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# On the $A_\alpha$ -Eigenvalues of Signed Graphs

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**Abstract:** For  $\alpha \in [0, 1]$ , let  $A_\alpha(G_\sigma) = \alpha D(G) + (1 - \alpha)A(G_\sigma)$ , where  $G$  is a simple undirected graph,  $D(G)$  is the diagonal matrix of its vertex degrees and  $A(G_\sigma)$  is the adjacency matrix of the signed graph  $G_\sigma$  whose underlying graph is  $G$ . In this paper, basic properties of  $A_\alpha(G_\sigma)$  are obtained, its positive semidefiniteness is studied and some bounds on its eigenvalues are derived—in particular, lower and upper bounds on its largest eigenvalue are obtained.

**Keywords:** signed graph; adjacency matrix; balanced signed graph; switching equivalent graphs; eigenvalue bounds

## 1. Introduction

The theory of signed graphs has attracted the attention of several researchers in recent decades (see Zaslavsky's dynamic survey [1] for a mathematical bibliography of signed graphs). Among the first contributions in this theory, we mention the works of Harary [2] and Zaslavsky [3] (1982). Among the most recent contributions, we highlight the works of Stanić in [4–7], in which the largest eigenvalue of signed graphs is studied. In Belardo et al. [8], there is a recent compendium of interesting open problems.

Throughout this paper  $G = (V(G), E(G))$  is a undirected simple graph on  $n$  vertices such that  $V(G)$  and  $E(G)$  denote the vertex set and edge set of  $G$ , respectively. Let  $A(G)$  be the adjacency matrix of  $G$  and let  $D(G)$  be the diagonal matrix whose  $(i, i)$ -entry is the degree of the  $i$ -th vertex of  $G$ . The matrices  $L(G) = D(G) - A(G)$  and  $Q(G) = D(G) + A(G)$  are the Laplacian and signless Laplacian matrices of  $G$ , respectively. Such matrices  $L(G)$  and  $Q(G)$  are both positive semidefinite. It is known that if  $G$  is a bipartite graph, then the spectra of  $L(G)$  and  $Q(G)$  are the same.

A signed graph is a pair  $G_\sigma = (G, \sigma)$  where  $\sigma : E(G) \rightarrow \{+1, -1\}$  is a sign function on the edges of  $G$ . Thus, a signed graph has positive and negative edges. The underlying graph  $G$  is interpreted as a signed graph where all its edges are positive. The adjacency matrix of  $G_\sigma$  is  $A(G_\sigma) = (a_{i,j})$  in which  $a_{i,j} = \sigma(v_i v_j)$  if  $v_i$  and  $v_j$  are adjacent and 0 otherwise. The Laplacian matrix of  $G_\sigma$  is  $L(G_\sigma) = D(G) - A(G_\sigma)$ . Then  $A(G_\sigma)$  and  $L(G_\sigma)$  are both real symmetric matrices. Then their eigenvalues are real, and they, counting multiplicities, define the spectrum of  $G_\sigma$  and the Laplacian spectrum of  $G_\sigma$ , respectively.

In [9], Nikiforov introduced the convex linear combination of  $D(G)$  and  $A(G)$  matrices:

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$$

where  $\alpha \in [0, 1]$ . Clearly,  $A_0(G) = A(G)$ ,  $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$ , and  $A_1(G) = D(G)$ . Then, for  $0 \leq \alpha \leq 1$ , the spectrum of  $A_\alpha(G)$  varies continuously from the spectrum of  $A(G)$  to the multiset of vertex degrees of  $G$  with the spectrum of  $\frac{1}{2}Q(G)$  at the middle of the range.

Several works on  $A_\alpha$  have already been published, some of which are [9–21]. In these articles, properties previously shown for the adjacency matrix ( $\alpha = 0$ ) or signless Laplacian matrix (essentially  $\alpha = \frac{1}{2}$ ) have been extended to all  $\alpha \in [0, 1]$ , among other results.

If  $G_\sigma$  is a signed graph, the matrix  $A_\alpha(G_\sigma)$  is defined in [22] as follows:



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**Definition 1.** For  $\alpha \in [0, 1]$ , let

$$A_\alpha(G_\sigma) = \alpha D(G) + (1 - \alpha)A(G_\sigma)$$

where  $D(G)$  is the diagonal matrix of the vertex degrees of  $G$  and  $A(G_\sigma)$  is the adjacency matrix of  $G_\sigma$ .

Clearly,  $A_\alpha(G_\sigma)$  is a real symmetric matrix. Let  $(-\sigma)(e) = -\sigma(e)$  for all  $e \in E(G)$ . Note that  $G_{-\sigma} = (G, -\sigma)$  is obtained from the signed graph  $G_\sigma = (G, \sigma)$  by reversing the sign of all edges. We see that  $A_0(G_\sigma) = A(G_\sigma)$ ,  $A_{\frac{1}{2}}(G_\sigma) = \frac{1}{2} \left( D(G) + A(G_\sigma) \right) = \frac{1}{2} \left( D(G) - A(G_{-\sigma}) \right) = \frac{1}{2}L(G_{-\sigma})$ , and  $A_1(G_\sigma) = D(G)$ . Thus, the spectrum of  $A_\alpha(G_\sigma)$  varies continuously from the spectrum of  $A(G_\sigma)$  to the multiset of vertex degrees of  $G$  with the spectrum of  $\frac{1}{2}L(G_{-\sigma})$  at the middle of the range as  $\alpha$  runs from 0 to 1.

To our knowledge, the first work on  $A_\alpha(G_\sigma)$  is the contribution of Belardo et al. [22], in which the results obtained in [11] on the multiplicity of  $\alpha$  as an eigenvalue of  $A_\alpha(G)$ , when the unsigned graph  $G$  under study has pendant vertices, are extended to  $A_\alpha(G_\sigma)$ .

The rest of the work is organized as follows. In Section 2, we give some preliminaries including some basic results of the spectral theory of signed graphs. In Section 3, we derive new basic properties of the  $A_\alpha$ -eigenvalues of signed graphs. In Section 4, we study the positive semidefiniteness of  $A_\alpha(G_\sigma)$ , and we derive some bounds on its eigenvalues. Finally, in Section 5, we obtain lower and upper bounds on the largest eigenvalue of  $A_\alpha(G_\sigma)$ .

**2. Preliminaries**

We recall that a complex square matrix  $M$  is said Hermitian if  $M = M^*$ , where  $M^*$  is the conjugate transpose of  $M$ . Throughout this work, the eigenvalues  $\lambda_j(M)$ ,  $1 \leq j \leq n$ , of a Hermitian matrix  $M$  of order  $n$  are arranged in non-increasing order, that is,

$$\lambda_1(M) \geq \dots \geq \lambda_{n-1}(M) \geq \lambda_n(M).$$

We recall a simplified version of Weyl’s inequalities for eigenvalues of Hermitian matrices (see, e.g., [23]). In [24], So establishes the conditions for the equality in the Weyl’s inequalities.

**Theorem 1.** Let  $A$  and  $B$  be Hermitian matrices of order  $n$  and let  $1 \leq j \leq n$ . Then

$$\lambda_j(A) + \lambda_n(B) \leq \lambda_j(A + B) \leq \lambda_j(A) + \lambda_1(B).$$

In either of these inequalities the equality holds if and only if there exists a nonzero  $n$ -vector that is an eigenvector to each of the three eigenvalues involved. In particular,

$$\lambda_1(A) + \lambda_n(B) \leq \lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B). \tag{1}$$

The next result is immediate from Theorem 1.

**Corollary 1.** Let  $A$  and  $B$  be Hermitian matrices of order  $n$  and let  $1 \leq j \leq n$ . If  $B$  is positive and semidefinite, then

$$\lambda_j(A) \leq \lambda_j(A + B).$$

Corollary 1 tells us that the eigenvalues of an Hermitian matrix do not decrease if a positive semidefinite matrix is added to it.

We recall some notions of the theory of signed graphs.

**Definition 2** ([2]). Let  $G_\sigma = (G, \sigma)$  be a signed graph.

- (i) An edge  $e \in E(G_\sigma)$  is positive (negative) if  $\sigma(e) = +1$  ( $\sigma(e) = -1$ ).
- (ii) A positive cycle of  $G_\sigma$  is one in which the number of negative edges is even. A negative cycle of  $G_\sigma$  is a non-positive cycle.
- (iii) The sign of a path in  $G_\sigma$  is the product of the signs of its edges.
- (iv) The signed graph  $G_\sigma$  is balanced if all its cycles are positive. Otherwise,  $G_\sigma$  is unbalanced.

We now recall the notions of switching equivalent graphs, switching isomorphic graphs and sign-symmetric graphs.

**Definition 3.** Let  $G_\sigma = (G, \sigma)$  be a signed graph. For  $U \subset V(G)$ , let  $G_\sigma^U$  be the signed graph obtained from  $G_\sigma$  by reversing the sign of each edge between a vertex in  $U$  and a vertex in  $V(G) \setminus U$ . The signed graph  $G_\sigma^U$  is said to be switching-equivalent to  $G_\sigma$ .

Note that the signature switching preserves the set of the positive cycles.

**Definition 4.** Let  $G_\sigma$  and  $G_\zeta$  be two signed graphs.

- (i)  $G_\sigma$  and  $G_\zeta$  are switching isomorphic if there is an isomorphism of the underlying graphs that preserves the signs of cycles.
- (ii)  $G_\sigma$  is sign-symmetric if  $G_\sigma$  is switching isomorphic to  $G_{-\sigma}$ .

Clearly the signature-reversal changes the sign of every odd cycle and maintains the sign of every even cycle. In particular, since unsigned bipartite graphs are odd-cycle-free, a special case of sign-symmetric signed graphs are the bipartite graphs. There are non-bipartite graphs that are sign-symmetric (see Figure 1, [8]).

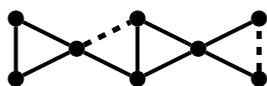


Figure 1. A sign-symmetric signed graph that is not an unsigned bipartite graph.

In the following theorem, we collect some of the basic results in the theory of signed graphs (see [2,5,8,25,26]).

**Theorem 2.** Let  $G_\sigma = (G, \sigma)$  be a signed graph. For  $U \subset V(G)$ , let  $G_\sigma^U$  as in Definition 3. Then:

- (i) There exists a diagonal matrix  $S$  with diagonal entries  $\pm 1$  such that  $A(G_\sigma^U) = S^{-1}A(G_\sigma)S$ .
- (ii)  $G_\sigma$  is balanced if and only if there is a diagonal matrix  $S$  with diagonal entries  $\pm 1$  such that  $S^{-1}A(G_\sigma)S = A(G)$ .
- (iii) Each bridge in a signed graph can be assumed to be a positive edge.
- (iv) Any signed tree is balanced.
- (v) If  $G_\sigma$  and  $G_{-\sigma}$  are sign-symmetric, then  $A(G_\sigma)$  and  $A(G_{-\sigma})$  share the same spectrum.
- (vi)  $L(G_\sigma)$  is a positive semidefinite matrix.
- (vii) For a connected graph  $G$  of order  $n$ ,  $G_\sigma$  is balanced if and only if  $\lambda_n(L(G_\sigma)) = 0$ .

From item (i) of Theorem 2, we see that switching equivalent signed graphs share the same spectrum.

For more results on signed graphs, the reader is referred to [2,3,27,28].

### 3. Some New Basic Results on the $A_\alpha$ -Matrix of Signed Graphs

We begin this section by mentioning that items (i), (ii), and (v) of Theorem 2 can be easily extended to the matrix  $A_\alpha(G_\sigma)$  for all  $\alpha \in [0, 1]$ .

**Theorem 3.** Let  $G_\sigma = (G, \sigma)$  be a signed graph. Let  $0 \leq \alpha \leq 1$ . For  $U \subset V(G)$ , let  $G_\sigma^U$  as in Definition 3. Then:

- (i) There exists a diagonal matrix  $S$  with diagonal entries  $\pm 1$  such that  $A_\alpha(G_\sigma^U) = S^{-1}A_\alpha(G_\sigma)S$ .

- (ii)  $G_\sigma$  is balanced if and only if there is a diagonal matrix  $S$  with diagonal entries  $\pm 1$  such that  $S^{-1}A_\alpha(G_\sigma)S = A_\alpha(G)$ .
- (iii) If  $G_\sigma$  and  $G_{-\sigma}$  are sign-symmetric, then  $A_\alpha(G_\sigma)$  and  $A_\alpha(G_{-\sigma})$  share the same spectrum.

For  $n \geq 3$ , let  $K_n$  and  $K_{1,n-1}$  be the complete graph and the star on  $n$  vertices, respectively.

**Theorem 4.** Let  $G_\sigma = (G, \sigma)$  be a signed graph on  $n$  vertices. Let  $\tilde{G}_\zeta = (\tilde{G}, \zeta)$  be the signed graph obtained from  $G_\sigma$  by removing an edge  $e \in E(G_\sigma)$ , where  $\zeta$  is the restriction of  $\sigma$  to  $E(\tilde{G})$ . If  $\frac{1}{2} \leq \alpha \leq 1$ , then

$$\lambda_j(A_\alpha(G_\sigma)) \geq \lambda_j(A_\alpha(\tilde{G}_\zeta))$$

for  $j = 1, \dots, n$ .

**Proof.** Without loss of generality, we may assume that  $e = v_1v_2$ . We have  $\sigma(v_1v_2) = \pm 1$ . Let  $\tilde{G}_\zeta$  be the graph obtained from  $G_\sigma$  by removing the edge  $v_1v_2$ . One can easily see that

$$A_\alpha(G_\sigma) = A_\alpha(\tilde{G}_\zeta) + C$$

where

$$C = \begin{bmatrix} \alpha & \pm(1-\alpha) & 0 & \dots & 0 \\ \pm(1-\alpha) & \alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}.$$

The eigenvalues of  $C$  are  $1, 2\alpha - 1$ , and  $0$  with multiplicity  $n - 2$ . Hence, if  $\alpha \geq \frac{1}{2}$ , then  $C$  is a positive semidefinite matrix and by Corollary 1 the result follows.  $\square$

There are two extremal cases for the sign function  $\sigma$ . They are  $\sigma^+(e) = +1$  and  $\sigma^-(e) = -1$ , for all  $e \in E(G)$ . Let  $G^+ = (G, \sigma^+)$  and  $G^- = (G, \sigma^-)$ . Clearly  $A_\alpha(G^+) = A_\alpha(G)$ .

**Corollary 2.** Let  $G$  be a graph on  $n$  vertices. Let  $\frac{1}{2} \leq \alpha \leq 1$ . Then

$$\lambda_1(A_\alpha(G^-)) \leq \alpha(n - 2) + 1.$$

**Proof.** By repeated applications of Theorem 4,  $\lambda_1(A_\alpha(G^-)) \leq \lambda_1(A_\alpha(K_n^-))$ . Since  $\lambda_1(A_\alpha(K_n^-)) = \lambda_1(\alpha(n - 1)I_n - (1 - \alpha)A(K_n)) = \alpha(n - 2) + 1$ , where  $I_n$  is the identity matrix of order  $n$  and the smallest eigenvalue of  $A(K_n)$  is equal to  $-1$ , the result follows.  $\square$

**Theorem 5.** Let  $G_\sigma$  be a connected signed graph on  $n$  vertices with only one negative edge  $e$ , which is not a bridge. Let  $G_\sigma - e$  be the graph obtained from  $G_\sigma$  by removing  $e$ . If  $\frac{1}{2} < \alpha \leq 1$ , then

$$\lambda_1(A_\alpha(G_\sigma)) > \lambda_1(A_\alpha(G_\sigma - e)) = \lambda_1(A_\alpha(G - e)).$$

**Proof.** We may assume that  $e = v_1v_2$ . Let  $G_\sigma - e$  be the graph obtained from  $G_\sigma$  by removing the edge  $v_1v_2$ . Since  $e$  is not a bridge,  $G_\sigma - e$  is a connected graph with only positive edges. We have

$$A_\alpha(G_\sigma) = A_\alpha(G_\sigma - e) + C$$

where

$$C = \begin{bmatrix} \alpha & -(1-\alpha) & 0 & \dots & 0 \\ -(1-\alpha) & \alpha & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix}.$$

The eigenvalues of  $C$  are  $1, 2\alpha - 1$ , and  $0$  with multiplicity  $n - 2$ . Since  $\alpha > \frac{1}{2}$ ,  $\lambda_n(C) = 0$ . Taking into account that  $A_\alpha(G_\sigma - e)$  and  $C$  are Hermitian matrices, by use of the Weyl's inequality (1) in  $A_\alpha(G_\sigma) = A_\alpha(G_\sigma - e) + C$ , we have

$$\lambda_1(A_\alpha(G_\sigma)) \geq \lambda_1(A_\alpha(G_\sigma - e)) + \lambda_n(C) = \lambda_1(A_\alpha(G_\sigma - e)). \tag{2}$$

Assume that

$$\lambda_1(A_\alpha(G_\sigma)) = \lambda_1(A_\alpha(G_\sigma - e)).$$

Under this assumption, we have an equality in (2). By the necessary and sufficient condition for the equality in the Weyl's inequality,  $\lambda_1(A_\alpha(G_\sigma))$ ,  $\lambda_1(A_\alpha(G_\sigma - e))$  and  $\lambda_n(C)$  have a common unit eigenvector  $\mathbf{x}$ . Since  $A_\alpha(G_\sigma - e)$  is a nonnegative irreducible matrix, from the Perron–Frobenius theory if nonnegative matrices, all the entries of  $\mathbf{x}$  are positive. Let  $\mathbf{x} = [x_1, \dots, x_n]^T$ . We have

$$\begin{aligned} \lambda_n(C) &= \mathbf{x}^T C \mathbf{x} = \alpha(x_1^2 + x_2^2) - 2(1-\alpha)x_1x_2 = \\ &= \alpha(x_1 + x_2)^2 - 2x_1x_2 > \frac{1}{2}(x_1 - x_2)^2 \geq 0, \end{aligned}$$

contradicting the fact that  $\lambda_n(C) = 0$ . Hence  $\lambda_1(A_\alpha(G_\sigma)) > \lambda_1(A_\alpha(G_\sigma - e))$ , and the proof is complete.  $\square$

The identity

$$A_\alpha(G) - A_\beta(G) = (\alpha - \beta)L(G) \tag{3}$$

plays a crucial role in [9]. Using the identity (3) together with the Weyl's inequalities and the fact that for a connected graph  $\lambda_n(L(G)) = 0$  is a simple eigenvalue with eigenvector the all ones vector, the following theorem is proved in [9]:

**Theorem 6** ([9], Proposition 4). *Let  $1 \geq \alpha > \beta \geq 0$ . If  $G$  is a graph of order  $n$ , then*

$$\lambda_k(A_\alpha(G)) - \lambda_k(A_\beta(G)) \geq \lambda_n(L(G)) = 0 \tag{4}$$

for any  $k = 1, \dots, n$ . If  $G$  is connected, then the inequality (4) is strict, unless  $k = 1$  and  $G$  is regular.

In the following theorem, the identity (3) and Theorem 6 are extended to signed graphs.

**Theorem 7.** *Let  $G_\sigma = (G, \sigma)$  where  $G$  is a graph of order  $n$ . Let  $1 \geq \alpha > \beta \geq 0$ . Then:*

(i)

$$A_\alpha(G_\sigma) - A_\beta(G_\sigma) = (\alpha - \beta)L(G_\sigma). \tag{5}$$

(ii) For  $1 \leq k \leq n$ ,

$$\lambda_k(A_\alpha(G_\sigma)) \geq \lambda_k(A_\beta(G_\sigma)). \tag{6}$$

If  $G_\sigma$  is a connected balanced signed graph, the inequality (6) is strict, unless  $k = 1$  and  $G_\sigma$  is regular.

(iii) If  $G_\sigma$  is a connected signed graph that is unbalanced, then all the inequalities in (6) are strict.

**Proof.**

(i) It is immediate that (5) holds. In fact,

$$A_\alpha(G_\sigma) - A_\beta(G_\sigma) = (\alpha - \beta)(D(G) - A(G_\sigma)) = (\alpha - \beta)L(G_\sigma).$$

(ii) Suppose  $1 \geq \alpha > \beta \geq 0$  and  $1 \leq k \leq n$ . From (5),

$$A_\alpha(G_\sigma) = A_\beta(G_\sigma) + (\alpha - \beta)L(G_\sigma).$$

Applying the Weyl's inequalities and the fact that  $\lambda_n(L(G_\sigma)) \geq 0$ , we obtain

$$\lambda_k(A_\alpha(G_\sigma)) \geq \lambda_k(A_\beta(G_\sigma)) + (\alpha - \beta)\lambda_n(L(G_\sigma)) \geq \lambda_k(A_\beta(G_\sigma)). \tag{7}$$

Thus, the inequality (6) is obtained. Suppose that  $G_\sigma$  is a connected balanced graph. Then, by item (vii) of Theorem 2,  $\lambda_n(L(G_\sigma)) = 0$ , and by item (ii) of Theorem 3, the matrices  $A_\alpha(G_\sigma)$ ,  $A_\beta(G_\sigma)$  and  $L(G_\sigma)$  are similar to  $A_\alpha(G)$ ,  $A_\beta(G)$  and  $L(G)$ , respectively. Thus, the condition of strict inequality is reduced to the case of Theorem 6, and the result follows.

(iii) Suppose  $1 \geq \alpha > \beta \geq 0$  and  $G_\sigma$  is a connected unbalanced graph. By item (vii) of Theorem 2,  $\lambda_n(L(G_\sigma)) > 0$ . Then the last inequality in (7) is strict, and the result follows.

This completes the proof.  $\square$

Since  $\sigma^+(e) = +1$  and  $\sigma^-(e) = -1$ , for all  $e \in E(G)$ , we have

- if  $\sigma = \sigma^+$  then  $A_\alpha(G^+) = A_\alpha(G)$  and  $L(G^+) = L(G)$ , and
- if  $\sigma = \sigma^-$  then  $L(G^-) = Q(G)$ .

**Corollary 3.** Let  $G$  be a graph of order  $n$ . Let  $1 \geq \alpha > \beta \geq 0$ . Then:

(i) If  $\sigma = \sigma^-$  and  $G$  is bipartite, then

$$\lambda_k(A_\alpha(G^-)) - \lambda_k(A_\beta(G^-)) \geq \lambda_n(L(G)) = 0 \tag{8}$$

for  $k = 1, \dots, n$ . If  $G$  is connected, then the inequality (8) is strict, unless  $k = 1$  and  $G$  is regular.

(ii) If  $\sigma = \sigma^-$  and  $G$  is a connected non-bipartite graph, then all the inequalities in (6) are strict.

**Proof.**

(i) Suppose  $\sigma = \sigma^-$  and  $G$  are bipartite. Then  $L(G_\sigma) = L(G^-) = Q(G)$ , all the edges of  $G^-$  are negative,  $\lambda_n(Q(G)) = \lambda_n(L(G)) = 0$ , and  $G$  does not contain odd cycles. Hence  $G^-$  is balanced. If, in addition,  $G$  is connected, then  $G^-$  is a connected balanced signed graph and the result follows from Theorem 7.

(ii) If  $\sigma = \sigma^-$  then  $L(G^-) = Q(G)$ . The result follows from Theorem 7 using the fact that the smallest eigenvalue of the signless Laplacian matrix of a connected non-bipartite graph is positive.

This completes the proof.  $\square$

In the following theorem, we collect some immediate consequences of the positive semidefiniteness of  $L(G_\sigma)$ , Equation (5) and Rayleigh's principle for Hermitian matrices, which extend the results given in [9] to the theory of signed graphs.

**Theorem 8.** Let  $G_\sigma = (G, \sigma)$  where  $G$  is a graph of order  $n$ . Let  $x = [x_1, \dots, x_n]^T$  be a column vector:

- (i) If  $1 \geq \alpha \geq \beta \geq 0$  then  $x^T A_\alpha(G_\sigma)x \geq x^T A_\beta(G_\sigma)x$ .
- (ii)  $\lambda_1(A_\alpha(G_\sigma)) = \max\{x^T A_\alpha(G_\sigma)x : \|x\|_2 = 1\}$ .

- (iii) If  $\mathbf{x}$  is a unit vector then  $\lambda_1(A_\alpha(G_\sigma)) = \mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x}$  if and only if  $\mathbf{x}$  is an eigenvector to  $\lambda_1(A_\alpha(G_\sigma))$ .
- (iv)  $\lambda_n(A_\alpha(G_\sigma)) = \min\{\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} : \|\mathbf{x}\|_2 = 1\}$ .
- (v) If  $\mathbf{x}$  is a unit vector then  $\lambda_n(A_\alpha(G_\sigma)) = \mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x}$  if and only if  $\mathbf{x}$  is an eigenvector to  $\lambda_n(A_\alpha(G_\sigma))$ .
- (vi)  $\lambda_1(A_\alpha(G_\sigma)) = \max\{\lambda_1(A_\alpha(H)) : H \text{ is a component of } G_\sigma\}$ .
- (vii)  $\lambda_n(A_\alpha(G_\sigma)) = \min\{\lambda_n(A_\alpha(H)) : H \text{ is a component of } G_\sigma\}$ .

#### 4. Positive Semidefiniteness of $A_\alpha(G_\sigma)$ and Some Bounds

We write  $d(u)$  for the degree of  $u \in V(G)$ . In order to continue searching for more basic properties of  $A_\alpha(G_\sigma)$ , we observe that the quadratic form  $\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x}$  can be represented as follows:

$$\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} = \sum_{uv \in E(G)} (\alpha x_u^2 + 2\sigma(uv)(1 - \alpha)x_u x_v + \alpha x_v^2).$$

$$\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} = (2\alpha - 1) \sum_{u \in V(G)} d(u)x_u^2 + (1 - \alpha) \sum_{uv \in E(G)} (x_u + \sigma(uv)x_v)^2. \tag{9}$$

$$\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} = \alpha \sum_{u \in V(G)} d(u)x_u^2 + 2(1 - \alpha) \sum_{uv \in E(G)} \sigma(uv)x_u x_v. \tag{10}$$

We note that the expressions (9) and (10) are obtained from  $\mathbf{x}^T A(G_\sigma) \mathbf{x}$ , the definition of  $A_\alpha(G_\sigma)$ , and some algebra.

**Theorem 9.** *If  $\frac{1}{2} \leq \alpha \leq 1$ , then  $A_\alpha(G_\sigma)$  is positive semidefinite. If  $\frac{1}{2} < \alpha \leq 1$  and  $G$  has no isolated vertices, then  $A_\alpha(G_\sigma)$  is positive and definite.*

**Proof.** Let  $\alpha \geq \frac{1}{2}$ . From (9), for any  $uv \in E(G)$ , we have

$$\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} \geq (2\alpha - 1)x_u^2 + (2\alpha - 1)x_v^2 + (1 - \alpha)(x_u + \sigma(uv)x_v)^2 \geq 0. \tag{11}$$

Hence  $A_\alpha(G_\sigma)$  is positive semidefinite. Let  $\alpha > \frac{1}{2}$ . Suppose that  $G$  has no isolated vertices. Let  $\mathbf{x} \neq \mathbf{0}$ . Then  $x_u \neq 0$  for some vertex  $u$ . There exists  $v \in V(G)$  such that  $uv \in E(G)$ . Since  $\alpha > \frac{1}{2}$ , from (11), it follows that  $\mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} > 0$  and thus  $A_\alpha(G_\sigma)$  is positive definite.  $\square$

Let  $G$  be a connected graph. Let  $f(\alpha) = \lambda_n(A_\alpha(G_\sigma)) = \lambda_{\min}(A_\alpha(G_\sigma))$ . Then  $f(0) = \lambda_{\min}(A(G_\sigma)) < 0$  and  $f(1) = \lambda_{\min}(D(G)) > 0$ . From Theorem 7,  $f$  is a strictly increasing function in  $[0, 1]$ . In addition,  $f$  is a continuous function. Hence there exists a smallest  $\alpha \in (0, 1)$  such that  $f(\alpha) = 0$ . Denote this value by  $\alpha_0(G_\sigma)$ . If  $G$  is a disconnected graph, we define

$$\alpha_0(G_\sigma) = \max\{\alpha_0(H_\zeta) : H \text{ is a component of } G\},$$

where  $\zeta$  is the restriction of  $\sigma$  to  $H$ .

Then  $A_\alpha(G_\sigma)$  is positive semidefinite if and only if  $1 \geq \alpha \geq \alpha_0(G_\sigma)$ . Moreover,  $\alpha_0(G_\sigma) \leq \frac{1}{2}$ .

**Remark 1.** *Since  $\alpha_0$  of any signed component of a signed graph  $G_\sigma$  does not exceed the value  $\frac{1}{2}$ , if there exists a component  $H_\zeta$  such that  $\alpha_0(H_\zeta) = \frac{1}{2}$ , then  $\alpha_0(G_\sigma) = \frac{1}{2}$ .*

The problem 8 posed by Nikiforov in [9] can be extended to the context of signed graphs as follows:

**Problem 1.** *Given a signed graph  $G_\sigma$ , find  $\alpha_0(G_\sigma)$ .*

We present below some progress regarding Problem 1.

**Theorem 10.** *If  $G$  is a  $d$ -regular graph of order  $n$ , then*

$$\alpha_0(G_\sigma) = \frac{-\lambda_{\min}(A(G_\sigma))}{d - \lambda_{\min}(A(G_\sigma))}. \tag{12}$$

**Proof.** Since  $G$  is  $d$ -regular of order  $n$ ,

$$A_\alpha(G_\sigma) = \alpha d I_n + (1 - \alpha) A(G_\sigma),$$

where  $I_n$  is the identity matrix of order  $n$ . Hence

$$\lambda_{\min}(A_\alpha(G_\sigma)) = \alpha d + (1 - \alpha) \lambda_{\min}(A(G_\sigma)).$$

The right side of this identity strictly increases with  $\alpha$ ; thus  $\alpha_0(G_\sigma)$  is the unique solution of the equation

$$\alpha d + (1 - \alpha) \lambda_{\min}(A(G_\sigma)) = 0,$$

which gives (12).  $\square$

We now recall a necessary and sufficient condition for an unsigned graph  $G$  to have  $\alpha_0(G) = \frac{1}{2}$ .

**Lemma 1** ([15], Corollary 7). *If  $G$  is a nonempty graph, then  $\alpha_0(G) = \frac{1}{2}$  if and only if  $G$  has a bipartite component.*

The following result gives us a necessary and sufficient condition for a signed graph  $G_\sigma$  to be bipartite.

**Lemma 2** ([29], Corollary 1.1). *Let  $G_\sigma$  be a signed graph. The signed graph  $G_{-\sigma}$  is balanced if and only if  $G_\sigma$  is bipartite.*

From now on,  $H_\zeta = (H, \zeta)$  is used to denote a signed subgraph of  $G_\sigma = (G, \sigma)$  where  $\zeta$  is the restriction of  $\sigma$  to the edge set of the graph  $H$  denoted by  $E(H)$ , that is,  $\zeta(e) = \sigma(e)$  for all  $e \in E(H)$ .

**Theorem 11.** *Let  $G_\sigma = (G, \sigma)$  be a signed graph. Then,*

- (i) *If  $G_\sigma$  contains a balanced bipartite component then  $\alpha_0(G_\sigma) = \frac{1}{2}$ .*
- (ii) *If  $G_\sigma$  contains a component  $H_\zeta$  such that  $H_{-\zeta}$  is balanced then  $\alpha_0(G_\sigma) = \frac{1}{2}$ .*
- (iii) *If  $\sigma = \sigma^-$ , then  $\alpha_0(G^-) = \frac{1}{2}$ .*

**Proof.**

- (i) Suppose that  $G_\sigma$  contains a balanced bipartite component  $H_\zeta$ . From item (ii) of Theorem 3;  $A_\alpha(H_\zeta)$  and  $A_\alpha(H)$  have the same spectrum. Then,  $\alpha_0(H_\zeta) = \alpha_0(H)$ . Since  $H$  is bipartite, by Lemma 1,  $\alpha_0(H) = \alpha_0(H_\zeta) = \frac{1}{2}$ . Hence  $\alpha_0(G_\sigma) = \frac{1}{2}$ .
- (ii) Suppose that  $G_\sigma$  contains a component  $H_\zeta$  such that  $H_{-\zeta}$  is balanced. By Lemma 2,  $H_\zeta$  is a bipartite component of  $G_\sigma$ . Hence  $\alpha_0(G_\sigma) = \frac{1}{2}$ .
- (iii) It is an immediate consequence of item (ii).

This completes the proof.  $\square$

**Remark 2.** *The converse of Theorem 11 (i) is not true. A counterexample is the signed graph  $G_\sigma$  depicted in Figure 2 in which the continuous lines are the positive edges and the dashed ones are the negative edges. This graph does not contain a balanced bipartite component. However, if  $H_\zeta$  is the component at the right,  $H_{-\zeta}$  is balanced, and then, from Theorem 11 (ii),  $\alpha_0(G_\sigma) = \frac{1}{2}$ .*

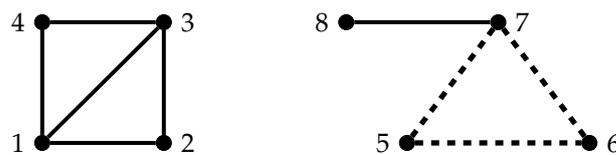


Figure 2.  $\alpha_0(G_\sigma) = \frac{1}{2}$ .

Let  $d_i(G) = d_i$  be the degree of the  $i$ -th vertex of  $G$ . Let  $\delta$  and  $\Delta$  be the minimum and maximum degree of  $G$ , respectively.

Applying Theorem 7, the results of Proposition 10 in [9] for unsigned graphs can be easily extended to signed graphs.

**Theorem 12.** Let  $G_\sigma = (G, \sigma)$  where  $G$  is a graph of order  $n$  with vertex degrees  $\Delta \geq d_2 \geq \dots \geq d_{n-1} \geq \delta$ . Let  $0 \leq \alpha \leq 1$ . If  $1 \leq j \leq n$  then

$$\lambda_j(A_\alpha(G_\sigma)) \leq d_j.$$

In particular,

$$\lambda_n(A_\alpha(G_\sigma)) \leq \delta \tag{13}$$

and

$$\lambda_1(A_\alpha(G_\sigma)) \leq \Delta. \tag{14}$$

We observe that  $\Delta$  is an upper bound on  $\lambda_1(A_\alpha(G_\sigma))$  for any  $\alpha \in [0, 1]$  and any sign function on  $E(G)$ .

The upper bound (13) on  $\lambda_n(A_\alpha(G_\sigma))$  can be improved as is shown in the next result.

**Theorem 13.** Let  $G_\sigma = (G, \sigma)$  where  $G$  is a graph of order  $n$ . Let  $0 \leq \alpha \leq 1$ . Then:

(i) For  $1 \leq j \leq n$ ,

$$\alpha\delta + (1 - \alpha)\lambda_j(A(G_\sigma)) \leq \lambda_j(A_\alpha(G_\sigma)) \leq \alpha\Delta + (1 - \alpha)\lambda_j(A(G_\sigma)).$$

(ii)

$$\lambda_n(A_\alpha(G_\sigma)) \leq \alpha\delta.$$

**Proof.** The proof of the result (i) is immediate from the Weyl’s inequalities. Let  $v$  be a vertex with a minimum degree  $\delta$ . Let  $\mathbf{y}$  be the  $n$ -vector whose entries are zeroes except the entry  $y_v = 1$ . Then

$$\lambda_n(A_\alpha(G_\sigma)) = \min_{\|\mathbf{x}\|_2=1} \mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} \leq \mathbf{y}^T A_\alpha(G_\sigma) \mathbf{y} = \alpha\delta,$$

which completes the proof of (ii).  $\square$

### 5. Bounds on the Largest Eigenvalue of $A_\alpha(G_\sigma)$

We remember that  $\lambda_1(M)$  is denoting the largest eigenvalue of a real symmetric matrix  $M$ . Here we present some bounds on the largest eigenvalue of  $A_\alpha(G_\sigma)$ . In particular, when  $\alpha = 0$ , we have the index of a signed graph, and some recent results on this index are given in [4–6] and [30].

**Theorem 14.** Let  $G_\sigma = (G, \sigma)$ , where  $G$  is a graph on  $n$  vertices. Let  $0 \leq \alpha < 1$ . Then

$$\lambda_1(A_\alpha(G_\sigma)) \leq \lambda_1(A_\alpha(G)).$$

**Proof.** Let  $\mathbf{x} = [x_1, \dots, x_n]^T$  be a unit eigenvector to  $\lambda_1(A_\alpha(G_\sigma))$ . Let  $|\mathbf{x}| = [|x_1|, \dots, |x_n|]^T$ . Then

$$\lambda_1(A_\alpha(G_\sigma)) = \mathbf{x}^T A_\alpha(G_\sigma) \mathbf{x} \leq |\mathbf{x}|^T |A_\alpha(G_\sigma)| |\mathbf{x}| = |\mathbf{x}|^T A_\alpha(G) |\mathbf{x}|$$

$$\leq \max_{\|z\|_2=1} z^T A_\alpha(G)z = \lambda_1(A_\alpha(G)).$$

This completes the proof.  $\square$

At this moment, we recall the following result.

**Lemma 3** ([7], Lemma 2.1). *For a connected signed graph  $G_\sigma$ , if  $\lambda_1(A(G_\sigma)) = \lambda_1(A(G))$ , then  $G_\sigma$  and  $G$  are switching equivalent.*

**Theorem 15.** *If  $0 \leq \alpha < 1$  and if  $G_\sigma = (G, \sigma)$ , where  $G$  is a graph on  $n$  vertices,  $m$  edges and maximum degree  $\Delta$ , then:*

(i) 
$$\lambda_1(A_\alpha(G_\sigma)) \leq \alpha\Delta + (1 - \alpha)\lambda_1(A(G)). \tag{15}$$

*The equality holds in (15) if and only if  $G_\sigma$  contains a  $\Delta$ -regular balanced component.*

(ii) *If, in addition,  $G$  is connected, then*

$$\lambda_1(A_\alpha(G_\sigma)) \leq \alpha\Delta + (1 - \alpha)\sqrt{2m - n + 1}. \tag{16}$$

*The equality holds in (16) if and only if  $G_\sigma = (K_n, \sigma)$  is balanced.*

(iii) 
$$\lambda_1(A_\alpha(G_\sigma)) < \alpha\Delta + \frac{1}{2}(1 - \alpha)(\sqrt{1 + 8m} - 1). \tag{17}$$

**Proof.**

(i) By definition  $A_\alpha(G_\sigma) = \alpha D(G) + (1 - \alpha)A(G_\sigma)$ . Using the Weyl’s inequality (1),  $\lambda_1(D(G)) = \Delta$  and Theorem 14, we get

$$\begin{aligned} \lambda_1(A_\alpha(G_\sigma)) &\leq \alpha\lambda_1(D(G)) + (1 - \alpha)\lambda_1(A(G_\sigma)) \\ &\leq \alpha\Delta + (1 - \alpha)\lambda_1(A(G)). \end{aligned}$$

Thus, the inequality (15) is obtained. Let  $H_\zeta$  be a component of  $G_\sigma$  such that

$$\lambda_1(A_\alpha(G_\sigma)) = \lambda_1(A_\alpha(H_\zeta)). \tag{18}$$

Suppose that  $H_\zeta$  is  $\Delta$ -regular and balanced. Then

$$\lambda_1(A_\alpha(H_\zeta)) = \lambda_1(A_\alpha(H)) = \Delta$$

and

$$\Delta = \lambda_1(A(H_\zeta)) = \lambda_1(A(H)) \leq \lambda_1(A(G)) \leq \Delta.$$

Hence

$$\lambda_1(A_\alpha(G_\sigma)) = \Delta = \alpha\lambda_1(D(G)) + (1 - \alpha)\lambda_1(A(G)).$$

Conversely, suppose that the equality in (15) holds. That is,

$$\lambda_1(A_\alpha(G_\sigma)) = \alpha\Delta + (1 - \alpha)\lambda_1(A(G)). \tag{19}$$

We will show that  $H_\zeta$  is  $\Delta$ -regular and balanced. We have  $A_\alpha(H_\zeta) = \alpha D(H) + (1 - \alpha)A(H_\zeta)$ . Applying the Weyl’s inequality (1),  $\lambda_1(D(H)) \leq \Delta$  and Theorem 14, we get

$$\lambda_1(A_\alpha(H_\zeta)) \leq \alpha\lambda_1(D(H)) + (1 - \alpha)\lambda_1(A(H_\zeta)) \leq \alpha\Delta + (1 - \alpha)\lambda_1(A(H)). \tag{20}$$

From (18)–(20), we obtain  $\lambda_1(A(G)) \leq \lambda_1(A(H))$ . Hence  $\lambda_1(A(G)) = \lambda_1(A(H))$ . This result, together with (18) and (19) imply

$$\lambda_1(A_\alpha(H_\zeta)) = \alpha\Delta + (1 - \alpha)\lambda_1(A(H))$$

and  $\lambda_1(D(H)) = \Delta$ . Hence  $\lambda_1(A(H_\zeta)) = \lambda_1(A(H))$ . Since  $H_\zeta$  is a connected graph, from Lemma 3,  $H_\zeta$  and  $H$  are switching equivalent. Therefore,  $H_\zeta$  is a balanced signed graph. Moreover, from Theorem 1, since

$$\lambda_1(A_\alpha(H_\zeta)) = \alpha\lambda_1(D(H)) + (1 - \alpha)\lambda_1(A(H_\zeta)),$$

$\lambda_1(A_\alpha(H_\zeta))$ ,  $\lambda_1(D(H))$  and  $\lambda_1(A(H))$  have a common eigenvector  $\mathbf{x}$ . Since  $H$  is a connected graph,  $A(H)$  is a nonnegative irreducible matrix, and then, from the Perron–Frobenius Theory for nonnegative matrices, all the entries of  $\mathbf{x}$  are positive. This fact and  $D(H)\mathbf{x} = \Delta\mathbf{x}$  imply that the degrees of  $H$  are equal, and thus  $H_\zeta$  is a  $\Delta$ -regular graph.

(ii) We recall the Yuan’s bound [31]: for any connected graph,

$$\lambda_1(A(G)) \leq \sqrt{2m - n + 1},$$

where the equality occurs if and only if  $G$  is a complete or star graph. We use this bound in (15) to obtain (16). Now, we use the information for the equality in (15) to conclude that the equality in (16) holds if and only if  $G_\sigma = (K_n, \sigma)$  is balanced.

(iii) We now recall the Stanley’s bound [32]:

$$\lambda_1(A(G)) \leq \frac{1}{2} \left( \sqrt{1 + 8m} - 1 \right),$$

where the equality occurs if and only if  $m = \binom{k}{2}$  and  $G$  is a disjoint union of the complete graph  $K_k$  and isolated vertices. We use this bound in (15) to obtain (17). Since the condition for the equality in the Stanley’s bound is given by a graph that is not a regular graph, the inequality in (17) is strict.

This completes the proof.  $\square$

**Remark 3.** Concerning upper bounds (16) and (17), for  $0 \leq \alpha < 1$ :

- if  $\sqrt{2m - n + 1} < \Delta$ , then (16) improves the upper bound on  $\lambda_1(A_\alpha(G_\sigma))$  obtained from (14), and
- if  $\sqrt{1 + 8m} < 2\Delta + 1$ , then (17) improves the upper bound on  $\lambda_1(A_\alpha(G_\sigma))$  obtained from (14).

The frustration index of an unbalanced signed graph  $G_\sigma$ , denoted by  $l(G_\sigma)$ , is the minimum number of edges in  $E(G_\sigma)$  that need to be removed in order to obtain a balanced signed graph, say  $H_\zeta$ . Thus, every cycle in  $H_\zeta$  is positive. If  $G_\sigma$  is unbalanced,  $l(G_\sigma) \geq 1$ .

We recall some upper bounds on  $\lambda_1(A(G_\sigma))$ .

**Theorem 16** ([5], Theorem 3.2). For an unbalanced signed graph  $G_\sigma$  with  $n$  vertices,  $m$  edges and frustration index  $l(G_\sigma)$ ,

$$\lambda_1(A(G_\sigma)) \leq \frac{1}{2} \left( \sqrt{1 + 8(m - l(G_\sigma))} \right) - 1.$$

If, in addition,  $G$  is connected, then

$$\lambda_1(A(G_\sigma)) \leq \sqrt{2(m - l(G_\sigma)) - n + 1}.$$

We know that

$$\lambda_1(A_\alpha(G_\sigma)) \leq \alpha\Delta + (1 - \alpha)\lambda_1(A(G_\sigma))$$

where  $\Delta$  is the maximum degree of  $G$ .

The upper bounds (16) and (17) in Theorem 15 can be improved by applying the upper bounds on  $\lambda_1(A(G_\sigma))$  given in Theorem 16.

**Theorem 17.** For an unbalanced signed graph  $G_\sigma$  with  $n$  vertices,  $m$  edges, frustration index  $l(G_\sigma)$ , and maximum degree  $\Delta$ ,

(i)

$$\lambda_1(A_\alpha(G_\sigma)) \leq \alpha\Delta + \frac{1}{2}(1 - \alpha)(\sqrt{1 + 8(m - l(G_\sigma))} - 1).$$

(ii) If, in addition,  $G$  is connected, then

$$\lambda_1(A_\alpha(G_\sigma)) < \alpha\Delta + (1 - \alpha)\sqrt{2(m - l(G_\sigma)) - n + 1}.$$

We now recall bounds for  $\alpha$ -index of  $G$  in terms of  $\alpha$  and the vertex degrees of  $G$ .

**Theorem 18** ([9], Theorem 20). If  $G$  is a graph with no isolated vertices, then

$$\lambda_1(A_\alpha(G)) \leq \max_{u \in V(G)} \left\{ \alpha d(u) + \frac{1 - \alpha}{d(u)} \sum_{uv \in E(G)} d(v) \right\}. \tag{21}$$

and

$$\lambda_1(A_\alpha(G)) \geq \min_{u \in V(G)} \left\{ \alpha d(u) + \frac{1 - \alpha}{d(u)} \sum_{uv \in E(G)} d(v) \right\}. \tag{22}$$

If  $\frac{1}{2} < \alpha < 1$  and  $G$  is connected, equality in (21) and (22) holds if and only if  $G$  is regular.

**Theorem 19.** Let  $G_\sigma$  be a connected unbalanced signed graph of order  $n$ . Let  $H_\zeta$  be the graph obtained from  $G_\sigma$  by removing  $l(G_\sigma)$  edges such that  $H_\zeta$  is balanced. If  $\frac{1}{2} < \alpha < 1$  then

$$\lambda_1(A_\alpha(G_\sigma)) > \lambda_1(A_\alpha(H_\zeta)) = \lambda_1(A_\alpha(H)). \tag{23}$$

**Proof.** We have  $l(G_\sigma) \geq 1$  and  $H_\zeta$  is a balanced spanning subgraph of  $G_\sigma$ . Suppose that  $l(G_\sigma) = 1$ . Then, without loss of generality, we may assume that  $G_\sigma$  has only one negative edge. We apply Theorem 5 to get (23). Suppose now that  $l(G_\sigma) > 1$ . Let  $G^1 = G_\sigma$ . Let  $G^t$  be the signed graph obtained from  $G^{t-1}$  by removing one negative edge,  $2 \leq t \leq l(G_\sigma)$ . Repeated applications of Theorem 4 yield

$$\lambda_1(A_\alpha(G_\sigma)) \geq \lambda_1(A_\alpha(G^2)) \geq \dots \geq \lambda_1(A_\alpha(G^{l(G_\sigma)})).$$

Observe that each graph  $G^t$  is connected and unbalanced. Moreover,  $G^{l(G_\sigma)}$  has only one negative edge. Removing this negative edge from  $G^{l(G_\sigma)}$  results in the balanced graph  $H_\zeta$ . Applying Theorem 5, we obtain

$$\lambda_1(A_\alpha(G^{l(G_\sigma)})) > \lambda_1(A_\alpha(H_\zeta)).$$

Thus, the strict inequality (23) is obtained.  $\square$

Applying the lower bound (22) to  $\lambda_1(A_\alpha(H))$  in Theorem 19, we obtain

**Theorem 20.** Let  $G_\sigma$  be a connected unbalanced signed graph. Let  $\frac{1}{2} \leq \alpha < 1$ . If  $H$  is as in Theorem 19, then

$$\lambda_1(A_\alpha(G_\sigma)) > \min_{u \in V(H)} \left\{ \alpha d(u) + \frac{1 - \alpha}{d(u)} \sum_{uv \in E(H)} d(v) \right\}.$$

The following result gives a lower bound on  $\lambda_1(A_\alpha(G))$  in terms of  $\alpha$  and the maximum degree of  $G$ .

**Theorem 21** ([9], Theorem 12). *If  $G$  is a graph with maximum degree  $\Delta$ , then*

$$\lambda_1(A_\alpha(G)) \geq \frac{1}{2} \left( \alpha(\Delta + 1) + \sqrt{\alpha^2(\Delta + 1)^2 + 4\Delta(1 - 2\alpha)} \right). \quad (24)$$

*If  $0 \leq \alpha \leq 1$  and  $G$  is connected, equality holds if and only if  $G = K_{1,\Delta}$ .*

Applying the lower bound (24) to  $\lambda_1(A_\alpha(H))$  in Theorem 19, we get

**Theorem 22.** *Let  $G_\sigma$  be a connected unbalanced signed graph. Let  $\frac{1}{2} < \alpha < 1$ . If  $H$  in Theorem 19 is a graph with maximum degree  $\Delta(H)$ , then*

$$\lambda_1(A_\alpha(G_\sigma)) > \frac{1}{2} \left( \alpha(\Delta(H) + 1) + \sqrt{\alpha^2(\Delta(H) + 1)^2 + 4\Delta(H)(1 - 2\alpha)} \right).$$

## 6. Conclusions

Since the spectrum of  $A_\alpha(G_\sigma)$  varies continuously from the spectrum of  $A(G_\sigma)$  to the multiset of vertex degrees of  $G$  with the spectrum of  $\frac{1}{2}L(G_{-\sigma})$  at the middle of the range as  $\alpha$  runs from 0 to 1, the study of  $A_\alpha(G_\sigma) = \alpha D(G) + (1 - \alpha)A(G_\sigma)$  allows us to extend results on  $A(G_\sigma)$  and  $L(G_{-\sigma})$  to  $A_\alpha(G_\sigma)$  to all  $\alpha$  in some subintervals of  $[0, 1]$ .

In this work, we obtained basic properties of  $A_\alpha(G_\sigma)$  together with results on its positive semidefiniteness and the derivation of some bounds on its eigenvalues. In particular, different lower and upper bounds on its largest eigenvalue (depending on whether the signed graph is balanced or unbalanced) have been derived in terms of parameters such as frustration index, maximum degree, number of edges, and number of vertices.

Since  $A_\alpha(G_\sigma)$  has been recently introduced, we believe that our results will be useful in future research such as extremal problems, new bounds on the eigenvalues of  $A_\alpha(G_\sigma)$ , new results about the positive semidefiniteness of  $A_\alpha(G_\sigma)$ .

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