{1}{p_1} - \frac{1}{p_2}} \|\cdot\|_{JN_{(p_2,q,s)_\alpha}(Q_0)}.$$

Proof. (i) is a direct corollary of the fact that for any $(u_2, v, s)_\alpha$-atom a on the cube Q,

$$|Q|^{\frac{1}{v_2} - \frac{1}{v_1}} a$$

is a $(u_1, v, s)_\alpha$-atom (see [36], Remark 5.5, for more details).

(ii) is a direct consequence of the Jensen inequality (see, for instance, [61], Remark 4.2(ii)). This finishes the proof of Proposition 9. □

Now, we consider the independence over the second sub-index, which strongly relies on the John–Nirenberg inequality as in the BMO case. The following John–Nirenberg-type inequality is just [61] (Theorem 4.3), which coincides with Theorem 2 when $\alpha = 0 = s$.

Theorem 4. *Let $p \in (1, \infty)$, $s \in \mathbb{Z}_+$, $\alpha \in [0, \infty)$, and Q_0 be a given cube of $\mathbb{R}^n$. If $f \in JN_{(p,1,s)_\alpha}(Q_0)$, then $f - P^{(s)}_{Q_0}(f) \in L^{p,\infty}(Q_0)$, and there exists a positive constant $C_{(n,p,s)}$, depending only on n, p, and s, but independent of f, such that*

$$\left\| f - P^{(s)}_{Q_0}(f) \right\|_{L^{p,\infty}(Q_0)} \le C_{(n,p,s)} |Q_0|^\alpha \|f\|_{JN_{(p,1,s)_\alpha}(Q_0)}.$$

It should be mentioned that the main tool used in the proof of Theorem 4 is the following *good-λ inequality* (namely, Lemma 2 below), which is just [61] (Lemma 4.6) (see also [30], Lemma 4.5, when $s = 0$). Recall that for any given cube Q_0 of $\mathbb{R}^n$, the *dyadic maximal operator* $\mathcal{M}^{(\mathrm{d})}_{Q_0}$ is defined by setting, for any given $g \in L^1(Q_0)$ and any $x \in Q_0$,

$$\mathcal{M}^{(\mathrm{d})}_{Q_0}(g)(x) := \sup_{Q \in \mathcal{D}_{Q_0},\, Q \ni x} \frac{1}{|Q|} \int_Q |g(x)|\,dx,$$

where $\mathcal{D}_{Q_0}$ is as in (20) with Q replaced by Q_0, and the supremum is taken over all dyadic cubes $Q \in \mathcal{D}_{Q_0}$ and $Q \ni x$.

Lemma 2. *Let $p \in (1, \infty)$, $s \in \mathbb{Z}_+$, $C_{(s)} \in [1, \infty)$ be as in (25), $\theta \in (0, 2^{-n} C_{(s)}^{-1})$, Q_0 be a given cube of $\mathbb{R}^n$, and $f \in JN_{(p,1,s)_0}(Q_0)$. Then, for any real number $\lambda > \frac{1}{\theta} \fint_{Q_0} |f - P^{(s)}_{Q_0}(f)|$,*

$$\begin{aligned} &\left| \left\{ x \in Q_0 : \ \mathcal{M}^{(\mathrm{d})}_{Q_0}\left(f - P^{(s)}_{Q_0}(f) \right)(x) > \lambda \right\} \right| \\ &\quad \le \frac{\|f\|_{JN_{(p,1,s)_0}(Q_0)}}{[1 - 2^n \theta C_{(s)}]\lambda} \left| \left\{ x \in Q_0 : \ \mathcal{M}^{(\mathrm{d})}_{Q_0}\left(f - P^{(s)}_{Q_0}(f) \right)(x) > \theta\lambda \right\} \right|^{\frac{1}{p'}}. \end{aligned}$$

Moreover, based on Theorem 4 in [61] (Proposition 4.1), Tao et al. further obtained the following independence over the second sub-index of $JN_{(p,q,s)_\alpha}(\mathcal{X})$.

Proposition 10. *Let $1 \le q < p < \infty$, $s \in \mathbb{Z}_+$, and $\alpha \in [0, \infty)$. Then,*

$$JN_{(p,q,s)_\alpha}(\mathcal{X}) = JN_{(p,1,s)_\alpha}(\mathcal{X})$$

with equivalent norms.

Furthermore, the following independence over the second sub-index of $HK_{(u,v,s)_\alpha}(\mathcal{X})$ is just [61] (Proposition 4.7), whose proof is based on Theorem 3 and Proposition 10.

Proposition 11. *Let $1 < u < v \le \infty$, $s \in \mathbb{Z}_+$, and $\alpha \in [0, \infty)$. Then,*

$$HK_{(u,v,s)_\alpha}(\mathcal{X}) = HK_{(u,\infty,s)_\alpha}(\mathcal{X})$$

with equivalent norms.

In particular, when $\alpha = 0 = s$, Propositions 10 and 11 were obtained, respectively, in [31] (Propositions 5.1 and 6.4).

Combining Theorem 3 and Propositions 10 and 11, we immediately have the following corollary; we omit the details here.

Corollary 3. *Let $p \in (1, \infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in [0, \infty)$. Then, $(HK_{(p',\infty,s)_\alpha}(\mathcal{X}))^* = JN_{(p,1,s)_\alpha}(\mathcal{X})$.*

Finally, we list some open questions.

Question 1. *For any given cube Q_0 of $\mathbb{R}^n$, by [61] (Remark 4.2(ii)) with slight modifications, we know that*

(i) *for any given $p \in [1,\infty)$ and $s \in \mathbb{Z}_+$,*

$$JN_{(p,q,s)_0}(Q_0) = \begin{cases} JN_{(p,1,s)_0}(Q_0), & q \in [1,p), \\ JN_{(q,q,s)_0}(Q_0), & q \in [p,\infty); \end{cases}$$

(ii) *for any given $p \in [1,\infty)$, $q \in [p,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$,*

$$JN_{(q,q,s)_\alpha}(Q_0) \subset JN_{(p,q,s)_\alpha}(Q_0)$$

and

$$\left[|Q_0|^{-\frac{1}{p}}\|f\|_{JN_{(p,q,s)_\alpha}(Q_0)}\right] \le \left[|Q_0|^{-\frac{1}{q}}\|f\|_{JN_{(q,q,s)_\alpha}(Q_0)}\right];$$

(iii) *for any given $p \in [1,\infty)$, $q \in [p,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in (\frac{s+1}{n},\infty)$,*

$$JN_{(q,q,s)_\alpha}(Q_0) = \mathcal{P}_s(Q_0) = JN_{(p,q,s)_\alpha}(Q_0).$$

However, letting $RM_{p,q,\alpha}(\mathcal{X})$ denote the Riesz–Morrey space in Definition 14, it is still unknown whether or not

(i) *for any given $p \in [1,\infty)$, $q \in [p,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in (-\infty,\frac{s+1}{n}] \setminus \{0\}$,*

$$JN_{(p,q,s)_\alpha}(Q_0) = JN_{(q,q,s)_\alpha}(Q_0) \text{ or } JN_{(p,q,s)_\alpha}(Q_0) = \left[RM_{p,q,\alpha}(Q_0)/\mathcal{P}_s(Q_0)\right]$$

holds true;

(ii) *for any given $p \in [1,\infty)$, $q \in [p,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$,*

$$JN_{(p,q,s)_\alpha}(\mathbb{R}^n) = JN_{(q,q,s)_\alpha}(\mathbb{R}^n) \text{ or } JN_{(p,q,s)_\alpha}(\mathbb{R}^n) = \left[RM_{p,q,\alpha}(\mathbb{R}^n)/\mathcal{P}_s(\mathbb{R}^n)\right]$$

holds true, where $\mathcal{P}_s(\mathbb{R}^n)$ denotes the set of all polynomials of degree not greater than s on $\mathbb{R}^n$.

Question 2. *Let $1 < u_1 < u_2 < \infty$, $v \in (1,\infty]$, $s \in \mathbb{Z}_+$, and Q_0 be a given cube of $\mathbb{R}^n$. From Proposition 9(i), we deduce that*

$$HK_{(u_2,v,s)_0}(Q_0) \subset HK_{(u_1,v,s)_0}(Q_0)$$

and

$$\|\cdot\|_{HK_{(u_1,v,s)_0}(Q_0)} \le \left[|Q_0|^{\frac{1}{u_1}-\frac{1}{u_2}}\|\cdot\|_{HK_{(u_2,v,s)_0}(Q_0)}\right].$$

Moreover, by [61] (Remark 4.2(iii)) and [36] (Proposition 5.7), we find that for any $u \in [1,\infty)$,

$$HK_{(u,v,s)_0}(Q_0) \subset H^{1,v,s}_{\mathrm{at}}(Q_0)$$

and for any $g \in \bigcup_{u\in[1,\infty)} HK_{(u,v,s)_0}(Q_0)$,

$$\|g\|_{H^{1,v,s}_{\mathrm{at}}(Q_0)} \le \liminf_{u\to 1^+}\|g\|_{HK_{(u,v,s)_0}(Q_0)},$$

where $H^{1,v,s}_{\mathrm{at}}(\mathcal{X})$ denotes the atomic Hardy space *(see Coifman and Weiss [5], and also [61], Remark 3.2(ii), for its definition). Here and thereafter, $u \to 1^+$ means $u \in (1,\infty)$ and $u \to 1$. However, for any given $v \in (1,\infty]$, $s \in \mathbb{Z}_+$, $\alpha \in [0,\infty)$, and any given cube Q_0 of $\mathbb{R}^n$,*

(i) *it is still unknown whether or not for any $g \in \bigcup_{u\in[1,\infty)} HK_{(u,v,s)_\alpha}(Q_0)$,*

$$\|g\|_{H^{\frac{1}{\alpha+1},v,s}_{\mathrm{at}}(Q_0)} = \lim_{u\to 1^+}\|g\|_{HK_{(u,v,s)_\alpha}(Q_0)}$$

holds true;

(ii) *it is interesting to clarify the relation between* $\bigcup_{u\in[1,\infty)} HK_{(u,v,s)_\alpha}(Q_0)$ *and* $H_{\rm at}^{\frac{1}{\alpha+1},v,s}(Q_0)$.

The last question in this subsection is on an interpolation result in [56]. We first recall some notation in [56]. Let $p \in (1,\infty)$, $\lambda \in \mathbb{R}$, and Q_0 be a given cube of $\mathbb{R}^n$. The *space* $N^{(p,\lambda)}(Q_0)$ is defined by setting

$$N^{(p,\lambda)}(Q_0) := \left\{u \in L^1(Q_0) :\ [u]_{N^{(p,\lambda)}(Q_0)} < \infty\right\},$$

where

$$[u]_{N^{(p,\lambda)}}(Q_0) := \sup\left\{\sum_i \left|\int_{Q_i} |u(x) - u_{Q_i}|\,dx\right|^p |Q_i|^{1-p-\lambda}\right\}^{1/p}$$

and the supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_i\}_i$ of Q_0, and u_{Q_i} is the mean of u over Q_i for any i. Let $\mathcal{F}(Q_0)$ denote the set of all simple functions on Q_0.

Definition 7 ([56], Definition 3.1)**.** *A linear operator T defined on $\mathcal{F}(Q_0)$ is said to be of* strong type $N[p,(q,\mu)]$ *if there exists a positive constant K, such that for any $u \in \mathcal{F}(Q_0)$,*

$$[Tu]_{N^{(q,\mu)}(Q_0)} \le K\|u\|_{L^p(Q_0)};$$

the smallest of the constant K for which the above inequality holds true is called the strong $N[p,(q,\mu)]$-norm.

Theorem 5 ([56], Theorem 3.1)**.** *Let $[p_i, q_i, \mu_i]$ be real numbers, such that p_i, $q_i \in [1,\infty)$ for any $i \in \{1,2\}$. If T is a linear operator which is simultaneously of strong type $N[p_i,(q_i,\mu_i)]$ with respective norms K_i $(i \in \{1,2\})$, then T is of strong type $N[p_t,(q_t,\mu)]$, where*

$$\begin{cases} \dfrac{1}{p_t} := \dfrac{1-t}{p_1} + \dfrac{t}{p_2}, \quad \dfrac{1}{q_t} := \dfrac{1-t}{q_1} + \dfrac{t}{q_2} \\ \dfrac{\mu}{q} = (1-t)\dfrac{\mu_1}{q_1} \quad \text{for} \quad t \in [0,1]. \end{cases}$$

Moreover, for any $t \in [0,1]$,

$$[Tu]_{N[p_t,(q_t,\mu)]} \le K_1^{1-t}K_2^t\|u\|_{L^p(Q_0)}.$$

The theorem also holds true in the limit case $p_1 = \infty$ and $\frac{1}{q_1} = \mu_1 = 0$.

Question 3. *In the proof of Theorem 5, lines 1–3 of [56] (p. 454), the author applied [56] (Lemma 2.3) with*

$$F[u,v,S] := \sum_i \int_{Q_i} \left[u(y) - u_{Q_i}\right] v\,dy |Q_i|^{-\lambda/p_t}$$

replaced by

$$\Phi(S,t) := \sum_i \int_{Q_i} \left[T(\widetilde{u}(y,t)) - (T\widetilde{u})_{Q_i}\right]\widetilde{v}(y,t)\,dy|Q_i|^{-\mu(t)\beta(t)}.$$

Therefore, by the proof of [56] (Lemma 2.3), we need to choose a function $\widetilde{v}$ satisfying that for any i, there exists some constant c_i, such that

$$\widetilde{v}(y,t) = c_i\overline{\left\{\mathrm{sign}\left[T(\widetilde{u}(y,t)) - (T\widetilde{u})_{Q_i}\right]\right\}} \tag{28}$$

in Q_i. Meanwhile, from the definition of $\widetilde{v}$ (see line 3 of [56], p. 452), it follows that

$$\widetilde{v}(y,t) = |v(y)|^{[1-\beta(t)]q_t'} e^{i\arg v(y)} \tag{29}$$

for some simple function $v \in \mathcal{F}(Q_0)$, where $1/q_t + 1/q_t' = 1$. To summarize, we need to find a simple function v, such that both (28) and (29) hold true, which seems unreasonable because $T\widetilde{u}$ may behave so badly even though both u and $\widetilde{u}$ are simple functions. Thus, the proof of Theorem 5 in [56] seems problematic. It is interesting to check whether or not Theorem 5 is really true.

3.2. Localized John–Nirenberg–Campanato Spaces

As a combination of the JNC space and the localized BMO space in Section 2.1, Sun et al. [36] studied the localized John–Nirenberg–Campanato space, which is new even in a special case: localized John–Nirenberg spaces. Now, we recall the definition of the localized Campanato space, which was first introduced by Goldberg in [50] (Theorem 5). In what follows, for any $s \in \mathbb{Z}_+$ and $c_0 \in (0, \ell(\mathcal{X}))$, let

$$P^{(s)}_{Q,c_0}(f) := \begin{cases} P^{(s)}_Q(f), & \ell(Q) < c_0, \\ 0, & \ell(Q) \geq c_0, \end{cases}$$

where $P^{(s)}_Q(f)$ is as in (24).

Definition 8. *Let $q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in [0,\infty)$. Fix $c_0 \in (0, \ell(\mathcal{X}))$. The* local Campanato space $\Lambda_{(\alpha,q,s)}(\mathcal{X})$ *is defined to be the set of all functions $f \in L^q_{\mathrm{loc}}(\mathcal{X})$, such that*

$$\|f\|_{\Lambda_{(\alpha,q,s)}(\mathcal{X})} := \sup |Q|^{-\alpha} \left[\fint_Q \left| f(x) - P^{(s)}_{Q,c_0}(f)(x) \right|^q dx \right]^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all cubes Q of $\mathcal{X}$.

Fix the constant $c_0 \in (0, \ell(\mathcal{X}))$. In Definition 3, if $P^{(s)}_{Q_j}(f)$ were replaced by $P^{(s)}_{Q_j,c_0}(f)$, then we obtain the following localized John–Nirenberg–Campanato space. As was mentioned in Remark 4, we naturally extend the ranges of α and p, similar to Section 3.1; we omit some identical proofs.

Definition 9. *Let p, $q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Fix the constant $c_0 \in (0, \ell(\mathcal{X}))$. The* local John–Nirenberg–Campanato space $jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X})$ *is defined to be the set of all functions $f \in L^q_{\mathrm{loc}}(\mathcal{X})$, such that*

$$\|f\|_{jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X})} := \sup \left[\sum_{j\in\mathbb{N}} |Q_j| \left\{ |Q_j|^{-\alpha} \left[\fint_{Q_j} \left| f(x) - P^{(s)}_{Q_j,c_0}(f)(x) \right|^q dx \right]^{\frac{1}{q}} \right\}^p \right]^{\frac{1}{p}}$$

is finite, where the supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_{j\in\mathbb{N}}$ of $\mathcal{X}$. Moreover, the dual space $(jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X}))^*$ *of $jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X})$ is defined to be the set of all continuous linear functionals on $jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X})$ equipped with the weak-$*$ topology.*

Remark 6. *Notice that the Campanato space and the John–Nirenberg–Campanato space are quotient spaces, while their localized versions are not.*

Furthermore, in [36] (Proposition 2.5), Sun et al. showed that $jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X})$ in Definition 9 is independent of the choice of the positive constant c_0. Therefore, in what follows, we write

$$jn_{(p,q,s)_\alpha}(\mathcal{X}) := jn_{(p,q,s)_{\alpha,c_0}}(\mathcal{X}).$$

In particular, if $q = 1$ and $s = 0 = \alpha$, then $jn_{(p,q,s)_\alpha}(\mathcal{X})$ becomes the *local John–Nirenberg space*

$$jn_p(\mathcal{X}) := jn_{(p,1,0)_0}(\mathcal{X}).$$

The following Banach structure of $jn_{(p,q,s)_\alpha}(\mathcal{X})$ is just [36] (Proposition 2.7).

Proposition 12. *Let $p, q \in [1, \infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then, $jn_{(p,q,s)_\alpha}(\mathcal{X})$ is a Banach space.*

In what follows, the *space* $jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0)$ is defined by setting

$$jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0) := \left\{f \in jn_{(p,q,s)_\alpha}(Q_0) : \|f\|_{jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0)} < \infty\right\},$$

where

$$\|f\|_{jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0)} := \inf_{a\in\mathcal{P}_s(Q_0)} \|f + a\|_{jn_{(p,q,s)_\alpha}(Q_0)};$$

the *space* $JN_{(p,q,s)_\alpha}(\mathcal{X}) \cap L^p(\mathcal{X})$ is defined by setting

$$JN_{(p,q,s)_\alpha}(\mathcal{X}) \cap L^p(\mathcal{X}) := \left\{f \in L^1_{\mathrm{loc}}(\mathcal{X}) : \|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})\cap L^p(\mathcal{X})} < \infty\right\},$$

where

$$\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})\cap L^p(\mathcal{X})} := \max\left\{\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}, \|f\|_{L^p(\mathcal{X})}\right\}.$$

Moreover, the relations between $jn_{(p,q,s)_\alpha}(\mathcal{X})$ and $JN_{(p,q,s)_\alpha}(\mathcal{X})$, namely the following Propositions 13 and 14, are just [36] (Propositions 2.9 and 2.10), respectively.

Proposition 13. *Let $p, q \in [1, \infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then,*
(i) $jn_{(p,q,s)_\alpha}(\mathcal{X}) \subset JN_{(p,q,s)_\alpha}(\mathcal{X})$;
(ii) *if Q_0 is a given cube of $\mathbb{R}^n$, then $JN_{(p,q,s)_\alpha}(Q_0) = jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0)$ with equivalent norms;*
(iii) $L^p(\mathbb{R}) \subsetneqq jn_p(\mathbb{R}) \subsetneqq JN_p(\mathbb{R})$ *if $p \in (1, \infty)$.*

Proposition 14. *Let $p \in [1, \infty)$, $q \in [1, p]$, $s \in \mathbb{Z}_+$, and $\alpha \in (0, \infty)$. Then,*

$$jn_{(p,q,s)_\alpha}(\mathcal{X}) = \left[JN_{(p,q,s)_\alpha}(\mathcal{X}) \cap L^p(\mathcal{X})\right] \tag{30}$$

with equivalent norms.

Furthermore, observe that Proposition 14 is the counterpart of [51] (Theorem 4.1), which says that for any $\alpha \in (0, \infty)$, $q \in [1, \infty)$, and $s \in \mathbb{Z}_+$,

$$\Lambda_{(\alpha,q,s)}(\mathcal{X}) = \left[C_{(\alpha,q,s)}(\mathcal{X}) \cap L^\infty(\mathcal{X})\right].$$

However, the case $q \in [p, \infty)$ in Proposition 14 is unclear so far (see Question 5 below). As an application of Propositions 13(ii) and 14, we have the following result.

Proposition 15. *Let $p \in [1, \infty)$, $q \in [1, p]$, $s \in \mathbb{Z}_+$, $\alpha \in (0, \infty)$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$JN_{(p,q,s)_\alpha}(Q_0) \subset [L^p(Q_0)/\mathcal{P}_s(Q_0)].$$

Proof. Let p, q, s, α, and Q_0 be as in this proposition. Then, by Propositions 13(ii) and 14, we obtain

$$\begin{aligned} JN_{(p,q,s)_\alpha}(Q_0) &= \left[jn_{(p,q,s)_\alpha}(Q_0)/\mathcal{P}_s(Q_0)\right] \\ &= \left\{JN_{(p,q,s)_\alpha}(Q_0) \cap [L^p(Q_0)/\mathcal{P}_s(Q_0)]\right\} \end{aligned}$$

and

$$\|\cdot\|_{JN_{(p,q,s)_\alpha}(Q_0)} \sim \inf_{a\in\mathcal{P}_s(Q_0)} \|\cdot + a\|_{jn_{(p,q,s)_\alpha}(Q_0)}$$
$$\sim \max\left\{\|\cdot\|_{JN_{(p,q,s)_\alpha}(Q_0)}, \inf_{a\in\mathcal{P}_s(Q_0)} \|\cdot + a\|_{L^p(Q_0)}\right\}.$$

This implies that $JN_{(p,q,s)_\alpha}(Q_0) \subset [L^p(Q_0)/\mathcal{P}_s(Q_0)]$ with

$$\inf_{a\in\mathcal{P}_s(Q_0)} \|\cdot + a\|_{L^p(Q_0)} \lesssim \|\cdot\|_{JN_{(p,q,s)_\alpha}(Q_0)},$$

which completes the proof of Proposition 15. □

Propositions 16 and 17 below are just, respectively, [36] (Propositions 2.12 and 2.13), which show that the localized Campanato space is the limit of the localized John–Nirenberg–Campanato space.

Proposition 16. *Let $q \in [1,\infty)$, $s \in \mathbb{Z}_+$, $\alpha \in [0,\infty)$, and Q_0 be a given cube of $\mathbb{R}^n$. Then, for any $f \in L^1(Q_0)$,*

$$\|f\|_{\Lambda_{(\alpha,q,s)}(Q_0)} = \lim_{p\to\infty} \|f\|_{jn_{(p,q,s)_\alpha}(Q_0)}.$$

Moreover,

$$\Lambda_{(\alpha,q,s)}(Q_0) = \left\{ f \in \bigcap_{p\in[1,\infty)} jn_{(p,q,s)_\alpha}(Q_0) : \lim_{p\to\infty} \|f\|_{jn_{(p,q,s)_\alpha}(Q_0)} < \infty \right\}.$$

Proposition 17. *Let $q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in [0,\infty)$. Then,*

$$\lim_{p\to\infty} jn_{(p,q,s)_\alpha}(\mathbb{R}^n) = \Lambda_{(\alpha,q,s)}(\mathbb{R}^n)$$

in the following sense: if $f \in jn_{(p,q,s)_\alpha}(\mathbb{R}^n) \cap \Lambda_{(\alpha,q,s)}(\mathbb{R}^n)$, then

$$f \in \bigcap_{r\in[p,\infty)} jn_{(r,q,s)_\alpha}(\mathbb{R}^n)$$

and

$$\|f\|_{\Lambda_{(\alpha,q,s)}(\mathbb{R}^n)} = \lim_{r\to\infty} \|f\|_{jn_{(r,q,s)_\alpha}(\mathbb{R}^n)}.$$

As in Proposition 10, the following invariance of $jn_{(p,q,s)_\alpha}(\mathcal{X})$ on its indices in the appropriate range is just [36] (Proposition 3.1).

Proposition 18. *Let $p \in (1,\infty)$, $q \in [1,p)$, $s \in \mathbb{Z}_+$, and $\alpha \in [0,\infty)$. Then,*

$$jn_{(p,q,s)_\alpha}(\mathcal{X}) = jn_{(p,1,s)_\alpha}(\mathcal{X})$$

with equivalent norms.

In other ranges of indices, namely $q \geq p$, the following relation between $jn_{(p,q,s)_\alpha}(\mathcal{X})$ and the Lebesgue space is just [36] (Proposition 3.4).

Proposition 19. *Let $s \in \mathbb{Z}_+$ and Q_0 be a given cube of $\mathbb{R}^n$.*

(i) *If $1 \leq p \leq q < \infty$, then $[|Q_0|^{\frac{1}{q}-\frac{1}{p}} jn_{(p,q,s)_0}(Q_0)] = L^q(Q_0)$ with equivalent norms.*
(ii) *If $p \in [1,\infty)$, then $jn_{(p,p,s)_0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with equivalent norms.*
(iii) *If $p, q \in [1,\infty)$, $\alpha \in (-\infty, \frac{1}{p} - \frac{1}{q})$, and $f \in jn_{(p,q,s)_\alpha}(\mathbb{R}^n)$, then $f = 0$ almost everywhere.*

Using the localized atom, Sun et al. [36] introduced the localized Hardy-type space and showed that this space is the predual of the localized John–Nirenberg–Campanato space. First, recall the definitions of localized atoms, localized polymers, and localized Hardy-type spaces in order as follows.

Definition 10. *Let $v, w \in [1,\infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Fix $c_0 \in (0, \ell(\mathcal{X}))$, and let Q denote a cube of $\mathbb{R}^n$. Then, a function a on $\mathbb{R}^n$ is called a* local $(v, w, s)_{\alpha,c_0}$-atom *supported in Q if*

(i) $\operatorname{supp}(a) := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset Q$;

(ii) $\|a\|_{L^w(Q)} \le |Q|^{\frac{1}{w}-\frac{1}{v}-\alpha}$;

(iii) *when $\ell(Q) < c_0$, $\int_Q a(x)x^\beta dx = 0$ for any $\beta \in \mathbb{Z}_+^n$ and $|\beta| \le s$.*

Definition 11. *Let $v, w \in [1,\infty]$, $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and $c_0 \in (0, \ell(\mathcal{X}))$. The* space $\widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$ *is defined to be the set of all $g \in (jn_{(v',w',s)_{\alpha,c_0}}(\mathcal{X}))^*$, such that*

$$g = \sum_{j\in\mathbb{N}} \lambda_j a_j$$

in $(jn_{(v',w',s)_{\alpha,c_0}}(\mathcal{X}))^*$, where $1/v + 1/v' = 1 = 1/w + 1/w'$, $\{a_j\}_{j\in\mathbb{N}}$ are local $(v, w, s)_{\alpha,c_0}$-atoms supported, respectively, in interior pairwise disjoint subcubes $\{Q_j\}_{j\in\mathbb{N}}$ of $\mathcal{X}$, and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ with $\|\{\lambda_j\}_{j\in\mathbb{N}}\|_{\ell^v} < \infty$ (see (27) for the definition of $\|\cdot\|_{\ell^v}$). Any $g \in \widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$ is called a local $(v, w, s)_{\alpha,c_0}$-polymer *on $\mathcal{X}$, and let*

$$\|g\|_{\widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})} := \inf\left\|\{\lambda_j\}_{j\in\mathbb{N}}\right\|_{\ell^v},$$

where the infimum is taken over all decompositions of g as above.

Definition 12. *Let $v, w \in [1,\infty]$, $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and $c_0 \in (0, \ell(\mathcal{X}))$. The* local Hardy-type space $hk_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$ *is defined to be the set of all $g \in (jn_{(v',w',s)_{\alpha,c_0}}(\mathcal{X}))^*$, such that there exists a sequence $\{g_i\}_{i\in\mathbb{N}} \subset \widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$ satisfying that $\sum_{i\in\mathbb{N}} \|g_i\|_{\widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})} < \infty$ and*

$$g = \sum_{i\in\mathbb{N}} g_i \tag{31}$$

in $(jn_{(v',w',s)_{\alpha,c_0}}(\mathcal{X}))^*$. For any $g \in hk_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$, let

$$\|g\|_{hk_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})} := \inf \sum_{i\in\mathbb{N}} \|g_i\|_{\widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})},$$

where the infimum is taken over all decompositions of g as in (31).

Correspondingly, $hk_{(v,w,s)_{\alpha,c_0}}(\mathcal{X})$ is independent of the choice of the positive constant c_0 as well, which is just [36] (Proposition 4.7).

Proposition 20. *Let $v \in (1,\infty)$, $w \in (1,\infty]$, $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and $0 < c_1 < c_2 < \ell(\mathcal{X})$. Then, $hk_{(v,w,s)_{\alpha,c_1}}(\mathcal{X}) = hk_{(v,w,s)_{\alpha,c_2}}(\mathcal{X})$ with equivalent norms.*

Henceforth, we simply write

$$\text{local } (v, w, s)_{\alpha,c_0}\text{–atoms}, \ \widetilde{hk}_{(v,w,s)_{\alpha,c_0}}(\mathcal{X}), \ \text{and } hk_{(v,w,s)_{\alpha,c_0}}(\mathcal{X}),$$

respectively, as

$$\text{local } (v, w, s)_{\alpha}\text{–atoms}, \ \widetilde{hk}_{(v,w,s)_{\alpha}}(\mathcal{X}), \ \text{and } hk_{(v,w,s)_{\alpha}}(\mathcal{X}).$$

The corresponding dual theorem (namely Theorem 6 below) is just [36] (Theorem 4.11). In what follows, the *space* $hk^{\rm fin}_{(v,w,s)_\alpha}(\mathcal{X})$ is defined to be the set of all finite linear combinations of local $(v,w,s)_\alpha$-atoms supported, respectively, in cubes of $\mathcal{X}$.

Theorem 6. *Let $v, w \in (1,\infty)$, $1/v+1/v' = 1 = 1/w+1/w' = 1$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. Then, $jn_{(v',w',s)_\alpha}(\mathcal{X}) = (hk_{(v,w,s)_\alpha}(\mathcal{X}))^*$ in the following sense:*

(i) *For any given $f \in jn_{(v',w',s)_\alpha}(\mathcal{X})$, the linear functional*

$$\mathcal{L}_f : g \longmapsto \left\langle \mathcal{L}_f, g\right\rangle := \int_{\mathcal{X}} f(x)g(x)\,dx, \qquad \forall\, g \in hk^{\rm fin}_{(v,w,s)_\alpha}(\mathcal{X})$$

can be extended to a bounded linear functional on $hk_{(v,w,s)_\alpha}(\mathcal{X})$. Moreover, it holds true that $\|\mathcal{L}_f\|_{(hk_{(v,w,s)_\alpha}(\mathcal{X}))^*} \le \|f\|_{jn_{(v',w',s)_\alpha}(\mathcal{X})}$.

(ii) *Any bounded linear functional $\mathcal{L}$ on $hk_{(v,w,s)_\alpha}(\mathcal{X})$ can be represented by a function $f \in jn_{(v',w',s)_\alpha}(\mathcal{X})$ in the following sense:*

$$\langle \mathcal{L}, g\rangle = \int_{\mathcal{X}} f(x)g(x)\,dx, \quad \forall\, g \in hk^{\rm fin}_{(v,w,s)_\alpha}(\mathcal{X}).$$

Moreover, there exists a positive constant C, depending only on s, such that $\|f\|_{jn_{(v',w',s)_\alpha}(\mathcal{X})} \le C\|\mathcal{L}\|_{(hk_{(v,w,s)_\alpha}(\mathcal{X}))^*}$.

As a corollary of Theorem 6, as well as a counterpart of Proposition 18, for any admissible (v,s,α), Proposition 21, which is just [36] (Proposition 5.1), shows that $hk_{(v,w,s)_\alpha}(\mathcal{X})$ is invariant on $w \in (v,\infty]$.

Proposition 21. *Let $v \in (1,\infty)$, $w \in (v,\infty]$, $s \in \mathbb{Z}_+$, and $\alpha \in [0,\infty)$. Then,*

$$hk_{(v,w,s)_\alpha}(\mathcal{X}) = hk_{(v,\infty,s)_\alpha}(\mathcal{X})$$

with equivalent norms.

The following proposition, which is just [36] (Proposition 5.6), might be viewed as a counterpart of Proposition 19.

Proposition 22. *Let $v \in (1,\infty)$ and $s \in \mathbb{Z}_+$.*

(i) *If $w \in (1,v]$, and Q_0 is a given cube of $\mathbb{R}^n$, then $hk_{(v,w,s)_0}(Q_0) = |Q_0|^{\frac{1}{v}-\frac{1}{w}}L^w(Q_0)$ with equivalent norms.*

(ii) *$L^v(\mathbb{R}^n) = hk_{(v,v,s)_0}(\mathbb{R}^n)$ with equivalent norms.*

Finally, the following relation between $hk_{(v,w,s)_\alpha}(\mathcal{X})$ and the atomic localized Hardy space is just [36] (Proposition 5.7).

Proposition 23. *Let $w \in (1,\infty]$ and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$\bigcup_{v\in[1,\infty)} hk_{(v,w,0)_0}(Q_0) \subset h^{1,w}_{at}(Q_0).$$

Moreover, if $g \in \bigcup_{v\in[1,\infty)} hk_{(v,w,0)_0}(Q_0)$, then

$$\|g\|_{h^{1,w}_{at}(Q_0)} \le \liminf_{v\to 1^+}\|g\|_{hk_{(v,w,0)_0}(Q_0)},$$

where $v \to 1^+$ means that $v \in (1,\infty)$ and $v \to 1$.

We also list some open questions at the end of this subsection.

Question 4. *There still exists something* unclear *in Proposition 13(iii). Precisely, let* $p \in (1, \infty)$,

$$jn_p(\mathbb{R})/\mathbb{C} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|f\|_{jn_p(\mathbb{R})/\mathbb{C}} := \inf_{c\in\mathbb{C}} \|f + c\|_{jn_p(\mathbb{R})} < \infty \right\}$$

and

$$L^p(\mathbb{R})/\mathbb{C} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|f\|_{L^p(\mathbb{R})/\mathbb{C}} := \inf_{c\in\mathbb{C}} \|f + c\|_{L^p(\mathbb{R})} < \infty \right\}.$$

Then, it is still unknown whether or not

$$\left[jn_p(\mathbb{R})/\mathbb{C}\right] \subsetneqq JN_p(\mathbb{R})$$

holds true; namely, it is still unknown whether or not there exists some non-constant *function h, such that* $h \in JN_p(\mathbb{R})$ *but* $h \notin jn_p(\mathbb{R})$*. Moreover, it is still unknown whether or not*

$$[L^p(\mathbb{R}^n)/\mathbb{C}] \subsetneqq \left[jn_p(\mathbb{R}^n)/\mathbb{C}\right] \subsetneqq JN_p(\mathbb{R}^n)$$

holds true.

The following question is on the case $q > p$ corresponding to Proposition 14.

Question 5. *Let* $p \in [1, \infty)$*,* $q \in (p, \infty)$*,* $s \in \mathbb{Z}_+$*, and* $\alpha \in (0, \infty)$*. Then, it is still unknown whether or not*

$$jn_{(p,q,s)_\alpha}(\mathcal{X}) = \left[JN_{(p,q,s)_\alpha}(\mathcal{X}) \cap L^p(\mathcal{X})\right]$$

still holds true.

Furthermore, the corresponding localized cases of Questions 1 and 2 are listed as follows. The following Question 6 is a modification of [36] (Remark 3.5), and Question 7 is just [36] (Remark 5.8).

Question 6. *Let* $p \in [1, \infty)$*,* $q \in [1, \infty)$*,* $s \in \mathbb{Z}_+$*, and* $\alpha \in [\frac{1}{p} - \frac{1}{q}, \infty)$*. Then, the relation between* $jn_{(p,q,s)_\alpha}(\mathbb{R}^n)$ *and the Riesz–Morrey space* $RM_{p,q,\alpha}(\mathbb{R}^n)$ *(see Section 4.1 for its definition) is still* unclear, *except the identity*

$$jn_{(p,p,s)_0}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = RM_{p,p,0}(\mathbb{R}^n)$$

due to Proposition 19(ii) and Theorem 8(ii), and the inclusion

$$jn_{(p,q,s)_\alpha}(\mathbb{R}^n) \supset RM_{p,q,\alpha}(\mathbb{R}^n) \quad \textit{with} \quad \|\cdot\|_{jn_{(p,q,s)_\alpha}(\mathbb{R}^n)} \lesssim \|\cdot\|_{RM_{p,q,\alpha}(\mathbb{R}^n)}$$

due to (25) *and their definitions, where the implicit positive constant is independent of the functions under consideration.*

Question 7. *Let* $v \in (1, \infty)$*,* $w \in (1, \infty]$*, and* Q_0 *be a given cube of* $\mathbb{R}^n$*.*

(i) *It is interesting to clarify the relation between* $\bigcup_{v\in(1,\infty)} hk_{(v,w,0)_0}(Q_0)$ *and* $h^{1,w}_{at}(Q_0)$*, and to find the condition on g, such that* $\|g\|_{h^{1,w}_{at}(Q_0)} = \lim_{v\to 1^+} \|g\|_{hk_{(v,w,0)_0}(Q_0)}$*.*

(ii) *Let* $\alpha \in (0, \infty)$ *and* $s \in \mathbb{Z}_+$*. As* $v \to 1^+$*, the relation between the localized atomic Hardy space (see [50] for the definition) and* $hk_{(v,w,s)_\alpha}(Q_0)$ *is still unknown.*

3.3. Congruent John–Nirenberg–Campanato Spaces

Inspired by the JNC space (see Section 3.1) and the space $\mathcal{B}$ (introduced and studied by Bourgain et al. [70]), Jia et al. [64] introduced the special John–Nirenberg–Campanato spaces via congruent cubes, which are of some amalgam features. This subsection is devoted to the main properties and some applications of congruent JNC spaces.

In what follows, for any $m \in \mathbb{Z}$, $\mathcal{D}_m(\mathbb{R}^n)$ denotes the set of all subcubes of $\mathbb{R}^n$ with side length 2^{-m}, $\mathcal{D}_m(Q_0)$ the set of all subcubes of Q_0 with side length $2^{-m}\ell(Q_0)$ for any given $m \in \mathbb{Z}_+$, and $\mathcal{D}_m(Q_0) := \emptyset$ for any given $m \in \mathbb{Z} \setminus \mathbb{Z}_+$; here and thereafter, $\ell(Q_0)$ denotes the side length of Q_0.

Definition 13. *Let $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. The* special John–Nirenberg–Campanato space via congruent cubes *(for short,* congruent JNC space*) $JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ is defined to be the set of all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, such that*

$$\|f\|_{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} := \sup_{m\in\mathbb{Z}}\left\{[f]^{(m)}_{(p,q,s)_\alpha,\mathcal{X}}\right\} < \infty,$$

where, for any $m \in \mathbb{Z}$, $[f]^{(m)}_{(p,q,s)_\alpha,\mathcal{X}}$ is defined to be

$$\sup_{\{Q_j\}_j\subset\mathcal{D}_m(\mathcal{X})}\left[\sum_j |Q_j|\left\{|Q_j|^{-\alpha}\left[\fint_{Q_j}\left|f(x) - P^{(s)}_{Q_j}(f)(x)\right|^q dx\right]^{\frac{1}{q}}\right\}^p\right]^{\frac{1}{p}}$$

with $P^{(s)}_{Q_j}(f)$ for any j as in (24) via Q replaced by Q_j and the supremum taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_j \subset \mathcal{D}_m(\mathcal{X})$. In particular, let

$$JN^{\mathrm{con}}_{p,q}(\mathcal{X}) := JN^{\mathrm{con}}_{(p,q,0)_0}(\mathcal{X}).$$

Remark 7. *Let $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. There exist some useful equivalent norms on $JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ as follows.*

(i) *(non-dyadic side length) $f \in JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ if and only if $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and*

$$\|f\|_{\widetilde{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} := \sup\left[\sum_j |Q_j|\left\{|Q_j|^{-\alpha}\left[\fint_{Q_j}\left|f(x) - P^{(s)}_{Q_j}(f)(x)\right|^q dx\right]^{\frac{1}{q}}\right\}^p\right]^{\frac{1}{p}} < \infty$$

if and only if $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and

$$\|f\|_{\widehat{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} := \sup\left[\sum_j |Q_j|\left\{|Q_j|^{-\alpha}\inf_{P\in\mathcal{P}_s(Q_j)}\left[\fint_{Q_j}|f(x) - P(x)|^q dx\right]^{\frac{1}{q}}\right\}^p\right]^{\frac{1}{p}} < \infty, \quad (32)$$

where the suprema are taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_j$ of $\mathcal{X}$ with the same side length; moreover, $\|\cdot\|_{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} \sim \|\cdot\|_{\widetilde{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} \sim \|\cdot\|_{\widehat{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})}$; see [64] (Remark 1.6(ii) and Propositions 2.6 and 2.7).

(ii) *(integral representation) In what follows, for any $y \in \mathbb{R}^n$ and $r \in (0,\infty)$, let*

$$B(y,r) := \{x \in \mathbb{R}^n : |x - y| < r\}.$$

Then $f \in JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)$ if and only if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_* := \sup_{r\in(0,\infty)}\left[\int_{\mathbb{R}^n}\left\{|B(y,r)|^{-\alpha}\left[\fint_{B(y,r)}\left|f(x) - P^{(s)}_{B(y,r)}(f)(x)\right|^q dx\right]^{\frac{1}{q}}\right\}^p dy\right]^{\frac{1}{p}} < \infty;$$

moreover, $\|\cdot\|_{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)} \sim \|\cdot\|_*$; *see [64] (Proposition 2.2) for this equivalence, which plays an essential role when establishing the boundedness of operators on congruent JNC spaces (see [71–73] for more details).*

The following proposition is just [64] (Proposition 2.10).

Proposition 24. *Let* $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, *and* Q_0 *be a given cube of* $\mathbb{R}^n$.

(i) *For any given* $p \in [1,\infty)$ *and* $q \in [1,\infty)$,

$$JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0) \subset \left[|Q_0|^{\frac{1}{p}-\frac{1}{q}-\alpha} L^q(Q_0)/\mathcal{P}_s(Q_0)\right].$$

Moreover, for any $f \in JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)$,

$$\|f\|_{|Q_0|^{\frac{1}{p}-\frac{1}{q}-\alpha} L^q(Q_0)/\mathcal{P}_s(Q_0)} \le \|f\|_{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)}.$$

(ii) *If* $\alpha \in (-\infty, 0]$, *then, for any given* $p \in [1,\infty)$ *and* $q \in [p,\infty)$,

$$JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0) = \left[|Q_0|^{\frac{1}{p}-\frac{1}{q}-\alpha} L^q(Q_0)/\mathcal{P}_s(Q_0)\right]$$

with equivalent norms.

(iii) *If* $q \in [1,\infty)$ *and* $1 \le p_1 \le p_2 < \infty$, *then* $JN^{\mathrm{con}}_{(p_2,q,s)_\alpha}(Q_0) \subset JN^{\mathrm{con}}_{(p_1,q,s)_\alpha}(Q_0)$. *Moreover, for any* $f \in JN^{\mathrm{con}}_{(p_2,q,s)_\alpha}(Q_0)$,

$$|Q_0|^{-\frac{1}{p_1}} \|f\|_{JN^{\mathrm{con}}_{(p_1,q,s)_\alpha}(Q_0)} \le |Q_0|^{-\frac{1}{p_2}} \|f\|_{JN^{\mathrm{con}}_{(p_2,q,s)_\alpha}(Q_0)}.$$

(iv) *If* $p \in [1,\infty)$ *and* $1 \le q_1 \le q_2 < \infty$, *then* $JN^{\mathrm{con}}_{(p,q_2,s)_\alpha}(\mathcal{X}) \subset JN^{\mathrm{con}}_{(p,q_1,s)_\alpha}(\mathcal{X})$. *Moreover, for any* $f \in JN^{\mathrm{con}}_{(p,q_2,s)_\alpha}(\mathcal{X})$,

$$\|f\|_{JN^{\mathrm{con}}_{(p,q_1,s)_\alpha}(\mathcal{X})} \le \|f\|_{JN^{\mathrm{con}}_{(p,q_2,s)_\alpha}(\mathcal{X})}.$$

The relation of congruent JNC spaces and Campanato spaces is similar to Proposition 6 and Corollary 2, and hence we omit the statement here; see [64] (Proposition 2.11) for details. The relation of congruent JNC spaces and the space $\mathcal{B}$ was discussed in [64] (Proposition 2.20 and Remark 2.21). Recall that the *local Sobolev space* $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by setting

$$W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) := \left\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : |\nabla f| \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\right\},$$

here and thereafter, $\nabla f := (\partial_1 f, \ldots, \partial_n f)$, where for any $i \in \{1,\ldots,n\}$, $\partial_i f$ denotes the *weak derivative* of f, namely a locally integrable function on $\mathbb{R}^n$, such that for any $\varphi \in C^\infty_c(\mathbb{R}^n)$ (the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support),

$$\int_{\mathbb{R}^n} f(x)\partial_i\varphi(x)\,dx = -\int_{\mathbb{R}^n} \varphi(x)\partial_i f(x)\,dx.$$

The following proposition is just [64] (Proposition 2.13).

Proposition 25. *Let* $p \in (1,\infty)$ *and* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. *Then,* $|\nabla f| \in L^p(\mathbb{R}^n)$ *if and only if*

$$\liminf_{m\to\infty} [f]^{(m)}_{(p,p,0)_{1/n},\mathbb{R}^n} < \infty,$$

where $[f]^{(m)}_{(p,p,0)_{1/n},\mathbb{R}^n}$ *is as in Definition 13. Moreover, for any given* $p \in [1,\infty)$*, there exists a constant* $C_{(n,p)} \in [1,\infty)$*, such that for any* $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$*,*

$$\frac{1}{C_{(n,p)}}\left[\int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx\right]^{\frac{1}{p}} \le \liminf_{m\to\infty} [f]^{(m)}_{(p,p,0)_{1/n},\mathbb{R}^n} \le C_{(n,p)}\left[\int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx\right]^{\frac{1}{p}}.$$

Remark 8. *Fusco et al. studied BMO-type seminorms and Sobolev functions in [74]. Indeed, in [74] (Theorem 2.2), Fusco et al. showed that Proposition 25 still holds true with cubes* $\{Q_j\}_j$*, in the supremum of* $[f]^{(m)}_{(p,p,0)_{1/n},\mathbb{R}^n}$*, having the same side length but an* arbitrary orientation*. Later, the main results of [74] were further extended by Di Fratta and Fiorenza in [75], via replacing a family of open cubes by a broader class of* tessellations *(from pentagonal and hexagonal tilings to space-filling polyhedrons and creative tessellations).*

The following nontriviality is just [64] (Propositions 2.16 and 2.19).

Proposition 26. *Let* $p \in (1,\infty)$ *and* $q \in [1,p)$*.*

(i) *Let* I_0 *be a given bounded interval of* $\mathbb{R}$*. Then,*

$$JN_{p,q}(I_0) \subsetneqq JN^{\mathrm{con}}_{p,q}(I_0) \quad \text{and} \quad JN_{p,q}(\mathbb{R}) \subsetneqq JN^{\mathrm{con}}_{p,q}(\mathbb{R}).$$

(ii) *Let* Q_0 *be a given cube of* $\mathbb{R}^n$*. Then,*

$$JN_{p,q}(Q_0) \subsetneqq JN^{\mathrm{con}}_{p,q}(Q_0).$$

Similar to Theorem 3, the following dual result is just [64] (Theorem 4.10). Recall that the congruent Hardy-type space $HK^{\mathrm{con}}_{(u,v,s)_\alpha}(\mathcal{X})$ is defined as in Definition 6 with the additional condition that all cubes of the polymer have the same side length (see [64], Definition 4.7, for more details).

Theorem 7. *Let* $p,\ q \in (1,\infty)$*,* $1/p = 1/p' = 1 = 1/q + 1/q'$*,* $s \in \mathbb{Z}_+$*, and* $\alpha \in \mathbb{R}$*. If* $JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ *is equipped with the norm* $\|\cdot\|_{\widetilde{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})}$ *in (32), then*

$$\left(HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X})\right)^* = JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$$

with equivalent norms in the following sense:

(i) *Any* $f \in JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ *induces a linear functional* $\mathcal{L}_f$ *which is given by setting, for any* $g \in HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X})$ *and* $\{g_i\}_i \subset \widetilde{HK}^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X})$ *with* $g = \sum_i g_i$ *in* $(JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X}))^*$*,*

$$\mathcal{L}_f(g) := \langle g, f\rangle = \sum_i \langle g_i, f\rangle.$$

Moreover, for any $g \in HK^{\mathrm{con-fin}}_{(p',q',s)_\alpha}(\mathcal{X})$*,*

$$\mathcal{L}(g) = \int_{\mathcal{X}} f(x)g(x)\,dx \quad \text{and} \quad \left\|\mathcal{L}_f\right\|_{(HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X}))^*} \le \|f\|_{\widetilde{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})}.$$

(ii) *Conversely, for any continuous linear functional* $\mathcal{L}$ *on* $HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X})$*, there exists a unique* $f \in JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$*, such that for any* $g \in HK^{\mathrm{con-fin}}_{(p',q',s)_\alpha}(\mathcal{X})$*,*

$$\mathcal{L}(g) = \int_{\mathcal{X}} f(x)g(x)\,dx \quad \text{and} \quad \|f\|_{\widetilde{JN}^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})} \le \|\mathcal{L}\|_{(HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathcal{X}))^*}.$$

Moreover, when $\mathcal{X} = Q_0$, we further have the VMO-H^1-type duality for the congruent Hardy-type space (see Theorem 25 below).

Recall that Essén et al. [76] introduced and studied the Q space on $\mathbb{R}^n$, which generalizes the space $\mathrm{BMO}(\mathbb{R}^n)$. Later, the Q space proved very useful in harmonic analysis, potential analysis, partial differential equations, and closely related fields (see, for instance, [77–79]). Thus, it is natural to consider some "new Q space" corresponding to the John–Nirenberg space JN_p. Based on Remark 7(ii), Tao et al. [80] introduced and studied the *John–Nirenberg-Q space* on $\mathbb{R}^n$ via congruent cubes, which contains the congruent John–Nirenberg space on $\mathbb{R}^n$ as special cases and also sheds some light on the mysterious John–Nirenberg space.

4. Riesz-Type Space

Observe that if we partially subtract integral means (or polynomials for high order cases) in $\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}$, namely dropping $P^{(s)}_{Q_i}(f)$ in

$$\left\{\sum_i |Q_i|\left[|Q_i|^{-\alpha}\left\{\fint_{Q_i}\left|f(x)-P^{(s)}_{Q_i}(f)(x)\right|^q dx\right\}^{\frac{1}{q}}\right]^p\right\}^{\frac{1}{p}}$$

for any i satisfying $\ell(Q_i) \geq c_0$, then we obtain the localized JNC space as in Definition 9. Thus, a natural question arises: what if we thoroughly drop all $\{P^{(s)}_{Q_i}(f)\}_i$ in $\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}$? In this section, we study the space with such a norm (subtracting all $\{P^{(s)}_{Q_i}(f)\}_i$ in the norm of the JNC space). As a bridge connecting Lebesgue and Morrey spaces via Riesz norms, it is called the "Riesz–Morrey space". For more studies on the well-known Morrey space, we refer the reader to, for instance, [81–84] and, in particular, the recent monographs by Sawano et al. [85,86].

4.1. Riesz–Morrey Spaces

As a suitable substitute of $L^\infty(\mathcal{X})$, the space $\mathrm{BMO}(\mathcal{X})$ proves very useful in harmonic analysis and partial differential equations. Recall that

$$\|f\|_{\mathrm{BMO}(\mathcal{X})} := \sup_{\text{cube } Q\subset\mathcal{X}} \fint_Q \left|f(x)-f_Q\right| dx.$$

Indeed, the only difference between them exists in subtracting integral means, which is just the following proposition. In what follows, for any $q \in (0,\infty)$ and any measurable function f, let

$$\|f\|_{L^q_*(\mathcal{X})} := \sup_{\text{cube } Q\subset\mathcal{X}} \left[\fint_Q |f(x)|^q\, dx\right]^{\frac{1}{q}}.$$

Proposition 27. *Let $q \in (0,\infty)$. Then, $f \in L^\infty(\mathcal{X})$ if and only if $f \in L^q_{\mathrm{loc}}(\mathcal{X})$ and $\|f\|_{L^q_*(\mathcal{X})} < \infty$. Moreover,*

$$\|\cdot\|_{L^\infty(\mathcal{X})} = \|\cdot\|_{L^q_*(\mathcal{X})}.$$

Proof. For the simplicity of the presentation, we only consider the case $q = 1$. On the one hand, for any $f \in L^\infty(\mathcal{X})$, it is easy to see that $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and

$$\|f\|_{L^1_*(\mathcal{X})} = \sup_{Q\subset\mathcal{X}} \fint_Q |f(x)|\,dx \le \sup_{Q\subset\mathcal{X}} \|f\|_{L^\infty(\mathcal{X})} = \|f\|_{L^\infty(\mathcal{X})}.$$

On the other hand, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $\|f\|_{L^1_*(\mathcal{X})} < \infty$, let x be any Lebesgue point of f. Then, from the Lebesgue differentiation theorem, we deduce that

$$|f(x)| = \lim_{|Q|\to 0^+,\, Q\ni x} \fint_Q |f(y)|\,dy \le \sup_{Q\subset\mathcal{X}} \fint_Q |f(y)|\,dy = \|f\|_{L^1_*(\mathcal{X})},$$

which, together with the Lebesgue differentiation theorem again, further implies that

$$\|f\|_{L^\infty(\mathcal{X})} \le \|f\|_{L^1_*(\mathcal{X})}$$

and hence $f \in L^\infty(\mathcal{X})$. Moreover, we have $\|\cdot\|_{L^\infty(\mathcal{X})} = \|\cdot\|_{L^1_*(\mathcal{X})}$. This finishes the proof of Proposition 27. □

Furthermore, if we remove integral means in the $JN_p(Q_0)$-norm

$$\|f\|_{JN_p(Q_0)} = \sup\left[\sum_i |Q_i|\left(\fint_{Q_i}\left|f(x) - f_{Q_i}\right|dx\right)^p\right]^{\frac{1}{p}},$$

where the supremum is taken over all collections of cubes $\{Q_i\}_i$ of Q_0 with pairwise disjoint interiors, then we obtain

$$\sup\left[\sum_i |Q_i|\left(\fint_{Q_i}\left|f(x)\right|dx\right)^p\right]^{\frac{1}{p}} =: \|f\|_{R_p(Q_0)}$$

which coincides with $\|f\|_{L^p(Q_0)}$ due to Riesz [41]. Corresponding to the JNC space, the following triple index Riesz-type space $R_{p,q,\alpha}(\mathcal{X})$, called the Riesz–Morrey space, was introduced and studied in [37] and, independently, by Fofana et al. [87] when $\mathcal{X} = \mathbb{R}^n$.

Definition 14. *Let $p \in [1,\infty]$, $q \in [1,\infty]$, and $\alpha \in \mathbb{R}$. The* Riesz–Morrey space *$RM_{p,q,\alpha}(\mathcal{X})$ is defined by setting*

$$RM_{p,q,\alpha}(\mathcal{X}) := \left\{f \in L^q_{\mathrm{loc}}(\mathcal{X}) :\ \|f\|_{RM_{p,q,\alpha}(\mathcal{X})} < \infty\right\},$$

where

$$\|f\|_{RM_{p,q,\alpha}(\mathcal{X})} := \begin{cases} \sup\left[\sum_i |Q_i|^{1-p\alpha-\frac{p}{q}}\|f\|^p_{L^q(Q_i)}\right]^{\frac{1}{p}} & \text{if } p \in [1,\infty),\ q \in [1,\infty], \\ \sup\sup_i |Q_i|^{-\alpha-\frac{1}{q}}\|f\|_{L^q(Q_i)} & \text{if } p = \infty,\ q \in [1,\infty] \end{cases}$$

and the suprema are taken over all collections of subcubes $\{Q_i\}_i$ of $\mathcal{X}$ with pairwise disjoint interiors. In addition, $R_{p,q,0}(\mathcal{X}) =: R_{p,q}(\mathcal{X})$.

Observe that the Riesz–Morrey norm $\|\cdot\|_{RM_{p,q,\alpha}(\mathcal{X})}$ is different from the JNC norm $\|\cdot\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})}$ with $s = 0$, only in subtracting mean oscillations (see [37], Remark 2, for more details). It is easy to see that $\|\cdot\|_{R_{p,1,0}(Q_0)} = \|\cdot\|_{R_p(Q_0)}$, and, as a generalization of the above equivalence in Riesz [41], the following proposition is just [37] (Proposition 1).

Proposition 28. *Let $p \in [1,\infty]$ and $q \in [1,p]$. Then, $f \in L^p(\mathcal{X})$ if and only if $f \in R_{p,q}(\mathcal{X})$. Moreover, $L^p(\mathcal{X}) = R_{p,q}(\mathcal{X})$ with equivalent norms, namely, for any $f \in L^q_{\rm loc}(\mathcal{X})$, $\|f\|_{L^p(\mathcal{X})} = \|f\|_{R_{p,q}(\mathcal{X})}$.*

As for the case $1 \le p < q \le \infty$, by [37] (Remark 2.3), we know that

$$R_{p,q}(\mathbb{R}^n) = \{0\} \neq L^q(\mathbb{R}^n) = R_{q,q}(\mathbb{R}^n),$$

and

$$\left[|Q_0|^{-\frac{1}{p}} R_{p,q}(Q_0)\right] = \left[|Q_0|^{-\frac{1}{q}} L^q(Q_0)\right] = \left[|Q_0|^{-\frac{1}{q}} R_{q,q}(Q_0)\right]$$

with equivalent norms.

Moreover, it is shown in [37] (Theorem 1 and Corollary 1) that the endpoint spaces of Riesz–Morrey spaces are Lebesgue spaces or Morrey spaces. In this sense, we regard the Riesz–Morrey space as a bridge connecting the Lebesgue space and the Morrey space. Thus, a natural question arises: whether or not Riesz–Morrey spaces are truly new spaces different from Lebesgue spaces or Morrey spaces. Very recently, Zeng et al. [88] gave an *affirmative* answer to this question via constructing two nontrivial functions over $\mathbb{R}^n$ and any given cube Q of $\mathbb{R}^n$. It should be pointed out that the nontrivial function on the cube Q is geometrically similar to the striking function constructed by Dafni et al. in the proof of [31] (Proposition 3.2). Furthermore, we have the following classifications of Riesz–Morrey spaces, which are just [88] (Corollary 3.7).

Theorem 8.

(i) *Let $p \in (1,\infty]$ and $q \in [1,p)$. Then,*

$$RM_{p,q,\alpha}(\mathbb{R}^n)\begin{cases} = L^q(\mathbb{R}^n) & \text{if } \alpha = \frac{1}{p}-\frac{1}{q}, \\ \supsetneqq L^{\frac{p}{1-p\alpha}}(\mathbb{R}^n) & \text{if } \alpha \in \left(\frac{1}{p}-\frac{1}{q}, 0\right), \\ = L^p(\mathbb{R}^n) & \text{if } \alpha = 0, \\ = \{0\} & \text{if } \alpha \in \left(-\infty, \frac{1}{p}-\frac{1}{q}\right) \cup (0,\infty). \end{cases}$$

In particular, if $\alpha \in (-\frac{1}{q}, 0)$, then $RM_{\infty,q,\alpha}(\mathbb{R}^n) = M_q^{-1/\alpha}(\mathbb{R}^n)$, which is just the Morrey space defined in Remark 3.

(ii) *Let $p \in [1,\infty]$ and $q \in [p,\infty]$. Then,*

$$RM_{p,q,\alpha}(\mathbb{R}^n)\begin{cases} = L^q(\mathbb{R}^n) & \text{if } \alpha = \frac{1}{p}-\frac{1}{q} = 0, \\ = \{0\} & \text{if } \alpha = \frac{1}{p}-\frac{1}{q} \neq 0, \\ = \{0\} & \text{if } \alpha \in \mathbb{R} \setminus \left\{\frac{1}{p}-\frac{1}{q}\right\}. \end{cases}$$

(iii) *Let $p \in (1,\infty]$, $q \in [1,p)$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$RM_{p,q,\alpha}(Q_0)\begin{cases} = L^q(Q_0) & \text{if } \alpha = \left(-\infty, \frac{1}{p}-\frac{1}{q}\right], \\ \supsetneqq L^{\frac{p}{1-p\alpha}}(Q_0) & \text{if } \alpha \in \left(\frac{1}{p}-\frac{1}{q}, 0\right), \\ = L^p(Q_0) & \text{if } \alpha = 0, \\ = \{0\} & \text{if } \alpha \in (0,\infty). \end{cases}$$

In particular, $RM_{\infty,q,\alpha}(Q_0) = M_q^{-1/\alpha}(Q_0)$ if $\alpha \in (-\frac{1}{q}, 0)$.

(iv) *Let $p \in [1,\infty]$, $q \in [p,\infty]$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$RM_{p,q,\alpha}(Q_0)\begin{cases} = L^q(Q_0) & \text{if } \alpha \in (-\infty, 0], \\ = \{0\} & \text{if } \alpha \in (0,\infty). \end{cases}$$

Recall that by [89] (Theorem 1), the predual space of the Morrey space is the so-called block space. Combining this with the duality of John–Nirenberg–Campanato spaces in [61] (Theorem 3.9), the authors in [37] introduced the block-type space which proves the predual of the Riesz–Morrey space. Observe that every (∞, v, α)-block in Definition 15(i) is exactly a $(v, \frac{\alpha}{n})$-*block* introduced in [89].

Definition 15. *Let* $u,\ v \in [1, \infty]$, $\frac{1}{u} + \frac{1}{u'} = 1 = \frac{1}{v} + \frac{1}{v'}$, *and* $\alpha \in \mathbb{R}$. *Let* $(RM_{u',v',\alpha}(\mathcal{X}))^*$ *be the* dual space *of* $RM_{u',v',\alpha}(\mathcal{X})$ *equipped with the weak-∗ topology.*

(i) *A function b is called a* (u, v, α)-block *if*

$$\operatorname{supp}(b) := \{x \in \mathcal{X} :\ b(x) \neq 0\} \subset Q \quad \text{and} \quad \|b\|_{L^v(Q)} \leq |Q|^{\frac{1}{v} - \frac{1}{u} - \alpha}.$$

(ii) *The* space of (u, v, α)-chains, $\widetilde{B}_{u,v,\alpha}(\mathcal{X})$, *is defined by setting*

$$\widetilde{B}_{u,v,\alpha}(\mathcal{X}) := \left\{h \in (RM_{u',v',\alpha}(\mathcal{X}))^* :\ h = \sum_j \lambda_j b_j \ \text{and} \ \left\|\left\{\lambda_j\right\}_j\right\|_{\ell^u} < \infty\right\},$$

where $\{b_j\}_j$ *are* (u, v, α)*-blocks supported, respectively, in subcubes* $\{Q_j\}$ *of* $\mathcal{X}$ *with pairwise disjoint interiors, and* $\{\lambda_j\}_j \subset \mathbb{C}$ *with* $\|\{\lambda_j\}_j\|_{\ell^u} < \infty$ *(see (27) for the definition of* $\|\cdot\|_{\ell^u}$*). Moreover, any* $h \in \widetilde{B}_{u,v,\alpha}(\mathcal{X})$ *is called a* (u, v, α)-chain, *and its norm is defined by setting*

$$\|h\|_{\widetilde{B}_{u,v,\alpha}(\mathcal{X})} := \inf\left\|\left\{\lambda_j\right\}_j\right\|_{\ell^u},$$

where the infimum is taken over all decompositions of h as above.

(iii) *The* block-type space $B_{u,v,\alpha}(\mathcal{X})$ *is defined by setting*

$$B_{u,v,\alpha}(\mathcal{X}) := \left\{g \in (RM_{u',v',\alpha}(\mathcal{X}))^* :\ g = \sum_i h_i \ \text{and} \ \sum_i \|h_j\|_{\widetilde{B}_{u,v,\alpha}(\mathcal{X})} < \infty\right\},$$

where $\{h_i\}_i$ *are* (u, v, α)*-chains. Moreover, for any* $g \in B_{u,v,\alpha}(\mathcal{X})$,

$$\|g\|_{B_{u,v,\alpha}(\mathcal{X})} := \inf \sum_i \|h_j\|_{\widetilde{B}_{u,v,\alpha}(\mathcal{X})},$$

where the infimum is taken over all decompositions of g as above.

(iv) *The* finite block-type space $B^{\mathrm{fin}}_{u,v,\alpha}(\mathcal{X})$ *is defined to be the set of all finite summations*

$$\sum_{m=1}^{M} \lambda_m b_m,$$

where $M \in \mathbb{N}$, $\{\lambda_m\}_{m=1}^M \subset \mathbb{C}$, *and* $\{b_m\}_{m=1}^M$ *are* (u, v, α)*-blocks.*

The following dual theorem is just [37] (Theorem 2).

Theorem 9. *Let* $p,\ q \in (1, \infty)$, $1/p + 1/p' = 1 = 1/q + 1/q'$, *and* $\alpha \in \mathbb{R}$. *Then,* $(B_{p',q',\alpha}(\mathcal{X}))^* = RM_{p,q,\alpha}(\mathcal{X})$ *in the following sense:*

(i) *If* $f \in RM_{p,q,\alpha}(\mathcal{X})$, *then* f *induces a linear functional* $\mathcal{L}_f$ *on* $B_{p',q',\alpha}(\mathcal{X})$ *with*

$$\|\mathcal{L}_f\|_{(B_{p',q',\alpha}(\mathcal{X}))^*} \leq C\|f\|_{RM_{p,q,\alpha}(\mathcal{X})},$$

where C is a positive constant independent of f.

(ii) *If* $\mathcal{L} \in (B_{p',q',\alpha}(\mathcal{X}))^*$, *then there exists some* $f \in RM_{p,q,\alpha}(\mathcal{X})$, *such that for any* $g \in B^{\mathrm{fin}}_{p',q',\alpha}(\mathcal{X})$,

$$\mathcal{L}(g) = \int_{\mathcal{X}} f(x)g(x)\,dx,$$

and

$$\|\mathcal{L}\|_{(B_{p',q',\alpha}(\mathcal{X}))^*} \sim \|f\|_{RM_{p,q,\alpha}(\mathcal{X})}$$

with the positive equivalence constants independent of f.

Furthermore, for the Riesz–Morrey space, there exist three open questions unsolved so far. The first question is on the relation between the Riesz–Morrey space and the weak Lebesgue space.

Question 8. *Let* $p \in (1,\infty)$, $q \in [1,p)$, *and* $\alpha \in (\frac{1}{p} - \frac{1}{q}, 0)$. *Then, Zeng et al. ([88], Remark 3.4) showed that*

$$RM_{p,q,\alpha}(\mathbb{R}^n) \nsubseteq L^{\frac{p}{1-p\alpha},\infty}(\mathbb{R}^n) \nsubseteq RM_{p,q,\alpha}(\mathbb{R}^n),$$

which implies that on $\mathbb{R}^n$, *the Riesz–Morrey space and the weak Lebesgue space do not cover each other. Furthermore, for a given cube* Q_0 *of* $\mathbb{R}^n$, *Zeng et al. ([88], Remark 3.6) showed that*

$$L^{\frac{p}{1-p\alpha},\infty}(Q_0) \nsubseteq RM_{p,q,\alpha}(Q_0).$$

However, it is still unknown whether or not

$$RM_{p,q,\alpha}(Q_0) \nsubseteq L^{\frac{p}{1-p\alpha},\infty}(Q_0)$$

still holds true. This question was posed in [88] (Remark 3.6), and is still unclear.

The following Questions 9 and 10 are just [37] (Remarks 4 and 5), respectively.

Question 9. *As a counterpart of* (26), *for any given* $p \in [1,\infty)$, $q \in [1,p)$, $s \in \mathbb{Z}_+$, *and* $\alpha \in [\frac{1}{p} - \frac{1}{q}, 0)$, *it is interesting to ask whether or not*

$$JN_{(p,q,s)_\alpha}(\mathcal{X}) = \left[RM_{p,q,\alpha}(\mathcal{X})/\mathcal{P}_s(\mathcal{X})\right]$$

and, for any $f \in JN_{(p,q,s)_\alpha}(\mathcal{X})$,

$$\|f\|_{JN_{(p,q,s)_\alpha}(\mathcal{X})} \sim \left\|f - \sigma(f)\right\|_{RM_{p,q,\alpha}(\mathcal{X})},$$

with the positive equivalence constants independent of f, *still hold true. This is still* unclear.

Question 10. *Recall that for any given* $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ *and any* $x \in \mathcal{X}$, *the* Hardy–Littlewood maximal function $\mathcal{M}(f)(x)$ *is defined by setting*

$$\mathcal{M}(f)(x) := \sup_{Q \ni x} \fint_Q |f(y)|\,dy, \tag{33}$$

where the supremum is taken over all cubes Q *containing* x. *Meanwhile,* $\mathcal{M}$ *is called the* Hardy–Littlewood maximal operator. *It is well known that* $\mathcal{M}$ *is bounded on* $L^q(\mathcal{X})$ *for any given* $q \in (1,\infty]$ *(see, for instance, [42], p. 31, Theorem 2.5). Moreover,* $\mathcal{M}$ *is also bounded on* $M_q^{-1/\alpha}(\mathcal{X})$ *for any given* $q \in (1,\infty]$ *and* $\alpha \in [-\frac{1}{q}, 0]$ *(see, for instance, [90], Theorem 1). To summarize, the boundedness of* $\mathcal{M}$ *on endpoint spaces of Riesz–Morrey spaces (Lebesgue spaces and Morrey spaces) has already been obtained. Therefore, it is very interesting to ask whether or not* $\mathcal{M}$ *is* *bounded on the Riesz–Morrey space $RM_{p,q,\alpha}(\mathcal{X})$ with $p \in (1,\infty]$, $q \in [1,p)$, and $\alpha \in (\frac{1}{p}-\frac{1}{q},0)$. This is a challenging and important problem which is still open.*

4.2. Congruent Riesz–Morrey Spaces

To obtain the boundedness of several important operators, we next consider a special Riesz–Morrey space via congruent cubes, denoted by $RM_{p,q,\alpha}(\mathbb{R}^n)$, as in Section 3.3. In this subsection, we first recall the definition of $RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$, and then review the boundedness of the Hardy–Littlewood maximal operator on this space.

Definition 16. *Let p, $q \in [1,\infty]$, and $\alpha \in \mathbb{R}$. The* special Riesz–Morrey space via congruent cubes *(for short,* congruent Riesz–Morrey space*) $RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions f on $\mathbb{R}^n$, such that*

$$\|f\|_{RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)} := \begin{cases} \sup\left[\sum_j |Q_j|^{1-p\alpha-\frac{p}{q}}\|f\|^p_{L^q(Q_i)}\right]^{\frac{1}{p}}, & p \in [1,\infty), \\ \sup_{\text{cube } Q\subset\mathbb{R}^n} |Q|^{-\alpha-\frac{1}{q}}\|f\|_{L^q(Q)}, & p = \infty \end{cases}$$

is finite, where the first supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_j$ of $\mathbb{R}^n$ with the same side length.

Remark 9.

(i) *If we do not require that $\{Q_j\}_j$ has the same size in the definition of congruent Riesz–Morrey spaces, then it is just the Riesz–Morrey space $RM_{p,q,\alpha}(\mathbb{R}^n)$ in Section 4.1.*

(ii) *If $p=\infty$, $q \in (0,\infty)$, and $\alpha \in [-\frac{1}{q},0)$, then $RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$ in Definition 16 coincides with the Morrey space $M_q^{-1/\alpha}(\mathbb{R}^n)$ in Remark 3.*

(iii) *Similar to Remark 7, for any given p, $q \in [1,\infty)$, and $\alpha \in \mathbb{R}$, $f \in RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$ if and only if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$\|f\|_{\widetilde{RM}^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)} := \sup_{r\in(0,\infty)}\left[\int_{\mathbb{R}^n}\left\{|B(y,r)|^{-\alpha}\left[\fint_{B(y,r)}\left|f(x)\right|^q dx\right]^{\frac{1}{q}}\right\}^p dy\right]^{\frac{1}{p}}$$

is finite; moreover,

$$\|\cdot\|_{RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)} \sim \|\cdot\|_{\widetilde{RM}^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)};$$

see [71] for more details. Recall that for any $y \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$B(y,r) := \{x \in \mathbb{R}^n : |x-y| < r\}.$$

(iv) *If $1 \le q < \alpha < p \le \infty$, then the space $RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$ coincides with the amalgam space $(L^q,\ell^p)^{\frac{p}{1-p\alpha}}(\mathbb{R}^n)$, which was introduced by Fofana [91]. (See [87,92–96] for more studies on the amalgam space.)*

The following boundedness of the Hardy–Littlewood maximal operator on congruent Riesz–Morrey spaces was obtained in [71].

Theorem 10. *Let p, $q \in (1,\infty)$, $\alpha \in \mathbb{R}$, and $\mathcal{M}$ be the Hardy–Littlewood maximal operator as in (33). Then $\mathcal{M}$ is bounded on $RM^{\mathrm{con}}_{p,q,\alpha}(\mathbb{R}^n)$.*

Moreover, via Theorem 10, Jia et al. [71] also established the boundedness of Calderón–Zygmund operators on congruent Riesz–Morrey spaces.

Finally, since a congruent Riesz–Morrey space is a *ball Banach function space*, we refer the reader to [49] for the equivalent characterizations of the boundedness and the compactness of Calderón–Zygmund commutators on ball Banach function spaces. It should be mentioned that a crucial assumption in [49] is the boundedness of $\mathcal{M}$, and hence Theorem 10 provides an essential tool when studying the boundedness of operators on congruent Riesz–Morrey spaces.

5. Vanishing Subspace

In this section, we focus on several vanishing subspaces of aforementioned John–Nirenberg-type spaces. In what follows, $C^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$; $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$; for any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$; for any given normed linear space $\mathcal{Y}$ and any given its subset $\mathcal{X}$, $\overline{\mathcal{X}}^{\mathcal{Y}}$ denotes the *closure* of the set $\mathcal{X}$ in $\mathcal{Y}$ in terms of the topology of $\mathcal{Y}$; and if $\mathcal{Y} = \mathbb{R}^n$, we then denote $\overline{\mathcal{X}}^{\mathcal{Y}}$ simply by $\overline{\mathcal{X}}$.

5.1. Vanishing BMO Spaces

We now recall several vanishing subspaces of the space $\mathrm{BMO}(\mathbb{R}^n)$.

- $\mathrm{VMO}(\mathbb{R}^n)$, introduced by Sarason [6], is defined by setting

$$\mathrm{VMO}(\mathbb{R}^n) := \overline{C_{\mathrm{u}}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)},$$

  where $C_{\mathrm{u}}(\mathbb{R}^n)$ denotes the set of all uniformly continuous functions on $\mathbb{R}^n$.
- $\mathrm{CMO}(\mathbb{R}^n)$, announced in Neri [97], is defined by setting

$$\mathrm{CMO}(\mathbb{R}^n) := \overline{C_{\mathrm{c}}^\infty(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)},$$

  where $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. In addition, by approximations of the identity, it is easy to find that

$$\mathrm{CMO}(\mathbb{R}^n) = \overline{C_{\mathrm{c}}(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)} = \overline{C_0(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)}, \tag{34}$$

  where $C_{\mathrm{c}}(\mathbb{R}^n)$ denotes the set of all functions on $\mathbb{R}^n$ with compact support, and $C_0(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$ which vanish at the infinity.
- $\mathrm{MMO}(\mathbb{R}^n)$, introduced by Torres and Xue [98], is defined by setting

$$\mathrm{MMO}(\mathbb{R}^n) := \overline{A_\infty(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)},$$

  where

$$A_\infty(\mathbb{R}^n) := \left\{ b \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) : \ \forall\, \alpha \in \mathbb{Z}_+^n \setminus \{\mathbf{0}\}, \lim_{|x|\to\infty} \partial^\alpha b(x) = 0 \right\}.$$

- $\mathrm{XMO}(\mathbb{R}^n)$, introduced by Torres and Xue [98], is defined by setting

$$\mathrm{XMO}(\mathbb{R}^n) := \overline{B_\infty(\mathbb{R}^n)}^{\mathrm{BMO}(\mathbb{R}^n)},$$

  where

$$B_\infty(\mathbb{R}^n) := \left\{ b \in C^\infty(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n) : \ \forall\, \alpha \in \mathbb{Z}_+^n \setminus \{\mathbf{0}\}, \lim_{|x|\to\infty} \partial^\alpha b(x) = 0 \right\}.$$

- $\mathrm{X_1MO}\,(\mathbb{R}^n)$, introduced by Tao et al. [99], is defined by setting

$$\mathrm{X_1MO}\,(\mathbb{R}^n) := \overline{B_1(\mathbb{R}^n)}^{\mathrm{BMO}\,(\mathbb{R}^n)},$$

where

$$B_1(\mathbb{R}^n) := \left\{ b \in C^1(\mathbb{R}^n) \cap \mathrm{BMO}\,(\mathbb{R}^n) : \lim_{|x|\to\infty} |\nabla b(x)| = 0 \right\}$$

with $C^1(\mathbb{R}^n)$ being the set of all functions f on $\mathbb{R}^n$ whose gradients $\nabla f := (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ are continuous.

The relation of these vanishing subspaces reads as follows.

Proposition 29. $\mathrm{CMO}\,(\mathbb{R}^n) \subsetneqq \mathrm{MMO}\,(\mathbb{R}^n) \subsetneqq \mathrm{XMO}\,(\mathbb{R}^n) = \mathrm{X_1MO}\,(\mathbb{R}^n) \subsetneqq \mathrm{VMO}\,(\mathbb{R}^n)$.

Indeed,

$$\mathrm{CMO}\,(\mathbb{R}^n) \subsetneqq \mathrm{MMO}\,(\mathbb{R}^n) \subsetneqq \mathrm{XMO}\,(\mathbb{R}^n)$$

was obtained in [98] (p. 5). Moreover,

$$\mathrm{XMO}\,(\mathbb{R}^n) = \mathrm{X_1MO}\,(\mathbb{R}^n) \subsetneqq \mathrm{VMO}\,(\mathbb{R}^n)$$

was obtained in [99] (Corollary 1.3), which completely answered the open question proposed in [98] (p. 6).

Next, we investigate the mean oscillation characterizations of these vanishing subspaces. Recall that, for any cube Q of $\mathbb{R}^n$, and any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the *mean oscillation* $O(f;Q)$ is defined by setting

$$O(f;Q) := \fint_Q \left| f(x) - f_Q \right| dx = \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f(y)\, dy \right| dx.$$

The earliest results of $\mathrm{VMO}\,(\mathbb{R}^n)$ were obtained by Sarason in [6], and Theorem 11 below is a part of [6] (Theorem 1). In what follows, $a \to 0^+$ means $a \in (0,\infty)$ and $a \to 0$.

Theorem 11. *$f \in \mathrm{VMO}\,(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}\,(\mathbb{R}^n)$ and*

$$\lim_{a\to 0^+} \sup_{|Q|=a} O(f;Q) = 0.$$

The following equivalent characterization of $\mathrm{CMO}\,(\mathbb{R}^n)$ is just Uchiyama ([7], p. 166).

Theorem 12. *$f \in \mathrm{CMO}\,(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}\,(\mathbb{R}^n)$ and satisfies the following three conditions:*

(i) $\lim_{a\to 0^+} \sup_{|Q|=a} O(f;Q) = 0$;
(ii) *for any cube Q of $\mathbb{R}^n$,* $\lim_{|x|\to\infty} O(f;Q+x) = 0$;
(iii) $\lim_{a\to \infty} \sup_{|Q|=a} O(f;Q) = 0$.

Very recently, Tao et al. obtained the following equivalent characterization of $\mathrm{XMO}\,(\mathbb{R}^n)$ and $\mathrm{X_1MO}\,(\mathbb{R}^n)$, which is just [99] (Theorem 1.2).

Theorem 13. *The following statements are mutually equivalent:*

(i) $f \in \mathrm{X_1MO}\,(\mathbb{R}^n)$;
(ii) *$f \in \mathrm{BMO}\,(\mathbb{R}^n)$ and enjoys the properties that*
  a) $\lim_{a\to 0^+} \sup_{|Q|=a} O(f;Q) = 0$;

b) *for any cube Q of $\mathbb{R}^n$, $\lim_{|x|\to\infty} \mathcal{O}(f;Q+x)=0$.*

(iii) $f \in \mathrm{XMO}\,(\mathbb{R}^n)$.

Remark 10. *Proposition 12*(ii) *can be replaced by*

(ii') $\lim_{M\to\infty} \sup_{Q\cap Q(\mathbf{0},M)=\emptyset} \mathcal{O}(f;Q)=0,$

where $Q(\mathbf{0},M)$ denotes the cube centered at $\mathbf{0}$ with the side length M. However, $(\mathrm{ii})_2$ *of Theorem 13*(ii) *can not be replaced by* (ii') *(see [99], Proposition 2.5, for more details).*

However, the equivalent characterization of $\mathrm{MMO}\,(\mathbb{R}^n)$ is still unknown (see [99], Proposition 2.5 and Remark 2.6, for more details on the following open question.)

Question 11. *It is interesting to find the equivalent characterization of* $\mathrm{MMO}\,(\mathbb{R}^n)$*, as well as its localized counterpart (see Question 14), via the mean oscillations.*

As for the applications of these vanishing subspaces, we know that the commutator $[b,T]$, generated by $b \in \mathrm{BMO}\,(\mathbb{R}^n)$ and the Calderón–Zygmund operator T, plays an important role in harmonic analysis, complex analysis, partial differential equations, and other fields in mathematics. Here, we only list several typical *bilinear* results; other *linear* and *multi-linear* results can be found, for instance, in [22,100,101] and their references.

In what follows, let $\mathbb{Z}_+^{3n} := (\mathbb{Z}_+)^{3n}$ and $L_c^\infty(\mathbb{R}^n)$ denote the set of all functions $f \in L^\infty(\mathbb{R}^n)$ with compact support. We now consider the following particular type of bilinear Calderón–Zygmund operator T, whose kernel K satisfies

(i) The standard *size* and *regularity* conditions: for any multi-index $\alpha := (\alpha_1,\ldots,\alpha_{3n}) \in \mathbb{Z}_+^{3n}$ with $|\alpha| := \alpha_1+\cdots+\alpha_{3n} \le 1$, there exists a positive constant $C_{(\alpha)}$, depending on α, such that for any $x,\ y,\ z \in \mathbb{R}^n$ with $x \ne y$ or $x \ne z$,

$$|\partial^\alpha K(x,y,z)| \le C_{(\alpha)}(|x-y|+|x-z|)^{-2n-|\alpha|}. \tag{35}$$

Here and thereafter, $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_{3n}})^{\alpha_{3n}}$.

(ii) The additional decay condition: there exist positive constants C and δ, such that for any $x,\ y,\ z \in \mathbb{R}^n$ with $|x-y|+|x-z|>1$,

$$|K(x,y,z)| \le C(|x-y|+|x-z|)^{-2n-2-\delta}, \tag{36}$$

and for any $f,\ g \in L_c^\infty(\mathbb{R}^n)$ and $x \notin \mathrm{supp}\,(f) \cap \mathrm{supp}\,(g)$, T is supposed to have the following usual representation:

$$T(f,g)(x) = \int_{\mathbb{R}^{2n}} K(x,y,z)f(y)g(z)\,dy\,dz,$$

here and thereafter, $\mathrm{supp}\,(f) := \{x \in \mathbb{R}^n :\ f(x) \ne 0\}$. Notice that the (inhomogeneous) Coifman–Meyer bilinear Fourier multipliers and the bilinear pseudodifferential operators with certain symbols satisfy the above two conditions (see, for instance, [98] and references therein).

Recall that, usually, a non-negative measurable function w on $\mathbb{R}^n$ is called a *weight* on $\mathbb{R}^n$. For any given $\mathbf{p} := (p_1,p_2) \in (1,\infty)\times(1,\infty)$, let p satisfy $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$. Following [10], we call $\mathbf{w} := (w_1,w_2)$ a *vector* $\mathbf{A_p}(\mathbb{R}^n)$ *weight*, denoted by $\mathbf{w} := (w_1,w_2) \in \mathbf{A_p}(\mathbb{R}^n)$, if

$$[\mathbf{w}]_{\mathbf{A_p}(\mathbb{R}^n)} := \sup_Q\left[\frac{1}{|Q|}\int_Q w(x)\,dx\right]\left\{\frac{1}{|Q|}\int_Q [w_1(x)]^{1-p_1'}\,dx\right\}^{\frac{p}{p_1'}} \times\left\{\frac{1}{|Q|}\int_Q [w_2(x)]^{1-p_2'}\,dx\right\}^{\frac{p}{p_2'}} < \infty,$$

where $w := w_1^{p/p_1} w_2^{p/p_2}$, $1/p_1 + 1/p_1' = 1 = 1/p_2 + 1/p_2'$, and the supremum is taken over all cubes Q of $\mathbb{R}^n$. In what follows, for any given weight w on $\mathbb{R}^n$ and any measurable subset $E \subsetneqq \mathbb{R}^n$, the *symbol* $L_w^p(E)$, with $p \in (0, \infty)$, denotes the set of all measurable functions f on E, such that

$$\|f\|_{L_w^p(E)} := \left[\int_E |f(x)|^p w(x)\, dx\right]^{\frac{1}{p}} < \infty,$$

and, when $w \equiv 1$, we write $L_w^p(E) =: L^p(E)$. Furthermore, $\|\cdot\|_{L^\infty(E)}$ represents the essential supremum on E.

In addition, recall that the *bilinear commutators* $[b, T]_1$ and $[b, T]_2$ are defined, respectively, by setting, for any $f,\ g \in L_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f) \cap \operatorname{supp}(g)$,

$$\begin{aligned} [b, T]_1(f, g)(x) &:= (bT(f, g) - T(bf, g))(x) \\ &= \int_{\mathbb{R}^{2n}} [b(x) - b(y)] K(x, y, z) f(y) g(z)\, dy\, dz \end{aligned} \tag{37}$$

and

$$\begin{aligned} [b, T]_2(f, g)(x) &:= (bT(f, g) - T(f, bg))(x) \\ &= \int_{\mathbb{R}^{2n}} [b(x) - b(z)] K(x, y, z) f(y) g(z)\, dy\, dz. \end{aligned} \tag{38}$$

The following theorem, obtained in [11] (Theorem 1) for any given $p \in (1, \infty)$ and in [102] (Theorem 1) for any given $p \in (\frac{1}{2}, 1]$, showed that the bilinear commutators $\{[b, T]_i\}_{i=1,2}$ are compact for $b \in \mathrm{CMO}(\mathbb{R}^n)$.

Theorem 14. *Let $(p_1, p_2) \in (1, \infty) \times (1, \infty)$, $p \in (\frac{1}{2}, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $b \in \mathrm{CMO}(\mathbb{R}^n)$, and T be a bilinear Calderón–Zygmund operator whose kernel satisfies (35). Then, for any $i \in \{1, 2\}$, the bilinear commutator $[b, T]_i$ as in (37) or (38) is compact from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

If we require an extra additional decay (36) for the Calderón–Zygmund kernel in Theorem 14, we can then replace $\mathrm{CMO}(\mathbb{R}^n)$ by $\mathrm{XMO}(\mathbb{R}^n)$, that is, delete condition (iii) in Theorem 12 of $\mathrm{CMO}(\mathbb{R}^n)$. This new compactness result was first obtained in [98] (Theorem 1.1) and then generalized into the weighted case, namely the following Theorem 15, which is just [99] (Theorem 1.4).

Theorem 15. *Let $\mathbf{p} := (p_1, p_2) \in (1, \infty) \times (1, \infty)$, $p \in (\frac{1}{2}, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\mathbf{w} := (w_1, w_2) \in \mathbf{A_p}(\mathbb{R}^n)$, $w := w_1^{p/p_1} w_2^{p/p_2}$, $b \in \mathrm{XMO}(\mathbb{R}^n)$, and T be a bilinear Calderón–Zygmund operator whose kernel satisfies (35) and (36). Then, for any $i \in \{1, 2\}$, the bilinear commutator $[b, T]_i$ as in (37) or (38) is compact from $L_{w_1}^{p_1}(\mathbb{R}^n) \times L_{w_2}^{p_2}(\mathbb{R}^n)$ to $L_w^p(\mathbb{R}^n)$.*

On the other hand, if the kernel behaves "good", such as the *Riesz transforms* $\{\mathcal{R}_j\}_{j=1}^n$:

$$\mathcal{R}_j(f)(x) := \mathrm{p.v.}\, \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x - y)\, dy,$$

then the reverse of Theorem 14 holds true as well (see, for instance, the following Theorem 16, which is just [103], Theorem 3.1). Moreover, it should be mentioned that the linear case of Theorem 16 was obtained by Uchiyama ([7], Theorem 2).

Theorem 16. *Let $(p_1, p_2) \in (1, \infty) \times (1, \infty)$ and $p \in (\frac{1}{2}, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, for any $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$, the bilinear commutator $[b, \mathcal{R}_j]_i$ is compact from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$.*

However, the corresponding equivalent characterization of $\mathrm{XMO}(\mathbb{R}^n)$ is still unknown. For simplicity, we state this question in the unweighted case.

Question 12. *Let $(p_1, p_2) \in (1, \infty) \times (1, \infty)$, and $p \in (\frac{1}{2}, \infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, it is interesting to find some bilinear Calderón–Zygmund operator T, such that for any $i \in \{1, 2\}$, the bilinear commutator $[b, T]_i$ is compact from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$* if and only if *$b \in \mathrm{XMO}(\mathbb{R}^n)$.*

Next, recall the Riesz transform characterizations of $\mathrm{BMO}(\mathbb{R}^n)$ and its vanishing subspaces.

Theorem 17. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then,*

(i) *([2], Theorem 3) $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if there exist functions $\{f_j\}_{j=0}^n \subset L^\infty(\mathbb{R}^n)$, such that*

$$f = f_0 + \sum_{j=1}^{n} \mathcal{R}_j(f_j)$$

and

$$C^{-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \sum_{j=0}^{n} \left\|f_j\right\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \tag{39}$$

for some positive constant C independent of f and $\{f_j\}_{j=0}^n$.

(ii) *([6], Theorem 1) $f \in \mathrm{VMO}(\mathbb{R}^n)$ if and only if there exist functions $\{f_j\}_{j=0}^n \subset [C_{\mathrm{u}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)]$, such that*

$$f = f_0 + \sum_{j=1}^{n} \mathcal{R}_j(f_j)$$

and (39) holds true in this case.

(iii) *([97], p. 185) $f \in \mathrm{CMO}(\mathbb{R}^n)$ if and only if there exist functions $\{f_j\}_{j=0}^n \subset C_0(\mathbb{R}^n)$, such that*

$$f = f_0 + \sum_{j=1}^{n} \mathcal{R}_j(f_j)$$

and (39) holds true in this case.

Question 13. *Since the Riesz transform is well defined on $L^\infty(\mathbb{R}^n)$, it is interesting to find the counterpart of Theorem 17 when $f \in \mathrm{MMO}(\mathbb{R}^n)$. Moreover, since the Riesz transform characterization is useful when proving the duality of the CMO-H^1 type, it is also interesting to find the dual spaces of $\mathrm{MMO}(\mathbb{R}^n)$ and $\mathrm{XMO}(\mathbb{R}^n)$.*

When $\mathbb{R}^n$ is replaced by some cube Q_0 with finite side length, we then have $\mathrm{VMO}(Q_0) = \mathrm{CMO}(Q_0)$ (see [104] for more details). Moreover, the vanishing subspace on the *spaces of homogeneous type*, denoted by $\mathfrak{X}$, was studied in Coifman et al. [5], and they proved $(\mathcal{VMO}(\mathfrak{X}))^* = H^1(\mathfrak{X})$, where $\mathcal{VMO}(\mathfrak{X})$ denotes the closure in $\mathrm{BMO}(\mathfrak{X})$ of continuous functions on $\mathfrak{X}$ with compact support. Notice that when $\mathfrak{X} = \mathbb{R}^n$, by (34), we have $\mathcal{VMO}(\mathfrak{X}) = \mathcal{VMO}(\mathbb{R}^n) = \mathrm{CMO}(\mathbb{R}^n)$.

Finally, we consider the localized version of these vanishing subspaces. The following characterization of local $\mathrm{VMO}(\mathbb{R}^n)$ is a part of [105] (Theorem 1).

Proposition 30. *Let* $\mathrm{vmo}\,(\mathbb{R}^n)$ *be the closure of* $C_u(\mathbb{R}^n) \cap \mathrm{bmo}\,(\mathbb{R}^n)$ *in* $\mathrm{bmo}\,(\mathbb{R}^n)$*. Then,* $f \in \mathrm{vmo}\,(\mathbb{R}^n)$ *if and only if* $f \in \mathrm{bmo}\,(\mathbb{R}^n)$ *and*

$$\lim_{a\to 0^+} \sup_{|Q|=a} \mathcal{O}(f;Q) = 0.$$

Moreover, the following localized result of $\mathrm{CMO}\,(\mathbb{R}^n)$ is just Dafni ([104], Theorem 6) (see also [105], Theorem 3).

Theorem 18. *Let* $\mathrm{cmo}\,(\mathbb{R}^n)$ *be the closure of* $C_0(\mathbb{R}^n)$ *in* $\mathrm{bmo}\,(\mathbb{R}^n)$*. Then,* $f \in \mathrm{cmo}\,(\mathbb{R}^n)$ *if and only if* $f \in \mathrm{bmo}\,(\mathbb{R}^n)$ *and*

$$\lim_{a\to 0^+} \sup_{|Q|=a} \mathcal{O}(f;Q) = 0 = \lim_{M\to\infty} \sup_{|Q|>1,\, Q\cap Q(\mathbf{0},M)=\emptyset} \fint_Q |f|.$$

In addition, the localized version of Theorem 17 can be found in [50] (Corollary 1) for $\mathrm{bmo}\,(\mathbb{R}^n)$, and in [105] (Theorems 1 and 3) for $\mathrm{vmo}\,(\mathbb{R}^n)$ and $\mathrm{cmo}\,(\mathbb{R}^n)$, respectively.

Question 14. *Let* $\mathrm{mmo}\,(\mathbb{R}^n)$*,* $\mathrm{xmo}\,(\mathbb{R}^n)$*, and* $\mathrm{x_1mo}\,(\mathbb{R}^n)$ *be, respectively, the closure in* $\mathrm{bmo}\,(\mathbb{R}^n)$ *of* $A_\infty(\mathbb{R}^n)$*,* $B_\infty(\mathbb{R}^n)$*, and* $B_1(\mathbb{R}^n)$*. It is interesting to find the counterparts of*

(i) *Theorem 18 with* $\mathrm{cmo}\,(\mathbb{R}^n)$ *replaced by* $\mathrm{xmo}\,(\mathbb{R}^n)$*;*
(ii) *Theorem 13 with* $\mathrm{XMO}\,(\mathbb{R}^n)$ *and* $\mathrm{X_1MO}\,(\mathbb{R}^n)$ *replaced, respectively, by* $\mathrm{xmo}\,(\mathbb{R}^n)$ *and* $\mathrm{x_1mo}\,(\mathbb{R}^n)$*;*
(iii) *Question 13 with* $\mathrm{MMO}\,(\mathbb{R}^n)$ *replaced by* $\mathrm{mmo}\,(\mathbb{R}^n)$*;*
(iv) *The dual result* $(\mathrm{cmo}\,(\mathbb{R}^n))^* = h^1(\mathbb{R}^n)$*, in ([104], Theorem 9), with* $\mathrm{cmo}\,(\mathbb{R}^n)$ *replaced by* $\mathrm{mmo}\,(\mathbb{R}^n)$ *or* $\mathrm{xmo}\,(\mathbb{R}^n)$*, where* $h^1(\mathbb{R}^n)$ *is the* localized Hardy space*;*
(v) *The equivalent characterizations for* $\mathrm{mmo}\,(\mathbb{R}^n)$ *and* $\mathrm{xmo}\,(\mathbb{R}^n)$ *via* localized Riesz transforms.

Remark 11. *For the studies of vanishing Morrey spaces, we refer the reader to [106–109].*

5.2. Vanishing John–Nirenberg–Campanato Spaces

Very recently, the vanishing subspaces of John–Nirenberg spaces were also studied in [60,110]. Indeed, as a counterpart of Section 5.1, the vanishing subspaces of JNC spaces enjoy similar characterizations, which are summarized in this subsection.

Definition 17. *Let* $p \in (1,\infty)$*,* $q \in [1,\infty)$*,* $s \in \mathbb{Z}_+$*, and* $\alpha \in \mathbb{R}$*. The* vanishing subspace $VJN_{(p,q,s)_\alpha}(\mathcal{X})$ *is defined by setting*

$$VJN_{(p,q,s)_\alpha}(\mathcal{X}) := \left\{ f \in JN_{(p,q,s)_\alpha}(\mathcal{X}) :\ \limsup_{a\to 0^+} \sup_{\mathrm{size}\le a} \widetilde{\mathcal{O}}_{(p,q,s)_\alpha}(f;\{Q_i\}_i) = 0 \right\},$$

where

$$\widetilde{\mathcal{O}}_{(p,q,s)_\alpha}(f;\{Q_i\}_i) := \left\{ \sum_i |Q_i| \left[|Q_i|^{-\alpha} \left\{ \fint_{Q_i} \left| f(x) - P^{(s)}_{Q_i}(f)(x) \right|^q dx \right\}^{\frac{1}{q}} \right]^p \right\}^{\frac{1}{p}}$$

and the supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_i\}_i$ *of* $\mathcal{X}$ *with side lengths no more than* a*. To simplify the notation, write* $VJN_{p,q}(\mathcal{X}) := VJN_{(p,q,0)_0}(\mathcal{X})$ *and* $VJN_p(\mathcal{X}) := VJN_{p,1}(\mathcal{X})$*.*

On the unit cube $[0,1]^n$, the space $VJN_{(p,q,s)_\alpha}([0,1]^n)$ was studied by A. Brudnyi and Y. Brudnyi in [60] with different symbols. The following characterization (Theorem 19) and duality (Theorem 20) are just, respectively, [60] (Theorem 3.14 and 3.7). Notice that when $\alpha \geq \frac{s+1}{n}$, from [60] (Lemma 4.1), we deduce that $JN_{(p,q,s)_\alpha}([0,1]^n) = \mathcal{P}_s([0,1]^n)$ is trivial.

Theorem 19. *Let $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in (-\infty, \frac{s+1}{n})$. Then,*

$$VJN_{(p,q,s)_\alpha}([0,1]^n) = \overline{C^\infty([0,1]^n) \cap JN_{(p,q,s)_\alpha}([0,1]^n)}^{JN_{(p,q,s)_\alpha}([0,1]^n)},$$

where $C^\infty([0,1]^n) := C^\infty(\mathbb{R}^n)|_{[0,1]^n}$ denotes the restriction of infinitely differentiable functions from $\mathbb{R}^n$ to $[0,1]^n$.

Theorem 20. *Let $p, q \in (1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in (-\infty, \frac{s+1}{n})$. Then,*

$$\left(VJN_{(p,q,s)_\alpha}([0,1]^n)\right)^* = HK_{(p',q',s)_\alpha}([0,1]^n),$$

where $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$.

It is obvious that Theorems 19 and 20 hold true with $[0,1]^n$ replaced by a given cube Q_0 of $\mathbb{R}^n$. As an application of the duality, Tao et al. ([110], Proposition 5.7) showed that for any $p \in (1,\infty)$ and any given cube Q_0 of $\mathbb{R}^n$,

$$[L^p(Q_0)/\mathbb{C}] \subsetneqq VJN_p(Q_0)$$

which proves the *nontriviality* of $VJN_p(Q_0)$, here and thereafter,

$$L^p(\mathcal{X})/\mathbb{C} := \left\{f \in L^1_{\mathrm{loc}}(\mathcal{X}) : \|f\|_{L^p(\mathcal{X})/\mathbb{C}} < \infty\right\}$$

with

$$\|f\|_{L^p(\mathcal{X})/\mathbb{C}} := \inf_{c\in\mathbb{C}} \|f + c\|_{L^p(\mathcal{X})}.$$

Remark 12. *There exists a gap in the proof of [110] (Proposition 5.7): we cannot deduce*

$$\left(VJN_p(Q_0)\right)^{**} = JN_p(Q_0), \tag{40}$$

namely [110] (5.2), directly from Theorems 20 and 3 because, in the statements of these dual theorems, q cannot equal 1. Indeed, (40) still holds true due to the equivalence of $JN_{p,q}(Q_0)$ with $q \in [1,p)$. Precisely, let $p \in (1,\infty)$ and $q \in (1,p)$. By Theorems 20 and 3, we obtain

$$\left(VJN_{p,q}(Q_0)\right)^{**} = JN_{p,q}(Q_0),$$

which, together with Theorems 10 and 21 below, further implies that

$$\left(VJN_p(Q_0)\right)^{**} = \left(VJN_{p,q}(Q_0)\right)^{**} = JN_{p,q}(Q_0) = JN_p(Q_0),$$

and hence (40) holds true. This fixes the gap in the proof of [110] (5.2).

Next, we consider the case $\mathcal{X} = \mathbb{R}^n$. The following proposition indicates that the convolution is a suitable tool when approximating functions in $JN_p(\mathbb{R}^n)$, which is a counterpart of [6] (Lemma 1). Indeed, the approximate functions in the proofs of both Theorems 21 and 22 are constructed via the convolution (see [110] for more details).

Proposition 31. *Let $p \in (1,\infty)$ and $\varphi \in L^1(\mathbb{R}^n)$ with compact support. If $f \in JN_p(\mathbb{R}^n)$, then $f * \varphi \in JN_p(\mathbb{R}^n)$ and*

$$\|f * \varphi\|_{JN_p(\mathbb{R}^n)} \leq 2\|\varphi\|_{L^1(\mathbb{R}^n)}\|f\|_{JN_p(\mathbb{R}^n)}.$$

Proof. Let p, φ, and f be as in this lemma. Then, for any cube Q of $\mathbb{R}^n$, by the Fubini theorem, we have

$$\begin{aligned} O(f*\varphi;Q) &= \fint_Q \left|f*\varphi(x)-(f*\varphi)_Q\right| dx \\ &= \fint_Q \left|\fint_Q \int_{\mathbb{R}^n} \varphi(z)[f(x-z)-f(y-z)]\,dz\,dy\right| dx \\ &\le \int_{\mathbb{R}^n} \fint_Q \fint_Q |\varphi(z)|\left|f(x-z)-f(y-z)\right| dy\,dx\,dz \\ &= \int_{\mathbb{R}^n} |\varphi(z)| \fint_{Q-z} \fint_{Q-z} \left|f(x)-f(y)\right| dy\,dx\,dz \\ &\le 2\int_{\mathbb{R}^n} |\varphi(z)| O(f;Q-z)\,dz, \end{aligned} \tag{41}$$

where $Q-z := \{w-z:\ w\in Q\}$. Therefore, for any interior pairwise disjoint subcubes $\{Q_i\}_i$ of $\mathbb{R}^n$, by (41) and the generalized Minkowski integral inequality, we conclude that

$$\begin{aligned} &\left\{\sum_i |Q_i|[O(f*\varphi;Q_i)]^p\right\}^{\frac{1}{p}} \\ &\quad\le 2\left\{\sum_i |Q_i|\left[\int_{\mathbb{R}^n} |\varphi(z)| O(f;Q-z)\,dz\right]^p\right\}^{\frac{1}{p}} \\ &\quad= 2\left\{\sum_i \left[\int_{\mathbb{R}^n} |Q_i|^{\frac{1}{p}} |\varphi(z)| O(f;Q_i-z)\,dz\right]^p\right\}^{\frac{1}{p}} \\ &\quad\le 2\int_{\mathbb{R}^n} \left\{\sum_i \left[|Q_i|^{\frac{1}{p}} |\varphi(z)| O(f;Q_i-z)\right]^p\right\}^{\frac{1}{p}} dz \\ &\quad= 2\int_{\mathbb{R}^n} |\varphi(z)| \left\{\sum_i |Q_i-z|[O(f;Q_i-z)]^p\right\}^{\frac{1}{p}} dz \\ &\quad\le 2\|\varphi\|_{L^1(\mathbb{R}^n)}\|f\|_{JN_p(\mathbb{R}^n)}, \end{aligned}$$

where $Q_i-z := \{w-z:\ w\in Q_i\}$ for any i. This further implies that

$$\|f*\varphi\|_{JN_p(\mathbb{R}^n)} \le 2\|\varphi\|_{L^1(\mathbb{R}^n)}\|f\|_{JN_p(\mathbb{R}^n)}$$

and hence finishes the proof of Proposition 31. □

The following equivalent characterization is just [110] (Theorem 3.2).

Theorem 21. *Let $p\in(1,\infty)$. Then, the following three statements are mutually equivalent:*

(i) $f \in \overline{D_p(\mathbb{R}^n)\cap JN_p(\mathbb{R}^n)}^{JN_p(\mathbb{R}^n)} =: VJN_p(\mathbb{R}^n)$, *where*

$$D_p(\mathbb{R}^n) := \{f\in C^\infty(\mathbb{R}^n):\ |\nabla f|\in L^p(\mathbb{R}^n)\}$$

and ∇f denotes the gradient of f;

(ii) $f\in JN_p(\mathbb{R}^n)$ *and, for any given* $q\in[1,p)$,

$$\lim_{a\to 0^+} \sup_{\{\{Q_i\}_i:\ \ell(Q_i)\le a,\ \forall i\}} \left\{\sum_i |Q_i|\left[\fint_{Q_i} \left|f(x)-f_{Q_i}\right|^q dx\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}} = 0,$$

where the supremum is taken over all collections $\{Q_i\}_i$ *of interior pairwise disjoint subcubes of* $\mathbb{R}^n$ *with side lengths no more than* a*;*

(iii) $f \in JN_p(\mathbb{R}^n)$ *and*

$$\lim_{a\to 0^+} \sup_{\{\{Q_i\}_i:\ \ell(Q_i)\le a,\ \forall\, i\}} \left\{\sum_i |Q_i| \left[\fint_{Q_i} \left|f(x) - f_{Q_i}\right| dx\right]^p\right\}^{\frac{1}{p}} = 0,$$

where the supremum is taken over all collections $\{Q_i\}_i$ *of interior pairwise disjoint subcubes of* $\mathbb{R}^n$ *with side lengths no more than* a*.*

Now, we recall another vanishing subspace of $JN_p(\mathbb{R}^n)$ introduced in [110], which is of the CMO type.

Definition 18. *Let* $p \in (1, \infty)$*. The* vanishing subspace $CJN_p(\mathbb{R}^n)$ *of* $JN_p(\mathbb{R}^n)$ *is defined by setting*

$$CJN_p(\mathbb{R}^n) := \overline{C_c^\infty(\mathbb{R}^n)}^{JN_p(\mathbb{R}^n)},$$

where $C_c^\infty(\mathbb{R}^n)$ *denotes the set of all infinitely differentiable functions on* $\mathbb{R}^n$ *with compact support.*

The following theorem is just [110] (Theorem 4.3).

Theorem 22. *Let* $p \in (1, \infty)$*. Then,* $f \in CJN_p(\mathbb{R}^n)$ *if and only if* $f \in JN_p(\mathbb{R}^n)$*, and* f *satisfies the following two conditions:*

(i)

$$\lim_{a\to 0^+} \sup_{\{\{Q_i\}_i:\ \ell(Q_i)\le a,\ \forall\, i\}} \left\{\sum_i |Q_i| \left[\fint_{Q_i} \left|f(x) - f_{Q_i}\right| dx\right]^p\right\}^{\frac{1}{p}} = 0,$$

where the supremum is taken over all collections $\{Q_i\}_i$ *of interior pairwise disjoint subcubes of* $\mathbb{R}^n$ *with side lengths* $\{\ell(Q_i)\}_i$ *no more than* a*;*

(ii)

$$\lim_{a\to\infty} \sup_{\{Q\subset\mathbb{R}^n:\ \ell(Q)\ge a\}} |Q|^{1/p} \fint_Q \left|f(x) - f_Q\right| dx = 0,$$

where the supremum is taken over all cubes Q *of* $\mathbb{R}^n$ *with side lengths* $\ell(Q)$ *no less than* a*.*

Moreover, Tao et al. ([110], Theorem 4.4) showed that Theorem 22(ii) can be replaced by the following statement:

$$\lim_{a\to\infty} \sup_{\{\{Q_i\}_i:\ \ell(Q_i)\ge a,\ \forall\, i\}} \left\{\sum_i |Q_i| \left[\fint_{Q_i} \left|f(x) - f_{Q_i}\right| dx\right]^p\right\}^{\frac{1}{p}} = 0,$$

where the supremum is taken over all collections $\{Q_i\}_i$ of interior pairwise disjoint subcubes of $\mathbb{R}^n$ with side lengths $\{\ell(Q_i)\}_i$ greater than a.

Furthermore, Tao et al. ([110], Corollary 4.5) showed that Theorem 22 holds true with

$$\fint_Q \left|f(x) - f_Q\right| dx \quad \text{and} \quad \fint_{Q_i} \left|f(x) - f_{Q_i}\right| dx$$

in (i) and (ii) replaced, respectively, by

$$\left[\fint_Q \left|f(x) - f_Q\right|^q dx\right]^{\frac{1}{q}} \quad \text{and} \quad \left[\fint_{Q_i} \left|f(x) - f_{Q_i}\right|^q dx\right]^{\frac{1}{q}}$$

for any $q \in [1, p)$.

However, there still exist some unsolved questions on the vanishing John–Nirenberg space. The first question is on the case $p = 1$.

Question 15. *The proof of [110] (Theorem 3.2) indicates that (i) and (iii) of Theorem 21 are equivalent when $p = 1$. However, the corresponding equivalent characterization of $CJN_1(\mathbb{R}^n)$ is still* unclear.

The following question is just [110] (Question 5.5).

Question 16.

(i) *It is still unknown whether or not Theorems 21 and 22 hold true with $JN_p(\mathbb{R}^n)$ replaced by $JN_{(p,q,s)_\alpha}(\mathbb{R}^n)$ when $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R} \setminus \{0\}$.*

(ii) *It is interesting to ask whether or not for any given $p \in (1,\infty)$, $q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$,*

$$\left(CJN_{(p,q,s)_\alpha}(\mathbb{R}^n)\right)^* = HK_{(p',q',s)_\alpha}(\mathbb{R}^n) \quad \text{or} \quad \left(CJN_{(p,q,s)_\alpha}(\mathbb{R}^n)\right)^{**} = JN_{(p,q,s)_\alpha}(\mathbb{R}^n)$$

still holds true, where $1/p + 1/p' = 1 = 1/q + 1/q'$, $CJN_{(p,q,s)_\alpha}(\mathbb{R}^n)$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ in $JN_{(p,q,s)_\alpha}(\mathbb{R}^n)$, and $HK_{(p',q',s)_\alpha}(\mathbb{R}^n)$ the Hardy-type space introduced in [61] (Definition 3.6).

Obviously, $[L^p(\mathbb{R}^n)/\mathbb{C}] \subset CJN_p(\mathbb{R}^n) \subset VJN_p(\mathbb{R}^n) \subset JN_p(\mathbb{R}^n)$. Then, the last question naturally arises, which is just [110] (Questions 5.6 and 5.8).

Question 17. *Let $p \in (1,\infty)$. It is interesting to ask whether or not*

$$[L^p(\mathbb{R}^n)/\mathbb{C}] \subsetneqq CJN_p(\mathbb{R}^n) \subsetneqq VJN_p(\mathbb{R}^n) \subsetneqq JN_p(\mathbb{R}^n)$$

holds true. This is still unclear.

5.3. Vanishing Congruent John–Nirenberg–Campanato Spaces

As a counterpart of Section 5.2, the vanishing subspace of congruent John–Nirenberg–Campanato spaces $VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ was studied in [64].

Definition 19. *Let $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in \mathbb{R}$. The* space *$VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})$ is defined by setting*

$$VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X}) := \overline{D_p(\mathcal{X}) \cap JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})}^{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathcal{X})},$$

where

$$D_p(\mathcal{X}) := \{f \in C^\infty(\mathcal{X}) : \ |\nabla f| \in L^p(\mathcal{X})\}.$$

Furthermore, simply write $VJN^{\mathrm{con}}_{p,q}(\mathcal{X}) := VJN^{\mathrm{con}}_{(p,q,0)_0}(\mathcal{X})$ and $VJN^{\mathrm{con}}_{p}(\mathcal{X}) := VJN^{\mathrm{con}}_{p,1}(\mathcal{X})$.

Remark 13. *Let $p, q \in [1,\infty)$, $s \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, and Q_0 be a given cube of $\mathbb{R}^n$. Then, the observation $D_p(Q_0) = C^\infty(Q_0)$ implies that*

$$VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0) = \overline{C^\infty(Q_0) \cap JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)}^{JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)}.$$

Recall that $\mathcal{D}_m(\mathcal{X})$ with $m \in \mathbb{Z}$ is defined in the beginning of Section 3.3. The following characterizations, namely Theorems 23 and 24, are just [64] (Theorems 3.5 and 3.9, respectively).

Theorem 23. *Let $p, q \in [1, \infty)$, $s \in \mathbb{Z}_+$, $\alpha \in (-\infty, \frac{s+1}{n})$, and Q_0 be a given cube of $\mathbb{R}^n$. Then, $f \in VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)$ if and only if $f \in L^q(Q_0)$ and*

$$\limsup_{m\to\infty} \sup_{\{Q_j\}_j\subset\mathcal{D}_m(Q_0)} \left[\sum_j |Q_j| \left\{ |Q_j|^{-\alpha} \left[\fint_{Q_j} \left| f - P^{(s)}_{Q_j}(f) \right|^q \right]^{\frac{1}{q}} \right\}^p \right]^{\frac{1}{p}} = 0, \tag{42}$$

where the second supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_j \subset \mathcal{D}_m(Q_0)$ for any $m \in \mathbb{Z}$.

Corollary 4. *Let $p = 1$, $q \in [1, \infty)$, $s \in \mathbb{Z}_+$, $\alpha = 0$, and Q_0 be a given cube of $\mathbb{R}^n$. Then, (42) holds true for any $f \in L^q(Q_0)$.*

Proof. By Proposition 24(ii) and the definition of $VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)$, we have

$$[L^q(Q_0)/\mathcal{P}_s(Q_0)] = VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0) = JN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0),$$

which, combined with Theorem 23, then completes the proof of Corollary 4. □

Theorem 24. *Let $p \in [1, \infty)$ and $q \in [1, p]$. Then, $f \in VJN^{\mathrm{con}}_{p,q}(\mathbb{R}^n)$ if and only if $f \in JN^{\mathrm{con}}_{p,q}(\mathbb{R}^n)$ and*

$$\limsup_{m\to\infty} \sup_{\{Q_j\}_j\subset\mathcal{D}_m(\mathbb{R}^n)} \left[\sum_j |Q_j| \left(\fint_{Q_j} \left| f - f_{Q_j} \right|^q \right)^{\frac{p}{q}} \right]^{\frac{1}{p}} = 0,$$

where the second supremum is taken over all collections of interior pairwise disjoint cubes $\{Q_j\}_j \subset \mathcal{D}_m(\mathbb{R}^n)$ for any $m \in \mathbb{Z}$.

We can partially answer Question 17 in the congruent JNC space as follows.

Proposition 32. *Let I_0 be a given bounded interval of $\mathbb{R}$, and Q_0 a given cube of $\mathbb{R}^n$.*

(i) *([64], Proposition 3.11) If $p \in (1, \infty)$ and $q \in [1, p)$, then $[L^p(\mathbb{R})/\mathbb{C}] \subsetneqq VJN^{\mathrm{con}}_{p,q}(\mathbb{R})$.*
(ii) *([64], Proposition 3.12) If $p \in (1, \infty)$ and $q \in [1, p)$, then $VJN^{\mathrm{con}}_{p,q}(\mathbb{R}) \subsetneqq JN^{\mathrm{con}}_{p,q}(\mathbb{R})$ and $VJN^{\mathrm{con}}_{p,q}(I_0) \subsetneqq JN^{\mathrm{con}}_{p,q}(I_0)$.*
(iii) *([64], Proposition 4.40) If $p \in (1, \infty)$ and $q \in (1, p)$, then $[L^p(Q_0)/\mathbb{C}] \subsetneqq VJN^{\mathrm{con}}_{p,q}(Q_0)$.*

Furthermore, it is easy to show that $[L^1(Q_0)/\mathbb{C}] = VJN^{\mathrm{con}}_1(Q_0) = JN^{\mathrm{con}}_1(Q_0)$ (see Remark 2(ii)).

The following VMO-H^1-type duality is just [64] (Theorem 4.39).

Theorem 25. *Let $p, q \in (1, \infty)$, $s \in \mathbb{Z}_+$, $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$, $\alpha \in (-\infty, \frac{s+1}{n})$, and Q_0 be a given cube of $\mathbb{R}^n$. Then,*

$$\left(VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)\right)^* = HK^{\mathrm{con}}_{(p',q',s)_\alpha}(Q_0)$$

in the following sense: there exists an isometric isomorphism

$$K: HK^{\mathrm{con}}_{(p',q',s)_\alpha}(Q_0) \longrightarrow \left(VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)\right)^*$$

such that for any $g \in HK^{\mathrm{con}}_{(p',q',s)_\alpha}(Q_0)$ and $f \in VJN^{\mathrm{con}}_{(p,q,s)_\alpha}(Q_0)$,

$$\langle Kg, f\rangle = \langle g, f\rangle.$$

Similar to Question 16(ii), the following question, posed in [64] (Remark 4.41), is still unsolved.

Question 18. *For any given $p, q \in (1,\infty)$, $s \in \mathbb{Z}_+$, and $\alpha \in (-\infty, \frac{s+1}{n})$, it is interesting to ask whether or not*

$$\left(CJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)\right)^* = HK^{\mathrm{con}}_{(p',q',s)_\alpha}(\mathbb{R}^n) \quad \text{and} \quad \left(CJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)\right)^{**} = JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)$$

hold true, where $CJN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)$ denotes the closure of $C^\infty_c(\mathbb{R}^n)$ in $JN^{\mathrm{con}}_{(p,q,s)_\alpha}(\mathbb{R}^n)$ and $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$. This is still unclear.

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References

1. John, F.; Nirenberg, L. On functions of bounded mean oscillation. *Commun. Pure Appl. Math.* **1961**, *14*, 415–426. [CrossRef]
2. Fefferman, C.; Stein, E.M. H^p spaces of several variables. *Acta Math.* **1972**, *129*, 137–193. [CrossRef]
3. Coifman, R.R.; Rochberg, R.; Weiss, G. Factorization theorems for Hardy spaces in several variables. *Ann. Math.* **1976**, *103*, 611–635. [CrossRef]
4. Coifman, R.R.; Weiss, G. *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes. (French) Étude de Certaines Intégrales Singulières*; Lecture Notes in Mathematics; Springer: Berlin, Germany; New York, NY, USA, 1971; Volume 242.
5. Coifman, R.R.; Weiss, G. Extensions of Hardy spaces and their use in analysis. *Bull. Am. Math. Soc.* **1977**, *83*, 569–645. [CrossRef]
6. Sarason, D. Functions of vanishing mean oscillation. *Trans. Am. Math. Soc.* **1975**, *207*, 391–405. [CrossRef]
7. Uchiyama, A. On the compactness of operators of Hankel type. *Tôhoku Math. J.* **1978**, *30*, 163–171. [CrossRef]
8. Nakai, E.; Yabuta, K. Pointwise multipliers for functions of bounded mean oscillation. *J. Math. Soc. Jpn.* **1995**, *37*, 207–218.
9. Iwaniec, T. L^p-theory of quasiregular mappings. In *Quasiconformal Space Mappings*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1992; Volume 1508, pp. 39–64.
10. Bényi, A.; Damián, W.; Moen, K.; Torres, R.H. Compact bilinear commutators: The weighted case. *Mich. Math. J.* **2015**, *64*, 39–51. [CrossRef]
11. Bényi, A.; Torres, R.H. Compact bilinear operators and commutators. *Proc. Am. Math. Soc.* **2013**, *141*, 3609–3621. [CrossRef]
12. Chang, D.-C.; Li, S.-Y. On the boundedness of multipliers, commutators and the second derivatives of Green's operators on H^1 and BMO. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1999**, *28*, 341–356.
13. Janson, S. Mean oscillation and commutators of singular integral operators. *Ark. Mat.* **1978**, *16*, 263–270. [CrossRef]
14. Li, W.; Nakai, E.; Yang, D. Pointwise multipliers on BMO spaces with non-doubling measures. *Taiwan. J. Math.* **2018**, *22*, 183–203. [CrossRef]
15. Nakai, E. Pointwise multipliers on several functions spaces—A survey. *Linear Nonlinear Anal.* **2017**, *3*, 27–59.
16. Nakai, E. The Campanato, Morrey and Hölder spaces on spaces of homogeneous type. *Stud. Math.* **2006**, *176*, 1–19. [CrossRef]
17. Nakai, E. Pointwise multipliers on weighted BMO spaces. *Stud. Math.* **1997**, *125*, 35–56. [CrossRef]
18. Nakai, E. Pointwise multipliers for functions of weighted bounded mean oscillation. *Stud. Math.* **1993**, *105*, 105–119. [CrossRef]
19. Nakai, E.; Yabuta, K. Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type. *Math. Japon.* **1997**, *46*, 15–28.

20. Chang, D.-C.; Dafni, G.; Stein, E.M. Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in $\mathbb{R}^n$. *Trans. Am. Math. Soc.* **1999**, *351*, 1605–1661. [CrossRef]
21. Nakai, E.; Yoneda, T. Bilinear estimates in dyadic BMO and the Navier–Stokes equations. *J. Math. Soc. Jpn.* **2012**, *64*, 399–422. [CrossRef]
22. Tao, J.; Yang, D.; Yang, D. Beurling–Ahlfors commutators on weighted Morrey spaces and applications to Beltrami equations. *Potential Anal.* **2020**, *53*, 1467–1491. [CrossRef]
23. Ambrosio, L.; Bourgain, J.; Brezis, H.; Figalli, A. BMO-type norms related to the perimeter of sets. *Commun. Pure Appl. Math.* **2016**, *69*, 1062–1086. [CrossRef]
24. Butaev, A.; Dafni, G. Approximation and extension of functions of vanishing mean oscillation. *J. Geom. Anal.* **2021**, *31*, 6892–6921. [CrossRef]
25. Chang, D.-C.; Dafni, G.; Sadosky, C. A div-curl lemma in BMO on a domain. In *Harmonic Analysis, Signal Processing, and Complexity*; Progress in Mathematics; Birkhäuser Boston: Boston, MA, USA, 2005; Volume 238.
26. Dafni, G.; Gibara, R. BMO on shapes and sharp constants. In *Advances in Harmonic Analysis and Partial Differential Equations*; Contemporary Mathematics; American Mathematical Society: Providence, RI, USA, 2020; Volume 748, pp. 1–33.
27. Dafni, G.; Gibara, R.; Lavigne, A. BMO and the John–Nirenberg inequality on measure spaces. *Anal. Geom. Metr. Spaces* **2020**, *8*, 335–362. [CrossRef]
28. Dafni, G.; Gibara, R.; Yue, H. Geometric maximal operators and BMO on product bases. *J. Geom. Anal.* **2021**, *31*, 5740–5765. [CrossRef]
29. Chang, D.-C.; Sadosky, C. Functions of bounded mean oscillation. *Taiwan. J. Math.* **2006**, *10*, 573–601. [CrossRef]
30. Aalto, D.; Berkovits, L.; Kansanen, O.E.; Yue, H. John–Nirenberg lemmas for a doubling measure. *Stud. Math.* **2011**, *204*, 21–37. [CrossRef]
31. Dafni, G.; Hytönen, T.; Korte, R.; Yue, H. The space JN_p: Nontriviality and duality. *J. Funct. Anal.* **2018**, *275*, 577–603. [CrossRef]
32. Berkovits, L.; Kinnunen, J.; Martell, J.M. Oscillation estimates, self-improving results and good-λ inequalities. *J. Funct. Anal.* **2016**, *270*, 3559–3590. [CrossRef]
33. Blasco, O.; Espinoza-Villalva, C. The norm of the characteristic function of a set in the John–Nirenberg space of exponent p. *Math. Methods Appl. Sci.* **2020**, *43*, 9327–9336. [CrossRef]
34. Hurri-Syrjänen, R.; Marola, N.; Vähäkangas, A.V. Aspects of local-to-global results. *Bull. Lond. Math. Soc.* **2014**, *46*, 1032–1042. [CrossRef]
35. Milman, M. Marcinkiewicz spaces, Garsia–Rodemich spaces and the scale of John–Nirenberg self improving inequalities. *Ann. Acad. Sci. Fenn. Math.* **2016**, *41*, 491–501. [CrossRef]
36. Sun, J.; Xie, G.; Yang, D. Localized John–Nirenberg–Campanato spaces. *Anal. Math. Phys.* **2021**, *11*, 29. [CrossRef]
37. Tao, J.; Yang, D.; Yuan, W. A bridge connecting Lebesgue and Morrey spaces via Riesz norms. *Banach J. Math. Anal.* **2021**, *15*, 20. [CrossRef]
38. Franchi, B.; Pérez, C.; Wheeden, R.L. Self-improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type. *J. Funct. Anal.* **1998**, *153*, 108–146. [CrossRef]
39. MacManus, P.; Pérez, C. Generalized Poincaré inequalities: Sharp self-improving properties. *Internat. Math. Res. Notices* **1998**, *1998*, 101–116. [CrossRef]
40. Marola, N.; Saari, O. Local to global results for spaces of BMO type. *Math. Z.* **2016**, *282*, 473–484. [CrossRef]
41. Riesz, F. Untersuchungen über systeme integrierbarer funktionen. *Math. Ann.* **1910**, *69*, 449–497. (In German) [CrossRef]
42. Duoandikoetxea, J. *Fourier Analysis*; Graduate Studies in Mathematics; American Mathematical Society: Providence, RI, USA, 2001; Volume 29.
43. Stein, E.M. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*; Princeton Mathematical Series, Monographs in Harmonic Analysis III; Princeton University Press: Princeton, NJ, USA, 1993; Volume 43.
44. Bényi, A.; Martell, J.M.; Moen, K.; Stachura, E.; Torres, R.H. Boundedness results for commutators with BMO functions via weighted estimates: A comprehensive approach. *Math. Ann.* **2020**, *376*, 61–102. [CrossRef]
45. Bennett, C.; Sharpley, R. *Interpolation of Operators*; Pure and Applied Mathematics; Academic Press Inc.: Boston, MA, USA, 1988; Volume 129.
46. Grafakos, L. *Morden Fourier Analysis*, 3rd ed.; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2014; Volume 250.
47. Grafakos, L. *Classical Fourier Analysis*, 3rd ed.; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2014; Volume 249.
48. Izuki, M.; Noi, T.; Sawano, Y. The John–Nirenberg inequality in ball Banach function spaces and application to characterization of BMO. *J. Inequal. Appl.* **2019**, *2019*, 268. [CrossRef]
49. Tao, J.; Yang, D.; Yuan, W.; Zhang, Y. Compactness characterizations of commutators on ball Banach function spaces. *Potential Anal.* **2021**, in press.
50. Goldberg, D. A local version of real Hardy spaces. *Duke Math. J.* **1979**, *46*, 27–42. [CrossRef]
51. Jonsson, A.; Sjögren, P.; Wallin, H. Hardy and Lipschitz spaces on subsets of $\mathbb{R}^n$. *Stud. Math.* **1984**, *80*, 141–166. [CrossRef]
52. Chang, D.-C. The dual of Hardy spaces on a bounded domain in $\mathbb{R}^n$. *Forum Math.* **1994**, *6*, 65–81. [CrossRef]
53. Dafni, G.; Liflyand, E. A local Hilbert transform, Hardy's inequality and molecular characterization of Goldberg's local Hardy space. *Complex Anal. Synerg.* **2019**, *5*, 10. [CrossRef]

54. Dafni, G.; Yue, H. Some characterizations of local bmo and h^1 on metric measure spaces. *Anal. Math. Phys.* **2012**, *2*, 285–318. [CrossRef]
55. Garsia, A.M.; Rodemich, E. Monotonicity of certain functionals under rearrangement. *Ann. Inst. Fourier (Grenoble)* **1974**, *24*, 67–116. [CrossRef]
56. Stampacchia, G. The spaces $L^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1965**, *19*, 443–462.
57. Campanato, S. Su un teorema di interpolazione di G. Stampacchia. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1966**, *20*, 649–652.
58. Myyryläinen, K. Median-type John–Nirenberg space in metric measure spaces. *arXiv* **2021**, arXiv:2104.05380.
59. Kinnunen, J.; Myyryläinen, K. Dyadic John–Nirenberg space. *arXiv* **2021**, arXiv:2107.00492.
60. Brudnyi, A.; Brudnyi, Y. On the Banach structure of multivariate BV spaces. *Diss. Math.* **2020**, *548*, 1–52. [CrossRef]
61. Tao, J.; Yang, D.; Yuan, W. John–Nirenberg–Campanato spaces. *Nonlinear Anal.* **2019**, *189*, 111584. [CrossRef]
62. Domínguez, Ó.; Milman, M. Sparse Brudnyi and John–Nirenberg spaces. *arXiv* **2021**, arXiv:2107.05117.
63. Astashkin, S.; Milman, M. Garsia–Rodemich spaces: Local maximal functions and interpolation. *Stud. Math.* **2020**, *255*, 1–26. [CrossRef]
64. Jia, H.; Tao, J.; Yuan, W.; Yang, D.; Zhang, Y. Special John–Nirenberg–Campanato spaces via congruent cubes. *Sci. China Math.* **2021**. [CrossRef]
65. Taibleson, M.H.; Weiss, G. The molecular characterization of certain Hardy spaces. In *Representation Theorems for Hardy Spaces, Astérisque 77*; Société Mathématique de France: France, Paris, 1980; pp. 67–149.
66. Lu, S. *Four Lectures on Real H^P Spaces*; World Scientific Publishing Co., Inc.: River Edge, NJ, USA, 1995.
67. Campanato, S. Proprietà di una famiglia di spazi funzionali. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1964**, *18*, 137–160.
68. Morrey, C.B. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Am. Math. Soc.* **1938**, *43*, 126–166. [CrossRef]
69. García-Cuerva, J.; Rubio de Francia, J. *Weighted Norm Inequalities and Related Topics*; North-Holland Mathematics Studies, Notas de Matemática [Mathematical Notes]; North-Holland Publishing Co.: Amsterdam, The Netherlands, 1985; Volume 116.
70. Bourgain, J.; Brezis, H.; Mironescu, P. A new function space and applications. *J. Eur. Math. Soc. (JEMS)* **2015**, *17*, 2083–2101. [CrossRef]
71. Jia, H.; Tao, J.; Yuan, W.; Yang, D.; Zhang, Y. Boundedness of Calderón–Zygmund operators on special John–Nirenberg–Campanato and Hardy-type spaces via congruent cubes. *Anal. Math. Phys.* **2021**, under review.
72. Jia, H.; Tao, J.; Yuan, W.; Yang, D.; Zhang, Y. Boundedness of fractional integrals on special John–Nirenberg–Campanato and Hardy-type spaces via congruent cubes. *Front. Math. China* **2021**, under review.
73. Jia, H.; Yuan, W.; Yang, D.; Zhang, Y. Estimates for Littlewood–Paley operators on special John–Nirenberg–Campanato spaces via congruent cubes. *arXiv* **2021**, arXiv:2108.01559.
74. Fusco, N.; Moscariello, G.; Sbordone, C. BMO-type seminorms and Sobolev functions. *ESAIM Control Optim. Calc. Var.* **2018**, *24*, 835–847. [CrossRef]
75. Di Fratta, G.; Fiorenza, A. BMO-type seminorms from Escher-type tessellations. *J. Funct. Anal.* **2020**, *279*, 108556. [CrossRef]
76. Essén, M.; Janson, S.; Peng, L.; Xiao, J. Q spaces of several real variables. *Indiana Univ. Math. J.* **2000**, *49*, 575–615. [CrossRef]
77. Koskela, P.; Xiao, J.; Zhang, Y.; Zhou, Y. A quasiconformal composition problem for the Q-spaces. *J. Eur. Math. Soc. (JEMS)* **2017**, *19*, 1159–1187. [CrossRef]
78. Xiao, J. *Q_α Analysis on Euclidean Spaces*; Advances in Analysis and Geometry 1; De Gruyter: Berlin, Germany, 2019.
79. Zhuo, C.; Sickel, W.; Yang, D.; Yuan, W. Characterizations of Besov-type and Triebel–Lizorkin-type spaces via averages on balls. *Canad. Math. Bull.* **2017**, *60*, 655–672. [CrossRef]
80. Tao, J.; Yang, Z.; Yuan, W. John–Nirenberg-Q spaces via congruent cubes. 2021, submitted.
81. Hakim, D.I.; Nakamura, S.; Sawano, Y. Complex interpolation of smoothness Morrey subspaces. *Constr. Approx.* **2017**, *46*, 489–563. [CrossRef]
82. Hakim, D.I.; Sawano, Y. Complex interpolation of various subspaces of Morrey spaces. *Sci. China Math.* **2020**, *63*, 937–964. [CrossRef]
83. Mastyło, M.; Sawano, Y. Complex interpolation and Calderón–Mityagin couples of Morrey spaces. *Anal. PDE* **2019**, *12*, 1711–1740. [CrossRef]
84. Mastyło, M.; Sawano, Y.; Tanaka, H. Morrey-type space and its Köthe dual space. *Bull. Malays. Math. Sci. Soc.* **2018**, *41*, 1181–1198. [CrossRef]
85. Sawano, Y.; Di Fazio, G.; Hakim, D. *Morrey Spaces: Introduction and Applications to Integral Operators and PDE's*; Chapman and Hall/CRC: New York, NY, USA, 2020; Volume I. Available online: https://www.taylorfrancis.com/books/mono/10.1201/9780429085925/morrey-spaces-yoshihiro-sawano-giuseppe-di-fazio-denny-ivanal-hakim (accessed on 8 September 2021).
86. Sawano, Y.; Di Fazio, G.; Hakim, D. *Morrey Spaces: Introduction and Applications to Integral Operators and PDE's*; Chapman and Hall/CRC: New York, NY, USA, 2020; Volume II. Available online: https://www.taylorfrancis.com/books/mono/10.1201/9781003029076/morrey-spaces-yoshihiro-sawano-giuseppe-di-fazio-denny-ivanal-hakim (accessed on 8 September 2021).
87. Fofana, I.; Faléa, F.R.; Kpata, B.A. A class of subspaces of Morrey spaces and norm inequalities on Riesz potential operators. *Afr. Mat.* **2015**, *26*, 717–739. [CrossRef]
88. Zeng, Z.; Chang, D.-C.; Tao, J.; Yang, D. Nontriviality of Riesz–Morrey spaces. *Appl. Anal.* **2021**. 021.1932836. [CrossRef]

89. Blasco, O.; Ruiz, A.; Vega, L. Non-interpolation in Morrey–Campanato and block spaces. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **1999**, *28*, 31–40.
90. Chiarenza, F.; Frasca, M. Morrey spaces and Hardy–Littlewood maximal function. *Rend. Mat. Appl.* **1987**, *7*, 273–279.
91. Fofana, I. Étude d'une classe d'espace de fonctions contenant les espaces de Lorentz. *Afr. Mat.* **1988**, *1*, 29–50.
92. Adama, A.; Feuto, J.; Fofana, I. A weighted inequality for potential type operators. *Adv. Pure Appl. Math.* **2019**, *10*, 413–426. [CrossRef]
93. Coulibaly, S.; Fofana, I. On Lebesgue integrability of Fourier transforms in amalgam spaces. *J. Fourier Anal. Appl.* **2019**, *25*, 184–209. [CrossRef]
94. Diarra, N.; Fofana, I. On preduals and Köthe duals of some subspaces of Morrey spaces. *J. Math. Anal. Appl.* **2021**, *496*, 124842. [CrossRef]
95. Dosso, M.; Fofana, I.; Sanogo, M. On some subspaces of Morrey–Sobolev spaces and boundedness of Riesz integrals. *Ann. Polon. Math.* **2013**, *108*, 133–153. [CrossRef]
96. Feuto, J.; Fofana, I.; Koua, K. Integrable fractional mean functions on spaces of homogeneous type. *Afr. Diaspora J. Math. (N.S.)* **2010**, *9*, 8–30.
97. Neri, U. Fractional integration on the space H^1 and its dual. *Stud. Math.* **1975**, *53*, 175–189. [CrossRef]
98. Torres, R.H.; Xue, Q. On compactness of commutators of multiplication and bilinear pesudodifferential operators and a new subspace of BMO. *Rev. Mat. Iberoam.* **2020**, *36*, 939–956. [CrossRef]
99. Tao, J.; Xue, Q.; Yang, D.; Yuan, W. XMO and weighted compact bilinear commutators. *J. Fourier Anal. Appl.* **2021**, *27*, 60. [CrossRef]
100. Li, K. Multilinear commutators in the two-weight setting. *arXiv* **2020**, arXiv:2006.09071.
101. Tao, J.; Yang, D.; Yang, D. Boundedness and compactness characterizations of Cauchy integral commutators on Morrey spaces. *Math. Methods Appl. Sci.* **2019**, *42*, 1631–1651. [CrossRef]
102. Torres, R.H.; Xue, Q.; Yan, J. Compact bilinear commutators: The quasi-Banach space case. *J. Anal.* **2018**, *26*, 227–234. [CrossRef]
103. Chaffee, L.; Chen, P.; Han, Y.; Torres, R.H.; Ward, L.A. Characterization of compactness of commutators of bilinear singular integral operators. *Proc. Am. Math. Soc.* **2018**, *146*, 3943–3953. [CrossRef]
104. Dafni, G. Local VMO and weak convergence in h^1. *Canad. Math. Bull.* **2002**, *45*, 46–59. [CrossRef]
105. Bourdaud, G. Remarques sur certains sous-espaces de $\mathrm{BMO}(\mathbb{R}^n)$ et de $\mathrm{bmo}(\mathbb{R}^n)$. (French. English, French summary) [Remarks on some subspaces of $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$]. *Ann. Inst. Fourier (Grenoble)* **2002**, *52*, 1187–1218. [CrossRef]
106. Alabalik, A.; Almeida, A.; Samko, S. On the invariance of certain vanishing subspaces of Morrey spaces with respect to some classical operators. *Banach J. Math. Anal.* **2020**, *14*, 987–1000. [CrossRef]
107. Alabalik, A.; Almeida, A.; Samko, S. Preservation of certain vanishing properties of generalized Morrey spaces by some classical operators. *Math. Methods Appl. Sci.* **2020**, *43*, 9375–9386. [CrossRef]
108. Almeida, A. Maximal commutators and commutators of potential operators in new vanishing Morrey spaces. *Nonlinear Anal.* **2020**, *192*, 111684. [CrossRef]
109. Almeida, A.; Samko, S. Approximation in Morrey spaces. *J. Funct. Anal.* **2017**, *272*, 2392–2411. [CrossRef]
110. Tao, J.; Yang, D.; Yuan, W. Vanishing John–Nirenberg spaces. *Adv. Calc. Var.* **2021**. [CrossRef]