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Special Functions of Fractional Calculus in the Form of Convolution Series and Their Applications

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Abstract: In this paper, we first discuss the convolution series that are generated by Sonine kernels from a class of functions continuous on a real positive semi-axis that have an integrable singularity of power function type at point zero. These convolution series are closely related to the general fractional integrals and derivatives with Sonine kernels and represent a new class of special functions of fractional calculus. The Mittag-Leffler functions as solutions to the fractional differential equations with the fractional derivatives of both Riemann-Liouville and Caputo types are particular cases of the convolution series generated by the Sonine kernel $\kappa(t) = t^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$. The main result of the paper is the derivation of analytic solutions to the single- and multi-term fractional differential equations with the general fractional derivatives of the Riemann-Liouville type that have not yet been studied in the fractional calculus literature.

Keywords: Sonine kernel; Sonine condition; general fractional derivative; general fractional integral; convolution series; fundamental theorems of fractional calculus; fractional differential equations

MSC: 26A33; 26B30; 44A10; 45E10

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1. Introduction

Special functions of mathematical physics are usually defined in the form of a power series, or as solutions to some differential equations, or via integral representations. Of course, for a given function, these three (and possibly other) forms coincide for all arguments and parameter values for which they exist. However, the validity domains of different representations can be unequal. Very often, the series representations of the special functions hold valid only on some restricted domains. To define the corresponding functions for other values of their arguments and parameters, analytical continuation of the series in the form of integral representations is usually employed.

For special functions of fractional calculus (FC), the situation is very similar to the one described above. For instance, one of the most important FC special functions – the two-parameter Mittag-Leffler function – is usually defined in the form of a power series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)},\ \alpha > 0,\ \beta, z \in \mathbb{C}. \tag{1}$$

Because the series is convergent for all $z \in \mathbb{C}$, this definition can be used for all $z \in \mathbb{C}$ without any analytical continuation. Still, the integral representations of the Mittag-Leffler function are very important, say, for derivation of its asymptotic behavior [1] and for its numerical calculation [2]. For $0 < \alpha < 2$ and $\Re(\beta) > 0$, the following integral representations of the Mittag-Leffler function in terms of the integrals over the Hankel-type contours were presented in [1]:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi\alpha i} \int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z}\, d\zeta,\ z \in G^{(-)}(\epsilon;\delta),$$

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi\alpha i} \int_{\gamma(\epsilon;\delta)} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \; z \in G^{(+)}(\epsilon;\delta),$$

where the integration contour $\gamma(\epsilon;\delta)$ ($\epsilon > 0$, $0 < \delta \leq \pi$) with non-decreasing $\arg\zeta$ consists of the following parts:

(1) the ray $\arg\zeta = -\delta$, $|\zeta| \geq \epsilon$;
(2) the arc $-\delta \leq \arg\zeta \leq \delta$ of the circumference $|\zeta| = \epsilon$;
(3) the ray $\arg\zeta = \delta$, $|\zeta| \geq \epsilon$.

For $0 < \delta < \pi$, the domain $G^{(-)}(\epsilon;\delta)$ is to the left of the contour $\gamma(\epsilon;\delta)$ and the domain $G^{(+)}(\epsilon;\delta)$ is to the right of this contour. If $\delta = \pi$, the contour $\gamma(\epsilon;\delta)$ consists of the circumference $|\zeta| = \epsilon$ and of the cut $-\infty < \zeta \leq -\epsilon$. In this case, the domain $G^{(-)}(\epsilon;\delta)$ is the circle $|\zeta| < \epsilon$ and $G^{(+)}(\epsilon;\alpha) = \{\zeta : |\arg\zeta| < \pi, \; |\zeta| > \epsilon\}$.

For some parameter values, the Mittag-Leffler function can be also introduced in terms of solutions to the fractional differential equations with the Riemann-Liouville or Caputo fractional derivatives. For instance, for $0 < \alpha \leq 1$, the equation

$$(D^{\alpha}_{0+} y)(t) = \lambda \, y(t) \tag{2}$$

has the general solution [3]

$$y(t) = C \, t^{\alpha-1} E_{\alpha,\alpha}(\lambda \, t^{\alpha}), \; C \in \mathbb{R}. \tag{3}$$

In Equation (2), the Riemann-Liouville fractional derivative D^{α}_{0+} is defined by

$$(D^{\alpha}_{0+} \, f)(t) = \frac{d}{dt} (I^{1-\alpha}_{0+} f)(t), \; t > 0, \tag{4}$$

where I^{α}_{0+} is the Riemann-Liouville fractional integral of order α ($\alpha > 0$):

$$(I^{\alpha}_{0+} f)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, \; t > 0. \tag{5}$$

The general solution to the equation

$$(\,_{*}D^{\alpha}_{0+} y)(t) = \lambda \, y(t) \tag{6}$$

with the Caputo fractional derivative

$$(\,_{*}D^{\alpha}_{0+} \, f)(t) = (D^{\alpha}_{0+} \, f)(t) - f(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \; t > 0 \tag{7}$$

has the form [4]

$$y(t) = C \, E_{\alpha,1}(\lambda \, t^{\alpha}), \; C \in \mathbb{R}. \tag{8}$$

As we can see, the solutions to the fractional differential Equations (2) and (6) are expressed in terms of the Mittag-Leffler functions. However, the arguments of these functions are $\lambda \, t^{\alpha}$ and not just $\lambda \, t$. Thus, these solutions are represented in the form of power series with the fractional and not integer exponents. For more advanced properties and applications of the Mittag-Leffler type functions, see [1] and the recent book [5].

In [6], the single- and multi-term fractional differential equations with the general fractional derivatives of the Caputo type have been studied. By definition, their solutions belong to the class of the FC special functions (as the ones represented in form of solutions to the fractional differential equations). Moreover, in [6], another representation of these new FC special functions was derived, namely in terms of the convolution series generated by the Sonine kernels.

The convolution series are a far-reaching generalization of the conventional power series and the power series with the fractional exponents including the Mittag-Leffler Functions (3) and (8). They represent a new class of the FC special functions worth for investigation. In [7], the convolution series were employed for derivation of two different forms of the generalized convolution Taylor formula for representation of a function as a convolution polynomial with a remainder in the form of a composition of the n-fold general fractional integral and the n-fold general sequential fractional derivative of the Riemann-Liouville and the Caputo types, respectively. In [7], the generalized Taylor series in form of convolution series were also discussed. In this paper, we employ the convolution series for derivation of analytical solutions to the single- and multi-terms fractional differential equations with the general fractional derivatives in the Riemann-Liouville sense. This type of the fractional differential equations has not yet been studied in the FC literature.

One of the main reasons for this situation is that until recently, it was not clear at all what type of initial conditions is required while dealing with fractional differential equations with general fractional derivatives of the Riemann-Liouville type. A solution to this problem was provided in a very recent publication [7], where an explicit form of the projector operator of the n-fold sequential general fractional derivative in the Riemann-Liouville sense has been derived for the first time. Another challenge for treatment of fractional differential equations with general fractional derivatives in the Riemann-Liouville sense is an absence of methods for derivation of their analytical solutions. In [6], fractional differential equations with general fractional derivatives of the Caputo type have been studied by means of an operational calculus developed for these derivatives. An operational calculus for general fractional derivatives of the Riemann-Liouville has not yet been constructed. Thus, in this paper, we employ another method for analytical treatment of fractional differential equations with general fractional derivatives of the Riemann-Liouville type, namely the method of convolution series. This method is introduced and applied to fractional differential equations for the first time in the FC literature.

The rest of this paper is organized as follows. In the next section, we introduce general fractional derivatives of the Riemann-Liouville and Caputo types with Sonine kernels from a special class of functions and discuss some of their properties needed for further discussion. In the third section, we first provide some results regarding the convolution series generated by Sonine kernels. Then, convolution series are applied for derivation of analytical solutions to single- and multi-term fractional differential equations with general fractional derivatives in the Riemann-Liouville sense. For a treatment of single- and multi-term fractional differential equations with general fractional derivatives in the Caputo sense, we refer interested readers to [6].

2. General Fractional Integrals and Derivatives

General fractional derivatives (GFDs) with kernel k in the Riemann-Liouville and in the Caputo sense, respectively, are defined as follows [8–13]:

$$(\mathbb{D}_{(k)}\, f)(t) = \frac{d}{dt}(k * f)(t) = \frac{d}{dt}\int_0^t k(t-\tau)f(\tau)\, d\tau, \tag{9}$$

$$({}_*\mathbb{D}_{(k)}\, f)(t) = (\mathbb{D}_{(k)}\, f)(t) - f(0)k(t), \tag{10}$$

where by $*$ the Laplace convolution is denoted:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau. \tag{11}$$

The Riemann-Liouville and the Caputo fractional derivatives of order α, $0 < \alpha < 1$, defined by (4) and (7), respectively, are particular cases of the GFDs (9) and (10) with the kernel

$$k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1,\ h_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)},\ \beta > 0. \tag{12}$$

The multi-term fractional derivatives and fractional derivatives of distributed order are also particular cases of the GFDs (9) and (10) with the kernels

$$k(t) = \sum_{k=1}^{n} a_k\, h_{1-\alpha_k}(t),\ \ 0 < \alpha_1 < \cdots < \alpha_n < 1,\ a_k \in \mathbb{R},\ k = 1, \ldots, n, \tag{13}$$

$$k(t) = \int_0^1 h_{1-\alpha}(t)\, d\rho(\alpha), \tag{14}$$

respectively, where ρ is a Borel measure defined on the interval $[0, 1]$.

Several useful properties of the Riemann-Liouville fractional integral and the Riemann-Liouville and Caputo fractional derivatives are based on the formula

$$(h_\alpha \, * \, h_\beta)(t) \; = \; h_{\alpha+\beta}(t),\ \alpha, \beta > 0,\ t > 0 \tag{15}$$

that immediately follows from the well-known representation of the Euler Beta-function in terms of the Gamma-function:

$$B(\alpha, \beta) := \int_0^1 (1-\tau)^{\alpha-1}\, \tau^{\beta-1}\, d\tau \; = \; \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \alpha, \beta > 0.$$

In Formula (15) and in what follows, the power function h_α is defined as in (12).

In our discussions, we employ the integer order convolution powers that for a function $f = f(t),\ t > 0$ are defined by the expression

$$f^{<n>}(t) := \begin{cases} 1, & n = 0, \\ f(t), & n = 1, \\ (\underbrace{f * \ldots * f}_{n \text{ times}})(t), & n = 2, 3, \ldots. \end{cases} \tag{16}$$

For the kernel $\kappa(t) = h_\alpha(t)$ of the Riemann-Liouville fractional integral, we apply Formula (15) and arrive at the important representation

$$h_\alpha^{<n>}(t) = h_{n\alpha}(t),\ n \in \mathbb{N}. \tag{17}$$

A well-known particular case of (17) is the formula

$$\{1\}^n(t) = h_1^n(t) = h_n(t) = \frac{t^{n-1}}{\Gamma(n)} = \frac{t^{n-1}}{(n-1)!},\ n \in \mathbb{N}, \tag{18}$$

where by $\{1\}$ we denoted the function that is identically equal to 1 for $t > 0$.

Now let us write down Formula (15) for $\beta = 1 - \alpha,\ 0 < \alpha < 1$:

$$(h_\alpha \, * \, h_{1-\alpha})(t) = h_1(t) = \{1\},\ 0 < \alpha < 1,\ t > 0. \tag{19}$$

In [14,15], Abel employed the relation (19) to derive an inversion formula for the operator that is presently referred to as the Caputo fractional derivative and obtained it in form of the Riemann-Liouville fractional integral (solution to the Abel model for the tautochrone problem).

By an attempt to extend the Abel solution method to more general integral equations of convolution type, Sonine introduced in [16] the relation

$$(\kappa \, * \, k)(t) = \{1\},\ t > 0 \tag{20}$$

that is presently referred to as the Sonine condition. The functions that satisfy the Sonine condition are called Sonine kernels. For a Sonine kernel κ, the kernel k that satisfies the

Sonine condition (20) is called an associated kernel to κ. Of course, κ is then an associated kernel to k. In what follows, we denote the set of the Sonine kernels by $\mathcal{S}$.

In [16], Sonine introduced a class of Sonine kernels in the form

$$\kappa(t) = h_\alpha(t) \cdot \kappa_1(t),\ \kappa_1(t) = \sum_{k=0}^{+\infty} a_k t^k,\ a_0 \neq 0,\ 0 < \alpha < 1, \tag{21}$$

$$k(t) = h_{1-\alpha}(t) \cdot k_1(t),\ k_1(t) = \sum_{k=0}^{+\infty} b_k t^k, \tag{22}$$

where $\kappa_1 = \kappa_1(t)$ and $k_1 = k_1(t)$ are analytical functions and the coefficients $a_k,\ b_k,\ k \in \mathbb{N}_0$ satisfy the following triangular system of linear equations:

$$a_0 b_0 = 1,\ \sum_{k=0}^{n} \Gamma(k+1-\alpha)\Gamma(\alpha+n-k)a_{n-k}b_k = 0,\ n \geq 1. \tag{23}$$

An important example of the kernels from $\mathcal{S}$ in the form (21), (22) was derived in [16] in terms of the Bessel function J_ν and the modified Bessel function I_ν:

$$\kappa(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}),\ k(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}),\ 0 < \alpha < 1, \tag{24}$$

where

$$J_\nu(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k (t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},\ I_\nu(t) = \sum_{k=0}^{+\infty} \frac{(t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

For other examples of Sonine kernels we refer readers to [8,12,13,17].

In this paper, we deal with general fractional integrals (GFIs) with kernels $\kappa \in \mathcal{S}$ defined by the formula

$$(\mathbb{I}_{(\kappa)} f)(t) := (\kappa * f)(t) = \int_0^t \kappa(t-\tau) f(\tau)\, d\tau,\ t > 0 \tag{25}$$

and with GFDs with associated Sonine kernels k in the Riemann-Liouville and Caputo senses defined by (9) and (10), respectively.

In our discussions, we restrict ourselves to a class of the Sonine kernels from space $C_{-1,0}(0,+\infty)$ that is an important particular case of the following two-parameter family of spaces [6,12,13]:

$$C_{\alpha,\beta}(0,+\infty) \ = \ \{f:\ f(t) = t^p f_1(t),\ t > 0,\ \alpha < p < \beta,\ f_1 \in C[0,+\infty)\}. \tag{26}$$

By $C_{-1}(0,+\infty)$ we mean the space $C_{-1,+\infty}(0,+\infty)$.

The set of such Sonine kernels will be denoted by $\mathcal{L}_1$ [13]:

$$(\kappa,\ k \in \mathcal{L}_1) \ \Leftrightarrow \ (\kappa,\ k \in C_{-1,0}(0,+\infty)) \wedge ((\kappa * k)(t) \ = \ \{1\}). \tag{27}$$

In the rest of this section, we present some important results for GFIs and GFDs with Sonine kernels from $\mathcal{L}_1$ on space $C_{-1}(0,+\infty)$ and its sub-spaces.

The basic properties of the GFI (25) on space $C_{-1}(0,+\infty)$ easily follow from the known properties of the Laplace convolution:

$$\mathbb{I}_{(\kappa)} :\ C_{-1}(0,+\infty) \ \to C_{-1}(0,+\infty), \tag{28}$$

$$\mathbb{I}_{(\kappa_1)}\, \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_2)}\, \mathbb{I}_{(\kappa_1)},\ \kappa_1,\ \kappa_2 \in \mathcal{L}_1, \tag{29}$$

$$\mathbb{I}_{(\kappa_1)}\, \mathbb{I}_{(\kappa_2)} = \mathbb{I}_{(\kappa_1 * \kappa_2)},\ \kappa_1,\ \kappa_2 \in \mathcal{L}_1. \tag{30}$$

For functions $f \in C^1_{-1}(0,+\infty) := \{f : f' \in C_{-1}(0,+\infty)\}$, GFDs of the Riemann-Liouville type can be represented as follows [12]:

$$(\mathbb{D}_{(k)} f)(t) = (k * f')(t) + f(0)k(t), \; t > 0. \tag{31}$$

Thus, for $f \in C^1_{-1}(0,+\infty)$, GFD (10) of the Caputo type takes the form

$$({}_*\mathbb{D}_{(k)} f)(t) = (k * f')(t), \; t > 0. \tag{32}$$

It is worth mentioning that in FC publications, the Caputo fractional derivative (7) is often defined as in Formula (32):

$$({}_*D^\alpha_{0+} f)(t) = (h_{1-\alpha} * f')(t) = (I^{1-\alpha}_{0+} f')(t), \; t > 0. \tag{33}$$

Now, following [7,12], we define the n-fold GFI and the n-fold sequential GFDs in the Riemann-Liouville and Caputo senses.

Definition 1 ([12])**.** *Let $\kappa \in \mathcal{L}_1$. The n-fold GFI ($n \in \mathbb{N}$) is a composition of n GFIs with the kernel κ:*

$$(\mathbb{I}^{<n>}_{(\kappa)} f)(t) := (\underbrace{\mathbb{I}_{(\kappa)} \ldots \mathbb{I}_{(\kappa)}}_{n \text{ times}} f)(t), \; t > 0. \tag{34}$$

It is worth mentioning that the index law (30) leads to a representation of the n-fold GFI (34) in the form of GFI with kernel $\kappa^{<n>}$:

$$(\mathbb{I}^{<n>}_{(\kappa)} f)(t) = (\kappa^{<n>} * f)(t) = (\mathbb{I}_{(\kappa)^{<n>}} f)(t), \; t > 0. \tag{35}$$

Kernel $\kappa^{<n>}$, $n \in \mathbb{N}$ belongs to space $C_{-1}(0,+\infty)$, but it is not always a Sonine kernel.

Definition 2 ([7])**.** *Let $\kappa \in \mathcal{L}_1$ and k be its associated Sonine kernel. The n-fold sequential GFDs in the Riemann-Liouville and in the Caputo sense, respectively, are defined as follows:*

$$(\mathbb{D}^{<n>}_{(k)} f)(t) := (\underbrace{\mathbb{D}_{(k)} \ldots \mathbb{D}_{(k)}}_{n \text{ times}} f)(t), \; t > 0, \tag{36}$$

$$({}_*\mathbb{D}^{<n>}_{(k)} f)(t) := (\underbrace{{}_*\mathbb{D}_{(k)} \ldots {}_*\mathbb{D}_{(k)}}_{n \text{ times}} f)(t), \; t > 0. \tag{37}$$

It is worth mentioning that in [6,12], the n-fold GFDs ($n \in \mathbb{N}$) were defined in a different form:

$$(\mathbb{D}^n_{(k)} f)(t) := \frac{d^n}{dt^n}(k^{<n>} * f)(t), \; t > 0, \tag{38}$$

$$({}_*\mathbb{D}^n_{(k)} f)(t) := (k^{<n>} * f^{(n)})(t), \; t > 0. \tag{39}$$

The n-fold sequential GFDs (36) and (37) are a far-reaching generalization of the Riemann-Liouville and the Caputo sequential fractional derivatives to the case of Sonine kernels from $\mathcal{L}_1$.

Some important connections between n-fold GFI (34) and n-fold sequential GFDs (36) and (37) in the Riemann-Liouville and Caputo senses are provided in the so-called first and second fundamental theorems of FC ([18]) formulated below.

Theorem 1 ([7])**.** *Let $\kappa \in \mathcal{L}_1$ and k be its associated Sonine kernel.*

Then, the n-fold sequential GFD (36) in the Riemann-Liouville sense is a left inverse operator to the n-fold GFI (34) on the space $C_{-1}(0,+\infty)$:

$$(\mathbb{D}_{(k)}^{<n>}\, \mathbb{I}_{(\kappa)}^{<n>}\, f)(t) = f(t),\ f \in C_{-1}(0,+\infty),\ t > 0, \tag{40}$$

and the n-fold sequential GFD (37) in the Caputo sense is a left inverse operator to the n-fold GFI (34) on the space $C^{n}_{-1,(k)}(0,+\infty)$:

$$(\,{}_*\mathbb{D}_{(k)}^{<n>}\, \mathbb{I}_{(\kappa)}^{<n>}\, f)(t) = f(t),\ f \in C^{n}_{-1,(k)}(0,+\infty),\ t > 0, \tag{41}$$

where $C^{n}_{-1,(k)}(0,+\infty) := \{f:\ f(t) = (\mathbb{I}_{(k)}^{<n>}\, \phi)(t),\ \phi \in C_{-1}(0,+\infty)\}$.

Theorem 2 ([7])**.** *Let $\kappa \in \mathcal{L}_1$ and k be its associated Sonine kernel.*

For a function $f \in C^{(n)}_{-1,(k)}(0,+\infty) = \{f \in C_{-1}(0,+\infty):\ (\mathbb{D}_{(k)}^{<j>}\, f) \in C_{-1}(0,+\infty),\ j = 1,\ldots,n\}$, the formula

$$(\mathbb{I}_{(\kappa)}^{<n>}\, \mathbb{D}_{(k)}^{<n>}\, f)(t) = f(t) - \sum_{j=0}^{n-1}\Big(k * \mathbb{D}_{(k)}^{<j>}\, f\Big)(0)\kappa^{<j+1>}(t) = \tag{42}$$

$$f(t) - \sum_{j=0}^{n-1}\Big(\mathbb{I}_{(k)}\, \mathbb{D}_{(k)}^{<j>}\, f\Big)(0)\kappa^{<j+1>}(t),\ t > 0$$

holds valid, where $\mathbb{I}_{(\kappa)}^{<n>}$ is the n-fold GFI (34) and $\mathbb{D}_{(k)}^{<n>}$ is the n-fold sequential GFD (36) in the Riemann-Liouville sense.

For a function $f \in C^{n}_{-1}(0,+\infty) := \{f:\ f^{(n)} \in C_{-1}(0,+\infty)\}$, the formula

$$(\mathbb{I}_{(\kappa)}^{<n>}\, {}_*\mathbb{D}_{(k)}^{<n>}\, f)(t) = f(t) - f(0) - \sum_{j=1}^{n-1}\Big({}_*\mathbb{D}_{(k)}^{<j>}\, f\Big)(0)\Big(\{1\} * \kappa^{<j>}\Big)(t) \tag{43}$$

holds valid, where $\mathbb{I}_{(\kappa)}^{<n>}$ is the n-fold GFI (34) and ${}_*\mathbb{D}_{(k)}^{<n>}$ is the n-fold sequential GFD (37).

For proofs of Theorems 1 and 2 and their particular cases we refer interested readers to [7].

3. Solutions to Fractional Differential Equations with GFDs in the Riemann-Liouville Sense in Terms of the Convolution Series

First, we introduce the convolution series and treat some of their properties needed for the further discussions.

Definition 3. *For a function $\kappa \in C_{-1}(0,+\infty)$, the series in form*

$$\Sigma_{\kappa}(t) = \sum_{j=0}^{+\infty} a_j\, \kappa^{<j+1>}(t),\ a_j \in \mathbb{R}\ (a_j \in \mathbb{C}) \tag{44}$$

is called convolution series generated by κ.

Convolution series generated by Sonine kernels $\kappa \in \mathcal{L}_1$ were introduced in [13] for analytical treatment of fractional differential equations with n-fold GFDs of the Caputo type by means of an operational calculus developed for these GFDs. In [7], some of the results presented in [13] were extended to convolution series in the form (44) generated by any function $\kappa \in C_{-1}(0,+\infty)$ (i.e., not necessarily a Sonine kernel).

A very important question regarding convergence of the convolution series (44) was answered in [6,7].

Theorem 3 ([7]). *Let a function $\kappa \in C_{-1}(0,+\infty)$ be represented in the form*

$$\kappa(t) = h_p(t)\kappa_1(t),\ t > 0,\ p > 0,\ \kappa_1 \in C[0,+\infty) \tag{45}$$

and the convergence radius of the power series

$$\Sigma(z) = \sum_{j=0}^{+\infty} a_j\, z^j,\ a_j \in \mathbb{C},\ z \in \mathbb{C} \tag{46}$$

be non-zero. Then the convolution series (44) is convergent for all $t > 0$ and defines a function from the space $C_{-1}(0,+\infty)$. Moreover, the series

$$t^{1-\alpha}\,\Sigma_\kappa(t) = \sum_{j=0}^{+\infty} a_j\, t^{1-\alpha}\,\kappa^{<j+1>}(t),\ \ \alpha = \min\{p,\,1\} \tag{47}$$

is uniformly convergent for $t \in [0,\,T]$ for any $T > 0$.

In what follows, we always assume that the coefficients of the convolution series satisfy the condition that the convergence radius of the corresponding power series is non-zero and thus Theorem 3 is applicable for these convolution series.

As an example, let us consider the geometric series

$$\Sigma(z) = \sum_{j=0}^{+\infty} \lambda^j z^j,\ \lambda \in \mathbb{C},\ z \in \mathbb{C}. \tag{48}$$

For $\lambda \neq 0$, the convergence radius r of this series is equal to $1/|\lambda|$ and thus we can apply Theorem 3 that says that the convolution series generated by a function $\kappa \in C_{-1}(0,+\infty)$ in form

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j \kappa^{<j+1>}(t),\ \lambda \in \mathbb{C} \tag{49}$$

is convergent for all $t > 0$ and defines a function from the space $C_{-1}(0,+\infty)$.

The convolution series $l_{\kappa,\lambda}$ defined by (49) plays a very important role in the operational calculus for GFD of Caputo type developed in [6]. It provides a far-reaching generalization of both the exponential function and the two-parameter Mittag-Leffler function in form (3).

Indeed, let us consider the convolution series (49) in the case of the kernel function $\kappa = \{1\}$. Due to the formula $\kappa^{<j+1>}(t) = \{1\}^{<j+1>}(t) = h_{j+1}(t)$ (see (17)), the convolution series (49) is reduced to the power series for the exponential function:

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j h_{j+1}(t) = \sum_{j=0}^{+\infty} \frac{(\lambda\, t)^j}{j!} = e^{\lambda\, t}. \tag{50}$$

For the kernel $\kappa(t) = h_\alpha(t)$ of the Riemann-Liouville fractional integral, the formula $\kappa^{<j+1>}(t) = h_\alpha^{<j+1>}(t) = h_{(j+1)\alpha}(t)$ (see (17)) holds valid. Thus, the convolution series (49) takes the form

$$l_{\kappa,\lambda}(t) = \sum_{j=0}^{+\infty} \lambda^j h_{(j+1)\alpha}(t) = t^{\alpha-1} \sum_{j=0}^{+\infty} \frac{\lambda^j\, t^{j\alpha}}{\Gamma(j\alpha+\alpha)} = t^{\alpha-1} E_{\alpha,\alpha}(\lambda\, t^\alpha) \tag{51}$$

that is the same as the two-parameter Mittag-Leffler Function (3).

For $\kappa \in \mathcal{L}_1$, another important convolution series was introduced in [6] as follows:

$$L_{\kappa,\lambda}(t) = (k \,*\, l_{\kappa,\lambda})(t) = 1 + \left(\{1\} * \sum_{j=1}^{+\infty} \lambda^j \kappa^{<j>}(\cdot)\right)(t),\ \lambda \in \mathbb{C}, \tag{52}$$

where k is Sonine kernel associated with the kernel κ. It is easy to see that in the case $\kappa = \{1\}$, the convolution series (52) coincides with the exponential function:

$$L_{\kappa,\lambda}(t) = 1 + \left(\{1\} * \sum_{j=1}^{+\infty} \lambda^j h_j(\cdot)\right)(t) = 1 + \sum_{j=1}^{+\infty} \lambda^j h_{j+1}(t) = e^{\lambda\, t}. \tag{53}$$

In the case of the kernel $\kappa(t) = h_\alpha(t),\ t > 0,\ 0 < \alpha < 1$, the convolution series $L_{\kappa,\lambda}$ is reduced to the two-parameter Mittag-Leffler Function (8):

$$L_{\kappa,\lambda}(t) \ = 1 + \left(\{1\} * \sum_{j=1}^{+\infty} \lambda^j h_{j\alpha}(\cdot)\right)(t) = 1 + \sum_{j=1}^{+\infty} \lambda^j h_{j\alpha+1}(t) = E_{\alpha,1}(\lambda\, t^\alpha). \tag{54}$$

Analytical solutions to single- and multi-term fractional differential equations with n-fold GFDs of the Caputo type were presented in [6] in terms of the convolution series $l_{\kappa,\lambda}$ and $L_{\kappa,\lambda}$. In the rest of this section, we treat linear single- and multi-term fractional differential equations with n-fold GFDs in the Riemann-Liouville sense.

We start with the following auxiliary result:

Theorem 4. *Two convolution series generated by the same Sonine kernel $\kappa \in \mathcal{L}_1$ coincide for all $t > 0$, i.e.,*

$$\sum_{j=0}^{+\infty} b_j\, \kappa^{<j+1>}(t) \equiv \sum_{j=0}^{+\infty} c_j\, \kappa^{<j+1>}(t),\ t > 0 \tag{55}$$

if and only if the corresponding coefficients of these series are equal:

$$a_j = b_j,\ j = 0, 1, 2, \ldots. \tag{56}$$

Proof. If the corresponding coefficients of two convolution series generated by the same Sonine kernel $\kappa \in \mathcal{L}_1$ are equal, then we have just one series and evidently the identity (55) holds valid.

The idea of the proof of the second part of this theorem is the same as the one for the proof of the analogous calculus result for the power series, i.e., under the condition that the identity (55) holds valid we first show that $b_0 = c_0$ and then apply the same arguments to prove that $b_1 = c_1$, $b_2 = c_2$, etc.

According to Theorem 3, the convolution series in the form (44) is uniformly convergent on any interval $[\epsilon,\ T]$, and thus we can apply the GFI $\mathbb{I}_{(k)}$ to this series term by term:

$$\left(\mathbb{I}_{(k)} \sum_{j=0}^{+\infty} a_j\, \kappa^{<j+1>}(\cdot)\right)(t) = \sum_{j=0}^{+\infty} \left(\mathbb{I}_{(k)}\, a_j\, \kappa^{<j+1>}(\cdot)\right)(t) = \sum_{j=0}^{+\infty} \left(a_j\, (k(\cdot) \,*\, \kappa^{<j+1>}(\cdot)\right)(t) =$$

$$a_0 + \sum_{j=1}^{+\infty} a_j \left(\{1\} \,*\, \kappa^{<j>}(\cdot)\right)(t) = a_0 + \left(\{1\} \,*\, \sum_{j=1}^{+\infty} a_j\, \kappa^{<j>}(\cdot)\right)(t) = a_0 + (\{1\} \,*\, f_1)(t),$$

where f_1 is the following convolution series:

$$f_1(t) = \sum_{j=1}^{+\infty} a_j\, \kappa^{<j>}(t) = \sum_{j=0}^{+\infty} a_{j+1}\, \kappa^{<j+1>}(t). \tag{57}$$

Summarizing the calculations from above, for the convolution series in form (44), the formula

$$\left(\mathbb{I}_{(k)}\,\sum_{j=0}^{+\infty} a_j\,\kappa^{<j+1>}(\cdot)\right)(t) = a_0 + \left(\{1\}\,*\,\sum_{j=0}^{+\infty} a_{j+1}\,\kappa^{<j+1>}(\cdot)\right)(t) \tag{58}$$

holds valid.

Because the convergence radius of the power series $\Sigma_1(t) = \sum_{j=0}^{+\infty} a_{j+1}\,z^j$ is the same as the convergence radius of the power series $\Sigma(t) = \sum_{j=0}^{+\infty} a_j\,z^j$, Theorem 3 ensures the inclusion $f_1 \in C_{-1}(0,+\infty)$, where f_1 is defined by Formula (57). As has been shown in [4], the definite integral of a function from $C_{-1}(0,+\infty)$ is a continuous function on the whole interval $[0,\,+\infty)$ that takes the value zero at the point zero:

$$(\{1\}\,*\,f_1)(t) = (I^1_{0+}\,f_1)(t) \in C[0,\,+\infty),\;\; (I^1_{0+}\,f_1)(0) = 0. \tag{59}$$

Now we act with the GFI $\mathbb{I}_{(k)}$ on the equality (55) and apply Formula (58) to obtain the relationship

$$b_0 + \left(\{1\}\,*\,\sum_{j=0}^{+\infty} b_{j+1}\,\kappa^{<j+1>}(\cdot)\right)(t) \equiv c_0 + \left(\{1\}\,*\,\sum_{j=0}^{+\infty} c_{j+1}\,\kappa^{<j+1>}(\cdot)\right)(t),\; t>0. \tag{60}$$

Substituting point $t = 0$ into equality (60) and using Formula (59), we deduce that $b_0 = c_0$. Now we differentiate equality (60) and obtain the following identity:

$$\sum_{j=0}^{+\infty} b_{j+1}\,\kappa^{<j+1>}(t) \equiv \sum_{j=0}^{+\infty} c_{j+1}\,\kappa^{<j+1>}(t),\; t>0. \tag{61}$$

This identity has exactly same structure as identity (55) from Theorem 4. Thus, we can apply the same arguments as above and derive the relationhship $b_1 = c_1$. By repeating the same reasoning repeatedly, we arrive at Formula (56) that we wanted to prove. □

Now we are ready to apply the method of convolution series for derivation of solutions to the fractional differential equations with GFDs, and start with the fractional relaxation equation with the GFD of the Riemann-Liouville type:

$$(\mathbb{D}_{(k)}\,y)(t) = \lambda y(t),\; \lambda \in \mathbb{R},\; t>0. \tag{62}$$

As in the case of the power series, we look for a general solution to this equation in the form of a convolution series generated by the Sonine kernel κ that is an associated kernel to the kernel k of the GFD from Equation (62):

$$y(t) = \sum_{j=0}^{+\infty} b_j\,\kappa^{<j+1>}(t),\; b_j \in \mathbb{R}. \tag{63}$$

To proceed, let us first calculate the image of the convolution series (63) by action of the GFD $\mathbb{D}_{(k)}$:

$$(\mathbb{D}_{(k)}\,y)(t) = \left(\mathbb{D}_{(k)}\,\sum_{j=0}^{+\infty} b_j\,\kappa^{<j+1>}(\cdot)\right)(t) = \frac{d}{dt}\left(\mathbb{I}_{(k)}\,\sum_{j=0}^{+\infty} b_j\,\kappa^{<j+1>}(\cdot)\right)(t).$$

In the proof of Theorem 4 we already calculated the image of the convolution series (63) by action of the GFI $\mathbb{I}_{(k)}$ (Formula (58)). Applying this formula, we arrive at the representation

$$(\mathbb{D}_{(k)}\, y)(t) = \frac{d}{dt}\left(b_0 + \left(\{1\} * \sum_{j=0}^{+\infty} b_{j+1}\, \kappa^{<j+1>}(\cdot)\right)(t)\right) = \sum_{j=0}^{+\infty} b_{j+1}\, \kappa^{<j+1>}(t). \tag{64}$$

In the next step, we substitute the right-hand side of (64) into Equation (62) and obtain an equality of two convolution series generated by the same kernel κ:

$$\sum_{j=0}^{+\infty} b_{j+1}\, \kappa^{<j+1>}(t) = \sum_{j=0}^{+\infty} \lambda\, b_j\, \kappa^{<j+1>}(t),\ t > 0.$$

Application of Theorem 4 to the above identity leads to the following relationships for the coefficients of the convolution series (63):

$$b_{j+1} = \lambda\, b_j,\ j = 0, 1, 2, \ldots. \tag{65}$$

The infinite system (65) of linear equations can be easily solved step by step and we arrive at the explicit solution in form

$$b_j = b_0\, \lambda^j,\ j = 1, 2, \ldots, \tag{66}$$

where $b_0 \in \mathbb{R}$ is an arbitrary constant. Summarizing the arguments presented above, we proved the following theorem:

Theorem 5. *The general solution to the fractional relaxation Equation (62) with GFD (9) in the Riemann-Liouville sense can be represented as follows:*

$$y(t) = \sum_{j=0}^{+\infty} b_0\, \lambda^j\, \kappa^{<j+1>}(t) = b_0\, l_{\kappa,\lambda}(t),\ b_0 \in \mathbb{R}, \tag{67}$$

where $l_{\kappa,\lambda}$ is the convolution series (49).

Remark 1. *The constant b_0 in the general solution (67) to Equation (62) can be determined from a suitably posed initial condition. The form of this initial condition is prescribed by Theorem 2 (see also Formula (58)). Indeed, setting $n = 1$ in the relation (42), we obtain the following representation of the projector operator of the GFD (9) in the Riemann-Liouville sense:*

$$(P\, f)(t) = f(t) - (\mathbb{I}_{(\kappa)}\, \mathbb{D}_{(k)}\, f)(t) = \left(\mathbb{I}_{(k)}\, f\right)(0)\kappa(t),\ f \in C^{(1)}_{-1,(k)}(0, +\infty). \tag{68}$$

Thus, the initial-value problem

$$\begin{cases} (\mathbb{D}_{(k)}\, y)(t) = \lambda y(t),\ \lambda \in \mathbb{R},\ t > 0, \\ \left(\mathbb{I}_{(k)}\, y\right)(0) = b_0 \end{cases} \tag{69}$$

has a unique solution given by Formula (67).

In the case of the Sonine kernel $k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1$, the Equation (62) is reduced to Equation (2) with the Riemann-Liouville fractional derivative and its solution

(67) is exactly the solution (3) of Equation (2) in terms of the two-parameter Mittag-Leffler function (see Formula (51)). The initial-value problem (69) takes the well-known form

$$\begin{cases} (D^{\alpha}_{0+}\, y)(t) = \lambda y(t),\ \lambda \in \mathbb{R},\ t > 0, \\ \left(I^{1-\alpha}_{0+}\, y\right)(0) = b_0. \end{cases} \tag{70}$$

Its unique solution is given by the formula $y(t) = b_0\, t^{\alpha-1} E_{\alpha,\alpha}(\lambda\, t^{\alpha})$.

Now we proceed with the inhomogeneous equation of type (62)

$$(\mathbb{D}_{(k)}\, y)(t) = \lambda y(t) + f(t),\ \lambda \in \mathbb{R},\ t > 0, \tag{71}$$

where the source function f is represented in form of a convolution series

$$f(t) = \sum_{j=0}^{+\infty} a_j\, \kappa^{<j+1>}(t),\ a_j \in \mathbb{R}. \tag{72}$$

Again, we look for solutions to Equation (71) in the form of the convolution series (63). Applying exactly the same reasoning as above, we arrive at the following infinite system of linear equations for the coefficients of the convolution series (63):

$$b_{j+1} = \lambda\, b_j + a_j,\ j = 0, 1, 2, \ldots. \tag{73}$$

The explicit form of solutions to this system of equations is as follows:

$$b_j = b_0\, \lambda^j + \sum_{i=0}^{j-1} a_i\, \lambda^{j-i-1},\ j = 1, 2, \ldots, \tag{74}$$

where $b_0 \in \mathbb{R}$ is an arbitrary constant. Then the general solution to Equation (71) can be written in form of the following convolution series:

$$y(t) = b_0\, \kappa(t) + \sum_{j=1}^{+\infty} \left(b_0\, \lambda^j + \sum_{i=0}^{j-1} a_i\, \lambda^{j-i-1} \right) \kappa^{<j+1>}(t) = b_0 \sum_{j=0}^{+\infty} \lambda^j\, \kappa^{<j+1>}(t) + \sum_{j=1}^{+\infty} \sum_{i=0}^{j-1} a_i\, \lambda^{j-i-1}\, \kappa^{<j+1>}(t).$$

By direct calculation, we verify that the second sum in the last formula can be written in a more compact form:

$$\sum_{j=1}^{+\infty} \sum_{i=0}^{j-1} a_i\, \lambda^{j-i-1}\, \kappa^{<j+1>}(t) = \sum_{i=0}^{+\infty} a_i \sum_{j=1}^{+\infty} \lambda^{j-1}\, \kappa^{<j+i+1>}(t) = (f \ast l_{\kappa,\lambda})(t),$$

where the convolution series $l_{\kappa,\lambda}$ is defined by (49). We thus have proved the following result:

Theorem 6. *The general solution to the inhomogeneous Equation (71) has the form*

$$y(t) = b_0\, l_{\kappa,\lambda}(t) + (f \ast l_{\kappa,\lambda})(t),\ b_0 \in \mathbb{R}, \tag{75}$$

where the convolution series $l_{\kappa,\lambda}$ is defined by (49).

The constant b_0 is uniquely determined by the initial condition

$$\left(\mathbb{I}_{(k)}\, y\right)(0) = b_0. \tag{76}$$

Applying Theorem 6 to the case of the Riemann-Liouville fractional derivative (kernel $k(t) = h_{1-\alpha}(t),\ 0 < \alpha < 1$), we obtain the well-known result ([3]):

The unique solution to the initial-value problem

$$\begin{cases} (D_{0+}^{\alpha}\, y)(t) = \lambda y(t) + f(t),\ \lambda \in \mathbb{R},\ t > 0, \\ \left(I_{0+}^{1-\alpha}\, y\right)(0) = b_0 \end{cases}$$

is given by the formula

$$y(t) = b_0\, t^{\alpha-1} E_{\alpha,\alpha}(\lambda\, t^{\alpha}) + \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\lambda\, \tau^{\alpha})\, f(t-\tau)\, d\tau.$$

Remark 2. *In [6], single- and multi-term fractional differential equations with general fractional derivatives of the Caputo type have been studied. In particular, the unique solution to the initial-value problem*

$$\begin{cases} ({}_*\mathbb{D}_{(k)}\, y)(t) = \lambda y(t) + f(t), & \lambda \in \mathbb{R},\ t > 0, \\ y(0) = b_0, & b_0 \in \mathbb{R} \end{cases} \tag{77}$$

with the GFD of the Caputo type defined by (10) was derived in the form

$$y(t) = (f \, * \, l_{\kappa,\lambda})(t) + b_0\, L_{\kappa,\lambda}(t), \tag{78}$$

where $\kappa \in \mathcal{L}_1$ is the Sonine kernel associated with the kernel k and $l_{\kappa,\lambda}$, $L_{\kappa,\lambda}$ are the convolution series (49) and (52), respectively.

In the case of the homogeneous initial condition ($y(0) = b_0 = 0$), Formula (10) says that GFDs of the Riemann-Liouville and Caputo types coincide. As we see, the solutions to the initial-value problems with the homogeneous initial conditions for Equations (71) and (77) are also identical.

Let us now consider a linear inhomogeneous multi-term fractional differential equation with the sequential GFDs (36) of the Riemann-Liouville type and with the constant coefficients:

$$\sum_{i=0}^{n} \lambda_i (\mathbb{D}_{(k)}^{<i>}\, y)(t) = f(t),\ \lambda_i \in \mathbb{R},\ i = 0, 1, \ldots, n,\ \lambda_n \neq 0,\ t > 0, \tag{79}$$

where the source function f is represented in form of the convolution series (72).

As in the case of the single-term Equation (71), we look for solutions to the multi-term Equation (79) in the form of the convolution series (63). First, we determine the images of the convolution series (63) by action of the sequential GFDs $\mathbb{D}_{(k)}^{<i>}$, $i = 1, 2, \ldots, n$. For $i = 1$, the image is provided by Formula (64). For $i = 2, \ldots, n$, Formula (64) is applied iteratively and we arrive at the following result:

$$(\mathbb{D}_{(k)}^{<i>}\, y)(t) = \sum_{j=0}^{+\infty} b_{j+i}\, \kappa^{<j+1>}(t),\ i = 1, 2, \ldots, n. \tag{80}$$

Now we substitute the convolution series (63), its images by action of the sequential GFDs $\mathbb{D}_{(k)}^{<i>}$, $i = 1, 2, \ldots, n$ provided by Formula (80), and the convolution series (72) for the source function into Equation (79) and arrive at the following identity:

$$\sum_{i=0}^{n} \lambda_i \left(\sum_{j=0}^{+\infty} b_{j+i}\, \kappa^{<j+1>}(t) \right) = \sum_{j=0}^{+\infty} a_j\, \kappa^{<j+1>}(t),\ t > 0.$$

Application of Theorem 4 to the above identity leads to the following infinite triangular system of linear equations for the coefficients of the convolution series (63):

$$\begin{cases} \lambda_0\, b_0 + \lambda_1\, b_1 + \cdots + \ \lambda_n\, b_n = a_0, \\ \lambda_0\, b_1 + \lambda_1\, b_2 + \cdots + \ \lambda_n\, b_{n+1} = a_1, \\ \ldots \\ \lambda_0\, b_n + \lambda_1\, b_{n+1} + \cdots + \ \lambda_n\, b_{2n} = a_n, \\ \lambda_0\, b_{n+1} + \lambda_1\, b_{n+2} + \cdots + \ \lambda_n\, b_{2n+1} = a_{n+1} \\ \ldots \end{cases} \tag{81}$$

In this system, the first n coefficients $(b_0,\ b_1, \ldots, b_{n-1})$ can be chosen arbitrarily and all other coefficients are determined step by step as solutions to the infinite triangular system (81) of linear equations:

$$b_{n+l} = (a_l - \lambda_0\, b_l - \cdots - \lambda_{n-1} b_{n+l-1})/\lambda_n,\ l = 0, 1, 2, \ldots \tag{82}$$

We thus proved the following theorem:

Theorem 7. *The general solution to the inhomogeneous multi-term fractional differential Equation (79) can be represented as the convolution series (63), where the first n coefficients $(b_0,\ b_1, \ldots, b_{n-1})$ are arbitrary real constants and other coefficients are calculated according to Formula (82).*

The constants $b_0,\ b_1, \ldots, b_{n-1}$ in the general solution to Equation (79) presented in Theorem (7) can be determined based on the suitably posed initial conditions. The form of these initial conditions is prescribed by Theorem 2. Indeed, Formula (42) can be rewritten as follows:

$$(P\, f)(t) = f(t) - (\mathbb{I}_{(\kappa)}^{<n>}\, \mathbb{D}_{(k)}^{<n>}\, f)(t) = \sum_{j=0}^{n-1} \left(\mathbb{I}_{(k)}\, \mathbb{D}_{(k)}^{<j>}\, f\right)(0)\kappa^{<j+1>}(t),\ t > 0,\ f \in C_{-1,(k)}^{(n)}(0, +\infty), \tag{83}$$

where P is the projector operator of the n-fold sequential GFD of the Riemann-Liouville type. Thus, to uniquely determine the constants $b_0,\ b_1, \ldots, b_{n-1}$ in the general solution, Equation (79) has to be equipped with the initial conditions in the form

$$\left(\mathbb{I}_{(k)}\, \mathbb{D}_{(k)}^{<j>}\, y\right)(0) = b_j,\ j = 0, 1, \ldots, n-1. \tag{84}$$

Finally, we mention that the inhomogeneous multi-term fractional differential equation of type (79) with sequential Riemann-Liouville fractional derivatives (the case of the kernel $k(t) = h_{1-\alpha}(t)$ in Equation (79)) was treated in [3,19] using operational calculus of the Mikusiński type for the Riemann-Liouville fractional derivative.

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