

Article

Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications

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Abstract: A plethora of sufficient convergence criteria has been provided for single-step iterative methods to solve Banach space valued operator equations. However, an interesting question remains unanswered: is it possible to provide unified convergence criteria for single-step iterative methods, which are weaker than earlier ones without additional hypotheses? The answer is yes. In particular, we provide only one sufficient convergence criterion suitable for single-step methods. Moreover, we also give a finer convergence analysis. Numerical experiments involving boundary value problems and Hammerstein-like integral equations complete this paper.

Keywords: single step methods; Banach space; convergence criterion



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1. Introduction

Numerous applications from mathematics, economics, engineering, physics, chemistry, biology, and medicine, to mention a few, can be modeled as follows:

$$F(x) = 0, \quad (1)$$

with operator $F : \Omega \subseteq T_1 \rightarrow T_2$ acting between T_1 and T_2 , which are Banach spaces, whereas Ω is nonempty. That is why determining a solution denoted by x^* of Equation (1) is of extreme importance. However, this task is difficult in general. Ideally, one desires x^* to be in closed form, but this task is only accomplished in some instances. Practitioners and researchers resort to mostly iterative methods, generating a sequence approximating x^* under certain conditions on the initial data. The most popular single step methods are as follows:

Newton's [1,2]

$$x_{m+1} = x_m - F'(x_m)^{-1}F(x_m). \quad (2)$$

Secant [3]

$$x_{m+1} = x_m - [x_m, x_{m-1}; F]^{-1}F(x_m), \quad (3)$$

where $[\cdot, \cdot; F] : \Omega \times \Omega \rightarrow L(T_1, T_2)$.

Steffensen's-like [4]

$$x_{m+1} = x_m - [x_m + \lambda_1 F(x_m), x_m + \lambda_2 F(x_m); F]^{-1}F(x_m), \quad (4)$$

for $T_1 = T_2$ and λ_1, λ_2 being parameters.

Newton's-type [5–8]

$$x_{m+1} = x_m - A_m^{-1}F(x_m), \quad (5)$$

where $A_m = A(x_m)$, $A : \Omega \rightarrow L(T_1, T_2)$.

Stirling's [9]

$$x_{m+1} = x_m - G'(x_m)^{-1}G(x_m), \quad (6)$$

where $T_1 = T_2$ and $G(y) = y - F(y)$ are used to find fixed points of equation $x = G(x)$.

Picard's [10,11]

$$x_{m+1} = G(x_m). \quad (7)$$

Numerous other single step methods can be found in [12–14] and the references therein. Clearly, all the preceding methods can be written in a unified way as follows:

$$x_{m+2} = \varphi(x_{m+1}, x_m, x_{m-1}), \text{ for each } m = 0, 1, 2, \dots, \quad (8)$$

where $\varphi : \Omega \times \Omega \times \Omega \longrightarrow T_1$ and $x_{-1}, x_0, x_1 \in \Omega$.

We usually study two types of convergence for iterative methods. The local convergence uses information about x^* to find the radii of convergence balls. The semilocal uses information about x_0 that guarantees convergence to x^* . Sufficient convergence criteria for these methods have been provided by many authors [2,12,13].

The following common questions (Q) arise in the semilocal study of these methods:

- Q1 Can the convergence region be extended since it is small in general?
- Q2 Can the estimates on $\|x_m - x^*\|$, $\|x_{m+1} - x_m\|$ become tighter? Otherwise, we compute more iterates than we should to reach a predecided error tolerance.
- Q3 Can the convergence criteria be weakened?
- Q4 Can the location of solution be more precise?
- Q5 Is there a uniform way of studying single-step methods?
- Q6 Are there uniform convergence criteria for single-step methods?

The novelty of our paper is that we answer positively to all these questions (Q), without additional conditions.

In order to deal with single-step methods, we first consider the following iteration:

$$t_{m+2} = \psi(t_{m+1}, t_m, t_{m-1}) \text{ for each } m = 0, 1, 2, \dots, \quad (9)$$

where $\psi : [0, \infty) \times [0, \infty) \times [0, \infty)$ is a function related to the initial data. The task of choosing ψ so that sequence $\{t_n\}$ is majorizing for all methods listed previously is very difficult in general.

We define a special case of sequences given by (9) as follows:

$$\begin{aligned} t_{-1} &= \alpha, t_0 = \beta, t_1 = \gamma = \beta + \eta, \\ t_2 &= t_1 + \frac{(\bar{a}_1(t_1 - t_0) + \bar{a}_2(t_0 - t_{-1}) + \bar{a}_3 t_1 + \bar{a}_4 t_0 + \bar{a}_5 t_{-1} + \bar{a}_6)}{1 - (\bar{b}_1(t_1 - t_0) + \bar{b}_2(t_0 - t_{-1}) + \bar{b}_3 t_1 + \bar{b}_4 t_0 + \bar{b}_5 t_{-1} + \bar{b}_6)} (t_1 - t_0) \\ t_{m+2} &= t_{m+1} \\ &\quad + \frac{(a_1(t_{m+1} - t_m) + a_2(t_m - t_{m-1}) + a_3 t_{m+1} + a_4 t_m + a_5 t_{m-1} + a_6)}{1 - (b_1(t_{m+1} - t_m) + b_2(t_m - t_{m-1}) + b_3 t_{m+1} + b_4 t_m + b_5 t_{m-1} + b_6)} \\ &\quad \times (t_{m+1} - t_m), \end{aligned} \quad (10)$$

for each $m = 1, 2, \dots$, where $\alpha, \beta, \gamma, \bar{a}_i, \bar{b}_i, a_i, b_i, i = 1, 2, \dots, 6$ are nonnegative parameters. We shall show that all majorizing sequences used to study the preceding methods are specializations of $\{t_m\}$ given by (10).

Similarly, in the case of local convergence we show all preceding methods can be studied using the estimate as follows:

$$e_{m+1} \leq \lambda_m e_m, \quad (11)$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are nonnegative parameters, $e_m = \|x_m - x^*\|$ and $\lambda_m = \frac{c_1 e_m + c_2 e_{m-1} + c_3}{1 - (d_1 e_m + d_2 e_{m-1} + d_3)}$.

We suppose from now on that $\{t_m\}$ is a majorizing sequence for $\{x_n\}$. Recall that an increasing real sequence $\{t_m\}$ is majorizing for a sequence $\{x_m\}$ in a Banach space T_1 if $\|x_{m+1} - x_m\| \leq t_{m+1} - t_m$, for each $m = 0, 1, 2, \dots$ [11]. Additional conditions are needed to show that $F(\rho) = 0$, where $\rho := \lim_{m \rightarrow \infty} x_m$.

The paper contains also the semi-local as well as the local convergence of method (10) in Section 2. The numerical experiments can be found in Section 3. Conclusions appear in Section 4.

2. Majorizing Sequences and Convergence Analysis

In this section, we use majorizing sequence (10) to deal first with the semi-local convergence analysis for sequence $\{x_n\}$.

We provide very general sufficient criteria for the convergence of sequence (10).

Theorem 1. Suppose that for each $m = 0, 1, 2, \dots$ and $\bar{b}_6, b_6 \in [0, 1)$,

$$\bar{b}_1(t_1 - t_0) + \bar{b}_2(t_0 - t_{-1}) + \bar{b}_3 t_1 + \bar{b}_4 t_0 + \bar{b}_5 t_{-1} + \bar{b}_6 < 1,$$

and

$$b_1(t_{m+1} - t_m) + b_2(t_m - t_{m-1}) + b_3 t_{m+1} + b_4 t_m + b_5 t_{m-1} + b_6 < 1. \quad (12)$$

Then, sequence $\{t_m\}$ developed by (10) exists, is nondecreasing, bounded from above by $t^{**} = \frac{1-b_6}{b_1+b_2+b_3+b_4+b_5}$ and converges to its unique least upper bounds denoted by t^* , which satisfies $t_1 \leq t^* \leq t^{**}$.

Proof. Using the definition of sequence $\{t_k\}$ we see that $0 \leq t_k \leq t_{k+1}$ holds for each $k = 0, 1, 2, \dots$. Moreover, by condition (12), $t_{k+1} < t^{**}$. So, sequence $\{t_k\}$ converges to t^* . $\square$

Remark 1. Condition (12) can be satisfied only in some special cases. Next, we provide stronger conditions, which can easily be verified.

It is convenient for the following convergence analysis to develop real functions, parameters and sequences. Define functions on the interval $[0, 1)$ for $\mu = t_2 - t_1$ by the following:

$$\begin{aligned} f_k(t) = & a_1 \mu t^{k-1} + a_2 \mu t^{k-2} + a_3 t_1 \\ & + a_3 \mu (1 + t + \dots + t^{k-1}) + a_4 t_1 + a_4 (1 + t + \dots + t^{k-2}) \\ & + a_5 t_1 + a_5 \mu (1 + t + \dots + t^{k-3}) + a_6 + b_1 \mu t^k \\ & + b_2 \mu t^{k-1} + b_3 \mu t_1 + b_3 \mu (1 + t + \dots + t^k) \\ & + b_4 t_1 t + b_4 \mu (t + t^2 + \dots + t^{k-1}) + b_5 t_1 t \\ & + b_5 \mu (t + t^2 + \dots + t^{k-2}) + b_6 t - t, \end{aligned} \quad (13)$$

$$\begin{aligned} h(t) = & (b_1 + b_3)t^3 + (a_1 + a_3 - b_1 + b_2 + b_4)t^2 \\ & + (a_2 - a_1 + a_4 - b_2 + b_5)t + a_5 - a_2, \end{aligned} \quad (14)$$

$$\begin{aligned} g(t) = & a_3 t_1 + \frac{a_3 \mu}{1-t} + a_4 t_1 + \frac{a_4 t_1}{1-t} \\ & + a_5 t_1 + \frac{a_5 t_1}{1-t} + a_6 + b_3 t_1 t + \frac{b_3 \mu t}{1-t} + b_4 t_1 t \\ & + \frac{b_4 \mu t}{1-t} + b_5 \mu t + \frac{b_5 \mu t_1}{1-t} + b_6 t - t \end{aligned} \quad (15)$$

and sequence

$$\delta_k = \frac{a_1(t_{k+1} - t_k) + a_2(t_k - t_{k-1}) + a_3 t_{k-1} + a_4 t_k + a_5 t_{k-1} + a_6}{1 - (b_1(t_{k+1} - t_k) + b_2(t_k - t_{k-1}) + b_3 t_{k-1} + b_4 t_k + b_5 t_{k-1} + b_6)}. \quad (16)$$

Suppose that equations

$$h(t) = 0 \quad (17)$$

and

$$g(t) = 0 \quad (18)$$

have minimal solutions δ and λ , respectively in the interval $(0, 1)$ satisfying the following:

$$0 \leq \delta_1 \leq \delta < \lambda. \quad (19)$$

Notice that

$$f_{i+1}(t) = f_i(t) + \mu h(t)t^{i-2} \quad (20)$$

and

$$f_{i+1}(\delta) = f_i(\delta). \quad (21)$$

Indeed, by the definition of sequence $\{f_i\}$ and function h , we obtain in turn by adding and subtracting $f_i(t)$ (in the definition of $f_{i+1}(t)$) the following:

$$\begin{aligned} f_{i+1}(t) &= f_i(t) + a_1\mu t^i - a_1\mu t^{i-1} + a_2\mu t^{i-1} - a_2\mu t^{i-2} + a_3\mu t^i \\ &\quad + a_4\mu t^{i-1} + a_5\mu t^{i-2} + b_1\mu t^{i+1} - b_1\mu t^i \\ &\quad + b_2\mu t^i - b_2\mu t^{i-1} + b_3\mu t^{i+1} + b_4\mu t^i + b_5\mu t^{i-1} \\ &= f_i(t) + h(t)\mu t^{i-2}. \end{aligned}$$

In particular, by the definition of δ and (20) we obtain (21) since $h(\delta) = 0$.

Remark 2. Functions h and g appear in the proof of Theorem 1. The former is related to two consecutive functions f_i and f_{i+1} (see (20)). Then, (21) is true if (17) holds for $t = \delta$. The latter relates to the limit of these recurrent functions f_i and is independent of i . This function g then should satisfy (27) and that happens if (18) holds. Condition $0 \leq \delta_1 \leq \delta$ (see (19)) is needed to show that (22) holds for $i = 1$, which will imply the following:

$$t_3 - t_2 \leq \delta(t_2 - t_1)$$

and the induction for $0 \leq \delta_i \leq \delta$ can begin. The condition $\delta < \lambda$ is needed to show (29).

Next, we show the convergence of sequence $\{t_n\}$ under conditions (17)–(19).

Theorem 2. Under conditions (17)–(19), the conclusions of Theorem 1 hold for sequence $\{t_k\}$ but t^{**} is replaced by $s = t_1 + \frac{\mu}{1-\delta}$.

Proof. We shall show by induction

$$0 \leq \delta_i \leq \delta. \quad (22)$$

Item (22) holds for $i = 1$ by (19). Then, the definition of sequence $\{t_i\}$ and (22) give

$$\begin{aligned} 0 < t_3 - t_2 \leq \delta(t_2 - t_1) &\Rightarrow t_3 \leq t_2 + \delta(t_2 - t_1) \\ &= t_2 + (1 + \delta)(t_2 - t_1) - (t_2 - t_1) \\ &\leq t_1 + \frac{1 - \delta^2}{1 - \delta}(t_2 - t_1) < s. \end{aligned}$$

Suppose (22) holds. Then, we have the following:

$$0 < t_{i+2} - t_{i+1} \leq \delta^i(t_2 - t_1) \quad (23)$$

and

$$t_{i+2} \leq t_1 + \frac{(1 - \delta^{i+1})(t_2 - t_1)}{1 - \delta} < s. \quad (24)$$

Item (22) holds, if

$$\delta_{i+1} \leq \delta \quad (25)$$

(since $\delta_{i+1} \geq 0$). Evidently, item (25) holds, by (23) and (24), if we have the following:

$$\begin{aligned} & a_1\mu\delta^{i-1} + a_2\mu\delta^{i-2} + a_3(t_1 + \frac{(1-\delta^i)\mu}{1-\delta}) + a_4(t_1 + \frac{(1-\delta^{i-1})\mu}{1-\delta}) \\ & + a_5(1 + \frac{(1-\delta^{i-2})\mu}{1-\delta}) + a_6 + b_1\mu\delta^i + b_2\mu\delta^{i-1} \\ & + b_3(t_1 + \frac{(1-\delta^i)\mu}{1-\delta}) + b_4\delta(t_1 + \frac{(1-\delta^{i-1})\mu}{1-\delta}) \\ & + b_5\delta(t_1 + \frac{(1-\delta^{i-2})\mu}{1-\delta}) + b_6\delta - \delta \leq 0 \end{aligned} \quad (26)$$

or

$$f_{i+1}(\delta) \leq 0$$

or

$$f_i(\delta) \leq 0 \quad (27)$$

by the definition of f_i , (20) and (21). In view of (21), one obtains the following:

$$g(t) = f_\infty(t) := \lim_{i \rightarrow \infty} f_i(t). \quad (28)$$

So, we can show instead of (27) that the following holds:

$$h(\delta) \leq 0, \quad (29)$$

which is true by the definition of λ and (19). Hence, the induction for (22) is completed. Then, items (23) and (24) hold. Consequently, sequence $\{t_k\}$ converges to t^* . $\square$

Remark 3. The conditions of Theorem 2 imply condition (12) of Theorem 1 but not necessarily vice versa.

Next, we specialize $\bar{a}_i, \bar{b}_i, a_i, b_i$ in some interesting cases, justifying the already stated advantages.

Case 1: Newton's method. Let us abbreviate what is known. Suppose the following conditions (C) hold:

$$\begin{aligned} & \|F'(x_0)^{-1}F(x_0)\| \leq \eta, \\ & \|F'(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_1\|v_2 - v_1\| \text{ for each } v_1, v_2 \in \Omega, \\ & H_1 = \ell_1\eta \leq \frac{1}{2} \end{aligned} \quad (30)$$

and

$$U[x_0, u^*] \subset \Omega,$$

where $u^* = \frac{1 - \sqrt{1 - 2\ell_1\eta}}{\ell_1}$.

Next, we present the celebrated Newton–Kantorovich theorem (NKT) [10].

Theorem 3. Suppose the conditions (C) hold. Then, Newton's method converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, u^*) \cap \Omega$, and

$$\|x_{k+1} - x_k\| \leq \frac{\ell_1\|x_k - x_{k-1}\|^2}{2(1 - \ell_1\|x_k - x_0\|)} \leq \frac{\ell_1(u_k - u_{k-1})^2}{2(1 - \ell_1u_k)} = u_{k+1} - u_k$$

and

$$\|x_k - x^*\| \leq u^* - u_k,$$

where $u_0 = \eta$ and

$$u_{k+1} = u_k + \frac{\ell_1(u_k - u_{k-1})^2}{2(1 - \ell_1u_k)} \text{ for each } k = 0, 1, 2, \dots$$

Let us see what we obtain under our conditions. Suppose the following conditions (A) hold:

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$\|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq \ell_0\|v - x_0\| \text{ for each } v \in \Omega.$$

$$\text{Set } U = \Omega \cap U(x_0, \frac{1}{\ell_0}).$$

$$\|F'(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell\|v_2 - v_1\| \text{ for each } v_1, v_2 \in U,$$

$$H = \bar{\ell}\eta \leq \frac{1}{2} \quad (31)$$

and

$$U[x_0, t^*] \subset \Omega,$$

$$\text{where } \bar{\ell} = \frac{1}{8}(4\ell_0 + \sqrt{\ell_0\ell + 8\ell_0^2} + \sqrt{\ell_0\ell}).$$

Remark 4. Notice that $\ell = \ell(\Omega, \ell_0)$ but $\ell_1 = \ell_1(\Omega)$. Hence, U is used to define ℓ . It is important to see that in practice the computation of Lipschitz constant ℓ_1 requires that of center Lipschitz constant ℓ_0 and that of restricted Lipschitz constant ℓ as special cases. Hence, the conditions involving ℓ_0 and ℓ are not additional to the one involving ℓ_1 . Moreover, they are also weaker. This is also verified in the numerical section. In other words, the condition involving ℓ_1 implies the other two but not necessarily vice versa.

Next, we present our extended version of the Newton–Kantorovich Theorem 3.

Theorem 4. Suppose the conditions (A) hold. Then, Newton’s method converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, \frac{1}{\ell_0}) \cap \Omega$, and the following:

$$\|x_1 - x_0\| \leq t_1 - t_0,$$

$$\|x_2 - x_1\| \leq \frac{\ell_0\|x_1 - x_0\|^2}{2(1 - \ell_0\|x_1 - x_0\|)} \leq \frac{\ell_0(t_1 - t_0)^2}{2(1 - \ell_0t_1)} = t_2 - t_1,$$

$$\|x_{k+2} - x_{k+1}\| \leq \frac{\ell\|x_{k+1} - x_k\|^2}{2(1 - \ell_0\|x_{k+1} - x_0\|)} \leq \frac{\ell(t_{k+1} - t_k)^2}{2(1 - \ell_0t_{k+1})}$$

for each $k = 1, 2, \dots$

Proof. Simply choose $t_{-1} = 0, t_0 = 0, \bar{a}_1 = \frac{\ell_0}{2}, \bar{a}_2 = \dots = \bar{a}_6 = 0, \bar{b}_1 = \ell_0, \bar{b}_2 = \dots = \bar{b}_6 = 0, a_1 = \frac{\ell}{2}, a_2 = 0 \dots = a_6 = 0, b_0 = \ell_0$ and $b_2 = \dots = b_6 = 0$. Then, (19) reduces to (31). In particular, we use the following estimates:

$$\begin{aligned} \|F'(x_0)^{-1}(F'(x_{i+1}) - F'(x_0))\| &\leq \ell_0\|x_{i+1} - x_0\| \\ &\leq \ell_0(t_{i+1} - t_0) \leq \ell_0t_{i+1} < 1, \end{aligned}$$

so $F'(x_{i+1})^{-1} \in L(T_2, T_1)$ by the Banach perturbation lemma on invertible linear operators [10] and the following:

$$\|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - \ell_0\|x_{i+1} - x_0\|}.$$

Then, since

$$\begin{aligned} F(x_{i+1}) &= F(x_{i+1}) - F(x_i) - F'(x_i)(x_{i+1} - x_i) \\ &= \int_0^1 (F'(x_i + \tau(x_{i+1} - x_i)) - F'(x_i))d\tau(x_{i+1} - x_i), \end{aligned}$$

we obtain the following:

$$\|x_{i+2} - x_{i+1}\| \leq \frac{\bar{\ell} \|x_{i+1} - x_i\|^2}{2(1 - \ell_0 \|x_{i+1} - x_i\|)} \leq t_{i+2} - t_{i+1},$$

where we also used the following:

$$\|x_{i+1} - x_0\| \leq \sum_{m=1}^m \|x_m - x_{m-1}\| \leq \sum_{m=1}^{i+1} (t_m - t_{m-1}) \leq t_{m+1} - t_0 = t_{m+1} \leq t^*,$$

and

$$\|x_i + \tau(x_{i+1} - x_i) - x_0\| \leq t_i + \tau(t_{i+1} - t_i) \leq t^*$$

for each $\tau \in [0, 1]$. So, the sequence $\{t_k\}$ is majorizing for $\{x_k\}$. Then, sequence $\{x_k\}$ is fundamental in T_1 , which is a Banach space, so $\lim_{k \rightarrow \infty} x_k = x^* \in U[x_0, t^*]$, which solves Equation (1), since

$$\|F'(x_0)^{-1}F(x_{i+1})\| \leq \frac{\bar{\ell}}{2} \|x_{i+1} - x_i\|^2 \leq \frac{\bar{\ell}}{2} (t_{i+1} - t_i)^2 \rightarrow 0$$

as $i \rightarrow \infty$. Then, we conclude that $F(x^*) = 0$ since F is a continuous operator, where

$\bar{\ell} = \begin{cases} \ell_0, & i = 0 \\ \ell, & i = 1, 2, \dots \end{cases}$ Let $x_* \in U(x_0, \frac{1}{\ell_0}) \cap \Omega$ with $F(x_*) = 0$. Set $M = \int_0^1 F'(x_* + \tau(x^* - x_*))d\tau$. Using the center Lipschitz condition, we have the following:

$$\|F'(x_0)^{-1}(M - F'(x_0))\| \leq \ell_0 \int_0^1 [(1 - \tau)\|x_* - x_0\| + \tau\|x^* - x_0\|]d\tau < \ell_0 \frac{1}{\ell_0} = 1,$$

so $x^* = x_*$ follows since M^{-1} exists and $M(x^* - x_*) = F(x^*) - F(x_*) = 0$. $\square$

Remark 5. (a) We have by the definition of U

$$U \subset \Omega, \quad (32)$$

so

$$\ell \leq \ell_1, \ell_0 \leq \ell_1 \quad (33)$$

and

$$\bar{\ell} \leq \ell_1. \quad (34)$$

Hence, we have

$$H_1 \leq \frac{1}{2} \Rightarrow H \leq \frac{1}{2} \quad (35)$$

$$t_{k+1} - t_k \leq u_{k+1} - u_k \quad (36)$$

and

$$t^* \leq u^*. \quad (37)$$

Estimates (35)–(37) justify the benefits as stated previously. In the numerical section, we provide examples where (32)–(34) are strict, and (31) holds but not (30).

(b) The proof in Theorem 3 used the less precise estimate as follows:

$$\|F'(x_{i+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - \ell_1 \|x_{i+1} - x_0\|}.$$

Our modification leads to (31) instead of (30). Moreover, in [15] we showed Theorem 4 but using the following:

$$H_2 = \ell_2 \eta \leq \frac{1}{2},$$

where

$$\ell_2 = \frac{1}{8}(4\ell_0 + \sqrt{\ell_0\ell_1 + 8\ell_0^2} + \sqrt{\ell_0\ell_1}) \geq \bar{\ell},$$

so

$$H_2 \leq \frac{1}{2} \Rightarrow H \leq \frac{1}{2}.$$

Hence, our results extend the ones in [15] too.

- (c) Let us see how parameters $\delta_1, \delta, \lambda$ and functions h, g look like in the case of Newton's method. We obtain by (17)–(19) the following:

$$\delta_1 = \frac{\ell\eta}{2(1-\ell_0\eta)}, \delta = \frac{2\ell}{\ell + \sqrt{\ell^2 + 8\ell_0\ell}}, \lambda = 1 - \frac{1}{2} \left(\frac{\ell_0\eta}{1-\ell_0\eta} \right)^2,$$

$$h(t) = 2\ell_0 t^2 + \ell t - \ell,$$

and

$$g(t) = \left(\frac{\ell_0}{1-t} \mu + \ell_0\eta - 1 \right) t.$$

Notice that δ, λ solve Equations (17) and (18), respectively. Then, if we solve inequality (19), we obtain (31).

Comments similar to the ones given in the previous five remarks can be made for the methods that follow in this Section.

Case 2: Secant method [14] Choose $t_{-1} = 0, t_0 = \beta, t_1 = \beta + \eta, \bar{a}_1 = \bar{a}_2, \bar{b}_1 = \bar{b}_2, \bar{a}_3 = \bar{a}_4 = \bar{a}_5 = \bar{a}_6 = b_3 = b_4 = b_5 = b_6 = 0, a_1 = a_2$ and $b_1 = b_2$.

The nonzero parameters are again connected to the following:

$$\|x_0 - x_{-1}\| \leq \beta, \|[x_0, x_{-1}; F]^{-1}F(x_0)\| \leq \eta$$

$$\|[x_0, x_{-1}; F]^{-1}([v_1, v_2; F] - [x_0, x_{-1}; F])\| \leq \frac{\ell_0}{2}(\|v_1 - x_0\| + \|v_2 - x_{-1}\|)$$

for each $v_1, v_2 \in \Omega$,

$$\|[x_0, x_{-1}; F]^{-1}([v_1, v_2; F] - [z, w; F])\| \leq \frac{\ell}{2}(\|v_1 - z\| + \|v_2 - w\|)$$

for each $v_1, v_2, z, w \in V$, provided that

$$[v_1, v_2; F] = \int_0^1 F'(v_2 + \tau(v_1 - v_2))d\tau.$$

The standard condition used in connection to the secant method [14] is the following:

$$\|[x_0, x_{-1}; F]^{-1}([v_1, v_2; F] - [z, w; F])\| \leq \frac{\ell_1}{2}(\|v_1 - z\| + \|v_2 - w\|)$$

for each $v_1, v_2, z, w \in \Omega$. Then, we have again the following:

$$\ell \leq \ell_1$$

and

$$\ell_0 \leq \ell_1.$$

The old majorizing sequence $\{u_n\}$ [14] is defined by the following:

$$\begin{aligned} u_{-1} &= 0, u_0 = \beta, u_1 = \beta + \eta \\ u_{k+2} &= u_{k+1} + \frac{\ell_1(u_{k+1} - u_{k-1})(u_{k+1} - u_k)}{2(1 - \frac{\ell_1}{2}(u_{k+1} + u_k + \beta))} \end{aligned}$$

with the following estimates:

$$\|x_{k+2} - x_{k+1}\| \leq \frac{\ell_1 \|x_{k+1} - x_{k-1}\| \|x_{k+1} - x_k\|}{2(1 - \frac{\ell_1}{2}(\|x_{k+1} - x_0\| + \|x_k - x_0\| + \beta))} \leq u_{k+2} - u_{k+1}.$$

However, ours is as follows:

$$\begin{aligned} t_{-1} &= 0, t_0 = \beta, t_1 = \beta + \eta \\ t_{k+2} &= t_{k+1} + \frac{\tilde{\ell}(t_{k+1} - t_{k-1})(t_{k+1} - t_k)}{2(1 - \frac{\ell_0}{2}(t_{k+1} + t_k + \beta))} \end{aligned}$$

with corresponding estimates

$$\|x_{k+2} - x_{k+1}\| \leq \frac{\tilde{\ell} \|x_{k+1} - x_{k-1}\| \|x_{k+1} - x_k\|}{2(1 - \frac{\ell_0}{2}(\|x_{k+1} - x_0\| + \|x_k - x_0\| + \beta))} \leq t_{k+2} - t_{k+1}$$

which are tighter, where

$$\tilde{\ell} = \begin{cases} \ell_0, & k = 0 \\ \ell, & k = 1, 2, \dots \end{cases}$$

The old sufficient convergence criterion [14] is $\beta + \sqrt{2\ell_1\eta} \leq 1$ but the new one is (for $\ell_0 = \ell$) $\beta + \sqrt{2\ell\eta} \leq 1$, which is weaker. Hence, we obtain the semi-local convergence of the secant method.

Theorem 5. Under the preceding conditions secant method $\{x_n\} \subset U[x_0, t^*]$ and $\lim k \rightarrow \infty x_k = x^* \in U[x_0, t^*]$ with $F(x^*) = 0$.

Proof. As in Theorem 4, we obtain the following:

$$\|[x_{i+1}, x_i; F]^{-1}[x_0, x_{-1}; F]\| \leq \frac{1}{1 - \frac{\ell_0}{2}(\|x_{i+1} - x_0\| + \|x_i - x_{-1}\|)}$$

and

$$\begin{aligned} \|x_{i+2} - x_{i+1}\| &\leq \|[x_{i+1}, x_i; F]^{-1}[x_0, x_{-1}; F]\| \\ &\quad \times \|[x_0, x_{-1}; F]^{-1}([x_{i+1}, x_i; F] - [x_i, x_{i-1}; F])\| \\ &\leq \frac{\tilde{\ell} \|x_{i+1} - x_{i-1}\| \|x_{i+1} - x_i\|}{2(1 - \frac{\ell_0}{2}(\|x_{i+1} - x_0\| + \|x_i - x_0\| + \beta))} \\ &\leq t_{i+2} - t_{i+1} \end{aligned}$$

(see also [14]). $\square$

Case 3: Newton-type method [8,16] Choose: $t_{-1} = 0, t_0 = 0, t_1 = \eta, \bar{a}_1 = \frac{\ell_0}{2}, \bar{a}_2 = 0, \bar{a}_3 = \ell_5, \bar{a}_4 = \bar{a}_5 = 0, \bar{a}_6 = \ell_6, a_1 = \frac{\ell_4}{2}, a_2 = 0, a_3 = \ell_5, a_4 = a_5 = 0, a_6 = \ell_6, \bar{b}_1 = \ell_2, \bar{b}_2 = 0, \bar{b}_3 = \ell_2, \bar{b}_4 = \bar{b}_5 = 0, \bar{b}_6 = \ell_3, b_1 = \ell_2, b_2 = 0, b_3 = \ell_2, b_4 = b_5 = 0$ and $b_6 = \ell_3$.

The parameters are connected to the following:

$$\|A(x_0)^{-1}F(x_0)\| \leq \eta,$$

$$\|A(x_0)^{-1}(F'(v) - F'(x_0))\| \leq \ell_0 \|v - x_0\| \text{ for each } v \in \Omega$$

and

$$\|A(x_0)^{-1}(A(v) - A(x_0))\| \leq \ell_2 \|v - x_0\| + \ell_3.$$

Set $V_1 = \Omega \cap U[x_0, \frac{1-\ell_3}{\ell_2}]$, $\ell_2 \neq 0, \ell_3 \in [0, 1)$.

$$\|A(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_4 \|v_2 - v_1\| \text{ for each } v_1, v_2 \in V_1$$

and

$$\|A(x_0)^{-1}(F'(v) - A(v))\| \leq \ell_5 \|v - x_0\| + \ell_6 \text{ for each } v \in V_1$$

The conditions in [8,16] use the following:

$$\|A(x_0)^{-1}(F'(v_2) - F'(v_1))\| \leq \ell_7 \|v_2 - v_1\| \text{ for each } v_1, v_2 \in \Omega$$

and

$$\|A(x_0)^{-1}(F'(v) - A(v))\| \leq \ell_8 \|v - x_0\| + \ell_9 \text{ for each } v \in \Omega.$$

We have the following:

$$V_1 \subseteq \Omega,$$

so

$$\ell_4 \leq \ell_7,$$

$$\ell_5 \leq \ell_8$$

and

$$\ell_6 \leq \ell_9.$$

The old majorizing sequence $\{u_n\}$ [8,16] is defined for $u_{-1} = 0, u_0 = 0, u_1 = \eta$, $\sigma_1 = \max\{\ell_7, \ell_8 + \ell_2\}$ by

$$u_{i+1} = u_i + \frac{\frac{\sigma_1}{2}(u_i - u_{i-1}) + \ell_8 u_{i-1} + \ell_9)(u_i - u_{i-1})}{1 - (\ell_2 u_i + \ell_3)}$$

with the following estimates:

$$\begin{aligned} \|x_{i+1} - x_i\| &\leq \|A_i(x_i)^{-1}A(x_0)\| \left(\int_0^1 \|A(x_0)^{-1}(F'(x_i + \tau(x_{i-1} - x_i)) - F'(x_{i-1}))\| d\tau \right. \\ &\quad \left. + \|A(x_0)^{-1}(F'(x_{i-1}) - A(x_{i-1}))\| \right) \|x_i - x_{i-1}\| \\ &\leq \frac{(\frac{\ell_7}{2}\|x_i - x_{i-1}\| + \ell_8\|x_{i-1} - x_0\| + \ell_9)\|x_i - x_{i-1}\|}{1 - (\ell_2\|x_i - x_0\| + \ell_3)} \\ &\leq \frac{(\frac{\sigma_1}{2}(u_i - u_{i-1}) + \ell_8 u_{i-1} + \ell_9)(u_i - u_{i-1})}{1 - (\ell_2 u_i + \ell_3)} \\ &= u_{i+1} - u_i. \end{aligned}$$

However, ours is for $t_{-1} = 0, t_0 = 0, t_1 = \eta, \sigma = \max\{\ell_4, \ell_5 + \ell_2\}$

$$t_{i+1} = t_i + \frac{(\frac{\sigma}{2}(t_i - t_{i-1}) + \ell_5 t_{i-1} + \ell_6)(t_i - t_{i-1})}{1 - (\ell_2 t_i + \ell_3)}$$

with the following estimates:

$$\begin{aligned} \|x_{i+1} - x_i\| &\leq \frac{(\frac{\ell_4}{2}\|x_i - x_{i-1}\| + \ell_5\|x_{i-1} - x_0\| + \ell_6)\|x_i - x_{i-1}\|}{1 - (\ell_2\|x_i - x_0\| + \ell_3)} \\ &\leq \frac{(\frac{\sigma}{2}(t_i - t_{i-1}) + \ell_5 t_{i-1} + \ell_6)(t_i - t_{i-1})}{1 - (\ell_2 t_i + \ell_3)} \\ &= t_{i+1} - t_i. \end{aligned}$$

The old sufficient convergence criterion [8,16] is the following:

$$C_1 = \sigma_1 \eta \leq \frac{1}{2}(1 - (\ell_3 + \ell_9))^2, \ell_3 + \ell_9 < 1.$$

The new one is the following:

$$C = \sigma \eta \leq \frac{1}{2}(1 - (\ell_3 + \ell_6))^2, \ell_3 + \ell_6 < 1.$$

However, $\sigma \leq \sigma_1$, so again condition C is weaker than C_1 .

Hence, we obtain the semilocal convergence of the Newton-type method.

Theorem 6. Under the preceding conditions Newton-type method $\{x_n\} \subset U[x_0, t^*]$ and $\lim_{k \rightarrow \infty} x_k = x^* \in U[x_0, t^*]$ with $F(x^*) = 0$.

Proof. It follows from the aforementioned estimates (see also [8,16]). Hence, again the results are extended.

Similar benefits are derived in the local convergence case.

Suppose the conditions (B) hold:

$x^* \in \Omega$ is a simple solution of equation $F(x) = 0$,

$$\|F'(x^*)^{-1}(F'(v_2) - F'(v_1))\| \leq L_1 \|v_2 - v_1\| \text{ for each } v_1, v_2 \in \Omega,$$

and

$$U[x^*, r] \subset \Omega,$$

where $r = \frac{2}{3L_1}$. Then, we have the following local convergence result arrived at independently by Rheinboldt [17] and Traub [18]. $\square$

Theorem 7. Suppose that the conditions (B) hold. Then, Newton's method converges to x^* so that the following holds:

$$\|x_{k+1} - x^*\| \leq \frac{L_1 \|x_k - x^*\|^2}{2(1 - L_1 \|x_k - x^*\|)}$$

for each $k = 0, 1, 2, \dots$, provided that $x_0 \in U(x^*, r)$.

In our case, we consider the conditions (D):

$x^* \in \Omega$ is a simple solution of equation $F(x) = 0$.

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq L_0 \|v - x^*\| \text{ for each } v \in \Omega.$$

Set $V = \Omega \cap U(x^*, \frac{1}{L_0})$.

$$\|F'(x^*)^{-1}(F'(v_2) - F'(v_1))\| \leq L \|v_2 - v_1\| \text{ for each } v_1, v_2 \in V.$$

$U(x^*, R) \subset \Omega$, where $R = \frac{2}{2L_0 + L}$.

Theorem 8. Suppose that the conditions (D) hold. Then, Newton's method converges to x^* so that the following holds:

$$\|x_{k+1} - x^*\| \leq \frac{\tilde{L} \|x_k - x^*\|^2}{2(1 - L_0 \|x_k - x^*\|)}$$

for each $k = 0, 1, 2, \dots$, provided that $x_0 \in U(x^*, R)$, where $\tilde{L} = \begin{cases} L_0, & k = 0 \\ L, & k = 1, 2, \dots \end{cases}$

Proof. Choose $c_1 = \frac{L}{2}, d_1 = L_0, c_2 = c_3 = d_2 = d_3 = 0$ in (11). Then, we obtain the following:

$$\begin{aligned}
\|x_{i+1} - x^*\| &= \|x_i - x^* - F'(x^*)^{-1}F(x_i)\| \\
&\leq \|F'(x_i)^{-1}F'(x^*)\| \int_0^1 \|F'(x^*)^{-1}(F'(x^* + \tau(x_i - x^*)) - F'(x_i))\| d\tau \|x_i - x^*\| \\
&\leq \frac{\tilde{L}\|x_i - x^*\|^2}{2(1 - L_0\|x_i - x^*\|)}.
\end{aligned}$$

□

Remark 6. We have again the following:

$$V \subset \Omega,$$

so

$$L_0 \leq L_1,$$

$$L \leq L_1,$$

$$\tilde{L} \leq L_1,$$

$$r \leq R,$$

$$\lambda_n \leq \lambda_n^1,$$

where $\lambda_k = \frac{L\|x_k - x^*\|}{2(1 - L_0\|x_k - x^*\|)}$ and $\lambda_k^1 = \frac{L_1\|x_k - x^*\|}{2(1 - L_1\|x_k - x^*\|)}$ (see also the numerical section).

The same benefits can be obtained for the other single-step methods. Moreover, our idea can similarly be extended to multi-step and multi-point methods [4,5,13,19–37].

3. Numerical Experiments

We contact some experiments showing that the old convergence criteria are not verified, but ours are. Hence, there is no assurance that the methods converge under the old conditions. However, under our approach, convergence can be established.

Example 1. Define function as the following:

$$f(x) = \theta_0 x + \theta_1 + \theta_2 \sin \theta_3 x, \quad x_0 = 0,$$

where θ_j , $j = 0, 1, 2, 3$ are parameters. Then, clearly for θ_3 large and θ_2 small, $\frac{\ell_0}{\ell_1}$ can be small (arbitrarily). Notice that as $\frac{\ell_0}{\ell_1} \rightarrow 0$, $\frac{H}{H_1} \rightarrow 0$ too. So, the utilization of Newton's method is extended numerous (infinitely many) times under the data $(\Omega, F, x_0, \ell_0, \ell, \eta)$.

Example 2. Let $T_1 = T_2 = \mathbb{R}$, $x_0 = 1$ and $\Omega = U[1, 1 - q]$ for $q \in (0, \frac{1}{2})$. Let function f on Ω as the following:

$$f(s) = s^3 - q.$$

We consider case 1 of Newton's method. Then, we obtain $\ell_0 = 3 - q$, $\ell = \ell_1 = 2(2 - q)$ and $\eta = \frac{1}{3}(1 - q)$. However, then, $H_1 > \frac{1}{2}$ for all $q \in (0, \frac{1}{2})$. So, the Newton–Kantorovich theorem cannot assure convergence. However, we have $H \leq \frac{1}{2}$ for all $q \in I = [0.4271907643, \frac{1}{2})$. Hence, our result guarantees convergence to $x^* = \sqrt[3]{q}$ as long as $q \in I$.

Example 3. Let $T_1 = T_2 = S([0, 1])$ the domain of functions given on $[0, 1]$ which are continuous. We consider the norm-max. Choose $\Omega = U(0, d)$, $d > 1$. Define F on Ω by the following:

$$F(x)(s) = x(s) - w(s) - \zeta \int_0^1 K(s, t)x^3(t)dt, \quad (38)$$

$x \in T_1, s \in [0, 1], w \in T_1, \zeta$ is a number and K is the Green's kernel given by the following:

$$K(s_2, s_1) = \begin{cases} (1 - s_2)s_1, & s_1 \leq s_2 \\ s_2(1 - s_1), & s_2 \leq s_1. \end{cases}$$

By (38), we have the following:

$$(F'(x)(z))(s) = z(s) - 3\zeta \int_0^1 K(s, t)x^2(t)z(t)dt,$$

$t \in T_1, s \in [0, 1]$. Consider $x_0(s) = w(s) = 1$ and $|\zeta| < \frac{8}{3}$. We obtain the following:

$$\|I - F'(x_0)\| < \frac{3}{8}|\zeta|, F'(x_0)^{-1} \in L(T_2, T_1),$$

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\zeta|}, \eta = \frac{|\zeta|}{8 - 3|\zeta|}, \ell_0 = \frac{12|\zeta|}{8 - 3|\zeta|},$$

$\ell_1 = \ell = \frac{6d|\zeta|}{8 - 3|\zeta|}$ and $H_1 = \frac{6d|\zeta|^2}{(8 - 3|\zeta|)^2}$. Let ζ^* stand for the positive solution of equation $3(4d - 3)t^2 + 48t - 64 = 0$. Then, if $\zeta > \zeta^*$, we have $H_1 > \frac{1}{2}$. Hence, the Newton–Kantorovich criterion (30) is not satisfied. In particular, Table 1 shows that our criterion (31) is satisfied but not (30).

Table 1. Comparison table of criteria (30) and (31).

d	ζ^*	$2H_1$	$2H$
2.09899	0.9976613778	1.007515200	0.9639223786
2.19897	0.9831766058	1.055505600	0.9678118280
2.29597	0.9698185659	1.102065600	0.9715205068
3.095467	0.87963113211	1.485824160	1.000082409

Example 4. Let T_1, T_2 and Ω be as in the Example 3. It is well known that the boundary value problem [2]

$$\begin{aligned} \varphi(0) &= 0, \varphi(1) = 1, \\ \varphi'' &= -\varphi - \lambda\varphi^2 \end{aligned}$$

can be given as a Hammerstein-like nonlinear integral equation as follows:

$$\varphi(s) = s + \int_0^1 K(s, t)(\varphi^3(t) + \lambda\varphi^2(t))dt$$

where λ is a parameter. Then, define $F : \Omega \rightarrow T_2$ by the following:

$$[F(x)](s) = x(s) - s - \int_0^1 K(s, t)(x^3(t) + \lambda x^2(t))dt.$$

Choose $\varphi_0(s) = s$ and $\Omega = U(\varphi_0, r_0)$. Then, clearly $U(\varphi_0, r_0) \subset U(0, r_0 + 1)$, since $\|\varphi_0\| = 1$. Suppose $2\lambda < 5$. Then, by conditions (C) they are satisfied for the following:

$$\ell_0 = \frac{2\lambda + 3r_0 + 6}{8}, \ell_1 = \ell = \frac{\lambda + 6r_0 + 3}{4}$$

and $\eta = \frac{1+\lambda}{5-2\lambda}$. Notice that $\ell_0 < \ell_1$.

The rest of the examples are given for the local convergence study of Newton's method.

Example 5. Let $T_1 = T_2 = \mathbb{R}^3$, $\Omega = U[0, 1]$ and $x^* = (0, 0, 0)^{tr}$. Define mapping E on Ω for $\lambda = (\lambda_1, \lambda_2, \lambda_3)^{tr}$ as

$$E(\lambda) = (e^{\lambda_1} - 1, \frac{e-1}{2}\lambda_2^2 + \lambda_1, \lambda_3)^{tr}.$$

Then, conditions (B) and (D) hold, provided that $L_0 = e - 1$, $L = e^{\frac{1}{L_0}}$ and $L_1 = e$, since $F'(x^*)^{-1} = F'(x^*) = \text{diag}\{1, 1, 1\}$. Notice that

$$L_0 < L < L_1$$

and

$$r = 0.24 < R = 0.38.$$

Hence, our radius of convergence is larger.

Example 6. Let T_1, T_2 and Ω be as in Example 3. Define F on Ω as

$$F(\varphi_1)(x) = \varphi_1(x) - \int_0^1 x\varphi_1(j)^3 dj.$$

By this definition, we obtain the following:

$$F'(\varphi_1(\psi_1))(x) = \psi_1(x) - 3 \int_0^1 xj\varphi_1(j)^2\psi_1(j) dj$$

for all $\psi_1 \in \Omega$. So, we can choose $\ell_0 = 1.5$, $\ell = \ell_1 = 3$. However, then, we again obtain the following:

$$r = \frac{2}{9} < R = \frac{1}{3}.$$

4. Conclusions

We have provided a single sufficient criterion for the semi-local convergence of single step methods. Upon specializing the parameters involved, we showed that although our majorizing sequence is more general than earlier ones, the convergence criteria are weaker (i.e., the utility of the methods is extended), the upper error estimates are more accurate (i.e., at least as few iterates are required to achieve a predecided error tolerance), and we have, at most, an as-small ball containing the solution. These benefits are obtained without additional hypotheses. According to our new technique, we locate a more accurate domain than the earlier ones containing the iterates, leading to a more accurate Lipschitz condition (at least as small).

Our theoretical results are further justified using numerical experiments. In the future, we plan to extend these results by replacing the Lipschitz constants by generalized functions along the same lines [2,12,13].

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