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New Conditional Symmetries and Exact Solutions of the Diffusive Two-Component Lotka–Volterra System

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Abstract: The diffusive Lotka–Volterra system arising in an enormous number of mathematical models in biology, physics, ecology, chemistry and society is under study. New Q-conditional (nonclassical) symmetries are derived and applied to search for exact solutions in an explicit form. A family of exact solutions is examined in detail in order to provide an application for describing the competition of two species in population dynamics. The results obtained are compared with those published earlier as well.

Keywords: diffusive Lotka–Volterra system; nonclassical symmetry; Q-conditional symmetry of the first type; exact solution

MSC: 35K57; 35B06; 35B09

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Citation: Cherniha, R.; Davydovych, V. New Conditional Symmetries and Exact Solutions of the Diffusive Two-Component Lotka–Volterra System. *Mathematics* **2021**, *9*, 1984. https://doi.org/10.3390/math9161984

Academic Editors: Andrei Dmitrievich Polyanin and Alexander V. Aksenov

Received: 14 July 2021
Accepted: 12 August 2021
Published: 19 August 2021

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1. Introduction

About 100 years ago, A. J. Lotka [1] and V. Volterra [2] independently developed a mathematical model, which nowadays serves as the mathematical background for population dynamics, ecology, chemical reactions, etc. Their model is based on a system of ordinary differential equations (ODEs) with quadratic nonlinearities (typically two equations). The natural generalization of this model in 1D space reads as follows:

$$\begin{aligned} u_t &= d_1 u_{xx} + u(a_1 + b_1 u + c_1 v), \\ v_t &= d_2 v_{xx} + v(a_2 + b_2 u + c_2 v), \end{aligned} \tag{1}$$

where the lower subscripts t and x mean differentiation with respect to (w.r.t.) these variables, $u = u(t,x)$ and $v = v(t,x)$ are two unknown functions, which usually represent densities, a_i, b_i and c_i are arbitrary constants (some of them can vanish and different types of interactions arise depending on signs of nonvanish constants), and d_1 and d_2 are diffusion coefficients. System (1) is called the diffusive Lotka–Volterra (DLV) system and is the main object of this work. If the diffusivities are such that $d_1 = d_2 = 0$, then (1) reduces to the classical Lotka–Volterra system.

In contrast to the classical Lotka–Volterra system, the DLV system attracted the attention of scholars much later. Its rigorous study started in the 1970s (see the pioneering works [3–6]). At the present time, there are many recent works devoted to qualitative analysis of the DLV system (1) and its multi-component analogs (see [7,8] and the works cited therein). However, the number of the papers devoted to construction of exact solutions of the nonlinear system (1) is relatively small. Exact solutions in the form of traveling waves were constructed in [9–12]. In the case of the three-component DLV system, some traveling waves were found in [13,14]. The existence of traveling wave solutions were examined in [7,8,15,16]. To the best of our knowledge, exact solutions with more complicated structure were derived only in papers [17,18] for the two- and three-component DLV systems, respectively. In [19], nontrivial exact solutions were derived for a natural generalization of system (1) involving additional linear and/or quadratic terms. We also point out that systems of nonlinear ODEs for finding exact solutions of (1) in the very special case when $a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2$ are presented in the handbook [20]. However, those systems are not solved therein.

Thus, the problem of the construction of exact solutions of the DLV system (1) and its multi-dimensional analogs, especially those with a biological, physical or chemical interpretation, is a hot topic.

From the very beginning, we point out that the DLV system (1) is nonlinear; hence, it cannot be integrated in a straightforward way. Here, we examine this system assuming that both equations involve diffusion, are nonlinear and are not autonomous, i.e.,

$$d_1 d_2 \neq 0,\ b_1^2 + c_1^2 \neq 0,\ b_2^2 + c_2^2 \neq 0,\ c_1^2 + b_2^2 \neq 0. \tag{2}$$

The most powerful methods for the construction of exact solutions for non-integrable nonlinear partial differential equations (PDEs) are symmetry-based methods. These methods originate from the Lie method, which was created by the prominent Norwegian mathematician Sophus Lie in the 1880s. The Lie method (the terminology 'the Lie symmetry analysis' and 'the group-theoretical analysis' are also used) still attracts the attention of many researchers, and new results are published on a regular basis (see the recent monographs [21,22] and the papers cited therein). However, it is well known that some nonlinear PDEs and systems of PDEs arising in applications have poor Lie symmetry. The Lie method is not efficient for such equations since it enables only those exact solutions to be constructed, which can be easily obtained without using this method. The DLV system (1) belongs to such systems because one possesses a nontrivial Lie symmetry only under unrealistic restrictions on parameters (see more details in [10]). As a result, Lie symmetries allow us to construct only traveling wave solutions for (1).

During recent decades, other symmetry-based methods were developed in order to solve nonlinear PDEs with poor Lie symmetry. The best known among them is the method of nonclassical symmetries proposed by G. Bluman and J. Cole in 1969 [23]. Notably, following Fushchych's proposal dating back to the 1980s [24,25], we use the terminology 'Q-conditional symmetry' instead of 'nonclassical symmetry' (see also a discussion concerning terminology in Chapter 3 of [22]). Although this method was suggested 50 years ago, its successful applications for solving *nonlinear systems of PDEs* were accomplished only in the 2000s, and the majority of such papers were published during the last 10 years (see [17,18,26–30]). This occurred because application of the nonclassical method (such a terminology was used in [23] instead of nonclassical symmetries) leads to very complicated nonlinear equations to-be-solved. As a result, one needs to solve a much more complicated PDE system (the so-called system of determining equations (DEs)), comparing with the system in question. In paper [27], a simpler algorithm was proposed in order to make essential progress in solving systems of DEs and to construct Q-conditional symmetry. The algorithm is based on the notion of *Q-conditional symmetry of the first type*. In paper [17], we successfully applied the new algorithm for finding new exact solutions of the DLV system (1).

In this work, using a modification of the algorithm for finding Q-conditional symmetry of the first type, we make further progress in the construction of new symmetries and exact solutions of the DLV system (1). Moreover, we demonstrate that some solutions can be useful in population dynamics.

The paper is organized as follows. In Section 2, we present some definitions and provide a complete description of Q-conditional symmetries of the first type of the DLV system (1) in the so-called no-go case. In Section 3, the symmetries obtained are applied to reduce the DLV system to the systems of ODEs and to construct exact solutions. In Section 4, the properties of particular exact solutions are examined with the aim to provide their biological interpretation. Finally, we present some conclusions in the last section.

2. *Q*-Conditional Symmetries of the DLV System

Let us consider the general form of the Q-conditional symmetry operator of system (1), namely the first-order operator:

$$Q = \xi^0(t,x,u,v)\partial_t + \xi^1(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v, \ \left(\xi^0\right)^2 + \left(\xi^1\right)^2 \neq 0, \quad (3)$$

where $\xi^i(t,x,u,v)$ and $\eta^k(t,x,u,v)$ are smooth functions that can be found, using the well known criterion. Given (3), one can calculate the second prolongation of the operator Q:

$$\underset{2}{Q} = Q + \rho^1_t\partial_{u_t} + \rho^1_x\partial_{u_x} + \rho^2_t\partial_{v_t} + \rho^2_x\partial_{v_x} + \sigma^1_{tt}\partial_{u_{tt}} + \sigma^1_{tx}\partial_{u_{tx}} + \sigma^1_{xx}\partial_{u_{xx}} + \sigma^2_{tt}\partial_{v_{tt}} + \sigma^2_{tx}\partial_{v_{tx}} + \sigma^2_{xx}\partial_{v_{xx}},$$

where the coefficients ρ and σ with relevant subscripts are expressed via the functions ξ^i and η^k by the well-known formulae (see [21,31]).

Definition 1. *Operator (3) is called the Q-conditional symmetry for the DLV system (1) if the following invariance conditions are satisfied:*

$$\begin{aligned} &\underset{2}{Q}\left(d_1u_{xx} - u_t + u(a_1 + b_1u + c_1v)\right)\Big|_{\mathcal{M}} = 0, \\ &\underset{2}{Q}\left(d_2v_{xx} - v_t + v(a_2 + b_2u + c_2v)\right)\Big|_{\mathcal{M}} = 0, \end{aligned} \quad (4)$$

where the manifold

$$\mathcal{M} = \{S_1 = 0, S_2 = 0, Q(u) = 0, Q(v) = 0, \tfrac{\partial}{\partial t}\,Q(u) = 0, \tfrac{\partial}{\partial x}\,Q(u) = 0, \tfrac{\partial}{\partial t}\,Q(v) = 0, \tfrac{\partial}{\partial x}\,Q(v) = 0\},$$

while

$$\begin{aligned} &S_1 \equiv\ d_1u_{xx} - u_t + u(a_1 + b_1u + c_1v),\ S_2 \equiv\ d_2v_{xx} - v_t + v(a_2 + b_2u + c_2v), \\ &Q(u) \equiv\ \xi^0u_t + \xi^1u_x - \eta^1, Q(v) \equiv\ \xi^0v_t + \xi^1v_x - \eta^2. \end{aligned}$$

Since the expressions S_1 and S_2 contain only the derivatives $u_t,\ v_t,\ u_{xx}$ and v_{xx}, one needs to consider the following coefficients:

$$\begin{aligned} &\rho^1_t = \eta^1_t - \xi^1_t u_x + \left(\eta^1_u - \xi^0_t - \xi^1_u u_x\right)u_t + \left(\eta^1_v - \xi^1_v u_x\right)v_t - \xi^0_v u_t v_t - \xi^0_u u_t^2, \\ &\rho^2_t = \eta^2_t - \xi^1_t v_x + \left(\eta^2_u - \xi^1_u v_x\right)u_t + \left(\eta^2_v - \xi^0_t - \xi^1_v v_x\right)v_t - \xi^0_u u_t v_t - \xi^0_v v_t^2, \\ &\sigma^1_{xx} = \eta^1_{xx} + \left(2\eta^1_{xu} - \xi^1_{xx}\right)u_x + 2\eta^1_{xv}v_x + \left(\eta^1_{uu} - 2\xi^1_{xu} - \xi^1_{uu}u_x - 2\xi^1_{uv}v_x\right)u_x^2 \\ &+\left(\eta^1_{vv} - \xi^1_{vv}u_x\right)v_x^2 + 2\left(\eta^1_{uv} - \xi^1_{xv}\right)u_x v_x + \left(\eta^1_u - 2\xi^1_x - 3\xi^1_u u_x - 2\xi^1_v v_x\right)u_{xx} \\ &+\left(\eta^1_v - \xi^1_v u_x\right)v_{xx} - \left(\xi^0_{xx} + 2\xi^0_{xu}u_x + 2\xi^0_{xv}v_x + \xi^0_{uu}u_x^2 + \xi^0_{vv}v_x^2 + 2\xi^0_{uv}u_x v_x\right)u_t \\ &-2\left(\xi^0_x + \xi^0_u u_x + \xi^0_v v_x\right)u_{tx} - \xi^0_u u_t u_{xx} - \xi^0_v u_t v_{xx}, \\ &\sigma^2_{xx} = \eta^2_{xx} + 2\eta^2_{xu}u_x + \left(2\eta^2_{xv} - \xi^1_{xx}\right)v_x + \left(\eta^2_{uu} - \xi^1_{uu}v_x\right)u_x^2 \\ &+\left(\eta^2_{vv} - 2\xi^1_{xv} - 2\xi^1_{uv}u_x - \xi^1_{vv}v_x\right)v_x^2 + 2\left(\eta^2_{uv} - \xi^1_{xu}\right)u_x v_x + \left(\eta^2_u - \xi^1_u v_x\right)u_{xx} \\ &+\left(\eta^2_v - 2\xi^1_x - 2\xi^1_u u_x - 3\xi^1_v v_x\right)v_{xx} - \left(\xi^0_{xx} + 2\xi^0_{xu}u_x + 2\xi^0_{xv}v_x + \xi^0_{uu}u_x^2\right. \\ &\left.+\xi^0_{vv}v_x^2 + 2\xi^0_{uv}u_x v_x\right)v_t - 2\left(\xi^0_x + \xi^0_u u_x + \xi^0_v v_x\right)v_{tx} - \xi^0_u v_t u_{xx} - \xi^0_v v_t v_{xx}. \end{aligned} \quad (5)$$

System (1) is the system of evolution equations. Therefore, the problem of constructing its Q-conditional symmetries of the form (3) essentially depends on the value of the function ξ^0. Thus, one should consider two different cases:

1. $\xi^0 \neq 0$.
2. $\xi^0 = 0,\ \xi^1 \neq 0$.

In ***Case 1***, one can set $\xi^0 = 1$ without loss of generality using the well known property stating that the Q-conditional symmetry operator can be multiplied by an arbitrary smooth function (see the proof in [31]). Moreover, in this case, the differential consequences of equations $Q(u) = 0$ and $Q(v) = 0$ (see the manifold $\mathcal{M}$) w.r.t. the variables t and x lead to the second-order PDEs, namely, the following:

$$\begin{array}{l}
\frac{\partial}{\partial t}\,Q(u) \equiv \eta^1_t + \eta^1_u u_t + \eta^1_v v_t - \xi^1_t u_x - \xi^1_u u_t u_x - \xi^1_v v_t u_x - \xi^1 u_{tx} - u_{tt} = 0,\\
\frac{\partial}{\partial x}\,Q(u) \equiv \eta^1_x + \eta^1_u u_x + \eta^1_v v_x - \xi^1_x u_x - \xi^1_u u_x u_x - \xi^1_v v_x u_x - \xi^1 u_{xx} - u_{tx} = 0,\\
\frac{\partial}{\partial t}\,Q(v) \equiv \eta^2_t + \eta^2_u u_t + \eta^2_v v_t - \xi^1_t v_x - \xi^1_u u_t v_x - \xi^1_v v_t v_x - \xi^1 v_{tx} - v_{tt} = 0,\\
\frac{\partial}{\partial x}\,Q(v) \equiv \eta^2_x + \eta^2_u u_x + \eta^2_v v_x - \xi^1_x v_x - \xi^1_u u_x v_x - \xi^1_v v_x v_x - \xi^1 v_{xx} - v_{tx} = 0.
\end{array}$$

We notethat the above equations involve the time derivatives u_{tt}, v_{tt} and the mixed derivatives u_{tx}, v_{tx}, which do not occur in the invariance conditions (4). As a result, the manifold $\mathcal{M}$ can be rewritten as $\{S_1 = 0, S_2 = 0, Q(u) = 0, Q(v) = 0\}$, i.e., the first-order differential consequences can be omitted. ***Case 1*** for the DLV system (1) was investigated in the work [17] (see also Chapter 3 in [31]).

Here, we examine ***Case 2***, for which the terminology '*no-go case*' is often used. Thus, we are looking for operators of the following form:

$$Q = \xi(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v,\ \xi \neq 0. \tag{6}$$

In this case, formulae (5) are essentially simplified and take the following forms:

$$\begin{array}{l}
\rho^1_t = \eta^1_t - \xi_t u_x + (\eta^1_u - \xi_u u_x)u_t + (\eta^1_v - \xi_v u_x)v_t,\\
\rho^2_t = \eta^2_t - \xi_t v_x + (\eta^2_u - \xi_u v_x)u_t + (\eta^2_v - \xi_v v_x)v_t,\\
\sigma^1_{xx} = \eta^1_{xx} + (2\eta^1_{xu} - \xi_{xx})u_x + 2\eta^1_{xv}v_x + 2(\eta^1_{uv} - \xi_{xv})u_x v_x + (\eta^1_{vv} - \xi_{vv}u_x)v_x^2\\
+(\eta^1_{uu} - 2\xi_{xu} - \xi_{uu}u_x - 2\xi_{uv}v_x)u_x^2 + (\eta^1_u - 2\xi_x - 3\xi_u u_x - 2\xi_v v_x)u_{xx} + (\eta^1_v - \xi_v u_x)v_{xx},\\
\sigma^2_{xx} = \eta^2_{xx} + 2\eta^2_{xu}u_x + (2\eta^2_{xv} - \xi_{xx})v_x + 2(\eta^2_{uv} - \xi_{xu})u_x v_x + (\eta^2_{uu} - \xi_{uu}v_x)u_x^2\\
+(\eta^2_{vv} - 2\xi_{xv} - 2\xi_{uv}u_x - \xi_{vv}v_x)v_x^2 + (\eta^2_u - \xi_u v_x)u_{xx} + (\eta^2_v - 2\xi_x - 2\xi_u u_x - 3\xi_v v_x)v_{xx}.
\end{array} \tag{7}$$

First of all, we note that the task of constructing the Q-conditional symmetries with $\xi^0 = 0$ for scalar evolution equations is equivalent to solving the equation in question [32]. For this reason, one can obtain only some particular results finding the Q-conditional symmetry operators of the form (6) for system (1).

Our aim is to construct *Q-conditional symmetries of the first type* for the DLV system (1) in ***Case 2***. The notion of Q-conditional symmetry of the first type was introduced in the paper [27] as a special case of Q-conditional symmetry for *systems of PDEs*. Each Q-conditional symmetry of the first type is automatically a Q-conditional symmetry (non-classical symmetry) but not vice versa.

Definition 2. *Operator (6) is called the Q-conditional symmetry of the first type for the DLV system (1) if the following invariance conditions are satisfied:*

$$\underset{2}{Q}\,(S_1)\Big|_{\mathcal{M}_1} = 0,\ \underset{2}{Q}\,(S_2)\Big|_{\mathcal{M}_1} = 0, \tag{8}$$

where the manifold $\mathcal{M}_1$ is either given by $\mathcal{M}^u_1 = \{S_1 = 0, S_2 = 0, Q(u) = 0, \frac{\partial}{\partial t}\,Q(u) = 0, \frac{\partial}{\partial x}\,Q(u) = 0\}$ or $\mathcal{M}^v_1 = \{S_1 = 0, S_2 = 0, Q(v) = 0, \frac{\partial}{\partial t}\,Q(v) = 0, \frac{\partial}{\partial x}\,Q(v) = 0\}$.

We point out that the definition was given in [27] for an arbitrary multi-component system of evolution PDEs, and differential consequences (see above $\frac{\partial}{\partial t}\,Q(u) = 0, \ldots, \frac{\partial}{\partial x}\,Q(v) = 0$) were not used therein. In fact, such equations do not play any role if one looks for operators of the form (3) with $\xi^0 \neq 0$. It was only indicated (see conclusions in [27]) that differential consequences should be taken into account in the case of arbitrary systems (e.g., involving hyperbolic equations). However, it was not noted in [27] that one may use differential consequences for searching operators (3) with $\xi^0 = 0$ occurring in ***Case 2***. Here, we show that such an approach leads to new results.

The problem of finding Q-conditional symmetries of the first type for some reaction–diffusion systems (in particular, two- and three-component DLV systems) are considered in monograph [31] (see Chapters 3 and 4). In paper [29], such symmetries were constructed in the no-go case for a wide class of reaction–diffusion systems with nonconstant diffusivities.

Let us apply Definition 2 to construct the system of DEs for finding the Q-conditional symmetry operators of the form (6). Firstly, we note that the DLV system (1) has a symmetric structure and admits the discrete transformation $u \to v,\ v \to u$. Thus, to obtain all Q-conditional symmetries of the first type for the DLV system (1), it is enough to examine only one manifold from Definition 2, say $\mathcal{M}_1^u$. Solving the corresponding system of DEs, the list of inequivalent (up to same specified local transformations) DLV systems and corresponding operators will be derived. In order to obtain a complete classification, the DLV systems from the list derived will be checked to confirm whether they admit additional conditional symmetries satisfying Definition 2 with the manifold $\mathcal{M}_1^v$.

Thus, the invariance conditions (8) corresponding to the manifold $\mathcal{M}_1^u$ take the forms:

$$\begin{aligned}\left(d_1\sigma^1_{xx} - \rho^1_t + (a_1 + 2b_1u + c_1v)\,\eta^1 + c_1u\,\eta^2\right)\Big|_{\mathcal{M}_1^u} &= 0,\\ \left(d_2\sigma^2_{xx} - \rho^2_t + b_2v\,\eta^1 + (a_2 + b_2u + 2c_2v)\,\eta^2\right)\Big|_{\mathcal{M}_1^u} &= 0,\end{aligned} \tag{9}$$

where ρ and σ with indices are calculated by the Formula (7).

Using the equations generating the manifold $\mathcal{M}_1^u$, one can exclude the derivatives $u_x,\ u_t,\ v_t$ and u_{xx}:

$$\begin{aligned}&u_x = \tfrac{\eta^1}{\xi},\ u_{xx} = \tfrac{\eta^1_v - \xi_v u_x}{\xi}\,v_x + \tfrac{1}{\xi}\left(\eta^1_x + \eta^1_u u_x - \xi_x u_x - \xi_u u_x^2\right),\\ &u_t = d_1u_{xx} + u(a_1 + b_1u + c_1v),\ v_t = d_2v_{xx} + v(a_2 + b_2u + c_2v).\end{aligned} \tag{10}$$

Note that the derivative u_{tx} (which can be defined from the equation $\frac{\partial}{\partial t}\,Q(u) = 0$) is not presented in conditions (9). Thus, to construct the system of DEs, one needs to substitute (10) into (9) and to split the equations obtained w.r.t. $v_{xx},\ v_xv_{xx},\ v_x^2$ and v_x. Omitting straightforward calculations, we present only the following result:

$$\xi_v = 0,\ \eta^1_{vv} = 0,\ \eta^2_{vv} = 0,\ (d_1 - d_2)\eta^1_v = 0, \tag{11}$$

$$\xi\eta^1_{xv} + \eta^1\eta^1_{uv} = 0,\ \eta^1\xi_u + \xi\xi_x = 0, \tag{12}$$

$$\begin{aligned}&2d_2\eta^2_{xv} + u(a_1 + b_1u + c_1v)\xi_u + \xi_t\\ &+\frac{1}{\xi}\left(d_1\xi_u\eta^1_x + 2d_2\eta^1\eta^2_{uv} + (d_2 - d_1)\eta^1_v\eta^2_u\right) + d_1\frac{\eta^1}{\xi^2}\,\xi_u\eta^1_u = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}&d_1\eta^1_{xx} - \eta^1_t - u(a_1 + b_1u + c_1v)\eta^1_u - v(a_2 + b_2u + c_2v)\eta^1_v + c_1u\eta^2 + (a_1 + 2b_1u + c_1v)\eta^1\\ &+\frac{\eta^1}{\xi}\left(2d_1\eta^1_{xu} + u(a_1 + b_1u + c_1v)\xi_u + \xi_t\right) + d_1\frac{\eta^1}{\xi^2}\left(\eta^1\eta^1_{uu} + \xi_u\eta^1_x\right) + d_1\frac{(\eta^1)^2}{\xi^3}\,\xi_u\eta^1_u = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}&d_2\eta^2_{xx} - \eta^2_t - u(a_1 + b_1u + c_1v)\eta^2_u - v(a_2 + b_2u + c_2v)\eta^2_v + b_2v\eta^1 + (a_2 + b_2u + 2c_2v)\eta^2\\ &+\frac{1}{\xi}\left(2d_2\eta^1\eta^2_{xu} + (d_2 - d_1)\eta^1_x\eta^2_u\right) + \frac{\eta^1}{\xi^2}\left(d_2\eta^1\eta^2_{uu} + (d_2 - d_1)\eta^1_u\eta^2_u\right) = 0.\end{aligned} \tag{15}$$

Note that Equations (13)–(15) were simplified (using equation $\eta^1\xi_u + \xi\xi_x = 0$ and its differential consequences w.r.t. x and u) by excluding the derivatives $\xi_x,\ \xi_{xx}$ and ξ_{xu}.

The result of integrating the system of DEs (11)–(15) can be formulated as follows.

Theorem 1. *The DLV system (1) is invariant under the Q-conditional symmetry operator(s) of the first type (6) if and only if the system and the corresponding operator(s) have the forms listed in Table 1. Any other DLV system (1) admitting a nontrivial Q-conditional symmetry of the first type and the relevant operator(s) are reduced to those listed in Table 1 by local transformations from the following set:*

$$t^* = t + t_0,\ x^* = e^{\gamma_0}(x + x_0),\ u^* = \beta_{11}\,e^{\gamma_1 t}u + \beta_{12}\,v,\ v^* = \beta_{22}\,e^{\gamma_2 t}v + \beta_{21}\,u, \tag{16}$$

where t_0, x_0, β_{ij} and γ_j are some correctly-specified constants.

Table 1. Q-conditional symmetries of the first type of the DLV system (1).

	DVL Systems	Restrictions and Operators
1.	$u_t = d_1 u_{xx} + u(a_1 + u + v)$ $v_t = d_2 v_{xx} + v(a_2 + u + v)$	$d_1 \neq d_2,\ Q_1^u = \partial_x + \frac{g_x^1}{g^1}\, u\, (\partial_u - \partial_v),$ $Q_1^v = \partial_x + \frac{g_x^2}{g^2}\, v\, (\partial_v - \partial_u)$
2.	$u_t = u_{xx} + u(a + u + 2v)$ $v_t = dv_{xx} + v(ad + dv)$	$Q_2^v = G(x,v)(\partial_x + F(x,v)(\partial_u - \partial_v))$
3.	$u_t = u_{xx} + uv$ $v_t = dv_{xx} + v(a_2 + c_2 v)$	$a_2 c_2 \neq 0, Q_3^u = \partial_x + r(t,x)\, u\, \partial_u,$ $Q_3^v = \big(h^1(\omega) - 2th^2(\omega)\big)\partial_x + \big((h^2(\omega)x + h^3(\omega))u + p(t,x,v)\big)\partial_u$
4.	$u_t = u_{xx} + uv$ $v_t = dv_{xx} + c_2 v^2$	$c_2 \neq 0,\ Q_3^u, Q_4^v = \big(h^1(\theta) - 2th^2(\theta)\big)\partial_x$ $+\big((h^2(\theta)x + h^3(\theta))u + p(t,x,v)\big)\partial_u$
5.	$u_t = u_{xx} + uv$ $v_t = v_{xx} + v\big(a_2 + \frac{v}{2}\big)$	$a_2 \neq 0\,,\ Q_3^u,\quad Q_3^v$ *with* $c_2 = 1/2,$ $Q_5^v = \partial_x + e^{a_2 t} u \partial_v + \Big(\alpha\, u - \frac{e^{-a_2 t}}{2}\, v^2 - a_2 e^{-a_2 t} v\Big)\partial_u$
6.	$u_t = u_{xx} + uv$ $v_t = v_{xx} + \frac{1}{2}\, v^2$	$Q_3^u,\ Q_4^v$ *with* $c_2 = 1/2,$ $Q_6^v = (\alpha_1 t + \alpha_0)\partial_x + (\alpha_1 t + \alpha_0) u \partial_v + \Big(\big(\alpha_2 - \frac{\alpha_1}{2}\, x\big)u - \frac{\alpha_1 t + \alpha_0}{2}\, v^2 - \alpha_1 v\Big)\partial_u$
7.	$u_t = u_{xx} + uv$ $v_t = v_{xx} + v(a_2 + v)$	$a_2 \neq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0, Q_3^u, Q_3^v$ *with* $c_2 = 1,$ $Q_7^u = \partial_x + \Big(-\frac{x}{2t}\, u + \frac{\alpha_1}{t} + \Big(\frac{\alpha_2 e^{-a_2 t}}{t} + \frac{\alpha_1}{a_2 t}\Big)v\Big)\partial_u$
8.	$u_t = u_{xx} + uv$ $v_t = v_{xx} + v^2$	$\alpha_1^2 + \alpha_2^2 \neq 0,\ Q_3^u, Q_4^v$ *with* $c_2 = 1,$ $Q_8^u = \partial_x + \big(-\frac{x}{2t}\, u + \frac{\alpha_1}{t} + \big(\frac{\alpha_2}{t} + \alpha_1\big)v\big)\partial_u$

The proof of the theorem is presented in Appendix A.

Remark 1. *In Table 1, the upper indexes u and v mean that the relevant Q-conditional symmetry operators satisfy Definition 2 for the manifolds $\mathcal{M}_1^u$ and $\mathcal{M}_1^v$, respectively.*

Remark 2. *In Table 1, $\omega = \frac{a_2 + c_2 v}{v}\, e^{a_2 t}$, $\theta = t + \frac{1}{c_2 v}$; h^1, h^2 and h^3 are arbitrary smooth functions of the corresponding variables, the function $p(t,x,v)$ is the general solution of the linear ODE*

$$p_t = p_{xx} - (a_2 v + c_2 v^2) p_v + vp,$$

the functions F and G are the general solution of the system

$$FF_v - F_x + av + v^2 = 0,\ G_x = FG_v, \tag{17}$$

and the function $r(t,x)$ is the general solution of the Burgers equation $r_t = r_{xx} + 2rr_x$, while

$$g^i(t,x) = \begin{cases} \alpha_0 \exp(d_i \kappa^2 t) + \alpha_1 \sin(\kappa\, x) + \alpha_2 \cos(\kappa\, x), & \text{if} \quad \frac{a_1 - a_2}{d_1 - d_2} > 0, \\ \alpha_0 \exp(-d_i \kappa^2 t) + \alpha_1 e^{\kappa x} + \alpha_2 e^{-\kappa x}, & \text{if} \quad \frac{a_1 - a_2}{d_1 - d_2} < 0, \\ \alpha_0 + \alpha_1 x + \alpha_2 x^2 + 2 d_i \alpha_2 t, & \text{if} \quad a_1 = a_2, \end{cases} \tag{18}$$

where $i = 1,2$, $\kappa = \sqrt{\left|\frac{a_1 - a_2}{d_1 - d_2}\right|}$, α_0, α_1 and α_2 are arbitrary constants.

Remark 3. *The general solution of the quasilinear first-order system (17) can be constructed in an implicit form using of computation program, say Maple. In order to avoid cumbersome formulae, here we present its solution in the case $F_x = 0$:*

$$F = \pm\sqrt{\alpha - av^2 - \frac{2}{3}v^3},\ G = G_0(\omega),\ \omega = x \pm \int \frac{1}{\sqrt{\alpha - av^2 - \frac{2}{3}v^3}}\,dv,$$

where α is an arbitrary constant, while G_0 is an arbitrary smooth function. The above integral is expressed in the terms of elliptic functions, which degenerate to elementary functions in particular cases. For example, $\omega = \sqrt{-a}\,x \pm 2\mathrm{arctanh}\sqrt{1 + \frac{2v}{3a}}$ if $\alpha = 0,\ a < 0$.

Remark 4. *All the systems arising in Table 1, excepting that in Case 1, are semi-coupled because the second equation is autonomous. Interestingly, the equation in Cases 2, 3, 5 and 7 is nothing else but the Fisher equation [33].*

The most interesting system from applicability point of view occurs in Case 1. Using the transformations (see (16))

$$u \to \beta_{11}u, \quad v \to \beta_{22}v,$$

the system can be generalized to the following form:

$$\begin{aligned} u_t &= d_1u_{xx} + u(a_1 + \beta_{11}u + \beta_{22}v), \\ v_t &= d_2v_{xx} + v(a_2 + \beta_{11}u + \beta_{22}v). \end{aligned} \tag{19}$$

Thus, we conclude that the DLV system (1) admits exactly two Q-conditional symmetries of the first type provided, $b_1 = b_2 = \beta_{11} \neq 0$ and $c_1 = c_2 = \beta_{22} \neq 0$, i.e., has the form (19). Depending on signs of the parameters, the DLV system (19) can describe competition or mutualism of two populations of species (cells). However, this system cannot describe the prey–predator interaction because the quadratic terms have the same signs in both equations (see [34] for the classification of interaction types).

Finally, the following observation should be highlighted. Because each Q-conditional symmetry of the first type is automatically a usual Q-conditional (nonclassical) symmetry, all operators listed in Table 1 are nonclassical symmetries. On the other hand, it can be noted that Cases 7 and 8 of Table 1 do not present new nonclassical symmetries because the operators Q^u_7 and Q^u_8 are particular cases of those Q^v_3 and Q^v_4 arising in Cases 3 and 4. In fact, setting $h^1 = h^3 = 0,\ h^2 = c_2 = 1$ and $p = -2\alpha_1 - 2(\alpha_2 + \alpha_1 t)v$ in Q^v_4, one obtains exactly the operator $-2tQ^u_8$. Similarly, $-2tQ^u_7$ is a particular case of Q^v_3. We remind the reader that any Q-conditional symmetry can be multiplied by an arbitrary smooth function in contrast to the Lie symmetry and Q-conditional symmetry of the first type. Thus, Cases 7 and 8 of Table 1 can be skipped if one considers the Q-conditional (nonclassical) symmetries.

3. Reduction and Exact Solutions

In this section, we present examples of reductions of the DLV system to ODE systems, using the Q-conditional symmetry from Theorem 1, and solve the ODE systems obtained in order to construct exact solutions of the DLV system. Our aim is to find such exact solutions of the DLV system (1) that are bounded, nonnegative and satisfy the zero Neumann boundary conditions in some correctly-specified domain (interval).

Consider the DLV system from Case 1 of Table 1, namely the following:

$$\begin{aligned} u_t &= d_1u_{xx} + u(a_1 + u + v), \\ v_t &= d_2v_{xx} + v(a_2 + u + v),\ d_1 \neq d_2, \end{aligned} \tag{20}$$

which is the most interesting from the applicability point of view. Since the Q-conditional symmetry operators Q^u_1 and Q^v_1 of system (20) lead to the equivalent solutions (up to discrete transformation $u \to v,\ v \to u$), we use only one of them, namely Q_1^u. According to the standard procedure, in order to construct the ansatz corresponding to the operator Q_1^u, one needs to solve the first-order PDE system (see $Q(u) = 0$ and $Q(v) = 0$ in Definition 1):

$$u_x = \frac{g_x^1}{g^1}\,u,\ v_x = -\frac{g_x^1}{g^1}\,u. \tag{21}$$

Integrating system (21) for each form of the function g^1 from (18), one constructs the ansatz as follows:

$$\begin{aligned} u &= \varphi(t)\big(\alpha_0 + \alpha_1 \exp(-d_1\kappa^2 t)\sin(\kappa x) + \alpha_2 \exp(-d_1\kappa^2 t)\cos(\kappa x)\big),\\ v &= \psi(t) - \varphi(t)\big(\alpha_0 + \alpha_1 \exp(-d_1\kappa^2 t)\sin(\kappa x) + \alpha_2 \exp(-d_1\kappa^2 t)\cos(\kappa x)\big), \end{aligned} \tag{22}$$

if $\frac{a_1 - a_2}{d_1 - d_2} > 0$; the ansatz

$$\begin{aligned} u &= \varphi(t)\big(\alpha_0 + \alpha_1 \exp(d_1\kappa^2 t + \kappa x) + \alpha_2 \exp(d_1\kappa^2 t - \kappa x)\big),\\ v &= \psi(t) - \varphi(t)\big(\alpha_0 + \alpha_1 \exp(d_1\kappa^2 t + \kappa x) + \alpha_2 \exp(d_1\kappa^2 t - \kappa x)\big), \end{aligned} \tag{23}$$

if $\frac{a_1 - a_2}{d_1 - d_2} < 0$; and the ansatz is as follows:

$$\begin{aligned} u &= \varphi(t)\big(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + 2d_1\alpha_2 t\big),\\ v &= \psi(t) - \varphi(t)\big(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + 2d_1\alpha_2 t\big), \end{aligned} \tag{24}$$

if $a_1 = a_2 \equiv a$. It can be noted that each ansatz derived above satisfies the simple functional relation $u + v = \psi(t)$.

Now three reductions of the PDE system in question to the ODE systems can be provided. Substituting the above ansatz into the DLV system (20), we arrive at the ODE system as follows:

$$\frac{d\varphi}{dt} = \varphi(a_1 + \psi),\ \frac{d\psi}{dt} = \psi(a_2 + \psi) + \alpha_0(a_1 - a_2)\,\varphi, \tag{25}$$

in the case of (22) and (23), while the system

$$\frac{d\varphi}{dt} = \varphi(a + \psi),\ \frac{d\psi}{dt} = \psi(a + \psi) + 2\alpha_2(d_1 - d_2)\,\varphi, \tag{26}$$

is obtained in the case of (24). Here, $\varphi(t)$ and $\psi(t)$ are new unknown functions.

It turns out that each of the ODE systems (25) and (26) can be integrated by reducing to a single second-order ODE. As a result, the general solution of system (25) is derived in the following form:

$$\varphi = \begin{cases} \frac{a_1 e^{a_1 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}}, & \text{if } a_1 a_2 \neq 0,\\ \frac{1}{C_1 - \alpha_0\, t + C_2 e^{a_2 t}}, & \text{if } a_1 = 0,\\ \frac{a_1 e^{a_1 t}}{C_1 + C_2\, t - \alpha_0 e^{a_1 t}}, & \text{if } a_2 = 0, \end{cases} \qquad \psi = \begin{cases} \frac{\alpha_0 a_1 e^{a_1 t} - C_2 a_2 e^{a_2 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}}, & \text{if } a_1 a_2 \neq 0,\\ \frac{\alpha_0 - C_2 a_2 e^{a_2 t}}{C_1 - \alpha_0\, t + C_2 e^{a_2 t}}, & \text{if } a_1 = 0,\\ \frac{\alpha_0 a_1 e^{a_1 t} - C_2}{C_1 + C_2\, t - \alpha_0 e^{a_1 t}}, & \text{if } a_2 = 0, \end{cases} \tag{27}$$

where C_1 and C_2 are arbitrary constants. Substituting the functions φ and ψ from (27) into formulae (22) and (23), one obtains the exact solutions of the DLV system (20) with $a_1 a_2 \neq 0$:

$$\begin{aligned} u(t,x) &= \frac{a_1 e^{a_1 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}}\big(\alpha_0 + \alpha_1 \exp(-d_1\kappa^2 t)\sin(\kappa x) + \alpha_2 \exp(-d_1\kappa^2 t)\cos(\kappa x)\big),\\ v(t,x) &= \frac{\alpha_0 a_1 e^{a_1 t} - C_2 a_2 e^{a_2 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}} - u(t,x), \end{aligned} \tag{28}$$

if $\frac{a_1-a_2}{d_1-d_2} > 0$, and

$$\begin{aligned} u(t,x) &= \frac{a_1 e^{a_1 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}} \left(\alpha_0 + \alpha_1 \exp(d_1 \kappa^2 t + \kappa x) + \alpha_2 \exp(d_1 \kappa^2 t - \kappa x)\right), \\ v(t,x) &= \frac{\alpha_0 a_1 e^{a_1 t} - C_2 a_2 e^{a_2 t}}{C_1 - \alpha_0 e^{a_1 t} + C_2 e^{a_2 t}} - u(t,x), \end{aligned} \tag{29}$$

if $\frac{a_1-a_2}{d_1-d_2} < 0$, where $\kappa = \sqrt{\left|\frac{a_1-a_2}{d_1-d_2}\right|}$.

Similarly, solving the ODE system (26), we arrive at the exact solutions

$$\begin{aligned} u(t,x) &= \frac{a^2}{2(d_1-d_2)} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + 2 d_1 \alpha_2 t}{C_1 - \alpha_2 a t + C_2 e^{-at}}, \\ v(t,x) &= \frac{a(\alpha_2 - C_1 + \alpha_2 a t)}{C_1 - \alpha_2 a t + C_2 e^{-at}} - u(t,x) \end{aligned}$$

and

$$\begin{aligned} u(t,x) &= \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + 2 d_1 \alpha_2 t}{C_1 + C_2 t - (d_1 - d_2)\alpha_2 t^2}, \\ v(t,x) &= \frac{2(d_1-d_2)\alpha_2 t - C_2}{C_1 + C_2 t - (d_1-d_2)\alpha_2 t^2} - u(t,x) \end{aligned}$$

of system (20) with $a_1 = a_2 \equiv a$ for $a \neq 0$ and $a = 0$, respectively.

To the best of our knowledge, all the exact solutions obtained above are new, although several papers are devoted to finding exact solutions of the two-component DLV system. In fact, a majority of these papers [9–11] present the plane wave solutions (traveling waves), which have the following structure:

$$u = U(x - ct), \; v = V(x - ct), \tag{30}$$

where c is the speed of the wave. The solutions derived herein possess a more complicated structure than traveling waves and cannot be presented in the form (30). So, we compare our results only with paper [17] (see also Chapter 3 in [31]), where also nontrivial solutions were constructed for the DLV system as follows:

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 + u + v), \\ \lambda_2 v_t &= v_{xx} + v(a_2 + u + v), \; \lambda_1 \neq \lambda_2. \end{aligned}$$

It can be easily noted that this system is reduced to the form

$$\begin{aligned} u_t &= d_1 u_{xx} + u(a_1^* + d_1 u + d_1 v), \\ v_t &= d_2 v_{xx} + v(a_2^* + d_2 u + d_2 v), \; d_1 \neq d_2, \end{aligned} \tag{31}$$

by the introduction diffusivities $d_1 = \lambda_1^{-1}$ and $d_2 = \lambda_2^{-1}$ (here $a_1^* = \lambda_1^{-1} a_1$ and $a_2^* = \lambda_2^{-1} a_2$). Now, one realizes that two nonlinear systems (20) and (31) are inequivalent, provided $d_1 \neq d_2$. It means that any solution derived in [17] cannot be transformed into a solution of the DLV system (20). We only point out that the exact solutions (28) and (29) with $C_1 = C_2 = 0$ have the same structure as those of (117)–(118) [17]. It means that these solutions could be used for description of similar processes. A possible application is presented in the next section.

4. Interpretation of the Solution Obtained

In this section, we present an example that demonstrates remarkable properties of some solutions constructed in the previous section. Obviously, using the transformation $u \to -bu, \; v \to -cv$ (see Formula (16)) and introducing the notation $\alpha_0 \to -b\,\alpha_0, \; \alpha_1 \to -b\,\alpha_1$, one reduces the DLV system (20) to the following form:

$$\begin{aligned} u_t &= d_1 u_{xx} + u(a_1 - b\,u - c\,v), \\ v_t &= d_2 v_{xx} + v(a_2 - b\,u - c\,v). \end{aligned} \tag{32}$$

The nonlinear system (32) with positive parameters a_1, a_2, b and c is widely used for describing the competition of two population of species (or cells) (see, e.g., [34,35]). Solution (28) (we set $\alpha_2 = 0$ just for simplicity) after the above transformation takes the following form:

$$\begin{aligned} u(t,x) &= \frac{a_1 e^{a_1 t}}{C_1+\alpha_0 b\, e^{a_1 t}+C_2 e^{a_2 t}}\left[\alpha_0 + \alpha_1 \exp\left(\frac{d_1(a_2-a_1)}{d_1-d_2}\, t\right) \sin\left(\sqrt{\frac{a_1-a_2}{d_1-d_2}}\, x\right)\right], \\ v(t,x) &= \frac{1}{c}\,\frac{\alpha_0 a_1 b\, e^{a_1 t}+C_2 a_2 e^{a_2 t}}{C_1+\alpha_0 b\, e^{a_1 t}+C_2 e^{a_2 t}} - \frac{b}{c}\, u(t,x). \end{aligned} \tag{33}$$

In order to provide a biological interpretation, the components u and v must be bounded and nonnegative in some domain because they describe densities of species. If we consider the domain $\Omega = \{(t,x) \in [0,+\infty) \times (-\infty,+\infty)\}$, then it can be easily shown that both components are bounded and nonnegative, provided the coefficient restrictions

$$\alpha_0 > |\alpha_1|,\ C_2 > \max\left\{-\alpha_0 b - C_1,\ \frac{b a_1 |\alpha_1|}{a_2}\right\}$$

hold. Moreover, solution (33) possesses the asymptotical behavior as follows:

$$\begin{array}{l} (u,\,v) \to \left(\frac{a_1}{b},\,0\right),\ \mathtt{if}\ a_1 > a_2, \\ (u,\,v) \to \left(0,\,\frac{a_2}{c}\right),\ \mathtt{if}\ a_1 < a_2, \end{array} \quad \mathtt{as}\ t \to +\infty. \tag{34}$$

Now, one realizes that $\left(\frac{a_1}{b},\,0\right)$ and $\left(\frac{a_2}{c}\,0\right)$ are steady state points of the competition model (32) and the asymptotical behavior (34) is in agreement with the qualitative theory of this model (see [8] and the papers cited therein).

In real-world applications, the competition takes place in some bounded domain, say, $\Omega_* = \{(t,x) \in [0,+\infty) \times (A,B)\},\ -\infty < A < B < +\infty$. Typically, the zero flux conditions are assumed at the boundary of Ω_*:

$$\begin{aligned} x = A:\ u_x = 0,\ v_x = 0, \\ x = B:\ u_x = 0,\ v_x = 0. \end{aligned} \tag{35}$$

Such boundary conditions reflect a natural assumption that the competing species cannot cross the boundaries (for example, the wide river is a natural obstacle). It can be easily checked that the exact solution (33) satisfies the boundary conditions only under the following requirement:

$$A = \frac{\pi}{\kappa}\left(\frac{1}{2} + m_1\right),\ B = \frac{\pi}{\kappa}\left(\frac{1}{2} + m_2\right),\ m_1 < m_2,$$

where m_1 and m_2 are arbitrary integer parameters and $\kappa = \sqrt{\left|\frac{a_1-a_2}{d_1-d_2}\right|}$. Thus, we conclude that our solution with correctly-specified parameters describe the competition of two population of species in the bounded domain. An example is presented in Figure 1.

It should be pointed out that traveling wave solutions, which are widely studied for any nonlinear model and play an important role in qualitative analysis, usually cannot be used for solving the relevant models involving the zero flux boundary conditions in the bounded domains. Let us consider the traveling wave of the DLV system (32) with $d_1 = d_2 = 1$, which was firstly constructed in [10] (see also Section 3.2.3 in [31]) and much later rediscovered in [12] (see formulae (18) and (24) therein)

$$\begin{aligned} u(t,x) &= \frac{a_1}{4b}\left(1 - \tanh\left(\sqrt{\frac{a_1-a_2}{24}}\, x - \frac{5(a_1-a_2)}{12}\, t\right)\right)^2, \\ v(t,x) &= \frac{a_2}{c} - \frac{a_2}{4c}\left(1 - \tanh\left(\sqrt{\frac{a_1-a_2}{24}}\, x - \frac{5(a_1-a_2)}{12}\, t\right)\right)^2. \end{aligned} \tag{36}$$

Clearly, the components u and v are bounded and nonnegative, provided that $a_1 > a_2$ and the asymptotical behavior is the same as in (34). However, solution (36) does not satisfy the boundary condition (35) for any finite values of A and B. It can be done only for $A \to -\infty$ and $B \to +\infty$. An example of solution (36) is presented in Figure 2.

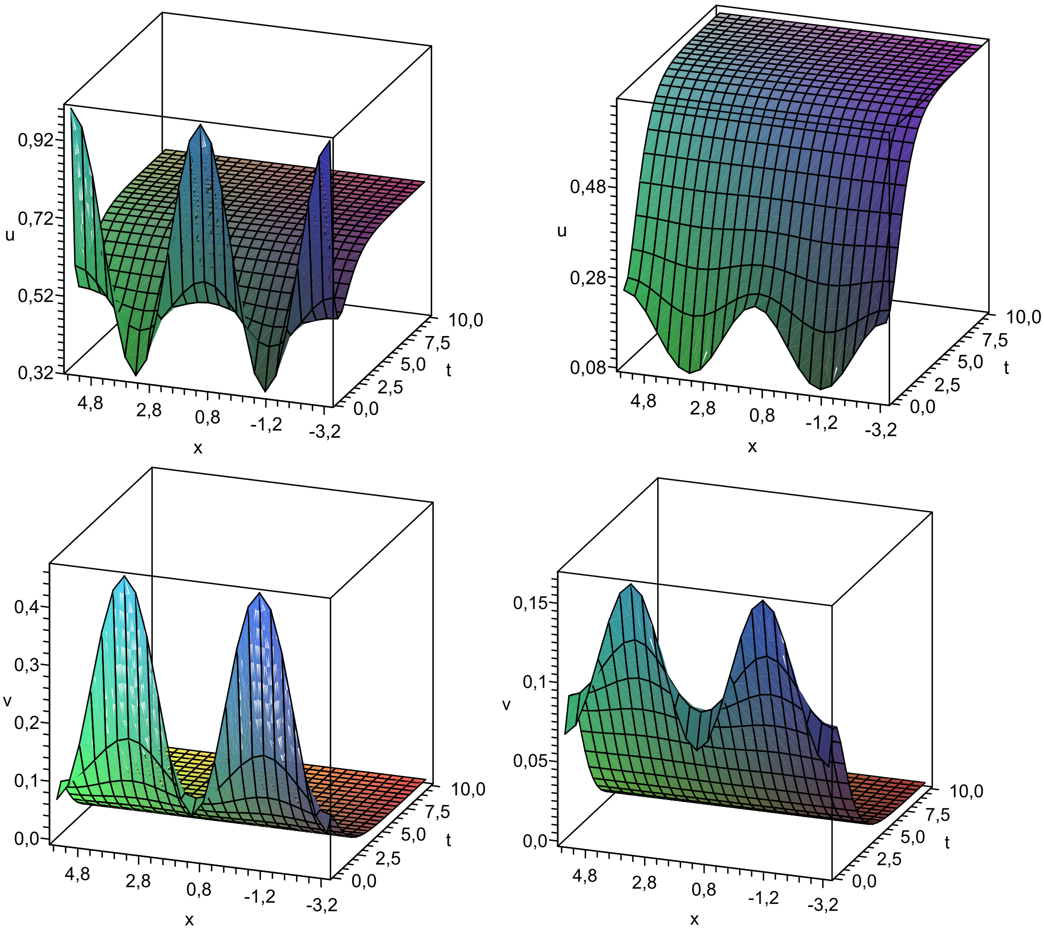

Figure 1. Surfaces representing the components u and v of solution (33) with $\alpha_0 = 1,\ \alpha_1 = 1/2,\ C_1 = -4,\ C_2 = 4$ (**left** surfaces) and $\alpha_0 = 1,\ \alpha_1 = 1/2,\ C_1 = 2,\ C_2 = 7$ (**right** surfaces) of the DLV system (32) with the parameters $d_1 = 3/2,\ d_2 = 1,\ a_1 = 2,\ a_2 = 1,\ b = 3,\ c = 5$. The functions u and v are defined in the domain Ω_* with $m_1 = -2$ and $m_2 = 2$.

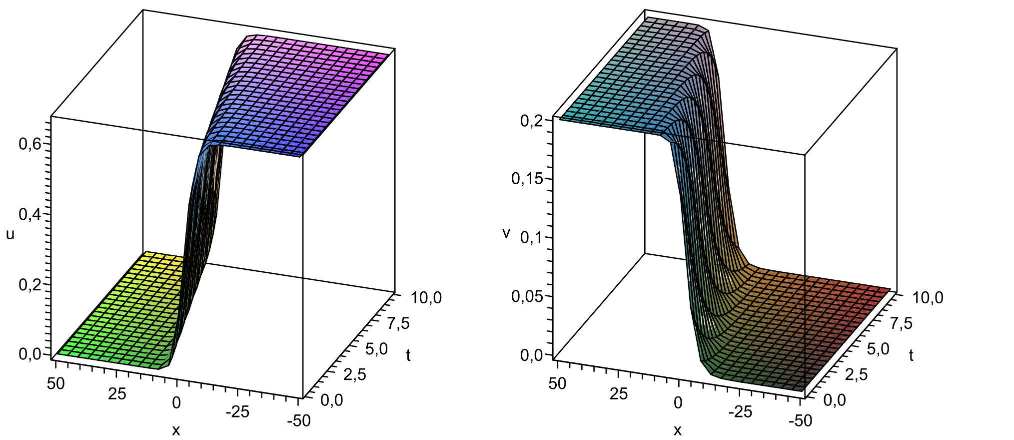

Figure 2. Surfaces representing the components u and v of the traveling wave solution (36) of the DLV system (32) with the parameters $d_1 = 1,\ d_2 = 1,\ a_1 = 2,\ a_2 = 1,\ b = 3,\ c = 5$.

5. Conclusions

In this paper, the two-component DLV system (1) was examined in order to find Q-conditional symmetries in the so-called no-go case (see (6)) and to construct exact solutions and provide their biological meaning.

From the very beginning, we modified the definition of Q-conditional symmetries of the first type [27] in the no-go case (see Definition 2). In contrast to the standard definition (see Definition 1), Definition 2 allows us to obtain the integrable system of DEs (11)–(15). Solving the system of DEs, the main theoretical result in the form of Theorem 1 was derived. The theorem presents an exhaustive list of Q-conditional symmetries of the first type, which the DLV system (1) admits depending on the parameters d_i, a_i, b_i and c_i $(i = 1, 2)$. All other cases, which are not listed in Table 1, are reducible to those in Table 1 by appropriate point transformations of the form (16).

We used the coefficient restrictions (2), which are motivated from the mathematical and applied point of view. For example, we excluded from the examination the systems of the form (1) involving a linear equation, i.e., the following:

$$u_t = d_1 u_{xx} + a_1 u, \; v_t = d_2 v_{xx} + v(a_2 + b_2 u + c_2 v). \tag{37}$$

It is not plausible that such a system can model any interaction between species (cells, chemicals) because the first equation is linear and autonomous. Interestingly, system (37) is reduced to the following:

$$u^*_t = d_1 u^*_{xx}, \; v^*_t = d_2 v^*_{xx} + c_2 e^{a_2 t} (v^*)^2 + b_2 e^{a_1 t} u^* v^*, \tag{38}$$

by the local transformation

$$u^* = e^{-a_1 t} u, \; v^* = e^{-a_2 t} v.$$

In the case $c_2 = 0$, all solutions of system (38) can be easily derived by substitution of the relevant solutions of the linear diffusion equation into the second equation, which becomes the form of a standard diffusion equation with a linear source (sink), i.e., it is again solvable via classical methods for linear PDEs. In the case $c_2 \neq 0$, the situation is more complicated because the second equation is nonlinear w.r.t. v^*.

The conditional symmetries from Table 1 allow us to construct exact solutions of the relevant systems. As a result, a variety of new exact solutions of the nonlinear DLV system (20), which is the most interesting from applicability point of view, were derived. Moreover, it was shown that all the solutions obtained are new.

Finally, we examined a model describing the competition of two populations of species. It was shown that the exact solution (33) with correctly-specified parameters are bounded, nonnegative and satisfies the zero Neumann boundary conditions at bounded space domains. Moreover, the solution possesses a realistic asymptotic behavior. Thus, we conclude that our solution with correctly-specified parameters describes the competition of two species. Interestingly, the solution is periodical in space (see Figure 1) in contrast to the known traveling wave (see Figure 2).

In conclusion, we want to highlight an unsolved problem. The nonlinear system (32) with positive parameters a_1, a_2 and negative b and c is the model describing mutualism or cooperation (see [34,36]). Obviously, the solutions constructed in this work can be used for these types of interaction in a quite similar way as that in Section 4. However, these solutions are not applicable for the third most common type of interactions between species (cells), prey–predator models. In the prey–predator model, the parameters should satisfy the conditions $a_1 a_2 < 0$, $c_1 b_2 < 0$, $b_1 \leq 0$ and $c_2 \leq 0$ (see the DLV system (1)). It can be seen that Table 1 does not contain such a type of systems; therefore, the relevant exact solutions cannot be found. Moreover, we noted that all the exact solutions derived in other papers [9–14,17,18] are not applicable for description of the prey–predator interaction as well. Thus, the problem of constructing exact solutions for the DLV system (1) modeling the interaction between the prey and predator is a hot topic.

Author Contributions: Conceptualization, methodology, formal analysis, writing—original draft preparation, R.C.; investigation, validation, software, writing—review and editing, V.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Theorem 1

In order to prove the theorem, one needs to solve the system of DEs (11)–(15) and to identify all inequivalent solutions depending on the parameters $a_i,\ b_i,\ c_i$ and d_i $(i = 1, 2)$.

First of all, we note that two essentially different cases occur (see (11)), namely: ***(i)*** $\eta^1_v = 0$ and diffusion coefficients d_1 and d_2 are arbitrary constants; ***(ii)*** $\eta^1_v \neq 0 \ \Rightarrow\ d_1 = d_2 \equiv d$.

Examination of *Case* ***(i)***. In this case, the functions $\xi,\ \eta^1$ and η^2 have the following form:

$$\xi = \xi(t,x,u),\ \eta^1 = r^1(t,x,u),\ \eta^2 = r^2(t,x,u) + q^2(t,x,u)\,v, \tag{A1}$$

where $\xi,\ r^1,\ r^2$ and q^2 are to-be-determined functions. Substituting the functions $\xi,\ \eta^1$ and η^2 from (A1) into Equations (12)–(15) and splitting the equations obtained w.r.t. the variable v, we arrive at the equation $c_1\xi_u = 0$ (see Equation (13)). So, two different subcases should be examined : ***(i1)*** $c_1 \neq 0 \ \Rightarrow\ \xi = \xi(t)$ (see the second equation in (12)); ***(i2)*** $c_1 = 0$.

Subcase ***(i1)***. Since $c_1 \neq 0$, we can set $c_1 = 1$ (using the transformation $v^* = c_1 v$). Thus, the system of DEs (11)–(15) is transformed to the following form:

$$uq^2_u - c_2q^2 = 0,\ ur^1_u - r^1 - uq^2 = 0, \tag{A2}$$

$$2d_2\left(r^1q^2_u + \xi q^2_x\right) + \xi\,\frac{d\xi}{dt} = 0, \tag{A3}$$

$$d_1r^1_{xx} - r^1_t + \frac{d_1\left(r^1\right)^2}{\xi^2}r^1_{uu} + \frac{r^1}{\xi}\left(2d_1r^1_{xu} + \frac{d\xi}{dt}\right) - u(a_1+b_1u)r^1_u + (a_1+2b_1u)r^1 + ur^2 = 0, \tag{A4}$$

$$\begin{aligned}&d_2r^2_{xx} - r^2_t + \frac{r^1}{\xi^2}\left(d_2r^1r^2_{uu} + (d_2-d_1)r^1_ur^2_u\right) - u(a_1+b_1u)r^2_u\\&+\frac{1}{\xi}\left(2d_2r^1r^2_{xu} + (d_2-d_1)r^1_xr^2_u\right) + (a_2+b_2u)r^2 = 0,\end{aligned} \tag{A5}$$

$$\begin{aligned}&d_2q^2_{xx} - q^2_t + \frac{r^1}{\xi^2}\left(d_2r^1q^2_{uu} + (d_2-d_1)r^1_uq^2_u\right) - u(a_1+b_1u)q^2_u\\&+\frac{1}{\xi}\left(2d_2r^1q^2_{xu} + (d_2-d_1)r^1_xq^2_u\right) - ur^2_u + b_2r^1 + 2c_2r^2 = 0.\end{aligned} \tag{A6}$$

Integrating Equation (A2), we have the following :

$$r^1 = \begin{cases} f^1(t,x)u + \frac{f^2(t,x)}{c_2}\,u^{c_2+1}, & \text{if} \quad c_2 \neq 0,\\ f^1(t,x)u + f^2(t,x)\,u\ln u, & \text{if} \quad c_2 = 0,\end{cases} \qquad q^2 = f^2(t,x)u^{c_2}, \tag{A7}$$

where f^1 and f^2 are to-be-determined functions. Substituting (A7) into Equation (A3), we arrive at the conditions $f^2 = 0$ and $\xi = const$. Therefore, one can set $\xi = 1$ without loss of generality.

Now we can find the function r^2 from Equation (A4) :

$$r^2 = -b_1uf^1 - 2d_1f^1f^1_x + f^1_t - d_1f^1_{xx},\ f^1 \neq 0. \tag{A8}$$

Substituting the functions r^1 and r^2 into Equations (A5) and (A6) and splitting the equations obtained w.r.t. the exponents of u, we arrive at the following system:

$$\begin{cases} b_1(b_1-b_2)=0, \\ 2b_1c_2-b_1-b_2=0, \end{cases} \Rightarrow \begin{cases} b_1=b_2\equiv b, \\ b(c_2-1)=0, \end{cases} \tag{A9}$$

$$c_2\Big(d_1f^1_{xx}-f^1_t+2d_1f^1f^1_x\Big)=0, \tag{A10}$$

$$b\Big((d_1+d_2)f^1_{xx}+2f^1_t-(d_1+3d_2)f^1f^1_x+(d_1-d_2)(f^1)^3+(a_1-a_2)f^1\Big)=0. \tag{A11}$$

Note that the case $c_2=0$ leads to the restrictions $b_1=b_2=0$ (see (A9)). Thus, one obtains the following system:

$$u_t=d_1u_{xx}+u(a_1+v),\ v_t=d_2v_{xx}+a_2v,$$

that is excluded from consideration (the second equation is linear).

In the case $b=0$, we immediately obtain the DLV system from Case 3 of Table 1 and the operator Q^u_3. To complete the examination of subcase ***(i1)*** one needs to solve the overdetermined nonlinear system of PDEs (A10) and (A11) with $b\neq 0$ (one can set $b=1$ using the transformation $u^*=bu$) and $c_2=1$. Taking into account the Burgers Equation (A10), Equation (A11) can be rewritten as follows:

$$(d_1-d_2)\big(f^1_{xx}+3f^1f^1_x+(f^1)^3\big)+(a_1-a_2)f^1=0. \tag{A12}$$

If $d_1=d_2\equiv d$, then $a_1=a_2\equiv a$, and the DLV system

$$u_t=du_{xx}+u(a+u+v),\ v_t=dv_{xx}+v(a+u+v) \tag{A13}$$

is obtained. Applying the transformation

$$x^*=\frac{1}{\sqrt{d}}\,x,\ u^*=e^{-at}u,\ v^*=u+v, \tag{A14}$$

system (A13) is reduced to the DLV system from Case 3 of Table 1 (with $d=1,\ a_2=a,\ c_2=1$) admitting the operator Q^u_3.

It is well known (see [37]) that Equation (A12) with $d_1\neq d_2$ is reducible to the following linear equation:

$$g^1_{xxx}+\frac{a_1-a_2}{d_1-d_2}\,g^1_x=0, \tag{A15}$$

by the nonlocal substitution

$$f^1=\frac{g^1_x}{g^1} \tag{A16}$$

(here $g^1(t,x)$ is a new smooth function). Integrating Equation (A15) and taking into account (A8), (A10) and (A16), we arrive at the operator Q^u_1 from Case 1 of Table 1. Since the DLV system from Case 1 of Table 1 has a symmetric structure, it admits additional operator Q^v_1, which satisfy Definition 2 on the manifold $\mathcal{M}^v_1$. Thus, subcase ***(i1)*** is completely examined.

Subcase ***(i2)*** is investigated in a quite similar way so that operators Q^v_2, Q^v_3 and Q^v_4 are derived (up to discreet transformation $u\to v,\ v\to u$).

Thus, Case ***(i)*** is completely examined and Cases 1–4 of Table 1 are obtained.

Examination of *Case* ***(ii)***. In this case, one can set $d=1$ using the transformation $x^*=\frac{1}{\sqrt{d}}\,x$. Since $\eta^1_v\neq 0$, Equations (12) and (13) lead to $\eta^1_{uv}=\eta^1_{xv}=0,\ \xi_u=\xi_x=0$ and $\eta^2_{uv}=0$, respectively. Integrating (11) and (13), one finds the functions $\xi,\ \eta^1$ and η^2 in the following form :

$$\xi=\xi(t),\ \eta^1=r^1(t,x,u)+q^1(t)\,v,\ \eta^2=r^2(t,x,u)+\left(q^2(t)-\frac{x}{2}\frac{d\xi}{dt}\right)v, \tag{A17}$$

where $\xi,\ r^1,\ r^2,\ q^1 \neq 0$ and q^2 are smooth functions, which should be determined from Equations (14) and (15).

Substituting (A17) into (14) and (15) and splitting the equations obtained w.r.t. the exponents of v, we arrive at the overdetermined system as follows:

$$q^1 r^1_{uu} = (c_2 - c_1)\xi^2, \quad \left(q^1\right)^2 r^2_{uu} = \left(\frac{c_2 x}{2}\frac{d\xi}{dt} - b_2 q^1 - c_2 q^2\right)\xi^2, \tag{A18}$$

$$\begin{aligned}&\frac{2q^1 r^1 r^1_{uu}}{\xi^2} + \frac{q^1}{\xi}\left(\frac{d\xi}{dt} + 2r^1_{xu}\right) + c_1 u\left(q^2 - \frac{x}{2}\frac{d\xi}{dt} - r^1_u\right)\\ &+c_1 r^1 + (a_1 - a_2 + 2b_1 u - b_2 u)q^1 - \frac{dq^1}{dt} = 0,\end{aligned} \tag{A19}$$

$$\frac{2q^1 r^1 r^2_{uu}}{\xi^2} + \frac{2q^1 r^2_{xu}}{\xi} - c_1 u r^2_u + b_2 r^1 + 2c_2 r^2 + \frac{x}{2}\frac{d^2\xi}{dt^2} - \frac{dq^2}{dt} = 0, \tag{A20}$$

$$r^1_{xx} - r^1_t + \frac{\left(r^1\right)^2}{\xi^2} r^1_{uu} + \frac{r^1}{\xi}\left(2r^1_{xu} + \frac{d\xi}{dt}\right) - u(a_1 + b_1 u)r^1_u + (a_1 + 2b_1 u)r^1 + c_1 u r^2 = 0, \tag{A21}$$

$$r^2_{xx} - r^2_t + \frac{\left(r^1\right)^2}{\xi^2} r^2_{uu} + \frac{2r^1}{\xi} r^2_{xu} - u(a_1 + b_1 u)r^2_u + (a_2 + b_2 u)r^2 = 0. \tag{A22}$$

Integrating Equation (A18) as two ODEs for r^1 and r^2, we find the following:

$$r^1 = \mu^0 + \mu^1 u + \mu^2 u^2,\ r^2 = \nu^0 + \nu^1 u + \nu^2 u^2, \tag{A23}$$

where $\mu^0,\ \mu^1,\ \nu^0$ and ν^1 are arbitrary smooth functions of variables t and x, while

$$\mu^2 = \frac{c_2 - c_1}{2q^1}\,\xi^2,\ \nu^2 = \frac{\xi^2}{2\left(q^1\right)^2}\left(\frac{c_2 x}{2}\frac{d\xi}{dt} - b_2 q^1 - c_2 q^2\right). \tag{A24}$$

Formula (A23) allows us to reduce Equations (A19)–(A22) to those without variable u. First of all, substituting the function r^1 from (A23) into Equation (A21), one immediately obtains $\mu^2 = 0 \Rightarrow c_1 = c_2 \equiv c$. Thus, taking into account (A23) and splitting Equations (A19)–(A22) w.r.t. the exponents of u, we arrive at the system as follows:

$$c\,\frac{d\xi}{dt} = 0,\ c\nu^2 = 0,\ (2b_1 - b_2)\nu^2 = 0,\ (2b_1 - b_2)q^1 + cq^2 = 0,\ b_1\mu^1 + c\nu^1 = 0, \tag{A25}$$

and

$$\begin{array}{l}\frac{4q^1}{\xi^2}\left(\xi\nu^2_x + \mu^1\nu^2\right) + b_2\mu^1 + c\nu^1 = 0,\\ \frac{dq^1}{dt} = \frac{q^1}{\xi}\left(2\mu^1_x + \frac{d\xi}{dt}\right) + (a_1 - a_2)q^1 + c\mu^0,\\ \frac{dq^2}{dt} = \frac{4q^1\mu^0\nu^2}{\xi^2} + \frac{2q^1\nu^1_x}{\xi} + \frac{x}{2}\frac{d^2\xi}{dt^2} + b_2\mu^0 + 2c\nu^0,\\ \mu^0_{xx} - \mu^0_t + \frac{\mu^0}{\xi}\left(2\mu^1_x + \frac{d\xi}{dt}\right) + a_1\mu^0 = 0,\\ \nu^0_{xx} - \nu^0_t + \frac{2\left(\mu^0\right)^2\nu^2}{\xi^2} + \frac{2\mu^0\nu^1_x}{\xi} + a_2\nu^0 = 0,\\ \mu^1_{xx} - \mu^1_t + \frac{\mu^1}{\xi}\left(2\mu^1_x + \frac{d\xi}{dt}\right) + 2b_1\mu^0 + c_1\nu^0 = 0,\\ \nu^1_{xx} - \nu^1_t + \frac{4\mu^0\mu^1\nu^2}{\xi^2} + \frac{2\left(2\mu^0\nu^2_x + \mu^1\nu^1_x\right)}{\xi} + b_2\nu^0 + (a_2 - a_1)\nu^1 = 0,\\ -\nu^2_t + \frac{2\left(\mu^1\right)^2\nu^2}{\xi^2} + \frac{4\mu^1\nu^2_x}{\xi} + (b_2 - b_1)\nu^1 + (a_2 - 2a_1)\nu^2 = 0.\end{array} \tag{A26}$$

Taking into account (A25), two different subcases should be examined : ***(ii1)*** $c \neq 0 \Rightarrow c = 1$; ***(ii2)*** $c = 0$.

Subcase ***(ii1)***. Formulas (A24) and (A25) lead to the conditions

$$\xi = const,\ \nu^2 = 0,\ b_1 = b_2 \equiv b,\ bq^1 + q^2 = 0,\ b\mu^1 + \nu^1 = 0. \tag{A27}$$

It turns out that the assumption $\mu_x^1 = \nu_x^1 = 0$ leads only to Lie symmetries, which is completely described in [10]. So, we assume $\left(\mu_x^1\right)^2 + \left(\nu_x^1\right)^2 \neq 0$ in what follows.

The detailed analysis of system (A26) with conditions (A27) leads to the restriction $a_1 = b = 0$. In fact, assuming $b \neq 0 \Rightarrow b = 1$, we additionally obtain $a_1 = a_2 \equiv a$. As a result, system (A13) with $d = 1$ is obtained, which is reduced to the DLV system (up to the notations)

$$u_t = u_{xx} + uv, \; v_t = v_{xx} + v(a_2 + v) \tag{A28}$$

by the transformation (A14). In the case $b = 0$, $a_1 \neq 0$, using the transformation $u^* = e^{-a_1 t} u$, we again arrive at the DLV system (A28).

Thus, to complete the examination of subcase ***(ii1)*** one needs to identify only Q-conditional symmetries of the form

$$Q = \partial_x + \left(\mu^0 + \mu^1 u + q^1 v\right)\partial_u, \; q^1 \neq 0, \; \mu_x^1 \neq 0$$

of the DLV system (A28), where the functions μ^0, μ^1 and q^1 satisfy the PDE system as follows:

$$\begin{array}{l} 2q^1\mu_x^1 - \frac{dq^1}{dt} - a_2 q^1 + \mu^0 = 0, \\ \mu_{xx}^1 - \mu_t^1 + 2\mu^1\mu_x^1 = 0, \; \mu_{xx}^0 - \mu_t^0 + 2\mu^0\mu_x^1 = 0. \end{array} \tag{A29}$$

System (A29) consists of three nonlinear PDEs; however, the function q^1 depends only on t and it allows us to treat the first equation as a linear first-order ODE. As a result, a relation between μ^0 and μ^1 is established; therefore, the 2nd and 3rd equations are solved. Finally, operators Q_7^u and Q_8^u arising in Cases 7 and 8 of Table 1 are identified.

Note that the DLV system from Case 7 is the subsystem of one from Case 3, i.e., one additionally admits the operators Q_3^u and Q_3^v provided $c_2 = 1$. A similar situation occurs for the DLV system from Case 8 of Table 1.

Thus, subcase ***(i1)*** is completely examined.

Subcase ***(ii2)*** is investigated in a quite similar way. In this case, the DLV system has the following form (see the fourth equation in (A25)):

$$u_t = u_{xx} + u(a_1 + b_1 u), \; v_t = v_{xx} + v(a_2 + 2b_1 u), \; b_1 \neq 0. \tag{A30}$$

Using the transformation

$$u^* = b_1 u, \; v^* = e^{-a_2 t} v, \tag{A31}$$

one can set $a_2 = 0$ and $b_1 = 1$ in system (A30). Solving system of DEs (A26) with $b_1 = 1$, $b_2 = \frac{1}{2}$, $a_2 = c = 0$, $\mu^0 = \mu^1 = 0$, $\nu^2 = -\frac{\xi^2}{q^1}$ and using the transformation $u^* = v$, $v^* = 2u$, the operators Q_5^v and Q_6^v arising in Cases 6 and 7 of Table 1 were identified.

Thus, Case ***(ii)*** is completely examined and Cases 5–8 of Table 1 are obtained.

Finally, it should be noted that several point transformations ((A14), (A31) and $u^* = v$, $v^* = 2u$ are examples) are used to simplify structures of the relevant DLV systems. These transformations can be united and presented in the form (16).

The proof is now complete.

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