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Noncanonical Neutral DDEs of Second-Order: New Sufficient Conditions for Oscillation

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Abstract: In this paper, new oscillation conditions for the 2nd-order noncanonical neutral differential equation $(a_0(t)((u(t) + a_1(t)u(g_0(t)))')^{\beta})' + a_2(t)u^{\beta}(g_1(t)) = 0$, where $t \geq t_0$, are established. Using Riccati substitution and comparison with an equation of the first-order, we obtain criteria that ensure the oscillation of the studied equation. Furthermore, we complement and improve the previous results in the literature.

Keywords: delay differential equation; neutral; oscillation; noncanonical case

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1. Introduction

Consider the 2nd-order delay differential equation (DDE) of the neutral type:

$$\left(a_0(t)\left(v'(t)\right)^{\beta}\right)' + a_2(t)u^{\beta}(g_1(t)) = 0, \tag{1}$$

where $t \in [t_0, \infty)$ and $v(t) := u(t) + a_1(t)u(g_0(t))$. In this paper, we obtain new sufficient criteria for the oscillation of solutions of (1) under the following hypotheses:

(A1) $\beta \geq 1$ is a ratio of odd integers;
(A2) $a_i \in C([t_0, \infty), [0, \infty))$ for $i = 0, 1, 2$, $a_0(t) > 0$, $a_1 \leq c_0$ a constant (this constant plays an important role in the results), and a_2 does not vanish identically on any half-line $[t_*, \infty)$ with $t_* \in [t_0, \infty)$;
(A3) $g_j \in C([t_0, \infty), \mathbb{R})$, $g_j(t) \leq t$, $g_0'(t) \geq g_0^* > 0$, $g_0 \circ g_1 = g_1 \circ g_0$, and $\lim_{t\to\infty} g_j(t) = \infty$, for $j = 0, 1$.

By a proper solution of (1), we mean a $u \in C^1([t_0, \infty))$ with $a_0 \cdot (v')^{\beta} \in C^1([t_0, \infty))$ and $\sup\{|u(t)| : t \geq t_*\} > 0$, for $t_* \in [t_0, \infty)$,and u satisfies (1) on $[t_0, \infty)$. A solution u of (1) is called *nonoscillatory* if it is eventually positive or eventually negative; otherwise, it is called *oscillatory*.

A *DDE* is a single-variable differential equation, usually called time, in which the derivative of the solution at a certain time is given in terms of the values of the solution at earlier times. Moreover, if the highest-order derivative of the solution appears both with and without delay, then the DDE is called of the *neutral* type.

The neutral DDEs have many interesting applications in various branches of applied science, as these equations appear in the modeling of many technological phenomena; see [1–4]. The problem of studying the oscillatory and nonoscillatory properties of DDEs has been a very active area of research in the past few decades, and many well-known and interesting results can be found in Agarwal et al. [5] and Saker [6].

In this work, we study the oscillatory properties of solutions of second-order neutral DDE (1) in the *noncanonical case*, that is:

$$\eta(t_0) < \infty, \tag{2}$$

where

$$\eta(t) := \int_t^\infty a_0^{-1/\beta}(\mu)\mathrm{d}\mu.$$

Although there are many works that have dealt with the study of the oscillation of this type of equation, in this work, we present a new approach that provides us with improved sufficient conditions for testing the oscillation of the studied equation. Contrary to the previous results, which studied the noncanonical case, our results test the oscillation of (1) when $c_0 \geq 1$ along with $c_0 < 1$.

The paper is organized as follows: Section 2 is concerned with presenting a review of the relevant literature. In Section 3, Section 3.1, we infer some qualitative properties of the positive solutions of (1). In Section 3.2, we use the new properties to obtain improved oscillation conditions. Finally, in Section 4, we summarize the main conclusions extracted from our present work and discuss potential applications and future extensions of this study.

2. Literature Review

It is easy to note the continuing and growing interest in the study of the oscillatory behavior of DDEs, and improved results, methods, and approaches can be found in [7–12]. In more detail, contrary to most previous results, Baculíková [7,8] attained the oscillation of the second-order DDE (not neutral) in the noncanonical case (2) via only one condition. While using an improved approach, Chatzarakis et al. [9] studied the oscillatory behavior of the second-order noncanonical DDE with an advanced argument. On the other hand, Jadlovská et al. [10] and Moaaz et al. [11,12] studied the oscillatory behavior of higher-order equations.

For the neutral DDEs, in the following theorem, Ye and Xu [13] investigated the oscillation of (1) in the noncanonical case (2).

Theorem 1. [13] *Assume that* $c_0 < 1$,

$$\int_{t_0}^\infty \left(Q(\mu)\eta^\beta(g_1(\mu)) - \frac{(\beta/(\beta+1))^{\beta+1} g_1'(\mu)}{\eta(g_1(\mu))a_0^{1/\beta}(g_1(\mu))} \right)\mathrm{d}\mu = \infty$$

and:

$$\int_{t_0}^\infty \left(Q(\mu)\eta^\beta(\mu) - \frac{(\beta/(\beta+1))^{\beta+1} a_0(\vartheta(\mu))}{\eta(\mu)\left(g_1'(\mu)\right)^\beta a_0^{(\beta+1)/\beta}(\mu)} \right)\mathrm{d}\mu = \infty,$$

where $Q(t) := a_2(t)(1 - a_1(g_1(t)))$. *Then,* (1) *is oscillatory.*

Later, in 2010, Han et al. [14] corrected and complemented some results in [13].

Theorem 2. [14] *Assume that* $c_0 < 1$ *and* $g_1(t) \leq g_0(t) = t - \tau_0$, $\tau_0 > 0$. *If there is a function* $\theta \in C^1([t_0,\infty),(0,\infty))$ *such that:*

$$\limsup_{t\to\infty} \int_{t_0}^t \left(Q(\mu)\theta(\mu) - \frac{(\theta_+'(\mu))^{\beta+1} a_0(g_1(\mu))}{(\beta+1)^{\beta+1}(\theta(\mu)g_1'(\mu))^\beta} \right)\mathrm{d}\mu = \infty$$

and:

$$\limsup_{t\to\infty} \int_{t_0}^t \left(\frac{1}{(1+a_1(\mu))^\beta} a_2(\mu)\eta^\beta(\mu) - \frac{(\beta/(\beta+1))^{\beta+1}}{\eta(\mu)a_0^{1/\beta}(\mu)} \right)\mathrm{d}\mu = \infty$$

then (1) *is oscillatory, where* $(\theta'_+(t) := \max\{\theta'(t), 0\}$.

By using a generalized Riccati substitution, Agarwal et al. [15] improved the result in [14].

Theorem 3. [15] *(Theorem 2.2) Assume that* $a_1(t) < \eta(t)/\eta(g_0(t))$ *and there are functions* $\rho, \sigma \in C^1([t_0, \infty), (0, \infty))$ *satisfying:*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\rho(\mu)Q(\mu) - \frac{(\rho'(\mu))^{\beta+1} r(g_1(\mu))}{(\beta+1)^{\beta+1} \rho^{\beta}(\mu) (g_1'(\mu))^{\beta}} \right) \mathrm{d}\mu = \infty$$

and:

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\psi(\mu) - \frac{\sigma(\mu) r(\mu) (\varphi_+(\mu))^{\beta+1}}{(\beta+1)^{\beta+1}} \right) \mathrm{d}\mu = \infty,$$

where:

$$\psi(t) := \sigma(t) \left(a_2(t) \left(1 - a_1(g_1(t)) \frac{\eta(g_0(g_1(t)))}{\eta(g_1(t))} \right)^{\beta} + \frac{1-\beta}{a_0^{1/\beta}(t) \eta^{\beta+1}(t)} \right)$$

and:

$$\varphi(t) := \frac{\sigma'(t)}{\sigma(t)} + \frac{1+\beta}{a_0^{1/\beta}(t)\eta(t)},$$

and $\varphi_+(t) := \max\{0, \varphi(t)\}$. *Then,* (1) *is oscillatory.*

In 2017, Bohner et al. [16] improved and simplified the result in [14,15]. They established the oscillation criteria of (1) via only one condition.

Theorem 4. [16] *If* $a_1(t) < \eta(t)/\eta(g_0(t))$ *and:*

$$\limsup_{t\to\infty} \eta^{\beta}(t) \int_{t_1}^{t} G(\mu)\mathrm{d}\mu > 1,$$

then (1) *is oscillatory, where:*

$$G(t) := a_2(t) \left(1 - a_1(g_1(t)) \frac{\eta(g_0(g_1(t)))}{\eta(g_1(t))} \right)^{\beta}.$$

Theorem 5. [16] *If* $a_1(t) < \eta(t)/\eta(g_0(t))$ *and:*

$$\liminf_{t\to\infty} \int_{g_1(t)}^{t} \widetilde{G}(\mu)\mathrm{d}\mu > \frac{1}{\mathrm{e}},$$

then (1) *is oscillatory, where:*

$$\widetilde{G}(t) := \left(\frac{1}{a_0(t)} \int_{t_0}^{t} G(\mu)\mathrm{d}\mu \right)^{1/\beta}.$$

On the other hand, in the *canonical case*:

$$\int_{t_0}^{\infty} a_0^{-1/\beta}(\mu)\mathrm{d}\mu = \infty,$$

Baculikova and Dzurina [17] obtained the oscillation conditions of (1). Very recently, by using Riccati substitution, Moaaz et al. [18,19] improved the results in [17].

Even though the establishment of the oscillation criteria for (1) in [13,14] and the insertion of the nonstandard Riccati substitution in [15,16] constitute significant progress in the subject of noncanonical neutral DDEs of second-order, the relationship between the corresponding function with delay and without delay is used in the traditional form and has not been improved, and none of these works took into account the case $c_0 \geq 1$.

The main goal of our present work is create a better estimate of the ratio $(v \circ g_1)/v$, which contributes to improving the oscillation criteria of (1). Moreover, our results take into account the case $c_0 \geq 1$, along with $c_0 < 1$.

3. Main Results

We begin with the following notations: U^+ is the set of all eventually positive solutions of (1), $V(t) := a_0^{1/\beta}(t)v'(t)$,

$$\widetilde{a}_2(t) = \min\{a_2(t), a_2(g_0(t))\}$$

$$\gamma_0 := \frac{2^{1-\beta}}{\widetilde{c}_0\beta}, \quad \widetilde{c}_0 := 1 + \frac{c_0^\beta}{g_0^*}$$

and:

$$\widehat{c}_0 := 1 + \frac{c_0^\beta}{(g_0^*)^2}$$

3.1. Auxiliary Lemmas

Below, we obtain some asymptotic properties of the positive solutions of (1). First, from the definition of η and the fundamental theorem of calculus, we obtain that $\eta(t) > 0$ for $t \geq t_0$, $\eta(t) = -a_0^{-1/\beta}(t)$ and $\lim_{t\to\infty} \eta(t) = 0$. Then, η is a decreasing function.

Lemma 1. *Assume that $v \in U^+$ and there exists a $\delta_0 \in (0,1)$ such that:*

$$\widetilde{a}_2(t)a_0^{1/\beta}(t)\eta^{\beta+1}(t) \geq \delta_0. \tag{3}$$

Then, v eventually satisfies:

$(\mathbf{C}_1)$ *v is decreasing and converges to zero;*

$(\mathbf{C}_2)$ *$v(t) \geq -\eta(t)V(t)$ and $\dfrac{v}{\eta}$ is increasing,*

and:

$$(\mathbf{C}_3)\ \ V'(t) + \frac{c_0^\beta}{g_0^*}(V(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^{\beta-1}(t)\widetilde{a}_2(t)v(g_1(t)) \leq 0.$$

Proof. Let $u \in U^+$. Then, we have that $u(t)$, $u(g_0(t))$, and $u(g_1(t))$ are positive for $t \geq t_1$, for some $t_1 \geq t_0$. Therefore, it follows from (1) that:

$$v(t) > 0 \text{ and } \left(V^\beta(t)\right)' \leq 0.$$

Using (1) and Lemma 1 in [17], we see that:

$$\begin{aligned} 0 &= \left(V^{\beta}(t)\right)' + \frac{c_0^{\beta}}{g_0'(t)}\left(V^{\beta}(g_0(t))\right)' + a_2(t)u^{\beta}(g_1(t)) \\ &\quad + c_0^{\beta}a_2(g_0(t))u^{\beta}(g_1(g_0(t))) \\ &\geq \left(V^{\beta}(t)\right)' + \frac{c_0^{\beta}}{g_0^*}\left(V^{\beta}(g_0(t))\right)' + \widetilde{a}_2(t)\left[u^{\beta}(g_1(t)) + c_0^{\beta}u^{\beta}(g_0(g_1(t)))\right] \\ &\geq \left(V^{\beta}(t) + \frac{c_0^{\beta}}{g_0^*}V^{\beta}(g_0(t))\right)' + 2^{1-\beta}\widetilde{a}_2(t)[u(g_1(t)) + c_0 u(g_0(g_1(t)))]^{\beta} \end{aligned}$$

and so:

$$\left(V^{\beta}(t) + \frac{c_0^{\beta}}{g_0^*}V^{\beta}(g_0(t))\right)' + 2^{1-\beta}\widetilde{a}_2(t)v^{\beta}(g_1(t)) \leq 0. \tag{4}$$

Integrating this inequality from t_1 to t and using the fact $\left(V^{\beta}(t)\right)' \leq 0$, we find:

$$\widetilde{c}_0 V^{\beta}(t) \leq \widetilde{c}_0 V^{\beta}(g_0(t_1)) - 2^{1-\beta}\int_{t_1}^{t} \widetilde{a}_2(\mu)v^{\beta}(g_1(\mu))\mathrm{d}\mu. \tag{5}$$

$(\mathbf{C}_1)$ Assume the contrary, that $v'(t) > 0$ for $t \geq t_1$. Thus, from (5), we have:

$$V^{\beta}(t) \leq V^{\beta}(g_0(t_1)) - \frac{2^{1-\beta}}{\widetilde{c}_0}v^{\beta}(g_1(t_1))\int_{t_1}^{t} \widetilde{a}_2(\mu)\mathrm{d}\mu.$$

This, from (3), implies:

$$\begin{aligned} V^{\beta}(t) &\leq V^{\beta}(g_0(t_1)) - \frac{2^{1-\beta}}{\widetilde{c}_0}\delta_0 v^{\beta}(g_1(t_1))\int_{t_1}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta^{\beta+1}(\mu)}\mathrm{d}\mu \\ &\leq V^{\beta}(g_0(t_1)) - \gamma_0\delta_0 v^{\beta}(g_1(t_1))\left(\frac{1}{\eta^{\beta}(t)} - \frac{1}{\eta^{\beta}(t_1)}\right). \end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $V^{\beta}(t) \to -\infty$, which contradicts the positivity of $V(t)$.

Next, since v is positive decreasing, we have that $\lim_{t\to\infty} v(t) = v_0 \geq 0$. Assume the contrary, that $v_0 > 0$. Then, $v(t) \geq v_0$ for all $t \geq t_2$, for some $t_2 \geq t_1$. Thus, from (3) and (5), we have:

$$\begin{aligned} V^{\beta}(t) &\leq V^{\beta}(g_0(t_1)) - \frac{2^{1-\beta}}{\widetilde{c}_0}v_0^{\beta}\int_{t_1}^{t} \widetilde{a}_2(\mu)\mathrm{d}\mu \\ &\leq -2^{1-\beta}\beta\delta_0 v_0^{\beta}\int_{t_1}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta^{\beta+1}(\mu)}\mathrm{d}\mu \\ &\leq -\gamma_0\delta_0 v_0^{\beta}\left(\frac{1}{\eta^{\beta}(t)} - \frac{1}{\eta^{\beta}(t_1)}\right), \end{aligned}$$

or

$$v'(t) \leq -\gamma_0^{1/\beta}\delta_0^{1/\beta}v_0\frac{1}{a_0^{1/\beta}(t)}\left(\frac{1}{\eta^{\beta}(t)} - \frac{1}{\eta^{\beta}(t_1)}\right)^{1/\beta},$$

and so,

$$v'(t) \leq -\gamma_0^{1/\beta}\delta_0^{1/\beta}v_0\frac{1}{a_0^{1/\beta}(t)\eta(t)}\left(1 - \frac{\eta^{\beta}(t)}{\eta^{\beta}(t_1)}\right)^{1/\beta}. \tag{6}$$

Using the fact that $\eta'(t) < 0$, we obtain that $\eta(t) < \eta'(t_2) < \eta'(t_1)$ for all $t \geq t_2 \geq t_1$. Hence, by integrating (6) from t_1 to t, we obtain:

$$\begin{aligned} v(t) &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta}v_0 \int_{t_2}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta(\mu)}\left(1 - \frac{\eta^\beta(\mu)}{\eta^\beta(t_1)}\right)^{1/\beta} \mathrm{d}\mu \\ &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta}v_0 \left(1 - \frac{\eta^\beta(t_2)}{\eta^\beta(t_1)}\right)^{1/\beta} \int_{t_2}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta(\mu)} \mathrm{d}\mu \\ &\leq v(t_2) - \gamma_0^{1/\beta}\delta_0^{1/\beta}v_0 \left(1 - \frac{\eta^\beta(t_2)}{\eta^\beta(t_1)}\right)^{1/\beta} \ln \frac{\eta(t_2)}{\eta(t)}. \end{aligned}$$

Letting $t \to \infty$ and taking the fact that $\eta(t) \to 0$ as $t \to \infty$, we obtain $v(t) \to -\infty$, which contradicts the positivity of $v(t)$. Therefore, $v_0 = 0$.

$(\mathbf{C}_2)$ Since $V(t)$ is decreasing, we obtain:

$$\begin{aligned} -a_0^{-1/\beta}(t)v(t) &\leq a_0^{-1/\beta}(t) \int_t^\infty a_0^{-1/\beta}(\mu)V(\mu)\mathrm{d}\mu \\ &\leq a_0^{-1/\beta}(t)V(t) \int_t^\infty a_0^{-1/\beta}(\mu)\mathrm{d}\mu \end{aligned}$$

and:

$$-a_0^{-1/\beta}(t)v(t) \leq v'(t)\eta(t). \tag{7}$$

Then, $(v/\eta)' \geq 0$.

$(\mathbf{C}_3)$ From (7), we obtain:

$$-\frac{v(g(t))}{\eta(t)} \leq -\frac{v(t)}{\eta(t)} \leq V(t).$$

Thus, from (4) and the fact $V'(t) \leq 0$, we obtain:

$$\beta V^{\beta-1}(t)V'(t) + \frac{c_0^\beta}{g_0^*}\beta V^{\beta-1}(g_0(t))(V(g_0(t)))' + 2^{1-\beta}\widetilde{a}_2(t)v^\beta(g_1(t)) \leq 0,$$

and then:

$$V'(t) + \frac{c_0^\beta}{g_0^*}(V(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^{\beta-1}(t)\widetilde{a}_2(t)v(g_1(t)) \leq 0.$$

The proof is complete. □

Lemma 2. *Assume that $u \in U^+$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. Then:*

$$(\mathbf{C}_4)\ \ \eta(t)V(t) \leq -\gamma_0\delta_0 v(t) \ \text{ and } \ v/\eta^{\gamma_0\delta_0} \text{ is decreasing.}$$

Proof. Let $u \in U^+$. From Lemma 1, we have that $(\mathbf{C}_1)$–$(\mathbf{C}_3)$ hold for $t \geq t_1$. Integrating $(\mathbf{C}_3)$ from t_1 to t, we arrive at:

$$V(t) \leq V(g_0(t_1)) - \gamma_0 \int_{t_1}^{t} \eta^{\beta-1}(\mu)\widetilde{a}_2(\mu)v(g_1(\mu))\mathrm{d}\mu.$$

From (3), we obtain:

$$V(t) \leq V(g_0(t_1)) - \gamma_0\delta_0 v(t) \int_{t_1}^{t} \frac{1}{a_0^{1/\beta}(\mu)\eta^2(\mu)}\mathrm{d}\mu,$$

and:
$$V(t) \leq V(g_0(t_1)) + \gamma_0\delta_0 v(t)\left(\frac{1}{\eta(t_1)} - \frac{1}{\eta(t)}\right). \tag{8}$$

Using $(\mathbf{C}_1)$, we eventually have:

$$V(g_0(t_1)) + \gamma_0\delta_0 \frac{v(t)}{\eta(t_1)} \leq 0,$$

Hence, (8) becomes:
$$a_0^{1/\beta}(t)v'(t) \leq -\gamma_0\delta_0 \frac{v(t)}{\eta(t)}.$$

This implies that $v/\eta^{\gamma_0\delta_0}$ is a decreasing function.

The proof is complete. □

3.2. Oscillation Theorems

In the next theorem, by using the principle of comparison with an equation of the first-order, we obtain a new criterion for the oscillation of (1).

Theorem 6. *Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If the delay differential equation:*

$$W'(t) + \frac{\gamma_0}{(1-\gamma_0\delta_0)}\eta^\beta(t)\widetilde{a}_2(t)W\left(g_0^{-1}(g_1(t))\right) = 0 \tag{9}$$

is oscillatory, then every solution of (1) is oscillatory.

Proof. Assume the contrary, that (1) has a solution $u \in U^+$. Then, we have that $u(t)$, $u(g_0(t))$, and $u(g_1(t))$ are positive for $t \geq t_1$, for some $t_1 \geq t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1)$–$(\mathbf{C}_4)$ hold for $t \geq t_1$.

Next, we define:
$$w(t) := \eta(t)V(t) + v(t).$$

From $(\mathbf{C}_1)$, $w(t) > 0$ for $t \geq t_1$. Thus,

$$w'(t) = \eta(t)V'(t) \leq 0.$$

Thus, it follows from $(\mathbf{C}_3)$ that:

$$w'(t) + \frac{c_0^\beta}{g_0^*}(w(g_0(t)))' + \frac{2^{1-\beta}}{\beta}\eta^\beta(t)\widetilde{a}_2(t)v(g_1(t)) \leq 0. \tag{10}$$

Using $(\mathbf{C}_4)$, we obtain that:

$$\begin{aligned} w(t) &= \eta(t)V(t) + v(t) \\ &\leq -\gamma_0\delta_0 v(t) + v(t) \\ &= (1-\gamma_0\delta_0)v(t), \end{aligned}$$

which with (10) gives:

$$w'(t) + \frac{c_0^\beta}{g_0^*}(w(g_0(t)))' + \frac{2^{1-\beta}}{\beta(1-\gamma_0\delta_0)}\eta^\beta(t)\widetilde{a}_2(t)w(g_1(t)) \leq 0. \tag{11}$$

Now, we set:

$$W(t) := w(t) + \frac{c_0^\beta}{g_0^*}w(g_0(t)) > 0.$$

Then, $W(t) \leq \widetilde{c}_0 w(g_0(t))$, and so, (11) becomes:

$$W'(t) + \frac{\gamma_0}{(1-\gamma_0\delta_0)}\eta^{\beta}(t)\widetilde{a}_2(t)W\Big(g_0^{-1}(g_1(t))\Big) \leq 0,$$

which has a positive solution. In view of [20] (Theorem 1), (9) also has a positive solution, which is a contradiction.

The proof is complete. □

Corollary 1. *Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:*

$$\liminf_{t\to\infty} \int_{g_0^{-1}(g_1(t))}^{t} \eta^{\beta}(\mu)\widetilde{a}_2(\mu)\mathrm{d}\mu > \frac{1-\gamma_0\delta_0}{\gamma_0 \mathrm{e}} \tag{12}$$

then every solution of (1) is oscillatory.

Proof. It follows from Theorem 2 in [21] that the condition (12) implies the oscillation of (9). □

Next, by introducing two Riccati substitution, we obtain a new oscillation criterion for (1).

Theorem 7. *Assume that $g_1(t) \leq g_0(t)$ and there exists a $\delta_0 \in (0,1)$ such that (3) holds. If:*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{2^{1-\beta}}{\beta}\eta^{\beta}(\mu)\widetilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))} - \frac{\widehat{c}_0}{4}\frac{1}{a_0^{1/\beta}(g_0(\mu))\eta(\mu)} \right)\mathrm{d}\mu = \infty, \tag{13}$$

then every solution of (1) is oscillatory.

Proof. Assume the contrary, that (1) has a solution $u \in U^+$. Then, we have that $u(t)$, $u(g_0(t))$, and $u(g_1(t))$ are positive for $t \geq t_1$, for some $t_1 \geq t_0$. From Lemmas 1 and 2, we have that $(\mathbf{C}_1)$–$(\mathbf{C}_4)$ hold for $t \geq t_1$.

Now, we define the functions:

$$\Theta_1 := \frac{V}{v},$$

and:

$$\Theta_2 := \frac{V \circ g_0}{v \circ g_0}.$$

Then, Θ_1 and Θ_2 are negative for $t \geq t_1$. From $(\mathbf{C}_4)$, we obtain:

$$\frac{v \circ g_1}{\eta^{\gamma_0\delta_0} \circ g_1} \geq \frac{v \circ g_0}{\eta^{\gamma_0\delta_0} \circ g_0} \geq \frac{v}{\eta^{\gamma_0\delta_0}}.$$

Hence,

$$\begin{aligned}
\Theta_1' &= \frac{V'}{v} - \frac{V}{v^2}v' = \frac{V'}{v \circ g_1}\frac{v \circ g_1}{v} - \frac{1}{a^{1/\beta}}\left(\frac{V}{v}\right)^2 \\
&\leq \frac{\eta^{\gamma_0\delta_0} \circ g_1}{\eta^{\gamma_0\delta_0}}\frac{V'}{v \circ g_1} - \frac{1}{a^{1/\beta}}\Theta_1^2, \\
&\leq \frac{\eta^{\gamma_0\delta_0} \circ g_1}{\eta^{\gamma_0\delta_0} \circ g_0}\frac{V'}{v \circ g_1} - \frac{1}{a^{1/\beta}}\Theta_1^2,
\end{aligned}$$

and:

$$\begin{aligned}\Theta_2' &= \frac{(V\circ g_0)'}{v\circ g_0} - \frac{V\circ g_0}{(v\circ g_0)^2}(v'\circ g_0)g_0' \\ &= \frac{(V\circ g_0)'}{v\circ g_1}\frac{v\circ g_1}{v\circ g_0} - \frac{g_0'}{a^{1/\beta}\circ g_0}\left(\frac{V\circ g_0}{v\circ g_0}\right)^2 \\ &\leq \frac{\eta^{\gamma_0\delta_0}\circ g_1}{\eta^{\gamma_0\delta_0}\circ g_0}\frac{(V\circ g_0)'}{v\circ g_1} - \frac{g_0^*}{(a^{1/\beta}\circ g_0)}\Theta_2^2.\end{aligned}$$

Then:

$$\eta(t)\Theta_1'(t) - \eta(t)\frac{\eta^{\gamma_0\delta_0}(g_1(t))}{\eta^{\gamma_0\delta_0}(g_0(t))}\frac{V'(t)}{v(g_1(t))} + \frac{\eta(t)}{a^{1/\beta}(t)}\Theta_1^2(t) \leq 0, \tag{14}$$

and:

$$\begin{aligned}0 &\geq \eta(g_0(t))\Theta_2'(t) - \eta(g_0(t))\frac{\eta^{\gamma_0\delta_0}(g_1(t))}{\eta^{\gamma_0\delta_0}(g_0(t))}\frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^*\eta(g_0(t))}{a^{1/\beta}(g_0(t))}\Theta_2^2(t) \\ &\geq \eta(g_0(t))\Theta_2'(t) - \eta(t)\frac{\eta^{\gamma_0\delta_0}(g_1(t))}{\eta^{\gamma_0\delta_0}(g_0(t))}\frac{(V(g_0(t)))'}{v(g_1(t))} + \frac{g_0^*\eta(g_0(t))}{a^{1/\beta}(g_0(t))}\Theta_2^2(t).\end{aligned} \tag{15}$$

Combining (14) and (15), we obtain:

$$\begin{aligned}0 &\geq \eta(t)\Theta_1'(t) - \eta(t)\frac{\eta^{\gamma_0\delta_0}(g_1(t))}{\eta^{\gamma_0\delta_0}(g_0(t))}\left(\frac{V'(t)}{v(g_1(t))} + \frac{c_0^\beta}{g_0^*}\frac{(V(g_0(t)))'}{v(g_1(t))}\right) \\ &\quad + \frac{\eta(t)}{a^{1/\beta}(t)}\Theta_1^2(t) + \frac{c_0^\beta}{g_0^*}\eta(g_0(t))\Theta_2'(t) + c_0^\beta\frac{\eta(g_0(t))}{a^{1/\beta}(g_0(t))}\Theta_2^2(t) \\ &\geq \eta(t)\Theta_1'(t) + \frac{2^{1-\beta}}{\beta}\eta^\beta(t)\widetilde{a}_2(t)\frac{\eta^{\gamma_0\delta_0}(g_1(t))}{\eta^{\gamma_0\delta_0}(g_0(t))} \\ &\quad + \frac{\eta(t)}{a^{1/\beta}(t)}\Theta_1^2(t) + \frac{c_0^\beta}{g_0^*}\eta(g_0(t))\Theta_2'(t) + c_0^\beta\frac{\eta(g_0(t))}{a^{1/\beta}(g_0(t))}\Theta_2^2(t).\end{aligned}$$

Integrating this inequality from t_1 to t, we have:

$$\begin{aligned}0 &\geq \eta(t)\Theta_1(t) - \eta(t_1)\Theta_1(t_1) + \int_{t_1}^t\left(a_0^{-1/\beta}(\mu)\Theta_1(\mu) + \frac{\eta(\mu)}{a^{1/\beta}(\mu)}\Theta_1^2(t)\right)\mathrm{d}\mu \\ &\quad + \frac{c_0^\beta}{g_0^*}(\eta(g_0(t))\Theta_2(t) - \eta(g_0(t_1))\Theta_2(t_1)) \\ &\quad + \frac{c_0^\beta}{g_0^*}\left(\int_{t_1}^t a_0^{-1/\beta}(g_0(t))\Theta_2(\mu) + \frac{g_0^*\eta(g_0(\mu))}{a^{1/\beta}(g_0(\mu))}\Theta_2^2(\mu)\right)\mathrm{d}\mu \\ &\quad + \frac{2^{1-\beta}}{\beta}\int_{t_1}^t \eta^\beta(\mu)\widetilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))}\mathrm{d}\mu.\end{aligned}$$

From $(\mathbf{C}_2)$, we obtain $\eta(t)\Theta_1(t) \geq -1$. Therefore,

$$\begin{aligned}0 &\geq -K - \frac{1}{4}\int_{t_1}^t\left(\frac{1}{a_0^{1/\beta}(\mu)\eta(\mu)} + \frac{c_0^\beta}{(g_0^*)^2}\frac{1}{a_0^{1/\beta}(g_0(\mu))\eta(g_0(\mu))}\right)\mathrm{d}\mu \\ &\quad + \frac{2^{1-\beta}}{\beta}\int_{t_1}^t \eta^\beta(\mu)\widetilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))}\mathrm{d}\mu,\end{aligned}$$

where:

$$K := \eta(t_1)\Theta_1(t_1) + \frac{c_0^{\beta}}{g_0^*}\eta(g_0(t_1))\Theta_2(t_1) + \left(1 + \frac{c_0^{\beta}}{g_0^*}\right).$$

Since $\eta'(t) < 0$ and $a'(t) \geq 0$, we find:

$$\int_{t_1}^{t}\left(\frac{2^{1-\beta}}{\beta}\eta^{\beta}(\mu)\widetilde{a}_2(\mu)\frac{\eta^{\gamma_0\delta_0}(g_1(\mu))}{\eta^{\gamma_0\delta_0}(g_0(\mu))} - \frac{\widehat{c}_0}{4}\frac{1}{a_0^{1/\beta}(g_0(\mu))\eta(\mu)}\right)\mathrm{d}\mu \leq K.$$

Taking $\limsup_{t\to\infty}$ and using (13), we arrive at a contradiction.

The proof is complete. □

3.3. Applications and Discussion

Remark 1. *It is easy to see that the previous works that dealt with the noncanonical case required either $a_1(t) < 1$ or $a_1(t) < \eta(t)/\eta(g_0(t))$. Since η is decreasing and $g_0(t) \leq t$, we have that $\eta(g_0(t)) \geq \eta(t)$. Then, the results of these works only apply when $a_1(t) \in (0,1)$.*

Example 1. *Consider the DDE:*

$$\left(t^2(u(t) + a_1^* u(\lambda t))\right)' + a_2^* u(\kappa t) = 0, \tag{16}$$

where $t \geq 1$, $a_1^* > 0$, $a_2^* \in (0,1)$, and $\kappa < \lambda \in (0,1)$. By choosing $\delta_0 = a_2^*$, the condition (12) becomes:

$$a_2^* \ln\frac{\lambda}{\kappa} > \frac{\lambda + a_1^* - \lambda a_2^*}{\mathrm{e}\lambda}. \tag{17}$$

Using Corollary 1, Equation (16) is oscillatory if (17) holds.

Remark 2. *To apply Theorems 3 and 4 on (16), we must stipulate that $a_1^* < 1$. Let a special case of (16), namely,*

$$\left(t^2\left(u(t) + a_1^* u\left(\frac{t}{2}\right)\right)\right)' + a_2^* u(\kappa t) = 0,$$

A simple computation shows that (16) is oscillatory if:

$$a_2^*(1 - 2a_1^*) > \frac{1}{4}\ (\textit{using Theorem 3}) \tag{18}$$

or:

$$a_2^*(1 - 2a_1^*) > 1\ (\textit{using Theorem 4}) \tag{19}$$

or:

$$a_2^*(1 - 2a_1^*)\ln\frac{1}{\kappa} > \frac{1}{\mathrm{e}}\ (\textit{using Theorem 5}). \tag{20}$$

Consider the following most specific special case:

$$\left(t^2\left(u(t) + \frac{2}{5}u\left(\frac{t}{2}\right)\right)\right)' + \frac{4}{5}u\left(\frac{t}{4}\right) = 0. \tag{21}$$

Note that (18)–(20) fail to apply. However, (17) reduces to:

$$\frac{4}{5}\ln 2 > \frac{1}{e}.$$

which ensures the oscillation of (21).

4. Conclusions

In this work, the oscillatory properties of the solutions of a class of second-order neutral DDEs were studied. Using the Riccati technique and comparison principles, we obtained new criteria that guarantee the oscillation of all solutions of the studied equation.

The new approach, taken in this work, relies on creating a better estimate of the ratio $(v \circ g_1)/v$ by establishing the new decreasing function $v/\eta^{\gamma_0 \delta_0}$. This new estimate enables us to obtain new oscillation conditions that directly improve the previous related results. Moreover, our results considered the case where $c_0 \geq 1$, which was not taken into account in the previous results.

An interesting issue is obtaining results that take into account all c_0 and do not adhere to the condition $g_0 \circ g_1 = g_1 \circ g_0$. It is also interesting to extend our results to higher-order equations. It is also interesting to extend the results of this paper to study the oscillatory behavior of some concrete examples that may appear in physics, astronomy, medicine, hydrodynamics, etc.; as an example, see [22].

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