# Molecular Characterizations of Anisotropic Mixed-Norm Hardy Spaces and Their Applications 

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#### Abstract

Let $\vec{p} \in(0, \infty)^{n}$ be an exponent vector and $A$ be a general expansive matrix on $\mathbb{R}^{n}$. Let $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ be the anisotropic mixed-norm Hardy spaces associated with $A$ defined via the nontangential grand maximal function. In this article, using the known atomic characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, the authors characterize this Hardy space via molecules with the best possible known decay. As an application, the authors establish a criterion on the boundedness of linear operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to itself, which is used to explore the boundedness of anisotropic Calderón-Zygmund operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. In addition, the boundedness of anisotropic Calderón-Zygmund operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is also presented. The obtained boundedness of these operators positively answers a question mentioned by Cleanthous et al. All of these results are new, even for isotropic mixed-norm Hardy spaces on $\mathbb{R}^{n}$.


Keywords: expansive matrix; (mixed-norm) Hardy space; molecule; Calderón-Zygmund operator

## 1. Introduction

This article is devoted to exploring the molecular characterization of the anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ from [1], where $\vec{p} \in(0, \infty)^{n}$ is an exponent vector and $A$ is a general expansive matrix on $\mathbb{R}^{n}$ (see Definition 1 below). Recall that, as a generalization of the classical Lebesgue space $L^{p}\left(\mathbb{R}^{n}\right)$, the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$, in which the constant exponent $p$ is replaced by an exponent vector $\vec{p} \in[1, \infty]^{n}$, was studied by Benedek and Panzone [2] in 1961, which can be traced back to Hörmander [3]. Moreover, based on the mixed-norm Lebesgue space, the real-variable theory of various mixed-norm function spaces has rapidly developed over the last two decades; as can be seen, for instance, in ref. [4] on mixed-norm $\alpha$-modulation spaces, in ref. [5] on mixed-norm Morrey spaces, in refs. [1,6-12] on mixed-norm Hardy spaces, as well as in [13-17] on mixed-norm Besov spaces and mixed-norm Triebel-Lizorkin spaces. For more details on the progress made with regard to the theory of mixed-norm function spaces, we refer the reader to [18-27] as well as to the survey article [28]. In particular, Cleanthous et al. [6] first introduced the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ associated with an anisotropic quasihomogeneous norm $|\cdot|_{\vec{a}}$, where $\vec{a} \in[1, \infty)^{n}$ and $\vec{p} \in(0, \infty)^{n}$, via the non-tangential grand maximal function, and then established its various maximal function characterizations. Later on, Huang et al. $[10,11]$ further completed the real-variable theory of $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$.

On the other hand, motivated by the important role of discrete groups of dilations in wavelet theory, Bownik [29] originally introduced the anisotropic Hardy space $H_{A}^{p}\left(\mathbb{R}^{n}\right)$, where $p \in(0, \infty)$. Nowadays, the anisotropic setting has proved useful not only in developing the function spaces arising in harmonic analysis, but also in some other areas such as the wavelet theory (see, for instance [29-32]) and partial differential equations
(see, for instance $[33,34]$ ). Very recently, inspired by the previous works on both the Hardy spaces $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ and $H_{A}^{p}\left(\mathbb{R}^{n}\right)$, Huang et al. [1] introduced the anisotropic mixednorm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ associated with $A$, via the non-tangential grand maximal function, and established its various real-variable characterizations, respectively, by means of the radial or the non-tangential maximal functions, atoms, finite atoms, the Lusin area function, the Littlewood-Paley $g$-function or $g_{\lambda}^{*}$-function. The space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ includes the aforementioned Hardy space $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ as a special case; see Remark 1(i) below.

However, the molecular characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, which can be conveniently used to study the boundedness of many important operators (for instance, Calderón-Zygmund operators) on the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, is still missing. Thus, to further complete the real-variable theory of anisotropic mixed-norm Hardy spaces $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, in this article, we characterize the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ via molecules, in which the range of the decay index $\varepsilon$ is in a sense the best possible known decay (see Remark 1(iv) below). As an application, we then obtain a criterion on the boundedness of linear operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (see Theorem 3 below), which is used to prove the boundedness of anisotropic Calderón-Zygmund operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. In addition, the boundedness of anisotropic Calderón-Zygmund operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is also presented. When $A$ is as in (6) below, the obtained boundedness of these Calderón-Zygmund operators positively answers a question mentioned by Cleanthous et al. in [6] (p.2760); see [1,10] and Remark 2(iv) for more details. All these results are new, even for the isotropic mixed-norm Hardy spaces on $\mathbb{R}^{n}$. Here, we should point out that a molecular characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ has also been independently established in [35], in which the range of the decay index $\varepsilon$ is just a proper subset of that from the present article. In this sense, the molecular characterization obtained in [35] is covered by the corresponding result of the present article.

The remainder of this article is organized as follows.
In Section 2, we present some notions on expansive matrices, homogeneous quasinorms, the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ and the anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (see Definitions 3 and 5 below).

Section 3 is devoted to characterizing the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ via molecules (see Theorem 1 below). To do this, we first give the notion of the anisotropic mixed-norm molecular Hardy space $H_{A}^{\vec{p} r, r, \varepsilon}\left(\mathbb{R}^{n}\right)$ (see Definition 7 below). Then, by the known atomic characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ from [1] (Theorem 4.7) (see also Lemma 2 below), we have $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \subset H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ with continuous inclusion. Therefore, to complete the proof of Theorem 1, we only need to show $H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right) \subset H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ and the inclusion is continuous. Observe that, to obtain the inclusion of this type, the general method is to decompose a molecule into an infinite linear combination of the related atoms (see, for instance [36] (7.4) or [37] (3.23)), which does not work in the present article since the uniformly upper bound estimate of the dual-bases of the natural projection of each molecule on the infinite annuli of a dilated ball (see [36] (7.2) or [37] (3.18)) is still unclear due to its anisotropic structure. To overcome this difficulty, the main idea is to directly estimate the non-tangential maximal function of a molecule on the infinite annuli of a dilated ball (see (16) below), in which we need fully use the integral size condition of a molecule (see Definition 6(i) below). Then, we prove that $H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ is continuously embedded into $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, which completes the proof of Theorem 1.

As applications, in Section 4, we present the boundedness of anisotropic CalderónZygmund operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (see Theorem 2 below) or to itself (see Theorem 3 below). For this purpose, by the known finite atomic characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, we first give the proof of Theorem 2. To prove Theorem 3, we then obtain a technical lemma, which shows that, if $T$ is an anisotropic Calderón-Zygmund operator of order $\ell$ as in Definition 11, then, for any $(\vec{p}, r, \ell)$-atom $\widetilde{a}, T(\widetilde{a})$ is a harmless constant multiple of a $\left(\vec{p}, q, s_{0}, \varepsilon\right)$-molecule with $s_{0}$ and $\varepsilon$, respectively, as in Definition 11 and (24)
below; see Lemma 8 below. In addition, the density of $H_{A, \text { fin }}^{\vec{p}, \infty}\left(\mathbb{R}^{n}\right) \cap C\left(\mathbb{R}^{n}\right)$ in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is also presented in Lemma 9 below. Using this density and the molecular characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ from Section 3, we establish a useful criterion on the boundedness of linear operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (see Theorem 4 below), which shows that, if a linear operator $T$ maps each atom to a related molecule, then $T$ has a unique bounded linear extension on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. Applying this criterion and Lemma 8, we then prove Theorem 3.

Finally, we make some conventions on notations. Let $\mathbf{0}$ be the origin of $\mathbb{R}^{n}, \mathbb{N}:=\{1,2, \ldots\}$ and $\mathbb{Z}_{+}:=\{0\} \cup \mathbb{N}$. We always use $C$ to denote a positive constant which is independent of the main parameters, but may vary from line to line. The notation $f \lesssim g$ means $f \leq C g$ and if $f \lesssim g \lesssim f$, then we write $f \sim g$. We also use the following convention: if $f \leq C g$ and $g=h$ or $g \leq h$, then we write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g=h$ or $f \lesssim g \leq h$. For each multi-index $\beta:=\left(\beta_{1}, \ldots, \beta_{n}\right) \in\left(\mathbb{Z}_{+}\right)^{n}=: \mathbb{Z}_{+}^{n}$, let $|\beta|:=\beta_{1}+\cdots+\beta_{n}$ and

$$
\partial^{\beta}:=\left(\frac{\partial}{\partial x_{1}}\right)^{\beta_{1}} \cdots\left(\frac{\partial}{\partial x_{n}}\right)^{\beta_{n}} .
$$

For each $r \in[1, \infty]$, we denote by $r^{\prime}$ its conjugate index, namely $1 / r+1 / r^{\prime}=1$. Moreover, if $\vec{r}:=\left(r_{1}, \ldots, r_{n}\right) \in[1, \infty]^{n}$, we denote by $\vec{r}^{\prime}:=\left(r_{1}^{\prime}, \ldots, r_{n}^{\prime}\right)$ its conjugate index. In addition, for each set $\Omega \subset \mathbb{R}^{n}$, we denote by $\Omega^{\complement}$ the set $\mathbb{R}^{n} \backslash \Omega$, by $\mathbf{1}_{\Omega}$ its characteristic function, and by $|\Omega|$ its n-dimensional Lebesgue measure. For any $s \in \mathbb{R}$, we denote by $\lfloor s\rfloor$ the largest integer not greater than $s$. Throughout this article, the symbol $C^{\infty}\left(\mathbb{R}^{n}\right)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^{n}$.

## 2. Preliminaries

In this section, we present some notions on expansive matrices, mixed-norm Lebesgue spaces and anisotropic mixed-norm Hardy spaces (see, for instance [1,2,29]).

We begin with recalling the notion of expansive matrices from [29] (p. 5, Definition 2.1).
Definition 1. An expansive matrix, i.e., a dilation, is a real $n \times n$ matrix $A$ satisfying:

$$
\min _{\lambda \in \sigma(A)}|\lambda|>1,
$$

and here and thereafter, $\sigma(A)$ denotes the set of all eigenvalues of $A$.
Let $b:=|\operatorname{det} A|$. Then, by [29] (p. 6, (2.7)), it is easy to see that $b \in(1, \infty)$. By [29] (p.5, Lemma 2.2), we know that there exists an open ellipsoid $\Delta$, with $|\Delta|=1$, and $r \in(1, \infty)$ such that $\Delta \subset r \Delta \subset A \Delta$. This further implies that, for any $j \in \mathbb{Z}, B_{j}:=A^{j} \Delta$ is open, $B_{j} \subset r B_{j} \subset B_{j+1}$ and $\left|B_{j}\right|=b^{j}$. For each $x \in \mathbb{R}^{n}$ and $j \in \mathbb{Z}$, an ellipsoid $x+B_{j}$ is called a dilated ball. Hereinafter, we always use $\mathfrak{B}$ to denote the collection of all such dilated balls, namely:

$$
\begin{equation*}
\mathfrak{B}:=\left\{x+B_{j}: x \in \mathbb{R}^{n} \text { and } j \in \mathbb{Z}\right\} \tag{1}
\end{equation*}
$$

and:

$$
\begin{equation*}
\omega:=\inf \left\{i \in \mathbb{Z}: r^{i} \geq 2\right\} \tag{2}
\end{equation*}
$$

The following notion of the homogeneous quasi-norm is just [29] (p.6, Definition 2.3).
Definition 2. For any given dilation $A$, a homogeneous quasi-norm, with respect to $A$, is a measurable mapping $\rho: \mathbb{R}^{n} \rightarrow[0, \infty)$ satisfying:
(i) If $x \neq \mathbf{0}$, then $\rho(x) \in(0, \infty)$;
(ii) For any $x \in \mathbb{R}^{n}, \rho(A x)=b \rho(x)$;
(iii) There exists some $R \in[1, \infty)$ such that, for any $x, y \in \mathbb{R}^{n}, \rho(x+y) \leq R[\rho(x)+\rho(y)]$.

Throughout this article, for a fixed dilation $A$, by [29] (p.6, Lemma 2.4), we can use the following step homogeneous quasi-norm $\rho$ defined by setting for any $x \in \mathbb{R}^{n}$ :

$$
\begin{equation*}
\rho(x):=\sum_{j \in \mathbb{Z}} b^{j} \mathbf{1}_{B_{j+1} \backslash B_{j}}(x) \text { when } x \neq \mathbf{0}, \quad \text { or else } \rho(\mathbf{0}):=0 \tag{3}
\end{equation*}
$$

for both simplicity and convenience.
For any $\vec{p}:=\left(p_{1}, \ldots, p_{n}\right) \in(0, \infty)^{n}$, let:

$$
\begin{equation*}
p_{-}:=\min \left\{p_{1}, \ldots, p_{n}\right\}, \quad p_{+}:=\max \left\{p_{1}, \ldots, p_{n}\right\} \quad \text { and } \quad \underline{p} \in\left(0, \min \left\{p_{-}, 1\right\}\right) \tag{4}
\end{equation*}
$$

The following definition of mixed-norm Lebesgue spaces is from [2].
Definition 3. Let $\vec{p}:=\left(p_{1}, \ldots, p_{n}\right) \in(0, \infty]^{n}$. The mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^{n}$ such that:

$$
\|f\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}:=\left\{\int_{\mathbb{R}} \cdots\left[\int_{\mathbb{R}}\left|f\left(x_{1}, \ldots, x_{n}\right)\right|^{p_{1}} d x_{1}\right]^{\frac{p_{2}}{p_{1}}} \cdots d x_{n}\right\}^{\frac{1}{p_{n}}}<\infty
$$

with the usual modifications made when $p_{i}=\infty$ for some $i \in\{1, \ldots, n\}$.
Obviously, when $\vec{p}:=(\overbrace{p, \ldots, p}^{n \text { times }})$ with some $p \in(0, \infty]$, the space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is just the classical Lebesgue space $L^{p}\left(\mathbb{R}^{n}\right)$.

Recall that a Schwartz function is a $C^{\infty}\left(\mathbb{R}^{n}\right)$ function $\varphi$ satisfying that, for any $v \in \mathbb{Z}_{+}$ and multi-index $\gamma \in \mathbb{Z}_{+}^{n}$,

$$
\|\varphi\|_{\gamma, v}:=\sup _{x \in \mathbb{R}^{n}}[\rho(x)]^{\nu}\left|\partial^{\gamma} \varphi(x)\right|<\infty .
$$

Denote by $\mathcal{S}\left(\mathbb{R}^{n}\right)$ the collection of all Schwartz functions as above, equipped with the topology determined by $\left\{\|\cdot\|_{\gamma, \nu}\right\}_{\gamma \in \mathbb{Z}_{+}^{n}, v \in \mathbb{Z}_{+}}$and $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ its dual space, equipped with the weak-* topology. For any $N \in \mathbb{Z}_{+}$, denote by $\mathcal{S}_{N}\left(\mathbb{R}^{n}\right)$ the following set:

$$
\left\{\varphi \in \mathcal{S}\left(\mathbb{R}^{n}\right):\|\varphi\|_{\mathcal{S}_{N}\left(\mathbb{R}^{n}\right)}:=\sup _{\gamma \in \mathbb{Z}_{+}^{n},|\gamma| \leq N} \sup _{x \in \mathbb{R}^{n}}\left[\left|\partial^{\gamma} \varphi(x)\right| \max \left\{1,[\rho(x)]^{N}\right\}\right] \leq 1\right\}
$$

Hereinafter, for any $\varphi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and $j \in \mathbb{Z}$, let: $\varphi_{j}(\cdot):=b^{-j} \varphi\left(A^{-j}\right)$.
Let $\lambda_{-}, \lambda_{+} \in(1, \infty)$ be two numbers such that:

$$
\lambda_{-} \leq \min \{|\lambda|: \lambda \in \sigma(A)\} \leq \max \{|\lambda|: \lambda \in \sigma(A)\} \leq \lambda_{+}
$$

We should point out that if $A$ is diagonalizable over $\mathbb{C}$, then we can let:

$$
\lambda_{-}:=\min \{|\lambda|: \lambda \in \sigma(A)\} \quad \text { and } \quad \lambda_{+}:=\max \{|\lambda|: \lambda \in \sigma(A)\}
$$

Otherwise, we may choose them sufficiently close to these equalities in accordance with what we need in our arguments.

Definition 4. For any fixed $N \in \mathbb{N}$, the non-tangential grand maximal function $M_{N}(f)$ of $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ is defined by setting, for any $x \in \mathbb{R}^{n}$ :

$$
M_{N}(f)(x):=\sup _{\varphi \in \mathcal{S}_{N}\left(\mathbb{R}^{n}\right)} \sup _{y \in x+B_{j}, j \in \mathbb{Z}}\left|f * \varphi_{j}(y)\right|
$$

We now recall the notion of anisotropic mixed-norm Hardy spaces as follows; see [1] (Definition 2.5).

Definition 5. Let $\vec{p} \in(0, \infty)^{n}$ and $N \in \mathbb{N} \cap\left[\left\lfloor\left(\frac{1}{\min \left\{1, p_{-}\right\}}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor+2, \infty\right)$ with $p_{-}$as in (4). The anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is defined as the set of all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $M_{N}(f) \in L^{\vec{p}}\left(\mathbb{R}^{n}\right)$. Moreover, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, let:

$$
\|f\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}:=\left\|M_{N}(f)\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} .
$$

Observe that, by [1] (Theorem 4.7), we know that the Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is independent of the choice of $N$ as in Definition 5.

## 3. Molecular Characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$

In this section, we characterize $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ via molecules. Recall that, for any $r \in(0, \infty]$ and measurable set $\Omega \subset \mathbb{R}^{n}$, the Lebesgue space $L^{r}(E)$ is defined as the set of all measurable functions $g$ on $\Omega$ such that, when $r \in(0, \infty)$,

$$
\|g\|_{L^{r}(\Omega)}:=\left[\int_{\Omega}|g(x)|^{r} d x\right]^{1 / r}<\infty
$$

and

$$
\|g\|_{L^{\infty}(\Omega)}:=\underset{x \in \Omega}{\operatorname{ess} \sup }|g(x)|<\infty .
$$

We now introduce the notion of anisotropic mixed-norm $(\vec{p}, r, s, \varepsilon)$-molecules as follows.
Definition 6. Let $\vec{p} \in(0, \infty)^{n}, r \in(1, \infty]$ :

$$
\begin{equation*}
s \in\left[\left\lfloor\left(\frac{1}{p_{-}}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor, \infty\right) \cap \mathbb{Z}_{+} \tag{5}
\end{equation*}
$$

and $\varepsilon \in(0, \infty)$, where $p_{-}$is as in (4). An anisotropic mixed-norm ( $\left.\vec{p}, r, s, \varepsilon\right)$-molecule, associated with some dilated ball $B:=x_{0}+B_{k_{0}} \in \mathfrak{B}$ with $x_{0} \in \mathbb{R}^{n}, k_{0} \in \mathbb{Z}$ and $\mathfrak{B}$ as in (1), is a measurable function $m$ satisfying the following two conditions:
(i) For any $k \in \mathbb{Z}_{+},\|m\|_{L^{r}\left(U_{k}(B)\right)} \leq b^{-k \varepsilon}|B|^{1 / r}\left\|\mathbf{1}_{B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)^{\prime}}^{-1}$ where $U_{0}(B):=B$ and, for any $k \in \mathbb{N}$,

$$
U_{k}(B)=U_{k}\left(x_{0}+B_{k_{0}}\right):=x_{0}+\left(A^{k} B_{k_{0}}\right) \backslash\left(A^{k-1} B_{k_{0}}\right) ;
$$

(ii) For any multi-index $\gamma \in \mathbb{Z}_{+}^{n}$ with $|\gamma| \leq s, \int_{\mathbb{R}^{n}} m(x) x^{\gamma} d x=0$.

Henceforth, we call an anisotropic mixed-norm $(\vec{p}, r, s, \varepsilon)$-molecule simply by a $(\vec{p}, r, s, \varepsilon)$ molecule. Via $(\vec{p}, r, s, \varepsilon)$-molecules, we give the following notion of anisotropic mixed-norm molecular Hardy spaces $H_{A}^{\vec{p} r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$.

Definition 7. Let $\vec{p} \in(0, \infty)^{n}, r \in(1, \infty]$, s be as in (5) and $\varepsilon \in(0, \infty)$. The anisotropic mixednorm molecular Hardy space $H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ is defined to be the set of all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfying that there exists a sequence $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s, \varepsilon)$-molecules, $\left\{m_{k}\right\}_{k \in \mathbb{N}}$, associated, respectively, with $\left\{B^{(k)}\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ such that:

$$
f=\sum_{k \in \mathbb{N}} \lambda_{k} m_{k} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

Moreover, for any $f \in H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$, let:

$$
\|f\|_{H_{A}^{\vec{p} r, s, \varepsilon}\left(\mathbb{R}^{n}\right)}:=\inf \left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}
$$

where the infimum is taken over all decompositions of $f$ as above and $\underline{p}$ as in (4).
The main result of this section is the subsequent Theorem 1.
Theorem 1. Let $\vec{p} \in(0, \infty)^{n}, r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right]$ with $p_{+}$as in (4), s be as in (5):

$$
N \in \mathbb{N} \cap\left[\left\lfloor\left(\frac{1}{\min \left\{1, p_{-}\right\}}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor+2, \infty\right) \text { with } p_{-} \text {as in (4) }
$$

and $\varepsilon \in\left((s+1) \log _{b}\left(\lambda_{+} / \lambda_{-}\right), \infty\right)$. Then, $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)=H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ with equivalent quasi-norms.
Remark 1. (i) When:

$$
A:=\left(\begin{array}{cccc}
2^{a_{1}} & 0 & \cdots & 0  \tag{6}\\
0 & 2^{a_{2}} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 2^{a_{n}}
\end{array}\right)
$$

with $\vec{a}:=\left(a_{1}, \ldots, a_{n}\right) \in[1, \infty)^{n}$, the Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ and the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ from [6] coincide with equivalent quasi-norms; see [1] (Remark 2(iv)). In this case, Theorem 1 is new. Moreover, if $A:=d \mathrm{I}_{n \times n}$ for some $d \in \mathbb{R}$ with $|d| \in(1, \infty)$, here and thereafter, $\mathrm{I}_{n \times n}$ denotes the $n \times n$ unit matrix, then $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ becomes the classical isotropic mixed-norm Hardy space from [7] which is just a special case of $H_{\vec{a}}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ from [6]; see [10] Remark 4.4(i) for more details. Even in this case, Theorem 1 is still new;
(ii) Let $\varphi: \mathbb{R}^{n} \times[0, \infty) \rightarrow[0, \infty)$ be an anisotropic growth function (see, for instance, ref. [38] (Definition 2.5)). Recall that, in [38] (Theorem 3.12), the authors established a molecular characterization of the anisotropic Musielak-Orlicz Hardy space $H_{A}^{\varphi}\left(\mathbb{R}^{n}\right)$; see also $[37,39]$ for the special cases. It follows from [40] (Remark 2.5(iii)), that the anisotropic Musielak-Orlicz Hardy space $H_{A}^{\varphi}\left(\mathbb{R}^{n}\right)$ and anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ in this article cannot cover each other, and hence neither do [38] (Theorem 3.12) and Theorem 1;
(iii) Let $p(\cdot): \mathbb{R}^{n} \rightarrow(0, \infty]$ be a variable exponent function satisfying the so-called globally log-Hölder continuous condition (see [40] (2.5) and (2.6))). Very recently, the molecular characterization of the variable anisotropic Hardy space $H_{A}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ was established by Liu [41] (Theorem 3.1) and, independently, by Wang et al. [42] (Theorem 2.9) with some stronger assumptions on the decay of molecules. As pointed out in [1] (Introduction), the variable anisotropic Hardy space $H_{A}^{p(\cdot)}\left(\mathbb{R}^{n}\right)$ in [41] or [42] and the anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ in this article cannot cover each other. Thus, Theorem 1 cannot be covered by [41] (Theorem 3.1) or [42] (Theorem 2.9);
(iv) When $A:=d \mathrm{I}_{n \times n}$ for some $d \in \mathbb{R}$ with $|d| \in(1, \infty)$ and $\vec{p}:=(\overbrace{p, \ldots, p}^{n \text { times }})$ with some $p \in(0, \infty)$, the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ becomes the classical isotropic Hardy space $H^{p}\left(\mathbb{R}^{n}\right)$ and $\log _{b}\left(\lambda_{+} / \lambda_{-}\right)=0$. In this case, Theorem 1 gives a molecular characterization of $H^{p}\left(\mathbb{R}^{n}\right)$ with the best possible known decay of molecules, namely, $\varepsilon \in(0, \infty)$.

To show Theorem 1, we need several technical lemmas. First, Lemma 1 is just [1] (Lemma 4.5).

Lemma 1. Let $\vec{p} \in(0, \infty)^{n}, i \in \mathbb{Z}$ and $r \in[1, \infty] \cap\left(p_{+}, \infty\right]$ with $p_{+}$as in (4). Assume that $\left\{t_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{C},\left\{B^{(k)}\right\}_{k \in \mathbb{N}}:=\left\{x_{k}+B_{\ell_{k}}\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ and $\left\{a_{k}\right\}_{k \in \mathbb{N}} \subset L^{r}\left(\mathbb{R}^{n}\right)$ satisfy that, for any $k \in \mathbb{N}, \operatorname{supp} a_{k} \subset x_{k}+A^{i} B_{\ell_{k}}$ :

$$
\left\|a_{k}\right\|_{L^{r}\left(\mathbb{R}^{n}\right)} \leq \frac{\left|B^{(k)}\right|^{1 / r}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}
$$

and:
where $p$ is as in (4). Then:

$$
\left\|\left[\sum_{k \in \mathbb{N}}\left|t_{k} a_{k}\right|^{\underline{p}}\right]^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \leq C\left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|t_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)},
$$

where $C$ is a positive constant independent of $\left\{t_{k}\right\}_{k \in \mathbb{N},},\left\{B^{(k)}\right\}_{k \in \mathbb{N}}$ and $\left\{a_{k}\right\}_{k \in \mathbb{N}}$.
The following notions of anisotropic mixed-norm $(\vec{p}, r, s)$-atoms and anisotropic mixed-norm atomic Hardy spaces $H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ are from [1].

Definition 8. Let $\vec{p} \in(0, \infty)^{n}, r \in(1, \infty]$ and $s$ be as in (5).
(i) A measurable function a on $\mathbb{R}^{n}$ is called an anisotropic mixed-norm $(\vec{p}, r, s)$-atom if:
(i) $1_{1} \quad \operatorname{supp} a \subset B$ with some $B \in \mathfrak{B}$, where $\mathfrak{B}$ is as in (1);
(i) $2_{2} \quad\|a\|_{L^{r}\left(\mathbb{R}^{n}\right)} \leq \frac{|B|^{1 / r}}{\left\|\mathbf{1}_{B}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}$;
(i) $)_{3} \quad$ For any $\alpha \in \mathbb{Z}_{+}^{n}$ with $|\alpha| \leq s, \int_{\mathbb{R}^{n}} a(x) x^{\alpha} d x=0$.
(ii) The anisotropic mixed-norm atomic Hardy space $H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ is defined to be the set of all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfying that there exists a sequence $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s)$ atoms, $\left\{a_{k}\right\}_{k \in \mathbb{N}}$, supported, respectively, in $\left\{B^{(k)}\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ such that:

$$
f=\sum_{k \in \mathbb{N}} \lambda_{k} a_{k} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) .
$$

Furthermore, for any $f \in H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$, let:

$$
\|f\|_{H_{A}^{\vec{p} r, s}\left(\mathbb{R}^{n}\right)}:=\inf \left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)},
$$

where the infimum is taken over all decompositions of $f$ as above.
We also need the atomic characterization of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ obtained in [1] (Theorem 4.7).
Lemma 2. Let $\vec{p}, r, s$ and $N$ be as in Theorem 1. Then:

$$
H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)=H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)
$$

with equivalent quasi-norms.
In addition, by [29] (p. 8, (2.11), p. 5, (2.1) and (2.2) and p.17, Proposition 3.10), we have the following conclusions.

Lemma 3. Let $A$ be some fixed dilation. Then:
(i) For any $i \in \mathbb{Z}$ :

$$
B_{i}+B_{i} \subset B_{i+\omega} \quad \text { and } \quad B_{i}+\left(B_{i+\omega}\right)^{\complement} \subset\left(B_{i}\right)^{\complement}
$$

where $\omega$ is as in (2);
(ii) There exists a positive constant $C$ such that, for any $x \in \mathbb{R}^{n}$, when $k \in \mathbb{Z}_{+}$:

$$
\frac{1}{\mathrm{C}}\left(\lambda_{-}\right)^{k}|x| \leq\left|A^{k} x\right| \leq C\left(\lambda_{+}\right)^{k}|x|
$$

and, when $k \in \mathbb{Z} \backslash \mathbb{Z}_{+}$:

$$
\frac{1}{C}\left(\lambda_{+}\right)^{k}|x| \leq\left|A^{k} x\right| \leq C\left(\lambda_{-}\right)^{k}|x|
$$

(iii) For any given $N \in \mathbb{N}$, there exists a constant $C_{(N)} \in(0, \infty)$, depending on $N$, such that, for any $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and $x \in \mathbb{R}^{n}$,

$$
M_{N}^{0}(f)(x) \leq M_{N}(f)(x) \leq C_{(N)} M_{N}^{0}(f)(x)
$$

where $M_{N}^{0}(f)$ denotes the radial grand maximal function of $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ defined by setting, for any $x \in \mathbb{R}^{n}$,

$$
M_{N}^{0}(f)(x):=\sup _{\varphi \in \mathcal{S}_{N}\left(\mathbb{R}^{n}\right)} \sup _{k \in \mathbb{Z}}\left|f * \varphi_{k}(x)\right|
$$

Denote by $L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ the set of all locally integrable functions on $\mathbb{R}^{n}$. Recall that the anisotropic Hardy-Littlewood maximal function $M_{\mathrm{HL}}(f)$ of $f \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{n}\right)$ is defined by setting, for any $x \in \mathbb{R}^{n}$ :

$$
\begin{equation*}
M_{\mathrm{HL}}(f)(x):=\sup _{k \in \mathbb{Z}} \sup _{y \in x+B_{k}} \frac{1}{\left|B_{k}\right|} \int_{y+B_{k}}|f(z)| d z=\sup _{x \in B \in \mathfrak{B}} \frac{1}{|B|} \int_{B}|f(z)| d z \tag{7}
\end{equation*}
$$

where $\mathfrak{B}$ is as in (1).
The two following lemmas are, respectively, from [1] (Lemma 4.4) and [16] (p.188).
Lemma 4. Let $\vec{p} \in(1, \infty)^{n}$ and $u \in(1, \infty]$. Then, there exists a positive constant $C$ such that, for any sequence $\left\{f_{k}\right\}_{k \in \mathbb{N}}$ of measurable functions:

$$
\left\|\left\{\sum_{k \in \mathbb{N}}\left[M_{\mathrm{HL}}\left(f_{k}\right)\right]^{u}\right\}^{1 / u}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \leq C\left\|\left(\sum_{k \in \mathbb{N}}\left|f_{k}\right|^{u}\right)^{1 / u}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}
$$

with the usual modification made when $u=\infty$, where $M_{\mathrm{HL}}$ denotes the Hardy-Littlewood maximal operator as in (7).

Lemma 5. Let $\vec{p} \in(0, \infty]^{n}$. Then, for any $t \in(0, \infty)$ and $f \in L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ :

$$
\left\||f|^{t}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}=\|f\|_{L^{t \vec{p}}\left(\mathbb{R}^{n}\right)}^{t}
$$

In addition, for any $\mu \in \mathbb{C}, t \in\left[0, \min \left\{p_{-}, 1\right\}\right]$ and $f, g \in L^{\vec{p}}\left(\mathbb{R}^{n}\right),\|\mu f\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}=$ $|\mu|\|f\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}$ and:

$$
\|f+g\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{t} \leq\|f\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{t}+\|g\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{t}
$$

We now prove Theorem 1.

Proof of Theorem 1. Let $\vec{p} \in(0, \infty)^{n}, r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right]$ with $p_{+}$as in (4) and $s$ be as in (5). Then, by the fact that a $(\vec{p}, r, s)$-atom is a $(\vec{p}, r, s, \varepsilon)$-molecule for any $\varepsilon \in(0, \infty)$, as well as the notions of both $H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ and $H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$, it is easy to see that $H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right) \subset H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ with continuous inclusion. In addition, by Lemma 2, we have $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)=H_{A}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ with equivalent quasi-norms. Therefore, $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \subset H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$ and this inclusion is continuous. Thus, to complete the proof of Theorem 1, it suffices to prove that:

$$
\begin{equation*}
H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right) \subset H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \tag{8}
\end{equation*}
$$

holds true with continuous inclusion. For this purpose, without loss of generality, for any $f \in H_{A}^{\vec{p} r, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$, we may assume that $f$ is not the zero element of $H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)$. Then, by Definition 7, we find that there exists a sequence $\left\{\lambda_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, r, s, \varepsilon)$-molecules $\left\{m_{k}\right\}_{k \in \mathbb{N}}$, associated, respectively, to $\left\{B^{(k)}\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ such that:

$$
\begin{equation*}
f=\sum_{k \in \mathbb{N}} \lambda_{k} m_{k} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \tag{9}
\end{equation*}
$$

and:

$$
\begin{equation*}
\|f\|_{H_{A}^{\vec{p} r, r, \varepsilon}\left(\mathbb{R}^{n}\right)} \sim\left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{\left.B^{(k}\right)}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \tag{10}
\end{equation*}
$$

with $\underline{p}$ as in (4). Take two sequences $\left\{x_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{R}^{n}$ and $\left\{i_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{Z}$ such that, for any $k \in \mathbb{N}, x_{k}+B_{i_{k}}=B^{(k)}$. From (9), we deduce that, for any $N \in \mathbb{N} \cap\left[\left\lfloor\left(\frac{1}{p}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor+2, \infty\right)$ and $x \in \mathbb{R}^{n}$ :

$$
\begin{align*}
M_{N}(f)(x) & \leq \sum_{k \in \mathbb{N}}\left|\lambda_{k}\right| M_{N}\left(m_{k}\right)(x) \mathbf{1}_{x_{k}+A^{\omega} B_{i_{k}}}(x)+\sum_{k \in \mathbb{N}}\left|\lambda_{k}\right| M_{N}\left(m_{k}\right)(x) \mathbf{1}_{\left(x_{k}+A^{\omega} B_{i_{k}}\right)}(x) \\
& =: \mathrm{J}_{1}+\mathrm{J}_{2} \tag{11}
\end{align*}
$$

where $\omega$ is an integer as in (2).
For the term $J_{1}$, by the boundedness of $M_{N}$ on $L^{q}\left(\mathbb{R}^{n}\right)$ with $q \in(1, \infty]$ (see [43] (Remark 2.10)) and the definition of ( $\vec{p}, r, s, \varepsilon$ )-molecules, we conclude that, for any $\varepsilon \in$ $\left((s+1) \log _{b}\left(\lambda_{+} / \lambda_{-}\right), \infty\right)$ and $k \in \mathbb{N}$ :

$$
\begin{aligned}
\left\|M_{N}\left(m_{k}\right)\right\|_{L^{r}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|m_{k}\right\|_{L^{r}\left(\mathbb{R}^{n}\right)} \lesssim \sum_{\ell \in \mathbb{Z}_{+}}\left\|m_{k}\right\|_{L^{r}\left(U_{\ell}\left(B^{(k)}\right)\right)} \\
& \lesssim \sum_{\ell \in \mathbb{Z}_{+}} b^{-\ell \varepsilon} \frac{\left|B^{(k)}\right|^{1 / r}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}} \sim \frac{\left|B^{(k)}\right|^{1 / r}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}},
\end{aligned}
$$

where $U_{0}\left(B^{(k)}\right):=B^{(k)}$ and, for each $\ell \in \mathbb{N}$ :

$$
U_{\ell}\left(B^{(k)}\right)=U_{\ell}\left(x_{k}+B_{i_{k}}\right):=x_{k}+\left(A^{\ell} B_{i_{k}}\right) \backslash\left(A^{\ell-1} B_{i_{k}}\right)
$$

This, together with the well-known inequality that, for any $\left\{\alpha_{k}\right\}_{k \in \mathbb{N}} \subset \mathbb{C}$ and $t \in(0,1]$ :

$$
\left[\sum_{k \in \mathbb{N}}\left|\alpha_{k}\right|\right]^{t} \leq \sum_{k \in \mathbb{N}}\left|\alpha_{k}\right|^{t}
$$

as well as Lemma 1 and (10), implies thatL

$$
\begin{align*}
\left\|J_{1}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\left\{\sum_{k \in \mathbb{N}}\left[\left|\lambda_{k}\right| M_{N}\left(m_{k}\right) \mathbf{1}_{x_{k}+A^{\omega} B_{i_{k}}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \sim\|f\|_{H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)} . \tag{12}
\end{align*}
$$

Then, we deal with $\mathrm{J}_{2}$. To this end, we assume that $Q$ is a polynomial with a degree not greater than $s$. Then, from Definition 6 and the Hölder inequality, it follows that, for any $N \in \mathbb{N}, \varphi \in \mathcal{S}_{N}\left(\mathbb{R}^{n}\right), v \in \mathbb{Z}$ and $x \in\left(x_{k}+B_{i_{k}+\omega}\right)^{\complement}$ with $k \in \mathbb{N}$ :

$$
\begin{align*}
& \left|\left(m_{k} * \varphi_{v}\right)(x)\right| \\
& \quad=b^{-v}\left|\int_{\mathbb{R}^{n}} m_{k}(z) \varphi\left(A^{-v}(x-z)\right) d z\right| \\
& \quad \leq b^{-v} \sum_{\ell \in \mathbb{Z}_{+}}\left|\int_{U_{\ell}\left(x_{k}+B_{i_{k}}\right)} m_{k}(z)\left[\varphi\left(A^{-v}(x-z)\right)-Q\left(A^{-v}(x-z)\right)\right] d z\right| \\
& \quad \leq b^{-v} \sum_{\ell \in \mathbb{Z}_{+}} \sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}}|\varphi(z)-Q(z)| \int_{U_{\ell}\left(x_{k}+B_{i_{k}}\right)}\left|m_{k}(z)\right| d z \\
& \quad \lesssim b^{i_{k} / r^{\prime}-v} \sum_{\ell \in \mathbb{Z}_{+}} b^{\ell / r^{\prime}} \sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}}|\varphi(z)-Q(z)|\left\|m_{k}\right\|_{L^{r}\left(U_{\ell}\left(x_{k}+B_{i_{k}}\right)\right)} \\
& \quad \lesssim b^{i_{k}-v}\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} \sum_{\ell \in \mathbb{Z}_{+}} b^{\left(1 / r^{\prime}-\varepsilon\right) \ell} \sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}}|\varphi(z)-Q(z)| . \tag{13}
\end{align*}
$$

For any $k \in \mathbb{N}$ and $x \in\left(x_{k}+B_{i_{k}+\omega}\right)^{\complement}$, it is easy to see that there exists some $j \in \mathbb{Z}_{+}$ such that $x \in\left[x_{k}+\left(B_{i_{k}+\omega+j+1} \backslash B_{i_{k}+\omega+j}\right)\right]$. Then, for any $v \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_{+}$, by Lemma 3(i), we have:

$$
\begin{align*}
A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v} & \subset A^{-v+\ell}\left(B_{i_{k}+\omega+j+1} \backslash B_{i_{k}+\omega+j}\right)+A^{\ell} B_{i_{k}-v} \\
& =A^{i_{k}-v+\ell}\left(\left[B_{\omega+j+1} \backslash B_{\omega+j}\right]+B_{0}\right) \subset A^{i_{k}-v+\ell}\left(B_{j}\right)^{\complement} . \tag{14}
\end{align*}
$$

When $i_{k} \geq v$, we pick $Q \equiv 0$. Then, by (14), the fact that $\varphi \in \mathcal{S}_{N}\left(\mathbb{R}^{n}\right)$ and (3), we find that, for any $N \in \mathbb{N}$ and $\ell \in \mathbb{Z}_{+}$:

$$
\begin{align*}
\sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}}|\varphi(z)-Q(z)| & \leq \sup _{z \in A^{i} k^{-v+\ell}\left(B_{j}\right)^{C}} \min \left\{1, \rho(z)^{-N}\right\} \\
& \leq b^{-N\left(i_{k}-v+\ell+j\right)} . \tag{15}
\end{align*}
$$

When $i_{k}<v$, we let $Q$ be the Taylor expansion of $\varphi$ at the point $A^{-v}\left(x-x_{k}\right)$ with order $s$. Then, from the Taylor remainder theorem, Lemma 3(ii) and (14), we deduce that, for any $N \in \mathbb{N} \cap[s+1, \infty)$ and $\ell \in \mathbb{Z}_{+}$:

$$
\begin{aligned}
& \sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}}|\varphi(z)-Q(z)| \\
& \lesssim\left(\lambda_{+}\right)^{\ell(s+1)}\left(\lambda_{-}\right)^{(s+1)\left(i_{k}-v\right)} \sup _{z \in A^{-v}\left(x-x_{k}\right)+A^{\ell} B_{i_{k}-v}} \min \left\{1, \rho(z)^{-N}\right\} \\
& \lesssim b^{\ell(s+1) \log _{b}\left(\lambda_{+}\right)}\left(\lambda_{-}\right)^{(s+1)\left(i_{k}-v\right)} \sup _{z \in A^{i} k^{-v+\ell}\left(B_{j}\right)^{C}} \min \left\{1, \rho(z)^{-N}\right\} \\
& \lesssim b^{\ell(s+1) \log _{b}\left(\lambda_{+}\right)}\left(\lambda_{-}\right)^{(s+1)\left(i_{k}-v\right)} \min \left\{1, b^{-N\left(i_{k}-v+\ell+j\right)}\right\} .
\end{aligned}
$$

This, combined with Lemma 3(iii), (13) and (15), further implies that, for any $k \in \mathbb{N}$, $N \in \mathbb{N} \cap[s+1, \infty)$ and $x \in\left[x_{k}+\left(B_{i_{k}+\omega+j+1} \backslash B_{i_{k}+\omega+j}\right)\right]$ with some $j \in \mathbb{Z}_{+}$:

$$
\begin{aligned}
& M_{N}\left(m_{k}\right)(x) \\
& \sim \sup _{\varphi \in \mathcal{S}_{N}\left(\mathbb{R}^{n}\right)} \sup _{v \in \mathbb{Z}}\left|\left(m_{k} * \varphi_{v}\right)(x)\right| \\
& \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} \sum_{\ell \in \mathbb{Z}_{+}} b^{\left(1 / r^{\prime}-\varepsilon\right) \ell} \max \left\{\sup _{v \in \mathbb{Z}, v \leq i_{k}} b^{i_{k}-v} b^{-N\left(i_{k}-v+\ell+j\right)},\right. \\
& \left.\sup _{v \in \mathbb{Z}, v>i_{k}} b^{i_{k}-v} b^{\ell(s+1) \log _{b}\left(\lambda_{+}\right)}\left(\lambda_{-}\right)^{(s+1)\left(i_{k}-v\right)} \min \left\{1, b^{-N\left(i_{k}-v+\ell+j\right)}\right\}\right\} .
\end{aligned}
$$

Notice that the supremum over $v \leq i_{k}$ has the largest value when $v=i_{k}$. Without loss of generality, we can take $s=\left\lfloor\left(1 / \min \left\{1, p_{-}\right\}-1\right) \ln b / \ln \lambda_{-}\right\rfloor$and $N=s+2$, which implies that $b \lambda_{-}^{s+1} \leq b^{N}$ and the above supremum over $v>i_{k}$ is attained when $i_{k}-v+\ell+j=0$. By this and the fact that $\varepsilon \in\left((s+1) \log _{b}\left(\lambda_{+} / \lambda_{-}\right), \infty\right)$, we conclude that:

$$
\begin{align*}
& M_{N}\left(m_{k}\right)(x) \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} \\
& \times \sum_{\ell \in \mathbb{Z}_{+}}\left\{b^{-\ell \varepsilon}+b^{-\ell\left[\varepsilon-(s+1) \log _{b}\left(\lambda_{+} / \lambda_{-}\right)\right]}\right\} \max \left\{b^{-N j},\left[b\left(\lambda_{-}\right)^{s+1}\right]^{-j}\right\} \\
& \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1}\left[b\left(\lambda_{-}\right)^{s+1}\right]^{-j} \\
& \sim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} b^{-j} b^{-(s+1) j \frac{\ln \lambda}{\ln b}} \\
& \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} b^{i_{k}\left[(s+1) \frac{\ln \lambda}{\ln b}+1\right]} b^{-\left(i_{k}+\omega+j\right)\left[(s+1) \frac{\ln \lambda}{\ln b}+1\right]} \\
& \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1} \frac{\left|x_{k}+B_{i_{k}}\right| \sigma}{\left[\rho\left(x-x_{k}\right)\right]^{\sigma}} \\
& \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1}\left[M_{\mathrm{HL}}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)(x)\right]^{\sigma} \\
&\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1}\left[M_{\mathrm{HL}}\left(\mathbf{1}_{B^{(k)}}\right)(x)\right]^{\sigma}, \tag{16}
\end{align*}
$$

where:

$$
\sigma:=\left(\frac{\ln b}{\ln \lambda_{-}}+s+1\right) \frac{\ln \lambda_{-}}{\ln b}>\frac{1}{\min \left\{1, p_{-}\right\}}
$$

By this and Lemmas 4 and 5, we obtain:

$$
\begin{aligned}
\left\|\mathrm{J}_{2}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\sum_{k \in \mathbb{N}} \frac{\left|\lambda_{k}\right|}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\left[M_{\mathrm{HL}}\left(\mathbf{1}_{B^{(k)}}\right)\right]^{\sigma}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \sim\left\|\left\{\sum_{k \in \mathbb{N}} \frac{\left|\lambda_{k}\right|}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\left[M_{H L}\left(\mathbf{1}_{B^{(k)}}\right)\right]^{\sigma}\right\}^{1 / \sigma}\right\|_{L^{\sigma \vec{p}}\left(\mathbb{R}^{n}\right)}^{\sigma} \\
& \lesssim\left\|\sum_{k \in \mathbb{N}} \frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right\| \|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{k \in \mathbb{N}}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{p}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \sim\|f\|_{H_{A}^{\vec{p} r, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)^{\prime}} .
\end{aligned}
$$

This, together with (11), (12) and Lemma 5 again, implies that:

$$
\|f\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}=\left\|M_{N}(f)\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)^{\prime}}
$$

which completes the proof of (8) and hence of Theorem 1.

## 4. Some Applications

In this section, as applications, we establish a criterion on the boundedness of linear operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, which further implies the boundedness of anisotropic CalderónZygmund operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. Moreover, the boundedness of these operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is also obtained.

We begin with the definition the notion of anisotropic Calderón-Zygmund operators from [29] (p. 60, Definition 9.1).

Definition 9. An anisotropic Calderón-Zygmund standard kernel is a locally integrable function $\mathcal{K}$ on $E:=\left\{(x, y) \in \mathbb{R}^{n} \times \mathbb{R}^{n}: x \neq y\right\}$ satisfying that there exist two positive constants $C$ and $\tau$ such that, for any $\left(x_{1}, y_{1}\right),\left(x_{1}, y_{2}\right),\left(x_{2}, y_{1}\right) \in E$ :

$$
\begin{gathered}
\left|\mathcal{K}\left(x_{1}, y_{1}\right)\right| \leq \frac{C}{\rho\left(x_{1}-y_{1}\right)^{\prime}}, \\
\left|\mathcal{K}\left(x_{1}, y_{1}\right)-\mathcal{K}\left(x_{1}, y_{2}\right)\right| \leq C \frac{\left[\rho\left(y_{1}-y_{2}\right)\right]^{\tau}}{\left[\rho\left(x_{1}-y_{1}\right)\right]^{1+\tau}} \quad \text { when } \quad \rho\left(x_{1}-y_{1}\right) \geq b^{2 \omega} \rho\left(y_{1}-y_{2}\right),
\end{gathered}
$$

and:

$$
\left|\mathcal{K}\left(x_{1}, y_{1}\right)-\mathcal{K}\left(x_{2}, y_{1}\right)\right| \leq C \frac{\left[\rho\left(x_{1}-x_{2}\right)\right]^{\tau}}{\left[\rho\left(x_{1}-y_{1}\right)\right]^{1+\tau}} \quad \text { when } \quad \rho\left(x_{1}-y_{1}\right) \geq b^{2 \omega} \rho\left(x_{1}-x_{2}\right)
$$

with $\omega$ as in (2). Moreover, an anisotropic Calderón-Zygmund operator is a linear operator $T$ satisfying that it is bounded on $L^{2}\left(\mathbb{R}^{n}\right)$ and there exists an anisotropic Calderón-Zygmund standard kernel $\mathcal{K}$ such that, for any $f \in L^{2}\left(\mathbb{R}^{n}\right)$ with compact support and $x \notin \operatorname{supp} f$,

$$
T(f)(x)=\int_{\operatorname{supp} f} \mathcal{K}(x, z) f(z) d z
$$

Hereinafter, for each $\ell \in \mathbb{N}$, let $C^{\ell}\left(\mathbb{R}^{n}\right)$ be the collection of all functions on $\mathbb{R}^{n}$ whose derivatives with order not greater than $\ell$ are continuous. The following no-
tion of anisotropic Calderón-Zygmund operator of order $\ell$ originates from [29] (p.61, Definition 9.2).

Definition 10. Let $\ell \in \mathbb{N}$. An anisotropic Calderón-Zygmund operator of order $\ell$ is an anisotropic Calderón-Zygmund operator $T$ whose kernel $\mathcal{K}$ is a $C^{\ell}\left(\mathbb{R}^{n}\right)$ function with respect to the second variable $y$ and satisfying that there exists a positive constant $C$ such that, for any $\gamma \in \mathbb{Z}_{+}^{n}$ with $1 \leq|\gamma| \leq \ell, t \in \mathbb{Z}$ and $(x, y) \in E$ with $\rho(x-y) \sim b^{t}:$

$$
\begin{equation*}
\left|\partial_{y}^{\gamma} \widetilde{\mathcal{K}}\left(x, A^{-t} y\right)\right| \leq C[\rho(x-y)]^{-1} \sim C b^{-t} \tag{17}
\end{equation*}
$$

where the implicit equivalent positive constants are independent of $x, y$, $t$ and, for any $x, y \in \mathbb{R}^{n}$ with $x \neq A^{t} y, \widetilde{\mathcal{K}}(x, y):=\mathcal{K}\left(x, A^{t} y\right)$.

Then, we first have the boundedness of anisotropic Calderón-Zygmund operators of order $\ell$ from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$.

Theorem 2. Let $\vec{p} \in(0, \infty)^{n}$ and $T$ be an anisotropic Calderón-Zygmund operator of order $\ell$ with $\ell \in\left[s_{0}+1, \infty\right)$, where $s_{0}:=\left\lfloor\left(1 / p_{-}-1\right) \ln b / \ln \lambda_{-}\right\rfloor$and $p_{-}$is as in (4). Then, there exists a positive constant $C$ such that, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ :

$$
\begin{equation*}
\|T(f)\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \tag{18}
\end{equation*}
$$

To prove this theorem, we need the finite atomic characterization of anisotropic mixednorm Hardy spaces $H_{A, f i n}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$; see [1] (Theorem 5.3). Denote by $C\left(\mathbb{R}^{n}\right)$ the set of all continuous functions on $\mathbb{R}^{n}$.

Lemma 6. Let $\vec{p} \in(0, \infty)^{n}$ and $s$ be as in (5):
(i) If $r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right)$ with $p_{+}$as in (4), then $\|\cdot\|_{H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)}$ and $\|\cdot\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}$ are two equivalent quasi-norms on $H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$;
(ii) $\|\cdot\|_{H_{A, \text { fin }}^{\vec{p}, \infty, s}\left(\mathbb{R}^{n}\right)}$ and $\|\cdot\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}$ are two equivalent quasi-norms on $H_{A, \text { fin }}^{\vec{p}, \infty, s}\left(\mathbb{R}^{n}\right) \cap C\left(\mathbb{R}^{n}\right)$.

Here and thereafter, $H_{A, \text {, fin }}^{\vec{p}, r,}\left(\mathbb{R}^{n}\right)$ denotes the anisotropic mixed-norm finite atomic Hardy space, namely the set of all $f \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfying that there exists $K \in \mathbb{N},\left\{\lambda_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{C}$ and a finite sequence of $(\vec{p}, r, s)$-atoms, $\left\{a_{k}\right\}_{k \in[1, K] \cap \mathbb{N}}$, supported, respectively, in $\left\{B^{(k)}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathfrak{B}$ such that:

$$
f=\sum_{k=1}^{K} \lambda_{k} a_{k} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)
$$

Moreover, for any $f \in H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$, let:

$$
\|f\|_{H_{A, f, \vec{p}, \vec{p}, s}\left(\mathbb{R}^{n}\right)}:=\inf \left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{B^{(k)}}}{\left\|\mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}
$$

where $\underline{p}$ is as in (4) and the infimum is taken over all decompositions of $f$ as above.

In addition, let $\vec{p} \in(1, \infty)^{n}$ and $i \in \mathbb{Z}_{+}$. Then, by Lemma 4 and the fact that, for any dilated ball $B \in \mathfrak{B}$ and $\epsilon \in(0, \underline{p}), \mathbf{1}_{A^{i} B} \leq b^{\frac{i}{\epsilon}}\left[M_{H L}\left(\mathbf{1}_{B}\right)\right]^{\frac{1}{\epsilon}}$, we know that there exists a positive constant $C$ such that, for any sequence $\left\{B^{(k)}\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ :

$$
\begin{equation*}
\left\|\sum_{k \in \mathbb{N}} \mathbf{1}_{A^{i} B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \leq C b^{\frac{i}{\epsilon}}\left\|\sum_{k \in \mathbb{N}} \mathbf{1}_{B^{(k)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \tag{19}
\end{equation*}
$$

Now, we show Theorem 2.
Proof of Theorem 2. Let $\vec{p}, r$ and $s$ be as in Lemma 6(i). We next prove this theorem in two steps.

Step (1). In this step, we prove that (18) holds true for any $f \in H_{A, \text { fin }}^{\vec{p} r, s}\left(\mathbb{R}^{n}\right)$. For this purpose, for any $f \in H_{A \text {, fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$, by Lemma 6 , we can find some $K \in \mathbb{N}$, three finite sequences $\left\{\lambda_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{C},\left\{x_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{R}^{n}$ and $\left\{i_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{Z}$, and a finite sequence of $(\vec{p}, r, s)$-atoms, $\left\{a_{k}\right\}_{k \in[1, K] \cap \mathbb{N}}$, supported, respectively, in $\left\{x_{k}+B_{i_{k}}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset$ $\mathfrak{B}$ such that $f=\sum_{k=1}^{K} \lambda_{k} a_{k}$ in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and:

$$
\begin{equation*}
\|f\|_{H_{A, \text { fin }}^{\vec{p} r, s}\left(\mathbb{R}^{n}\right)} \sim\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} . \tag{20}
\end{equation*}
$$

From the linearity of $T$ and Lemma 5, we obtain:

$$
\begin{align*}
\|T(f)\|_{L^{\bar{p}}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\sum_{k=1}^{K}\left|\lambda_{k}\right| T\left(a_{k}\right) \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}+\left\|\sum_{k=1}^{K}\left|\lambda_{k}\right| T\left(a_{k}\right) \mathbf{1}_{\left(x_{k}+B_{i_{k}+\omega}\right)^{c}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& =: \mathrm{J}_{1}+\mathrm{J}_{2} . \tag{21}
\end{align*}
$$

We first deal with $\mathrm{J}_{2}$. To do this, by a similar argument to that used in the proof of [44] (4.13), we conclude that, for each $k \in[1, K] \cap \mathbb{N}$ and $x \in\left(x_{k}+B_{i_{k}+\omega}\right)^{C}$ :

$$
T\left(a_{k}\right)(x) \lesssim\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-1}\left[M_{\mathrm{HL}}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)(x)\right]^{u}
$$

where:

$$
u:=\left(\frac{\ln b}{\ln \lambda_{-}}+s_{0}+1\right) \frac{\ln \lambda_{-}}{\ln b}>\frac{1}{\underline{p}} .
$$

This, together with Lemmas 5 and 4, and (20), implies that:

$$
\begin{align*}
\left\|\mathrm{J}_{2}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} & \lesssim\left\|\sum_{k=1}^{K} \frac{\left|\lambda_{k}\right|}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\left[M_{H L}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)\right]^{u}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \sim\left\|\left\{\sum_{k=1}^{K} \frac{\left|\lambda_{k}\right|}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\left[M_{H L}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)\right]^{u}\right\}^{1 / u}\right\|_{L^{u \vec{p}}}^{u} \\
& \lesssim\left\|\sum_{k=1}^{K} \frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L_{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L \vec{p}\left(\mathbb{R}^{n}\right)} \\
& \sim\|f\|_{H_{A, \vec{p}, r, s i n}\left(\mathbb{R}^{n}\right)} . \tag{22}
\end{align*}
$$

For $\mathrm{J}_{1}$, take $g \in L^{(\vec{p} / \underline{p})^{\prime}}\left(\mathbb{R}^{n}\right)$ such that $\|g\|_{\left.L^{(\vec{p} / p}\right)^{\prime}}\left(\mathbb{R}^{n}\right) \leq 1$ and:

$$
\begin{aligned}
& \left\|\sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left[T\left(a_{k}\right)\right]^{\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{\vec{p}} / \underline{p}\left(\mathbb{R}^{n}\right)} \\
& \quad=\int_{\mathbb{R}^{n}} \sum_{k=1}^{K}\left|\lambda_{k}\right|^{p}\left[T\left(a_{k}\right)(x)\right]^{\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}(x) g(x) d x .
\end{aligned}
$$

From this, Lemma 5 and the Hölder inequality, it follows that, for any $q \in(1, \infty)$ satisfying $p_{+}<q \underline{p}<r$ :

$$
\begin{aligned}
\left(\mathrm{J}_{1}\right)^{\underline{p}} & \lesssim\left\|\sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left[T\left(a_{k}\right)\right]^{\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{\vec{p}} / \underline{p}\left(\mathbb{R}^{n}\right)} \\
& \sim \int_{\mathbb{R}^{n}} \sum_{k=1}^{K}\left|\lambda_{k}\right|^{p}\left[T\left(a_{k}\right)(x)\right]^{\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}(x) g(x) d x . \\
& \lesssim \sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left\|\left[T\left(a_{k}\right)\right]^{p} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{q}\left(\mathbb{R}^{n}\right)}\left\|\mathbf{1}_{x_{k}+B_{i_{k}+\omega}} g\right\|_{L^{q^{\prime}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim \sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left\|T\left(a_{k}\right)\right\|_{L^{r}\left(\mathbb{R}^{n}\right)}\left\|\mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{r /(r-q \underline{p})}\left(\mathbb{R}^{n}\right)}^{1 / q}\left\|\mathbf{1}_{x_{k}+B_{i_{k}+\omega}} g\right\|_{L^{q^{\prime}\left(\mathbb{R}^{n}\right)}} .
\end{aligned}
$$

This, combined with the boundedness of $T$ on $L^{t}\left(\mathbb{R}^{n}\right)$ for any $t \in(1, \infty)$ (see [29] (p.60)), Definition 8(i) and the Hölder inequality again, further implies that:

$$
\begin{aligned}
\left(\mathrm{J}_{1}\right)^{\underline{p}} & \lesssim \sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-\underline{p}}\left|B_{i_{k}}\right|^{\underline{p} / r}\left|B_{i_{k}+\omega}\right|^{(r-q \underline{p}) / r q}\left\|\mathbf{1}_{x_{k}+B_{i_{k}+\omega}} g\right\|_{L^{q^{\prime}}\left(\mathbb{R}^{n}\right)} \\
& \sim \sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-\underline{p}}\left|B_{i_{k}+\omega}\right|\left[\frac{1}{\left|B_{i_{k}+\omega}\right|} \int_{x_{k}+B_{i_{k}+\omega}}[g(x)]^{q^{\prime}} d x\right]^{1 / q^{\prime}} \\
& \lesssim \sum_{k=1}^{K}\left|\lambda_{k}\right|^{\underline{p}}\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-\underline{p}} \int_{\mathbb{R}^{n}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}(x)\left[M_{\mathrm{HL}}\left(g^{q^{\prime}}\right)(x)\right]^{1 / q^{\prime}} d x \\
& \lesssim\left\|\sum_{k=1}^{K}\left|\lambda_{k}\right| \underline{p}\right\| \mathbf{1}_{x_{k}+B_{i_{k}}}\left\|_{L^{-\vec{p}\left(\mathbb{R}^{n}\right)}}^{-\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}+\omega}}\right\|_{L^{\vec{p} / p}\left(\mathbb{R}^{n}\right)}\left\|\left[M_{\mathrm{HL}}\left(g^{q^{\prime}}\right)\right]^{1 / q^{\prime}}\right\|_{\left.L^{(\vec{p} / p}\right)^{\prime}\left(\mathbb{R}^{n}\right)} .
\end{aligned}
$$

Note that $p_{+} / \underline{p} \in(0, q)$, we know that $(\vec{p} / \underline{p})^{\prime} \in\left(q^{\prime}, \infty\right]$. By this, (19), the boundedness of $M_{\mathrm{HL}}$ on $\vec{L}^{\vec{v}} \overline{\mathbb{R}^{n}}$ ) with $\vec{v} \in(1, \infty]^{n}$ (see [10] (Lemma 3.5)), Lemma 5, the fact that $\|g\|_{L^{(\vec{p} / \underline{p})^{\prime}}{ }_{\left(\mathbb{R}^{n}\right)} \leq 1 \text { and (20), we conclude that: }}$

$$
\begin{aligned}
& \mathrm{J}_{1} \lesssim\left\|\sum_{k=1}^{K}\left|\lambda_{k}\right| \underline{\underline{p}}\right\| \mathbf{1}_{x_{k}+B_{i_{k}}}\left\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{-\underline{p}} \mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p} / \underline{p}}\left(\mathbb{R}^{n}\right)}^{1 / \underline{p}}\|g\|_{L^{(\bar{p}} / \underline{p}^{\prime}}^{1 /\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \sim\|f\|_{H_{A, \text { fin }}^{\vec{p}, r, s}}\left(\mathbb{R}^{n}\right) .
\end{aligned}
$$

From this, (22), (21) and Lemma 6(i), we deduce that (18) holds true for any $f \in$ $H_{A, \text { fin }}^{\vec{p} r, s}\left(\mathbb{R}^{n}\right)$, which completes the proof of Step (1).

Step (2). This step aims to show that (18) holds true for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. To this end, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, by the obvious density of $H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, with respect to the quasi-norm $\|\cdot\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}$, we find that there exists a Cauchy sequence $\left\{f_{i}\right\}_{i \in \mathbb{N}} \subset H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ such that

$$
\lim _{i \rightarrow \infty}\left\|f_{i}-f\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}=0
$$

By this and the linearity of $T$, it is easy to see that, as $i, \iota \rightarrow \infty$ :

$$
\begin{aligned}
\left\|T\left(f_{i}\right)-T\left(f_{l}\right)\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} & =\left\|T\left(f_{i}-f_{l}\right)\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|f_{i}-f_{l}\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \rightarrow 0 .
\end{aligned}
$$

Therefore, $\left\{T\left(f_{i}\right)\right\}_{i \in \mathbb{N}}$ is also a Cauchy sequence in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. By this and the completeness of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, we know that there exists some $h \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ such that $h=\lim _{i \rightarrow \infty} T\left(f_{i}\right)$ in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. Let $T(f):=h$. Then, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ :

$$
\begin{align*}
\|T(f)\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} & \lesssim \limsup _{i \rightarrow \infty}\left[\left\|T(f)-T\left(f_{i}\right)\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}+\left\|T\left(f_{i}\right)\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}\right] \\
& \sim \limsup _{i \rightarrow \infty}\left\|T\left(f_{i}\right)\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \lesssim \lim _{i \rightarrow \infty}\left\|f_{i}\right\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \sim\|f\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \tag{23}
\end{align*}
$$

This finishes the proof of Step (2) and hence of Theorem 2.
Motivated by [29] (p. 64, Definition 9.4), we introduce the vanishing moment condition as follows.

Definition 11. Let $\vec{p} \in(0, \infty)^{n}, \ell \in \mathbb{N}$ satisfy:

$$
\frac{1}{p_{-}}-1<\frac{\left(\ln \lambda_{-}\right)^{2}}{\ln b \ln \lambda_{+}} \ell
$$

and $s_{0}:=\left\lfloor\left(1 / p_{-}-1\right) \ln b / \ln \lambda_{-}\right\rfloor$, where $p_{-}$is as in (4). An anisotropic Calderón-Zygmund operator $T$ of order $\ell$ is said to satisfy $T^{*}\left(x^{\gamma}\right)=0$ for any $\gamma \in \mathbb{Z}_{+}^{n}$ with $|\gamma| \leq s_{0}$ if, for any $g \in L^{2}\left(\mathbb{R}^{n}\right)$ with compact support and satisfying that, for each $\beta \in \mathbb{Z}_{+}^{n}$ with $|\beta| \leq \ell$, $\int_{\mathbb{R}^{n}} g(x) x^{\beta} d x=0$, the equality $\int_{\mathbb{R}^{n}} T(g)(x) x^{\gamma} d x=0$ holds true for each $\gamma \in \mathbb{Z}_{+}^{n}$ satisfying $|\gamma| \leq s_{0}$.

We have the following boundedness of anisotropic Calderón-Zygmund operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$.

Theorem 3. Let $\vec{p}, \ell, s_{0}$ be as in Definition 11. Assume that $T$ is an anisotropic CalderónZygmund operator of order $\ell$ and satisfies $T^{*}\left(x^{\gamma}\right)=0$ for any $\gamma \in \mathbb{Z}_{+}^{n}$ with $|\gamma| \leq s_{0}$. Then, there exists a positive constant $C$ such that, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$,

$$
\|T(f)\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}
$$

By [1] (Lemma 6.8) and [45] (Lemma 2.3), we easily obtain the succeeding Lemma 7; the details are omitted.

Lemma 7. Assume that $E \subset \mathbb{R}^{n}, F \in \mathfrak{B}$ with $\mathfrak{B}$ as in (1), $E \subset F$ and there exists a constant $c_{0} \in(0,1]$ such that $|E| \geq c_{0}|F|$. Then, for any $\vec{p} \in(0, \infty)^{n}$, there exists a positive constant $C$, independent of $E$ and $F$, such that:

$$
\frac{\left\|\mathbf{1}_{F}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}{\left\|\mathbf{1}_{E}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}} \leq C .
$$

To prove Theorem 3, we need the following technical lemma, which is motivated by [44] (Lemma 4.10) and [39] (Lemma 4.13).

Lemma 8. Let $\vec{p}, \ell, s_{0}$ be as in Definition 11. Assume that $r \in(1, \infty]$ and $T$ is an anisotropic Calderón-Zygmund operator of order $\ell$ satisfying $T^{*}\left(x^{\gamma}\right)=0$ for any $\gamma \in \mathbb{Z}_{+}^{n}$ with $|\gamma| \leq s_{0}$. Then, there exists a positive constant $C$ such that, for any $(\vec{p}, r, \ell)$-atom $\tilde{a}$ supported in some dilated ball $x_{0}+B_{i_{0}} \in \mathfrak{B}$ with $x_{0} \in \mathbb{R}^{n}, i_{0} \in \mathbb{Z}$ and $\mathfrak{B}$ as in $(1), \frac{1}{C} T(\widetilde{a})$ is a $\left(\vec{p}, r, s_{0}, \varepsilon\right)$-molecule associated with $x_{0}+B_{i_{0}+\omega}$, where:

$$
\begin{equation*}
\varepsilon:=\ell \log _{b}\left(\lambda_{-}\right)+1 / r^{\prime} \tag{24}
\end{equation*}
$$

and $\omega$ is as in (2).
Proof. Let $T$ be an anisotropic Calderón-Zygmund operator of order $\ell$ satisfying:

$$
T^{*}\left(x^{\gamma}\right)=0 \text { for any } \gamma \in \mathbb{Z}_{+}^{n} \text { with }|\gamma| \leq s_{0}
$$

For any $(\vec{p}, r, \ell)$-atom $\widetilde{a}$ supported in some dilated ball $x_{0}+B_{i_{0}} \in \mathfrak{B}$, without losing generality, we may assume that $x_{0}=\mathbf{0}$. Then, by the vanishing moments of $\tilde{a}$ and Definition 11, we find that $T(\widetilde{a})$ has vanishing moments up to an order of $s_{0}$.

Let $U_{0}\left(B_{i_{0}}\right):=B_{i_{0}+\omega}$ and, for any $k \in \mathbb{N}$ :

$$
U_{k}\left(B_{i_{0}}\right):=\left(A^{k} B_{i_{0}+\omega}\right) \backslash\left(A^{k-1} B_{i_{0}+\omega}\right)
$$

To show that $T(\widetilde{a})$ is a harmless constant multiple of a $\left(\vec{p}, r, s_{0}, \varepsilon\right)$-molecule associated with $B_{i_{0}+\omega}$, it suffices to prove that, for any $k \in \mathbb{Z}_{+}$:

$$
\begin{equation*}
\|T(\widetilde{a})\|_{L^{r}\left(U_{k}\left(B_{i_{0}}\right)\right)} \lesssim \frac{b^{-k \varepsilon}\left|B_{i_{0}+\omega}\right|^{1 / r}}{\left\|\mathbf{1}_{B_{i_{0}+\omega}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}} \tag{25}
\end{equation*}
$$

where $\varepsilon$ is as in (24).
Indeed, from the boundedness of $T$ on $L^{r}$, the fact that $\operatorname{supp} \widetilde{a} \subset B_{i_{0}}$, the size condition of $\widetilde{a}$ and Lemma 7, it follows that:

$$
\|T(\widetilde{a})\|_{L^{r}\left(U_{k}\left(B_{i_{0}}\right)\right)} \lesssim\|\widetilde{a}\|_{L^{r}\left(B_{i_{0}}\right)} \lesssim \frac{\left|B_{i_{0}}\right|^{1 / r}}{\left\|\mathbf{1}_{B_{i_{0}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}} \lesssim \frac{\left|B_{i_{0}+\omega}\right|^{1 / r}}{\left\|\mathbf{1}_{B_{i_{0}+\omega}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}
$$

and hence (25) holds true for $k=0$.
On another hand, for any $(\vec{p}, r, \ell)$-atom $\widetilde{a}, k \in \mathbb{N}, x \in U_{k}\left(B_{i_{0}}\right)$ and $y \in B_{i_{0}}$, by Lemma 3 (i), we know that $x-y \in B_{i_{0}+k+2 \omega} \backslash B_{i_{0}+k-1}$, which implies that $\rho(x-y) \sim b^{i_{0}+k}$. From this and (17), we deduce that, for any $\gamma \in \mathbb{Z}_{+}^{n}$ with $1 \leq|\gamma| \leq \ell$ :

$$
\begin{equation*}
\left|\partial_{y}^{\gamma}\left[\mathcal{K}\left(\cdot, A^{i_{0}+k} \cdot\right)\right]\left(x, A^{-i_{0}-k} y\right)\right| \lesssim[\rho(x-y)]^{-1} \lesssim b^{-i_{0}-k} . \tag{26}
\end{equation*}
$$

Note that supp $\widetilde{a} \subset B_{i_{0}}$. Then, we have:

$$
\begin{equation*}
T(\widetilde{a})(x)=\int_{B_{i_{0}}} \mathcal{K}(x, y) \widetilde{a}(y) d y=\int_{B_{i_{0}}} \widetilde{\mathcal{K}}\left(x, A^{-i_{0}-k} y\right) \widetilde{a}(y) d y \tag{27}
\end{equation*}
$$

where $\widetilde{\mathcal{K}}(x, y):=\mathcal{K}\left(x, A^{i_{0}+k} y\right)$ for any $x, y \in \mathbb{R}^{n}$ with $x \neq A^{i_{0}+k} y$. Moreover, by Taylor expansion theorem for the variable $y$ at the point $(x, 0)$, we easily obtain:

$$
\begin{equation*}
\widetilde{\mathcal{K}}(x, \widetilde{y})=\sum_{\gamma \in \mathbb{Z}_{+}^{n},|\gamma| \leq \ell-1} \frac{\partial_{y}^{\gamma} \widetilde{\mathcal{K}}(x, \mathbf{0})}{\gamma!}(\widetilde{y})^{\gamma}+R_{\ell}(\widetilde{y}) \tag{28}
\end{equation*}
$$

where $\tilde{y}:=A^{-i_{0}-k} y$ for any $y \in B_{i_{0}}$. This, combined with (26), further implies that:

$$
\left|R_{\ell}(\widetilde{y})\right| \lesssim \sup _{t \in B_{-k}} \sup _{\gamma \in \mathbb{Z}_{+}^{n},|\gamma|=\ell}\left|\partial_{y}^{\gamma} \widetilde{\mathcal{K}}(x, t)\right||\widetilde{y}|^{\ell} \lesssim b^{-i_{0}-k} \sup _{t \in B_{-k}}|t|^{\ell} .
$$

By the fact that, for any $t \in B_{-k}, \rho(t)<b^{-k}<1$ and [29] (p.11, Lemma 3.2), we conclude that, for any $\ell \in \mathbb{N}$ as in Definition 11,

$$
\sup _{t \in B_{-k}}|t|^{\ell} \lesssim \sup _{t \in B_{-k}}[\rho(t)]^{\ell \frac{\ln \lambda_{-}}{\ln b}} \lesssim b^{-k \ell \log _{b}\left(\lambda_{-}\right)} .
$$

Thus, we have:

$$
\left|R_{\ell}(\widetilde{y})\right| \lesssim b^{-i_{0}-k} b^{-k \ell \log _{b}\left(\lambda_{-}\right)}
$$

From this, (27), (28), the vanishing moments of atoms and the Hölder inequality, it follows that, for any $(\vec{p}, r, \ell)$-atom $\widetilde{a}, k \in \mathbb{N}$ and $x \in U_{k}\left(B_{i_{0}}\right)$ :

$$
\begin{aligned}
|T(\widetilde{a})(x)| & \leq \int_{B_{i_{0}}}\left|R_{\ell}\left(A^{-i_{0}-k} y\right) \widetilde{a}(y)\right| d y \\
& \lesssim b^{-i_{0}-k} b^{-k \ell \log _{b}\left(\lambda_{-}\right)} \int_{B_{i_{0}}}|\widetilde{a}(y)| d y \\
& \lesssim b^{-i_{0}-k} b^{-k \ell \log _{b}\left(\lambda_{-}\right)}\left|B_{i_{0}}\right|^{1 / r^{\prime}}\|\widetilde{a}\|_{L^{r}\left(B_{i_{0}}\right)} \\
& \sim b^{-k\left[1+\ell \log _{b}\left(\lambda_{-}\right)\right]} b^{-i_{0} / r}\|\widetilde{a}\|_{L^{r}\left(B_{i_{0}}\right)} .
\end{aligned}
$$

This, together with the size condition of $\widetilde{a}$, (24) and Lemma 7, imply that, for any $k \in \mathbb{N}$ :

$$
\begin{aligned}
\|T(\widetilde{a})\|_{L^{r}\left(U_{k}\left(B_{i_{0}}\right)\right)} & \lesssim b^{-k\left[1+\ell \log _{b}(\lambda-)\right]} b^{-i_{0} / r}\|\widetilde{a}\|_{L^{r}\left(B_{i_{0}}\right)}\left|B_{i_{0}+k+\omega}\right|^{1 / r} \\
& \lesssim b^{-k\left[1+\ell \log _{b}(\lambda-)\right]} b^{k / r} \frac{\left|B_{i_{0}}\right|^{1 / r}}{\left\|\mathbf{1}_{B_{i_{0}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}} \\
& \lesssim \frac{b^{-k \varepsilon}\left|B_{i_{0}+\omega}\right|^{1 / r}}{\left\|\mathbf{1}_{B_{i_{0}+\omega}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}},
\end{aligned}
$$

which completes the proof of (25) for $k \in \mathbb{N}$ and hence of Lemma 8.
In addition, we also need the subsequent density of $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$.
Lemma 9. Let $\vec{p} \in(0, \infty)^{n}$. Then:
(i) $\quad H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \cap C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ is dense in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$; here and thereafter, $C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ denotes the set of all infinitely differentiable functions with compact support on $\mathbb{R}^{n}$;
(ii) For any s as in (5), $H_{A, \text { fin }}^{\vec{p}, \infty, s}\left(\mathbb{R}^{n}\right) \cap C\left(\mathbb{R}^{n}\right)$ is dense in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$.

Proof. To prove (i), we first show that, for any $\varphi \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ with $\int_{\mathbb{R}^{n}} \varphi(x) d x \neq 0$ and $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, as $k \rightarrow-\infty$,

$$
\begin{equation*}
f * \varphi_{k} \rightarrow f \quad \text { in } \quad H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \tag{29}
\end{equation*}
$$

For this purpose, we first assume that $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$. In this case, to prove (29), we only need to show that, for almost every $x \in \mathbb{R}^{n}$, as $k \rightarrow-\infty$ :

$$
\begin{equation*}
M_{N}\left(f * \varphi_{k}-f\right)(x) \rightarrow 0 \quad \text { for almost every } x \in \mathbb{R}^{n} \text { as } k \rightarrow-\infty \tag{30}
\end{equation*}
$$

where $N:=N_{\vec{p}}+2$ with $N_{\vec{p}}:=\left\lfloor\left(\frac{1}{\min \left\{1, p_{-}\right\}}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor+2$. Indeed, note that, for any $k \in \mathbb{Z}, f * \varphi_{k}-f \in L^{2}\left(\mathbb{R}^{n}\right)$. Then, by [29] (p.13, Theorem 3.6), we know that, for any $k \in \mathbb{Z}, M_{N}\left(f * \varphi_{k}-f\right) \in L^{2}\left(\mathbb{R}^{n}\right)$. From this, ref. [29] (p.39, Lemma 6.6), (30) and the Lebesgue-dominated convergence theorem, it follows that, (29) holds true for any $f \in$ $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$.

Subsequently, we prove (30). To this end, let $g$ be a continuous function with compact support. Then, $g$ is uniformly continuous on $\mathbb{R}^{n}$. Thus, for any $\delta \in(0, \infty)$, there exists some $\eta \in(0, \infty)$ such that, for any $y \in \mathbb{R}^{n}$ satisfying $\rho(y)<\eta$ and $x \in \mathbb{R}^{n}$,

$$
|g(x-y)-g(x)|<\frac{\delta}{2\|\varphi\|_{L^{1}\left(\mathbb{R}^{n}\right)}}
$$

Without loss of generality, we can assume that $\int_{\mathbb{R}^{n}} \varphi(x) d x=1$. Then, for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^{n}$, we have:

$$
\begin{align*}
\left|g * \varphi_{k}(x)-g(x)\right| & \leq \int_{\rho(y)<\eta}|g(x-y)-g(x)|\left|\varphi_{k}(y)\right| d y+\int_{\rho(y) \geq \eta} \ldots \\
& <\frac{\delta}{2}+2\|g\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \int_{\rho(y) \geq b^{-k} \eta}|\varphi(y)| d y . \tag{31}
\end{align*}
$$

By the integrability of $\varphi$, we can find a $K \in \mathbb{Z}$ such that, for any $k \in(-\infty, K] \cap \mathbb{Z}$ :

$$
2\|g\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \int_{\rho(y) \geq b^{-k} \eta}|\varphi(y)| d y<\frac{\delta}{2} .
$$

From this and (31), we deduce that, for any $x \in \mathbb{R}^{n}$ :

$$
\lim _{k \rightarrow-\infty}\left|g * \varphi_{k}(x)-g(x)\right|=0 \quad \text { holds true uniformly. }
$$

Therefore, $\left\|g * \varphi_{k}-g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \rightarrow 0$ as $k \rightarrow-\infty$. This, together with [29] (p.13, Theorem 3.6), again implies that:

$$
\begin{equation*}
\left\|M_{N}\left(g * \varphi_{k}-g\right)\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \lesssim\left\|g * \varphi_{k}-g\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \rightarrow 0 \quad \text { as } k \rightarrow-\infty . \tag{32}
\end{equation*}
$$

For any given $\epsilon \in(0, \infty)$, there exists a continuous function $g$ with compact support such that:

$$
\|f-g\|_{L^{2}\left(\mathbb{R}^{n}\right)}^{2}<\epsilon
$$

By (32) and [29] (p. 39, Lemma 6.6), we again know that there exists a positive constant $\kappa$ such that, for any $x \in \mathbb{R}^{n}$ :

$$
\begin{aligned}
& \limsup _{k \rightarrow-\infty} M_{N}\left(f * \varphi_{k}-f\right)(x) \\
& \quad \leq \sup _{k \in \mathbb{Z}} M_{N}\left((f-g) * \varphi_{k}\right)(x)+\limsup _{k \rightarrow-\infty} M_{N}\left(g * \varphi_{k}-g\right)(x)+M_{N}(g-f)(x) \\
& \quad \leq \kappa M_{N_{\vec{p}}}(g-f)(x)
\end{aligned}
$$

Thus, for any $\lambda \in(0, \infty)$, we have:

$$
\begin{aligned}
& \left|\left\{x \in \mathbb{R}^{n}: \limsup _{k \rightarrow-\infty} M_{N}\left(f * \varphi_{k}-f\right)(x)>\lambda\right\}\right| \\
& \quad \leq\left|\left\{x \in \mathbb{R}^{n}: M_{N_{\vec{p}}}(g-f)(x)>\frac{\lambda}{\kappa}\right\}\right| \lesssim \frac{\|f-g\|_{L^{2}\left(\mathbb{R}^{n}\right)}^{2}}{\lambda^{2}} \lesssim \frac{\epsilon}{\lambda^{2}} .
\end{aligned}
$$

This implies that, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \cap L^{2}\left(\mathbb{R}^{n}\right)$,(30) holds true.

When $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, by an argument similar to that used in [43] (p.1700), it is easy to see that (29) also holds true.

Moreover, if $f \in H_{A, \text { fin }}^{\vec{p} r, s}\left(\mathbb{R}^{n}\right)$ and $\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\int_{\mathbb{R}^{n}} \varphi(x) d x \neq 0$, then, for any $k \in \mathbb{Z}$,

$$
f * \varphi_{k} \in C_{c}^{\infty}\left(\mathbb{R}^{n}\right) \cap H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)
$$

and, by (29),

$$
f * \varphi_{k} \rightarrow f \quad \text { in } \quad H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right) \quad \text { as } \quad k \rightarrow-\infty
$$

This, combined with the density of the set $H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)$ in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, further implies that $C_{c}^{\infty}\left(\mathbb{R}^{n}\right) \cap H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is dense in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, which completes the proof of (i).

We now prove (ii). By (i) and the proof of [43] (Theorem 6.13 (ii)) with some slight modifications, we conclude that $H_{A, \text { fin }}^{\vec{p}, \infty, s}\left(\mathbb{R}^{n}\right) \cap C\left(\mathbb{R}^{n}\right)$ is dense in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. This finishes the proof of (ii) and hence of Lemma 9.

Applying Lemmas 6, 7 and 9 as well as Theorem 1, we obtain a criterion on the boundedness of linear operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ as follows, which plays a key role in the proof of Theorem 3.

Theorem 4. Let $T$ be a linear operator defined on the set of all measurable functions. Assume that $\vec{p} \in(0, \infty)^{n}, r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right]$ with $p_{+}$as in (4) and $\widetilde{s}$ is as in (5) with s replaced by $\widetilde{s}$. If there exists some $i_{0} \in \mathbb{Z}$ and a positive constant $C$ such that, for any $(p(\cdot), r, \widetilde{s})$-atom $\widetilde{a}$ supported in some dilated ball $x_{0}+B_{k_{0}} \in \mathfrak{B}$ with $x_{0} \in \mathbb{R}^{n}, k_{0} \in \mathbb{Z}$ and $\mathfrak{B}$ as in (1), $\frac{1}{C} T(\widetilde{a})$ is a $(p(\cdot), r, s, \varepsilon)$-molecule associated with $x_{0}+B_{k_{0}+i_{0}}$, where s and $\varepsilon$ are as in Theorem 1 , then $T$ has a unique bounded linear extension on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$.

Proof. Let $\vec{p} \in(0, \infty)^{n}, r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right]$ and

$$
\widetilde{s} \in\left[\left\lfloor\left(\frac{1}{p_{-}}-1\right) \frac{\ln b}{\ln \lambda_{-}}\right\rfloor, \infty\right) \cap \mathbb{Z}_{+}
$$

with $p_{-}$as in (4). We next show Theorem 4 by considering two cases.
Case (1). $r \in\left(\max \left\{p_{+}, 1\right\}, \infty\right)$. For this case, let $f \in H_{A, f i n}^{\vec{p}, r, \widetilde{s}}\left(\mathbb{R}^{n}\right)$. Then, by the notion of $H_{A, \text { fin }}^{\vec{p}, r, \widetilde{s}}\left(\mathbb{R}^{n}\right)$ in Lemma 6 , we find that there exists some $K \in \mathbb{N}$, three finite sequences $\left\{\lambda_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{C},\left\{x_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{R}^{n}$ and $\left\{i_{k}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathbb{Z}$, and a finite sequence of $(\vec{p}, r, s)$-atoms, $\left\{a_{k}\right\}_{k \in[1, K] \cap \mathbb{N}}$, supported, respectively, in $\left\{x_{k}+B_{i_{k}}\right\}_{k \in[1, K] \cap \mathbb{N}} \subset \mathfrak{B}$ such that:

$$
\begin{equation*}
f=\sum_{k=1}^{K} \lambda_{k} a_{k} \quad \text { in } \quad \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \tag{33}
\end{equation*}
$$

and:

$$
\begin{equation*}
\|f\|_{H_{A, \text { fin }}^{\vec{p}, r, s}\left(\mathbb{R}^{n}\right)} \sim\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \tag{34}
\end{equation*}
$$

This, together with (33) and the linearity of $T$, implies that $T(f)=\sum_{k=1}^{K} \lambda_{k} T\left(a_{k}\right)$ in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, where, for any $k \in[1, K] \cap \mathbb{N}, \frac{1}{C} T\left(a_{k}\right)$ with $C$ being a positive constant independent of $k$ is a $(\vec{p}, r, s, \varepsilon)$-molecule associated with $x_{k}+B_{i_{k}+i_{0}}$ with $s, \varepsilon$ and $i_{0}$ as in Theorem 4.

From this, Theorem 1, Definition 7, as well as Lemmas 7, 4 and 5, (34) and Lemma 6, we further deduce that, for any $f \in H_{A, \text { fin }}^{\vec{p} r, \widetilde{s}}\left(\mathbb{R}^{n}\right)$ :

$$
\begin{align*}
\|T(f)\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)} & \sim\|T(f)\|_{H_{A}^{\vec{p}, r, s, \varepsilon}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}+i_{0}}}}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}+i_{0}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right]^{\underline{p}}\right\}^{1 / \underline{p}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)} \\
& \lesssim b^{i_{0} / \tau} \|\left[\sum _ { k = 1 } ^ { K } \left\{\frac{\left|\lambda_{k}\right|\left[M_{H L}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)\right]^{1 / \tau}}{\left.\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{p}\right\}^{1 / \underline{p}}\| \|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right.\right. \\
& \sim b^{i_{0} / \tau}\left\|\left\{\sum_{k=1}^{K}\left[\frac{\left|\lambda_{k}\right|^{\tau} M_{H L}\left(\mathbf{1}_{x_{k}+B_{i_{k}}}\right)}{\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}^{\tau}}\right]^{\underline{p} / \tau}\right\}^{\tau / \underline{p}}\right\|_{L^{\vec{p} / \tau}\left(\mathbb{R}^{n}\right)}^{1 / \tau} \\
& \lesssim \|\left\{\sum _ { k = 1 } ^ { K } \left[\frac{\left|\lambda_{k}\right| \mathbf{1}_{x_{k}+B_{i_{k}}}}{\left.\left.\left\|\mathbf{1}_{x_{k}+B_{i_{k}}}\right\|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}\right]^{\underline{p}}\right\}^{1 / \underline{p}} \|_{L^{\vec{p}}\left(\mathbb{R}^{n}\right)}}\right.\right. \\
& \sim\|f\|_{H_{A, \vec{p}, \tilde{f}}^{p, \tilde{s}}} \mathbb{R}_{\left.\mathbb{R}^{n}\right)}^{\sim \|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)^{\prime}}} \tag{35}
\end{align*}
$$

where $\tau \in(0, \underline{p})$ is a constant.
Moreover, by the obvious density of $H_{A, f i n}^{p(\cdot), r \widetilde{s}}\left(\mathbb{R}^{n}\right)$ in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ with respect to the quasinorm $\|\cdot\|_{H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)}$ and a proof similar to the estimation of (23), we conclude that, for any $f \in H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, (35) also holds true. This finishes the proof of Theorem 4 in Case (1).

Case (2). $r=\infty$. In this case, by Lemma 9(ii), we know that $H_{A, \text { fin }}^{\vec{p}, \infty, \widetilde{s}}\left(\mathbb{R}^{n}\right) \cap C\left(\mathbb{R}^{n}\right)$ is dense in $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. From this, repeating the proof of Case (1) with some slight modifications, it follows that Theorem 4 also holds true when $r=\infty$, which completes the proof of Theorem 4.

We now prove Theorem 3.
Proof of Theorem 3. Indeed, Theorem 3 is an immediate corollary of Theorem 4 and Lemma 8. This finishes the proof of Theorem 3.

Remark 2. (i) Assume that $\ell \in \mathbb{N}, p \in(0,1]$ and:

$$
\begin{equation*}
\frac{1}{p}-1 \leq \frac{\left(\ln \lambda_{-}\right)^{2}}{\ln b \ln \lambda_{+}} \ell . \tag{36}
\end{equation*}
$$

When $\vec{p}:=(\overbrace{p, \ldots, p}^{n \text { times }})$ with some $p \in(0, \infty)$, the spaces $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ and $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ are just, respectively, the anisotropic Hardy space $H_{A}^{p}\left(\mathbb{R}^{n}\right)$ of Bownik [29] and the Lebesgue space $L^{p}\left(\mathbb{R}^{n}\right)$. In this case, Theorems 2 and 3 implies that, for any $\ell \in \mathbb{N}$ and $p \in(0,1]$ as in (36), the anisotropic Calderón-Zygmund operator of order $\ell$ (see Definition 10) is bounded from $H_{A}^{p}\left(\mathbb{R}^{n}\right)$ to $L^{p}\left(\mathbb{R}^{n}\right)$ (or to itself), which are just, respectively, ref. [29] (p.69, Theorem 9.9 and $p .68$, Theorem 9.8). Moreover, let $A:=d \mathrm{I}_{n \times n}$ for some $d \in \mathbb{R}$ with $|d| \in(1, \infty), \ell=1$. Then, $\frac{\left(\ln \lambda_{-}\right)^{2}}{\ln b \ln \lambda_{+}} \ell=\frac{1}{n}$ and $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ becomes the classical isotropic Hardy space $H^{p}\left(\mathbb{R}^{n}\right)$. In this case, by Theorems 2 and 3 and [37] ((i) and (ii) of Remark 4.4), we further know that, for any $p \in\left(\frac{n}{n+1}, 1\right]$, the classical Calderón-Zygmund operator is bounded from $H^{p}\left(\mathbb{R}^{n}\right)$ to $L^{p}\left(\mathbb{R}^{n}\right)$ (or to itself), which is a well-known result (see, for instance [46]).
(ii) When $A:=d \mathrm{I}_{n \times n}$ for some $d \in \mathbb{R}$ with $|d| \in(1, \infty)$, the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ becomes the mixed-norm Hardy space $H^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (see [7]). In this case, Theorems 2 and 3 are new.
(iii) Very recently, Bownik et al. [47] introduced a kind of more general anisotropic CalderónZygmund operators (see [47] (Definition 5.4)) and established the boundedness of these operators from the anisotropic Hardy space $H^{p}(\Theta)$ to the Lebesgue space $L^{p}\left(\mathbb{R}^{n}\right)$ or to itself (see, respectively, ref. [47] (Theorems 5.12 and 5.11)), where $\Theta$ is a continuous multi-level ellipsoid cover of $\mathbb{R}^{n}$ (see [47] (Definition 2.1)). Here, we should point out that the space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, in this article, is not covered by the space $H^{p}(\Theta)$, since the exponent $p$ in $H^{p}(\Theta)$ is only a constant. Thus, Theorems 2 and 3 are covered by neither [47] (Theorems 5.12 or 5.11).
(iv) Recall that Huang et al. also introduced another sort of anisotropic non-convolutional $\beta$-order Calderón-Zygmund operators (see [1] (Definition 8.3)) and obtained the boundedness of these Calderón-Zygmund operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ (or to itself), where $\beta \in(0, \infty)$ and $\vec{p} \in(0,2)^{n}$ with:

$$
p_{-} \in\left(\frac{\ln b}{\ln b+\beta \ln \lambda_{-}}, \frac{\ln b}{\ln b+(\lceil\beta\rceil-1) \ln \lambda_{-}}\right]
$$

where the symbol $\lceil\beta\rceil$ denotes the least integer not less than $\beta$; see [1] (Theorem 8.5). Observe that the Calderón-Zygmund operator in [1] (Definition 8.3) is different from the one used in the present article (see Definition 10) and ref. [1] (Theorem 8.5) requires the integrable exponent $\vec{p}$ which belongs to $(0,2)^{n}$; however, this restriction is removed in Theorems 2 and 3. Thus, Theorems 2 and 3 cannot be covered by [1] (Theorem 8.5).

## 5. Conclusions

In this article, we characterize the anisotropic mixed-norm Hardy space $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ via molecules, in which the range of the decay index $\varepsilon$ is the known best possible in some sense. As an application, we then obtain a criterion on the boundedness of linear operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$, which is used to prove the boundedness of the anisotropic Calderón-Zygmund operators on $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$. In addition, the boundedness of anisotropic Calderón-Zygmund operators from $H_{A}^{\vec{p}}\left(\mathbb{R}^{n}\right)$ to the mixed-norm Lebesgue space $L^{\vec{p}}\left(\mathbb{R}^{n}\right)$ is also presented. When $A$ is as in (6), the obtained boundedness of these Calderón-Zygmund operators positively answers a question formulated by Cleanthous et al. in [6] (p. 2760). All these results are new, even for the isotropic mixed-norm Hardy spaces on $\mathbb{R}^{n}$.

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