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Kadomtsev–Petviashvili Hierarchy: Negative Times

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Abstract: The Kadomtsev–Petviashvili equation is known to be the leading term of a semi-infinite hierarchy of integrable equations with evolutions given by times with positive numbers. Here, we introduce new hierarchy directed to negative numbers of times. The derivation of such systems, as well as the corresponding hierarchy, is based on the commutator identities. This approach enables introduction of linear differential equations that admit lifts up to nonlinear integrable ones by means of the special dressing procedure. Thus, one can construct not only nonlinear equations, but corresponding Lax pairs as well. The Lax operator of this evolution coincides with the Lax operator of the "positive" hierarchy. We also derive (1 + 1)-dimensional reductions of equations of this hierarchy.

Keywords: commutator identities; integrable hierarchies; reductions

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1. Introduction

The main examples of (2 + 1) dimensional integrable hierarchies appear due to Zakharov–Shabat systems [1], or the approach of Miwa–Jimbo–Date [2], as semi-infinite sets of equations with a common Lax operator. These sequences start with the lowest (first) equations, and then the numbers of times grow together with the order of the second Lax operators. Thus, all the times associated to such a hierarchy can be called positive, as consequent numbers of these times are positive. Here, by means of the Kadomtsev–Petviashvili (KP) equation [3], we derive new kinds of integrable hierarchies that can be associated to negative numbers of times. This approach was suggested in [4], where example of the Davey–Stewartson (DS) hierarchy [5] was considered. Construction of such hierarchies gives an essential extension of the set of integrable equations because the approach of [1,2] is not fully applicable here.

In [6], we suggested a method for derivation of (2 + 1)-dimensional nonlinear integrable equations based on commutator identities on associative algebras. In [7], this method was extended to the standard hierarchies of integrable equations. Here, we apply it to hierarchies of negative numbers of times. Taking the algebraic similarity of operator commutators and time derivatives into account, we associate commutator identities and linear partial differential equations. Let A and B denote arbitrary elements of an arbitrary associative algebra $\mathcal{A}$. Then it is easy to check that we have the following commutator identity [6]:

$$4[A^3,[A,B]]-3[A^2,[A^2,B]]-[A,[A,[A,[A,B]]]]=0. \tag{1}$$

Let element B now depend on three times:

$$B_{t_m}=[A^m,B], \tag{2}$$

where $m=1,2,3$. Identity (1) readily proves that with respect to variables t_1, t_2, and t_3, this function obeys the linear equation

$$4\frac{\partial^2 B(t)}{\partial t_1\partial t_3}-3\frac{\partial^2 B(t)}{\partial t_2^2}-\frac{\partial^4 B(t)}{\partial t_1^4}=0, \tag{3}$$

which is the linearized version of the Kadomtsev–Petviashvili (KP) equation [3],

$$4\frac{\partial^2 u(t)}{\partial t_1 \partial t_3} - 6u(t)u_{t_1}(t) - \frac{\partial^4 u(t)}{\partial t_1^4} = 3\frac{\partial^2 u(t)}{\partial t_2^2}. \tag{4}$$

In order to present the expression for higher linearized equations of hierarchy, we introduce adjoint actions

$$\mathrm{ad}_m B = [A^m, B], \tag{5}$$

so that (1) takes the form

$$4\mathrm{ad}_3\mathrm{ad}_1 - 3\mathrm{ad}_2^2 - \mathrm{ad}_1^4 = 0. \tag{6}$$

It is easy to check that we have the following commutator identities [7],

$$2^m \mathrm{ad}_m \mathrm{ad}_1^m = \left(\mathrm{ad}_2 + \mathrm{ad}_1^2\right)^m - \left(\mathrm{ad}_2 - \mathrm{ad}_1^2\right)^m, \tag{7}$$

where $m \geq 1$. We see that these identities can be formulated as expressions of higher adjoint operations in terms of the lowest ones: ad_1 and ad_2. Thus, they can be understood as relations on the commutative algebra of adjoint actions. Thanks to (2), relation (7) reduces to the linear difference equation

$$2^m \partial_{t_m} \partial_{t_1}^m B = \left(\partial_{t_2} + \partial_{t_1}^2\right)^m B - \left(\partial_{t_2} - \partial_{t_1}^2\right)^m B, \tag{8}$$

that is, to higher equations of the linearised KP hierarchy.

The characteristic property of these linear equations is the possibility of dressing them to nonlinear integrable ones. This was proved for many different equations, including differential-difference and non-Abelian ones. In what follows, we use the special dressing procedure [6], to demonstrate that any linear equation that results from the commutator identity can be lifted up to a nonlinear integrable one. Here, in analogy to the [4], our aim is to extend the class of commutator identities and corresponding linear differential equations to negative values of m in (7). We again consider ad_1 and ad_2 as generating and start with derivations of ad_{-1} in their terms. Thus, we assume that the associative algebra $\mathcal{A}$ contains unity and that element A is invertible, so that

$$\mathrm{ad}_{-1} B = [A^{-1}, B]. \tag{9}$$

Taking associativity into account, it is easy to check directly that we have the commutator identity

$$\mathrm{ad}_{-1}\left(\mathrm{ad}_2^2 - \mathrm{ad}_1^4\right) + 4\mathrm{ad}_1^3 = 0. \tag{10}$$

In analogy, we derive higher versions of this equation, that is,

$$\mathrm{ad}_{-m}\left(\mathrm{ad}_2^2 - \mathrm{ad}_1^4\right)^m + 2^m \mathrm{ad}_1^m \left[(\mathrm{ad}_2 + \mathrm{ad}_1^2)^m - (\mathrm{ad}_2 - \mathrm{ad}_1^2)^m\right] = 0, \tag{11}$$

that gives (10) in the case of $m = 1$. Now, thanks to (2), we get by (10) a linear equation for element B:

$$\partial_{t_{-1}}\left(\partial_{t_2}^2 - \partial_{t_1}^4\right)B + 4\partial_{t_1}^3 B = 0, \tag{12}$$

and its "higher" analogs by (11) for $m \geq 1$

$$\partial_{t_{-m}}\left(\partial_{t_2}^2 - \partial_{t_1}^4\right)^m B + 2^n \partial_{t_1}^m \left[(\partial_{t_2} + \partial_{t_1}^2)^m - (\partial_{t_2} - \partial_{t_1}^2)^m\right] B = 0. \tag{13}$$

Next, we consider a dressing procedure that lifts up these equations to integrable nonlinear ones.

2. Operator Realisation of the Elements of Associative Algebra

In order to develop a dressing procedure, we need to introduce a special realization of elements of an associative algebra $\mathcal{A}$, see [6]. Our construction here is close to the standard definition of the pseudo-differential operators. Let us denote the symbol of $F \in \mathcal{A}$ as $\widetilde{F}(t,z)$. Here, t denotes a finite subset of real variables $t = \{\dots, t_{-2}, t_{-1}, t_1, t_2, \dots\}$, that is, times, and $z \in \mathbb{C}$ denotes a complex parameter. Subset t includes variables t_1, t_2, and at least one of the other variables of this list. On this set of symbols, we define the symbol of composition of two elements of the algebra:

$$\widetilde{FG}(t,z) = \frac{1}{2\pi}\int dp \int dy\, \widetilde{F}(t, z+ip)e^{ip(t_1-y)}\widetilde{G}(y,t',z), \tag{14}$$

where t' denotes subset t without variable t_1. We see that variable t_1 plays here a special role: the composition with respect to other variables is pointwise. In what follows, we consider elements of algebra $\mathcal{A}$, such that their symbols belong to the space of tempered distributions of their arguments. The symbol of the unity operator is equal to 1, and we choose the symbol of operator A as

$$\widetilde{A}(t,z) = z. \tag{15}$$

Thanks to (14), we have that for any F,

$$\widetilde{A^m F}(t,z) = (z+\partial_{t_1})^m \widetilde{F}(t,z), \qquad \widetilde{FA^m}(t,z) = z^m\widetilde{F}(t,z), \tag{16}$$

where A^m is understood as the m-th power in the sense of composition (14), where now $m \in \mathbb{Z}$. Then for $m = 1$, we get $[A,F] = \partial_{t_1}F$ in correspondence to (2), and then for any $m \in \mathbb{Z}$, we have in terms of symbols,

$$\widetilde{B}_{t_m}(t,z) = \big((z+\partial_{t_1})^m - z^m\big)\widetilde{B}(t,z). \tag{17}$$

Because of our assumption, the symbol $\widetilde{B}(t,z)$ admits Fourier transform with respect to the variable t_1, so the above relations show

$$\widetilde{B}(t,z) = \int dp\, \exp\left(\sum_m \big((z+ip)^m - z^m\big)t_m\right) f(p,z), \tag{18}$$

where $m \in \mathbb{Z}$ and $f(p,z)$ are arbitrary functions independent of all t_m. Let us mention that here, we do not specify the set of "times" t_i involved in (18). Set t can include more times than three, but t_1, t_2, and every third time gives the evolution equation, generated by the commutator identity. In (18), the summation in the exponent goes over a finite number of terms, corresponding to times that are "switched on", while other times are equated to zero.

It is natural to impose on $\widetilde{B}(t,z)$ in (18) conditions of convergence of the integral and boundedness of the limits of $\widetilde{B}(t,z)$ when t tends to infinity. We list two obvious conditions that are enough for this. The first one is given by the choice $f(p,z) = \delta(p + 2z_{\mathrm{Im}})g(z)$, so that (18) takes the form

$$\widetilde{B}(t,z) = \exp\left(\sum_m (\overline{z}^m - z^m)t_m\right) g(z), \tag{19}$$

where $g(z)$ is an arbitrary bounded function of its argument. The second case is given by reduction $f(p,z) = \delta(z_{\mathrm{Re}})h(p, z_{\mathrm{Im}})$, where $h(p, z_{\mathrm{Im}})$ is an arbitrary function. Then, (18) takes the form

$$\widetilde{B}(t,z) = \int dp\, \exp\left(\sum_m i^m\big((z_{\mathrm{Im}}+p)^m - z_{\mathrm{Im}}^m\big)t_m\right) h(p, z_{\mathrm{Im}})\delta(z_{\mathrm{Re}}). \tag{20}$$

Finally, in this case, in order to make $\widetilde{B}(t,z)$ bounded, we have to make the substitution

$$t_{2m} \to it_{2m}. \tag{21}$$

Below, we show that the choice of (19), or (20), results in two kinds of dynamical systems.

3. Dressing Procedure

The specific property of the above set of operators is the possibility to define the operation of $\bar{\partial}$-differentiation with respect to the complex variable z, $F \to \bar{\partial}F$. In terms of symbols, it is defined (see [6]) as

$$(\widetilde{\bar{\partial}F})(t,z) = \frac{\partial \widetilde{F}(t,z)}{\partial \bar{z}}, \tag{22}$$

where the derivative is understood in the sense of distributions. Thanks to (15), we get equality

$$\bar{\partial}A = 0, \tag{23}$$

that plays an essential role in what follows.

In terms of these definitions, we introduce (see [6]) the dressing operator K with symbol $\widetilde{K}(t,z)$ by means of the $\bar{\partial}$-problem

$$\bar{\partial}K = KB, \tag{24}$$

where the product on the r.h.s. is understood in the sense of composition law (14). We normalize solution K of the Equation (24) by the asymptotic condition

$$\widetilde{K}(t,x,z) \to 1, \quad z \to \infty. \tag{25}$$

Thanks to (14) and (22), the equality (24) takes the explicit form

$$\frac{\partial \widetilde{K}(t,z)}{\partial \bar{z}} = \widetilde{K}(t,\bar{z}) \exp\left(\sum_m (\bar{z}^m - z^m)t_m\right) g(z), \tag{26}$$

for time evolutions given by (19) and the form

$$\frac{\partial \widetilde{K}(t,z)}{\partial \bar{z}} = \delta(z_{\mathrm{Re}}) \int dp\, \widetilde{K}(t,ip) \exp\left(\sum_m i^m (p^m - z^m_{\mathrm{Im}})t_m\right) h'(p,z_{\mathrm{Im}}), \tag{27}$$

$h'(p,z_{\mathrm{Im}}) = h(p - z_{\mathrm{Im}}, z_{\mathrm{Im}})$, for time evolutions given by (20). Thus, in the case (26), Equation (24) gives the $\bar{\partial}$-problem, while in the case (27), we get the Riemann–Hilbert problem.

An essential assumption for the following construction is the condition of unique solvability of the problem (24), (25). The time evolution of the dressing operator follows from this assumption. Say, due to (2), we get

$$\bar{\partial}K_{t_m} = K_{t_m}B + K[A^m, B]. \tag{28}$$

Correspondingly,

$$\bar{\partial}K_{t_m t_n} = K_{t_m t_n}B + K_{t_n}[A^m, B] + K_{t_m}[A^n, B] + K[A^m, [A^n, B]],$$

so that taking the commutativity of A^m and A^n into account, we get by (24), $\bar{\partial}(K_{t_m t_n} - K_{t_n t_m}) = (K_{t_m t_n} - K_{t_n t_m})B$. Thus, the commutativity of derivatives

$$K_{t_m t_n} = K_{t_n t_m} \tag{29}$$

follows thanks to the unique solvability of the problem (24), (25).

In [7], time derivatives of the dressing operator for positive times ($m > 0$ in (2)) were calculated in terms of the asymptotic decomposition of the dressing operator K

$$\widetilde{K}(t,z) = 1 + u(t)z^{-1} + v(t)z^{-2} + w(t)z^{-3} + o(z^{-3}), \tag{30}$$

where u, v, and w are multiplication operators, that is, their symbols are independent of z. Say, by means of (28) for $m = 1$, we get $\bar{\partial}K_{t_1} = K_{t_1}B + K[A,B]$. This can be written in the form $\bar{\partial}(K_{t_1} + KA) = (K_{t_1} + KA)B$, where (23) and (24) were used. Due to the condition of unique solvability of (24), we derive by (25) that there exists such a multiplication operator X that $K_{t_1} + KA = (A + X)K$. Thanks to (30), it is easy to see that it equals to zero, so we have

$$K_{t_1} = [A,K], \tag{31}$$

in correspondence to (2) for $m = 1$. However, the situation with K_{t_2} is more involved. By (24), we derive $\bar{\partial}K_{t_2} = K_{t_2}B + K[A^2,B]$ that, thanks to (24), gives $\bar{\partial}\big(K_{t_2} + KA^2\big) = \big(K_{t_2} + KA^2\big)B$, so that by (30), we get

$$K_{t_2} + KA^2 = A^2K - 2u_{t_1}K. \tag{32}$$

Our aim here is to follow the approach of [4], and chose t_{-1} as the third time starting with times t_1 and t_2. Thus, we consider time evolutions given by (2)

$$B_{t_1} = [A,B], \qquad B_{t_2} = [A^2,B], \qquad B_{t_{-1}} = [A^{-1},B]. \tag{33}$$

The derivative with respect to t_{-1} of the dressing operator is given by (24):

$$\bar{\partial}K_{t_{-1}} = K_{t_{-1}}B + K[A^{-1},B], \tag{34}$$

so that $\bar{\partial}K_{t_{-1}} = K_{t_{-1}}B + KA^{-1}B - KBA^{-1}$, that is, thanks to (23),

$$\bar{\partial}(K_{t_{-1}}A + K) = (K_{t_{-1}}A + K)A^{-1}BA. \tag{35}$$

The situation here is more involved than in the case of positive numbers of times. In that case, we were able to reduce equations to the form $\bar{\partial}(K_{t_m} + KA^m) = (K_{t_m} + KA^m)B$, $m > 0$, due to (23). However, for negative m, this equality gives an additional delta-term. Thus, in order to use relation (35), we apply the substitution for $A^{-1}BA$ suggested in [4].

We consider symbols of operator B, K, and so forth, depending on the discrete variable $n \in \mathbb{Z}$ besides variables t and z:

$$B^{(1)} = ABA^{-1}, \qquad B^{(-1)} = A^{-1}BA, \tag{36}$$

where we denote $\widetilde{B}^{(\pm1)}(t,n,z) = \widetilde{B}(t,n\pm1,z)$, $\widetilde{K}^{(\pm1)}(t,n,z) = \widetilde{K}(t,n\pm1,z)$. It is easy to see that these shifts commute with times t: $(B^{(1)})_{t_j} = (B_{t_j})^{(1)}$, and so forth, and we extend the definition of composition law (14) pointwise to symbols depending on n. Now because of (24) $\bar{\partial}K^{(1)} = K^{(1)}ABA^{-1}$ so that due to the unique solvability of the problem (24), (25) there exists multiplication operator ψ such that

$$K^{(1)}A = (A + \psi)K, \tag{37}$$

and thanks to (30), we derive

$$\psi = u^{(1)} - u, \tag{38}$$

where $u^{(1)}(t,n) = u(t,n+1)$. Let us perform the shift $n \to n+1$ of (35) that due to (36) gives $\bar{\partial}(K^{(1)}_{t_{-1}}A + K^{(1)}) = (K^{(1)}_{t_{-1}}A + K^{(1)})B$, so that thanks to (25) there exists the multiplication operator Z such that $K^{(1)}_{t_{-1}}A + K^{(1)} = ZK$. Due to (30), we derive that $Z = 1 + u^{(1)}_{t_{-1}}$, that is,

$$K^{(1)}_{t_{-1}}A + K^{(1)} = 1 + u^{(1)}_{t_{-1}}K. \tag{39}$$

Thus, we constructed a $(3+1)$-dimensional integrable system with independent variables t_1, t_2, t_{-1}, and n. It is clear that this system is a combination of three integrable systems with variables t_1, t_2, n (see (37)), t_1, t_{-1}, n and t_1, t_2, t_{-1}. Set t_1, t_2, n gives no negative numbers of times. Set t_1, t_{-1}, n generates a two-dimensional Toda lattice, see [8–10]:

$$\frac{\partial^2 \phi_n}{\partial_{t_1} \partial_{t_{-1}}} = e^{\phi_{n+1}-\phi_n} - e^{\phi_n - \phi_{n-1}}, \qquad n \in \mathbb{Z}. \tag{40}$$

We see that the combination of time with negative numbers and discrete variables does not lead to problems. This is different to choices of t_1, t_2, t_{-1} as the set of independent variables: one cannot omit the dependence of K on n either in (37), or in (39). However, in this case, we can exclude the shift of K with respect to n. Indeed, substituting $K^{(1)}$ for K in (39) by means of (37) and using ψ in (38) as the new dependent variable, we get

$$K_{t_1 t_{-1}} + K_{t_1} A^{-1} + K_{t_{-1}} A + \psi (K_{t_{-1}} + K A^{-1}) - u_{t_{-1}} K = 0. \tag{41}$$

Compatible evolutions (41) admit higher (in fact, lower) versions that involve times t_{-m}, $m > 1$, see (2). In analogy to (2) we get, for this case,

$$\bar{\partial} K_{t_{-m}} = K_{t_{-m}} B + K[A^{-m}, B]. \tag{42}$$

Multiplying this equality by A^m from the right, we use a m-fold application of (36): $B^{[-m]} = A^{-m} B A^m$. Thus, (42) takes the form

$$\bar{\partial}(K_{t_{-m}} A^m + K) = (K_{t_{-m}} A^m + K) B^{[-m]},$$

cf. (35). Again, thanks to the assumed unique solvability of the Inverse problem (24), (25) we get that there exist such multiplication operators $\alpha_0, \ldots, \alpha_{m-1}$, that

$$K^{[m]}_{t_{-m}} A^m + K^{[m]} = \sum_{j=0}^{m-1} \alpha_j A^j K, \tag{43}$$

where we applied an m-fold operation of shift. Operators α_j are given in terms of operators u, v, and so forth in (30). We omit these calculations here.

Next, we perform a $(m-1)$-fold shift of discrete variables in Equation (37) that gives

$$K^{[m]} A^m = \left(A + \psi^{[m-1]}\right)\left(A + \psi^{[m-2]}\right) \cdots (A + \psi) K, \tag{44}$$

where the multiplication operator ψ was defined in (38). The final expression follows as a result of insertion of $K^{[m]}$ from (44) to (43), that again cancels dependence on the auxiliary variable n.

4. Lax Pair and Nonlinear Equations

Equation (29) proves that the commutativity of evolutions (31) and (37) is a direct consequence of commutativity of evolutions (2) and (36) and the consequence of unique solvability of the problem (24), (25). This results in nonlinear equations of motion. In order to simplify them, it is reasonable to use the Jost solutions defined by means of the symbol of the dressing operator:

$$\varphi(t,z) = \widetilde{K}(t,z) e^{z t_1 + z^2 t_2 + z^{-1} t_{-1}}. \tag{45}$$

We omit here the dependence on n, as it was excluded from (41).

Due to this substitution, coefficients of Equations (32) and (41) become independent on z:

$$\varphi_{t_2} = \varphi_{t_1 t_1} - 2 u_{t_1} \varphi, \tag{46}$$

$$\varphi_{t_1 t_{-1}} = -\psi \varphi_{t_{-1}} + (1 + u_{t_{-1}}) \varphi, \tag{47}$$

where the first equation is the famous heat conductivity equation.

One can also rewrite (24) in terms of the Jost solutions. Say, by means of (19), we get

$$\frac{\partial\varphi(t,z)}{\partial\bar{z}}=\varphi(t,\bar{z})g(z), \tag{48}$$

and by means of (20),

$$\frac{\partial\varphi(t,z)}{\partial\bar{z}}=\delta(z_{\mathrm{Re}})\int dp\varphi(t,ip)h'(p,z_{\mathrm{Im}}). \tag{49}$$

We see that equations on the Jost solutions are independent on all "time" variables, t. Dependence on them, as well as on z in (46) and (47), is given by (25), that thanks to (45), takes the form

$$\lim_{z\to\infty}\varphi(t,z)e^{-zt_1-z^2t_2-z^{-1}t_{-1}}=1. \tag{50}$$

Notice that (48) is the standard $\bar{\partial}$-problem with normalization condition (50), where we have to perform the substitution mentioned in (19). At the same time, (49) shows that the Jost solution in this case is analytic in the left and right half-planes of z with discontinuity on the imaginary axis. Thus, here, the inverse problem is given in terms of the Riemann–Hilbert problem; that is, we define boundary values of the Jost solution as $\varphi^{\pm}(t,iz_{\mathrm{Im}})=\lim_{z_{\mathrm{Re}}\to\pm 0}\varphi(t,x,z)$ and set

$$\varphi^{+}(t,iz_{\mathrm{Im}})-\varphi^{-}(t,iz_{\mathrm{Im}})=\int dp\varphi^{-}(t,ip)h'(p,z_{\mathrm{Im}}), \tag{51}$$

under condition (50) and the substitution given in (21). The difference between these two formulations of the inverse problem results from the condition of boundedness of the symbol of operator B in (19) and (20). In the case of (48), t_m are real, while in the case of (49), t_m with odd m are real and are pure imaginary for even m, see (21). In summary, we have here the two standard forms of the Lax operator: the heat conductivity equation and non-stationary Schroedinger equation.

Time evolution with respect to t_{-1} results from the compatibility of (46) and (47):

$$u_{t_2t_{-1}}+u_{t_1t_1t_{-1}}+2\psi u_{t_1t_{-1}}+2\psi_{t_1}(1+u_{t_{-1}})=0, \tag{52}$$
$$\psi_{t_2}-\psi_{t_1t_1}+2\psi_{t_1}\psi-2u_{t_1t_1}=0. \tag{53}$$

We have here the nonlinear evolution Equation (52) and the auxiliary function ψ obeying (53). Results for higher differential operators follow as compatibility conditions of (32) with (43), (44). Equations (52) and (53) and the Lax pair (46), (47) were derived in [11], see the discussion in Section 6. The version of the system that results from substitution (21) was not studied in the literature to our knowledge.

5. Dimensional Reductions

The dimensional reductions of the above integrable equations follow by delta-functional behavior of functions $g(z)$ and $h'(p,z_{\mathrm{Im}})$ in (19), (20). Taking the fact that both these functions are independent on t into account, we get that such reductions are preserved under time evolution. Due to the Inverse problem (24), (25) reductions of time-dependence of the operator B are inherited by the dressing operator. In this way, we derive $(1+1)$-dimensional integrable systems and their Lax pairs.

Say, for the operator $\widetilde{B}(t,z)$ in (19) depending on times t_1, t_2, and t_{-1}, we can cancel dependence on t_2 by imposing condition

$$g(z)=\delta(z_{\mathrm{Re}})G(z_{\mathrm{Im}}). \tag{54}$$

Thanks to (19), this gives

$$\widetilde{B}(t,z) = \exp\left(-2i\left[z_{\mathrm{Im}}t_1 - \frac{t_{-1}}{z_{\mathrm{Im}}}\right]\right)\delta(z_{\mathrm{Re}})G(z_{\mathrm{Im}}). \tag{55}$$

Thus, the symbol of the operator K is also independent on t_2, and now it is an analytic function for $z_{\mathrm{Re}} \neq 0$. In order to preserve independence of the Jost solution on t_2, we have to change its definition (cf. (45)):

$$\varphi(t_1, t_{-1}, z) = \widetilde{K}(t,z)e^{zt_1 + z^{-1}t_{-1}}. \tag{56}$$

Thus, thanks to (46) and (47), this solution obeys the Lax pair, where the first equation reads as

$$\varphi_{t_1t_1} - 2u_{t_1}\varphi = z^2\varphi, \tag{57}$$

cf. (46), and the second equation coincides with (47).

In the same way, we derive from (52) and (53) the condition of compatibility for these equations:

$$u_{t_1t_{-1}} + 2\psi(1 + u_{t_{-1}}) = 0, \tag{58}$$
$$\psi_{t_1} - \psi^2 + 2u_{t_1} = 0, \tag{59}$$

where both equations were integrated once with respect to t_1. We see that the $\bar{\partial}$-problem in this case is the Riemann–Hilbert problem for function analytics in the right and left half-planes on the complex z-plane with discontinuity given by (55) on the imaginary axis. Function φ is normalized by conditions (25) and (56) at $z \to \infty$.

This is not the only reduction applicable to (19). Setting there

$$g(z) = \delta(|z| - 1)\tilde{g}(z), \tag{60}$$

we get scattering data, that is, the symbol of operator B, depending on two variables $t_1 - t_{-1}$ and t_2:

$$\begin{aligned}\widetilde{B}(t,z) = {} & \delta\left(z_{\mathrm{Re}} - \sqrt{1 - z_{\mathrm{Im}}^2}\right)e^{-2iz_{\mathrm{Im}}\left(t_1 - t_{-1} + 2\sqrt{1 - z_{\mathrm{Im}}^2}t_2\right)}\tilde{g}_+(z_{\mathrm{Im}}) \\ & + \delta\left(z_{\mathrm{Re}} + \sqrt{1 - z_{\mathrm{Im}}^2}\right)e^{-2iz_{\mathrm{Im}}\left(t_1 - t_{-1} - 2\sqrt{1 - z_{\mathrm{Im}}^2}t_2\right)}\tilde{g}_-(z_{\mathrm{Im}}).\end{aligned} \tag{61}$$

Thus, after shifting $t_1 \to t_1 + t_{-1}$, we exclude dependence on t_{-1} from B, and then from K. Now, due to the delta-function in (61), we reduce the inverse problem (24) to the Riemann–Hilbert problem on the circle $|z| = 1$ and normalization condition (25). The Jost solution is defined here by means of relation

$$\varphi(t_1, t_2, z) = \widetilde{K}(t_1 + t_{-1}, t_{-1}, t_2, z)e^{zt_1 + z^2t_2}, \tag{62}$$

where the r.h.s. is independent on t_{-1}. Thanks to this substitution, we reduce Equations (32) and (41) to

$$\varphi_{t_1t_1} = (\lambda - \psi)\varphi_{t_1} + (\lambda\psi - 1 + u_{t_1})\varphi, \tag{63}$$
$$\varphi_{t_2} = (\lambda - \psi)\varphi_{t_1} + (\lambda\psi - 1 - u_{t_1})\varphi, \tag{64}$$

where we denoted $\lambda = z + 1/z$ and integrated (63) with respect to t_1. Considering the fact that (53) is unchanged under this reduction, we substituted $\varphi_{t_1t_1}$ into it by means of (63), that gave (64).

The integrable equation follows either from compatibility of (63) and (64), or from (52) after integration with respect to t_1:

$$u_{t_2} + u_{t_1t_1} - 2\psi(1 - u_{t_1}) = 0, \tag{65}$$

where the second Equation (53) is left unchanged.

In analogy, we can consider reductions of the other equations of this hierarchy.

6. Concluding Remarks

In the above, we introduced the hierarchy of integrable equations that can be called a "negative KP hierarchy". Lax operators of this hierarchy coincide with operators of the "positive" one, while their time evolutions are essentially different. Indeed, if $m > 0$, we get, by analogy to (32), that there exist operators P_m such that $K_{t_m} + KA^m = P_mK$, where symbols of P_m are polynomials with respect to z. Let us introduce K^{-1} as an inversion of K in correspondence to (14), $KK^{-1} = I$. Then, see [1], $P_m = (KA^mK^{-1})_+$, where index $+$ denotes the entire (with respect to z) part of the symbol in parentheses. It is clear that in the case of $m < 0$, this relation gives zero. Moreover, the direct application of such a construction to Equation (39) is senseless, as all terms there are of zero order. This was the reason to develop the construction above with the inclusion of an auxiliary function ψ. Thanks to (53), this function is defined by means of initial data $u|_{t_{-1}=0}$, that makes problem (52), (53) closed.

We already mentioned that the Lax pair (46), (47) and system (52), (53) are known in the literature [11]. Direct and inverse problems for this system were resolved in [12]. However, it is necessary to mention that the operators of the Lax pair were exchanged. The linear problem was considered to be given by (47), and t_2 was a time variable. Correspondingly, spectral data of these two problems happen to be very different. In [12] it was shown that there, we had two sets of spectral data because the Jost solution had a nonzero $\bar{\partial}$-derivative and discontinuity on the real axis, while in the case here, the solution of the heat conductivity equation in (46) has singularity of the first kind only. We also derived $(1+1)$-integrable systems presented in Section 5.

Consideration here was close to [4], where the Davey–Stewartson hierarchy was used as an example. Existence of both these hierarchies shows that this approach can be applied to the construction of other new, integrable hierarchies.

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