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Hermite–Hadamard–Fejér-Type Inequalities and Weighted Three-Point Quadrature Formulae

Mihaela Ribičić Penava

Department of Mathematics, Josip Juraj Strossmayer University of Osijek, Trg Ljudevita Gaja 6, 31000 Osijek, Croatia; mihaela@mathos.hr

Abstract: The goal of this paper is to derive Hermite–Hadamard–Fejér-type inequalities for higher-order convex functions and a general three-point integral formula involving harmonic sequences of polynomials and w-harmonic sequences of functions. In special cases, Hermite–Hadamard–Fejér-type estimates are derived for various classical quadrature formulae such as the Gauss–Legendre three-point quadrature formula and the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

Keywords: Hermite–Hadamard–Fejér inequalities; weighted three-point formulae; higher-order convex functions; w-harmonic sequences of functions; harmonic sequences of polynomials

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1. Introduction

The Hermite–Hadamard inequalities and their weighted versions, the so-called Hermite-Hadamard-Fejér inequalities, are the most well-known inequalities related to the integral mean of a convex function (see [1] (p. 138)).

Theorem 1 (The Hermite–Hadamard–Fejér inequalities). *Let $h : [a,b] \to \mathbb{R}$ be a convex function. Then*

$$h\left(\frac{a+b}{2}\right)\int_a^b u(x)\,dx \le \int_a^b u(x)h(x)\,dx \le \left[\frac{1}{2}h(a)+\frac{1}{2}h(b)\right]\int_a^b u(x)\,dx, \tag{1}$$

where $u : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$. If h is a concave function, then the inequalities in (1) are reversed.

If $u \equiv 1$, then we are talking about the Hermite–Hadamard inequalities.

Hermite–Hadamard and Hermite–Hadamard–Fejér-type inequalities have many applications in mathematical analysis, numerical analysis, probability and related fields. Their generalizations, refinements and improvements have been an important topic of research (see [1–13], and the references listed therein). In the past few years, Hermite–Hadamard–Fejér-type inequalities for superquadratic functions [2], GA-convex functions [7], quasi-convex functions [11] and convex functions [13] have been largely investigated in the literature.

The importance and significance of our paper are reflected in the way in which we prove new Hermite–Hadamard–Fejér-type inequalities for higher-order convex functions and the general weighted three-point quadrature formula by using inequality (1), and a weighted version of the integral identity expressed by w-harmonic sequences of functions.

For this purpose, let us introduce the notations and terminology used in relation to w-harmonic sequences of functions (see [14]).

Let us consider a subdivision $\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ of the segment $[a,b]$, $m \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an arbitrary integrable function. For each segment $[x_{j-1}, x_j]$, $j = 1, \ldots, m$, we define w-harmonic sequences of functions $\{w_{jk}\}_{k=1,\ldots,n}$ by:

$$\begin{aligned} w'_{j1}(t) &= w(t),\ t \in [x_{j-1}, x_j], \\ w'_{jk}(t) &= w_{j,k-1}(t),\ t \in [x_{j-1}, x_j],\ k = 2,3,\ldots,n. \end{aligned} \tag{2}$$

Further, the function $W_{n,w}$ is defined as follows:

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t), & t \in [a, x_1], \\ w_{2n}(t), & t \in (x_1, x_2], \\ \cdot \\ \cdot \\ \cdot \\ w_{mn}(t), & t \in (x_{m-1}, b]. \end{cases} \tag{3}$$

The following theorem gives a general integral identity (see [14]).

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is piecewise continuous on $[a,b]$. Then, the following holds:*

$$\begin{aligned} \int_a^b w(t) f(t)\, dt &= \sum_{k=1}^{n} (-1)^{k-1} \Big[w_{mk}(b) f^{(k-1)}(b) \\ &+ \sum_{j=1}^{m-1} \Big[w_{jk}(x_j) - w_{j+1,k}(x_j) \Big] f^{(k-1)}(x_j) - w_{1k}(a) f^{(k-1)}(a) \Big] \\ &+ (-1)^n \int_a^b W_{n,w}(t,\sigma) f^{(n)}(t)\, dt. \end{aligned} \tag{4}$$

In [15], the authors proved the following Fejér-type inequalities by using identity (4).

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be $(n+2)$-convex on $[a,b]$ and $f^{(n)}$ piecewise continuous on $[a,b]$. Further, let us suppose that the function $W_{n,w}$, defined in (3), is nonnegative and symmetric about $\frac{a+b}{2}$ (i.e., $W_{n,w}(t,\sigma) = W_{n,w}(a+b-t,\sigma)$). Then*

$$\begin{aligned} & U_n(\sigma) \cdot f^{(n)}\left(\frac{a+b}{2}\right) \\ & \leq (-1)^n \Bigg\{ \int_a^b w(t) f(t)\, dt - \sum_{k=1}^{n} (-1)^{k-1} \Big[w_{mk}(b) f^{(k-1)}(b) \\ & + \sum_{j=1}^{m-1} \Big[w_{jk}(x_j) - w_{j+1,k}(x_j) \Big] f^{(k-1)}(x_j) - w_{1k}(a) f^{(k-1)}(a) \Big] \Bigg\} \\ & \leq U_n(\sigma) \cdot \left[\frac{1}{2} f^{(n)}(a) + \frac{1}{2} f^{(n)}(b) \right], \end{aligned} \tag{5}$$

where

$$\begin{aligned} U_n(\sigma) = \frac{(-1)^n}{n!} \int_a^b w(t) \cdot t^n\, dt - (-1)^n \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(n-k+1)!} \\ \cdot \left(w_{mk}(b) b^{n-k+1} + \sum_{j=1}^{m-1} \Big(w_{jk}(x_j) - w_{j+1,k}(x_j) \Big) x_j^{n-k+1} - w_{1k}(a) a^{n-k+1} \right). \end{aligned} \tag{6}$$

If $W_{n,w}(t,\sigma) \leq 0$ or f is an $(n+2)$-concave function on $[a,b]$, then the inequalities in (5) hold with reversed inequality signs.

Further, let us recall the definition of the divided difference and the definition of an n-convex function (see [1] (p. 15)).

Definition 1. *Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,\dots,x_n \in [a,b]$ is defined recursively by*

$$f[x_i] = f(x_i), \quad (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of points $x_0,\dots,x_n$.

Definition 2. *A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if, for all choices of $(n+1)$ distinct points $x_0,\dots,x_n \in [a,b]$, the n-th order divided difference in f satisfies*

$$f[x_0,\dots,x_n] \geq 0.$$

From the previous definitions, the following property holds: if f is an $(n+2)$-convex function, then there exists the n-th order derivative $f^{(n)}$, which is a convex function (see, e.g., [1] (pp. 16, 293)).

The paper is organized as follows. After this introduction, in Section 2, we establish Hermite–Hadamard–Fejér-type inequalities for weighted three-point quadrature formulae by using the integral identity with w-harmonic sequences of functions, the properties of harmonic sequences of polynomials and the properties of n-convex functions. Since we deal with three-point quadrature formulae that contain values of the function in nodes x, $\frac{a+b}{2}$ and $a+b-x$ and values of higher-ordered derivatives in inner nodes, the level of exactness of these quadrature formulae is retained. In Section 3, we derive Hermite–Hadamard–Fejér-type estimates for a generalization of the Gauss–Legendre three-point quadrature formula, and a generalization of the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

Throughout the paper, the symbol B denotes the beta function defined by

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\,ds,$$

Γ denotes the gamma function defined as:

$$\Gamma(x) = 2\int_0^\infty s^{2x-1}e^{-s^2}\,ds,$$

and

$$F(\alpha,\beta,\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt$$

is a hypergeometric function with $\gamma > \beta > 0$, $z < 1$.

In the paper, we assume that all considered integrals exist and that they are finite.

2. Hermite–Hadamard–Fejér-Type Inequalities for Three-Point Quadrature Formulae

In this section, we establish Hermite–Hadamard–Fejér-type inequalities for the weighted three-point formula using a weighted version of the integral identity expressed by w-

harmonic sequences of functions that are given in Theorem 2 and the method that originated in [15].

In [16] (p. 54), the authors proved the following theorem.

Theorem 4. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function, $x \in [a, \frac{a+b}{2})$, and let $\{L_{j,x}\}_{j=0,1,\dots,n}$, $n \in \mathbb{N}$, be a sequence of harmonic polynomials such that deg $L_{j,x} \le j-1$ and $L_{0,x} \equiv 0$. Further, let us suppose that $\{w_{jk}\}_{k=1,\dots,n}$ are w-harmonic sequences of functions on $[x_{j-1}, x_j]$, for $j = 1,2,3,4$, defined by the following relations:*

$$w_{1k}(t) = \frac{1}{(k-1)!}\int_a^t (t-s)^{k-1} w(s)\, ds, \quad t \in [a,x],$$

$$w_{2k}(t) = \frac{1}{(k-1)!}\int_x^t (t-s)^{k-1} w(s)\, ds + L_{k,x}(t), \quad t \in \left(x, \frac{a+b}{2}\right],$$

$$w_{3k}(t) = -\frac{1}{(k-1)!}\int_t^{a+b-x} (t-s)^{k-1} w(s)\, ds + (-1)^k L_{k,x}(a+b-t), \quad t \in \left(\frac{a+b}{2}, a+b-x\right],$$

$$w_{4k}(t) = -\frac{1}{(k-1)!}\int_t^b (t-s)^{k-1} w(s)\, ds, \quad t \in (a+b-x, b].$$

If $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is piecewise continuous on $[a,b]$, then we have

$$\begin{aligned} \int_a^b w(t) f(t)\, dt &= \sum_{k=1}^n A_k(x)\Big(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\Big) \\ &+ \sum_{k=1}^n B_k(x) f^{(k-1)}\left(\frac{a+b}{2}\right) + (-1)^n \int_a^b W_{n,w}(t,x) f^{(n)}(t)\, dt, \end{aligned} \tag{7}$$

where

$$A_k(x) = (-1)^{k-1}\left[\frac{1}{(k-1)!}\int_a^x (x-s)^{k-1} w(s)\, ds - L_{k,x}(x)\right], \quad k \ge 1, \tag{8}$$

$$B_k(x) = 2\left[\frac{1}{(k-1)!}\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - s\right)^{k-1} w(s)\, ds + L_{k,x}\left(\frac{a+b}{2}\right)\right], \quad \text{for odd } k \ge 1, \tag{9}$$

and

$$B_k(x) = 0, \quad \text{for even } k \ge 1,$$

such that

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t), & t \in [a,x], \\ w_{2n}(t), & t \in \left(x, \frac{a+b}{2}\right], \\ w_{3n}(t), & t \in \left(\frac{a+b}{2}, a+b-x\right], \\ w_{4n}(t), & t \in (a+b-x, b]. \end{cases} \tag{10}$$

Remark 1. *If we assume $w(t) = w(a+b-t)$, for each $t \in [a,b]$, then the following symmetry conditions hold for $k = 1,\dots,n$:*

$$w_{1k}(t) = (-1)^k w_{4k}(a+b-t), \quad \text{for } t \in [a,x],$$

and

$$w_{2k}(t) = (-1)^k w_{3k}(a+b-t), \quad \text{for } t \in \left(x, \frac{a+b}{2}\right].$$

Using Theorems 1 and 4, the properties of both n-convex functions and w-harmonic sequences of functions, and the method that originated in [15], in the next theorem, we derive new Hermite–Hadamard–Fejér-type inequalities for the weighted three-point quadrature Formula (7).

Theorem 5. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a+b-t)$, for each $t \in [a,b]$ and $x \in [a, \frac{a+b}{2})$. Let the function $W_{2n,w}$, defined by (10), be nonnegative. If $f : [a,b] \to \mathbb{R}$ is $(2n+2)$-convex on $[a,b]$ and $f^{(2n)}$ is piecewise continuous on $[a,b]$, then*

$$\begin{aligned} & U_{n,w}(x) \cdot f^{(2n)}\left(\frac{a+b}{2}\right) \\ & \le \int_a^b w(t)f(t)\,dt - \sum_{k=1}^{2n} A_k(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ & - \sum_{k=1, k\,odd}^{2n} B_k(x) f^{(k-1)}\left(\frac{a+b}{2}\right) \le U_{n,w}(x) \cdot \left[\frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b)\right], \end{aligned} \tag{11}$$

where

$$\begin{aligned} U_{n,w}(x) &= \frac{1}{(2n)!}\int_a^b w(t)\cdot t^{2n}\,dt \\ &- \sum_{k=1}^{2n} A_k(x) \frac{x^{2n-k+1} + (-1)^{k-1}(a+b-x)^{2n-k+1}}{(2n-k+1)!} \\ &- \sum_{k=1, k\,odd}^{2n} B_k(x) \frac{(a+b)^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}, \end{aligned} \tag{12}$$

and A_k and B_k are defined as in Theorem 4. If $W_{2n,w}(t,x) \le 0$ or f is a $(2n+2)$-concave function, then inequalities (11) hold with reversed inequality signs.

Proof. Let us observe that the function f is $(2n+2)$-convex. Hence, $f^{(2n)}$ is a convex function. It follows from Remark 1 that the function $W_{2n,w}$ is symmetric about $\frac{a+b}{2}$, i.e., $W_{2n,w}(t,x) = W_{2n,w}(a+b-t,x)$. Thus, inequalities (11) follow directly from Theorem 1, replacing a nonnegative and symmetric function u by a nonnegative and symmetric function $W_{2n,w}$, and a convex function h by a convex function $f^{(2n)}$, and then using identity (7) in $\int_a^b W_{2n,w}(t,x) f^{(2n)}(t)\,dt$.

Identity (7) yields $U_{n,w}(x)$ by substituting n with $2n$ and putting $f(t) = \frac{t^{2n}}{(2n)!}$. Then, $f^{(2n)}(t) = 1$ and $f^{(k-1)}(t) = \frac{1}{(2n-k+1)!} \cdot t^{2n-k+1}$. On the other hand, if $W_{2n,w}(t,x)$ is nonpositive, then $-W_{2n,w}(t,x)$ is nonnegative, from where there follow reversed signs in (11).

Further, let us assume that f is a $(2n+2)$-concave function. Hence, the function $-f^{(2n)}$ is convex. Reversed signs in (11) are obtained by putting $-f^{(2n)}$ and the nonnegative function $W_{2n,w}(t,x)$ in (1). This completes the proof. □

Remark 2. *The value of $U_{n,w}(x)$ can be obtained from Theorem 3 by taking an appropriate subdivision of the segment $[a,b]$ and applying the properties of functions w_{1k}, w_{2k}, w_{3k} and w_{4k}.*

To get a maximum degree of exactness of quadrature Formula (7) for fixed $x \in \left[a, \frac{a+b}{2}\right)$, we consider a sequence of harmonic polynomials $\{L_{j,x}\}_{j=0,1,\ldots,n}$ defined as follows:

$$\begin{aligned}
L_{0,x}(t) &= 0, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \\
L_{1,x}(x) &= \int_a^x w(s)\,ds - \frac{2}{(a+b-2x)^2}\int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s)\,ds, \\
L_{j,x}(x) &= \frac{1}{(j-1)!}\int_a^x (x-s)^{j-1} w(s)\,ds, \ j = 2,3,4,5,6, \\
L_{j,x}(t) &= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \ \text{for } t \in \left[x, \frac{a+b}{2}\right], \ j = 1,\ldots,n.
\end{aligned} \tag{13}$$

Therefore, we have

$$A_1(x) = \frac{2}{(a+b-2x)^2}\int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s)\,ds, \tag{14}$$

$$B_1(x) = \int_a^b w(s)\,ds - 2A_1(x),$$

$A_k(x) = 0$, for $k = 2,3,4,5,6$ and $B_k(x) = 0$, for $k = 2,3,4$.

Finally, from identity (7), for $x \in \left[a, \frac{a+b}{2}\right)$, we obtain the following three-point weighted integral formula:

$$\begin{aligned}
\int_a^b w(t)f(t)\,dt &= A_1(x)[f(x) + f(a+b-x)] + \left(\int_a^b w(s)\,ds - 2A_1(x)\right) f\left(\frac{a+b}{2}\right) \\
&+ T_{n,w}(x) + (-1)^n \int_a^b W_{n,w}(t,x) f^{(n)}(t)\,dt,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
T_{n,w}(x) &= \sum_{k=7}^{n} A_k(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\
&+ \sum_{k=5, odd\,k}^{n} B_k(x) f^{(k-1)}\left(\frac{a+b}{2}\right).
\end{aligned} \tag{16}$$

Now, applying results from Theorem 5 to identity (15), we get the following results.

Corollary 1. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function such that $w(t) = w(a+b-t)$, for each $t \in [a,b]$ and let $x \in [a, \frac{a+b}{2})$. Let the function $W_{2n,w}$, defined by (10), be nonnegative and let $L_{j,x}$ be defined by (13). If $f : [a,b] \to \mathbb{R}$ is $(2n+2)$-convex on $[a,b]$ and $f^{(2n)}$ is piecewise continuous on $[a,b]$, then*

$$
\begin{aligned}
& U_{n,w}(x) \cdot f^{(2n)}\left(\frac{a+b}{2}\right) \\
& \leq \int_a^b w(t) f(t)\, dt - A_1(x)[f(x) + f(a+b-x)] \\
& - \left(\int_a^b w(s)\, ds - 2A_1(x)\right) f\left(\frac{a+b}{2}\right) - T_{2n,w}(x) \\
& \leq U_{n,w}(x) \cdot \left[\frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b)\right],
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
U_{n,w}(x) &= \frac{1}{(2n)!}\int_a^b w(t) \cdot t^{2n}\, dt - A_1(x)\frac{x^{2n} + (a+b-x)^{2n}}{(2n)!} \\
&- \left(\int_a^b w(s)\, ds - 2A_1(x)\right)\frac{(a+b)^{2n}}{2^{2n}(2n)!} \\
&- \sum_{k=7}^{2n} A_k(x)\frac{x^{2n-k+1} + (-1)^{k-1}(a+b-x)^{2n-k+1}}{(2n-k+1)!} \\
&- \sum_{k=5, k\, odd}^{2n} B_k(x)\frac{(a+b)^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}.
\end{aligned} \tag{18}
$$

If $W_{2n,w}(t,x) \leq 0$ or f is a $(2n+2)$-concave function, then inequalities (17) hold with reversed inequality signs.

Proof. The proof follows from Theorem 5 for the special choice of the polynomials $L_{j,x}$. □

Remark 3. *If we assume $B_5(x) = 0$, then we get*

$$
x = \frac{a+b}{2} - \sqrt{\frac{\int_a^b \left(s - \frac{a+b}{2}\right)^4 w(s)\, ds}{\int_a^b \left(s^2 - \left(\frac{a+b}{2}\right)^2\right) w(s)\, ds}}.
$$

Therefore, for such a choice of x, we obtain the quadrature formula with three nodes, which is accurate for the polynomials of degree at most 5, and the approximation formula includes derivatives of order 6 and more.

3. Special Cases

Considering some special cases of the weight function w, in our results given in the previous section, we obtain estimates for the Gauss–Legendre three-point quadrature formula and for the Gauss–Chebyshev three-point quadrature formula of the first and of the second kind.

3.1. Gauss–Legendre Three-Point Quadrature Formula

Let us assume that $w(t) = 1$, $t \in [a,b]$ and $x \in \left[a, \frac{a+b}{2}\right)$.

Now, from Theorem 4, we calculate

$$W_n^{GL}(t,x) = \begin{cases} w_{1n}(t) = \frac{(t-a)^n}{n!}, & t \in [a,x], \\ w_{2n}(t) = \frac{(t-x)^n}{n!} + L_{n,x}(t), & t \in \left(x, \frac{a+b}{2}\right], \\ w_{3n}(t) = \frac{(t-a-b+x)^n}{n!} + (-1)^n L_{n,x}(a+b-t), & t \in \left(\frac{a+b}{2}, a+b-x\right], \\ w_{4n}(t) = \frac{(t-b)^n}{n!}, & t \in (a+b-x, b], \end{cases} \tag{19}$$

and

$$A_k^{GL}(x) = (-1)^{k-1}\left[\frac{(x-a)^k}{k!} - L_{k,x}(x)\right], \quad \text{for } k \geq 1,$$

$$B_k^{GL}(x) = 2\left[\frac{\left(\frac{a+b}{2}-x\right)^k}{k!} + L_{k,x}\left(\frac{a+b}{2}\right)\right], \quad \text{for odd } k \geq 1,$$

and

$$B_k^{GL}(x) = 0, \quad \text{for even } k > 1.$$

Corollary 2. *Let $w_{2,2n}(t) \geq 0$, for all $t \in \left(x, \frac{a+b}{2}\right]$ and for $n \in \mathbb{N}$. If $f : [a,b] \to \mathbb{R}$ is a $(2n+2)$-convex function and $f^{(2n)}$ is piecewise continuous on $[a,b]$, then*

$$\begin{aligned} & U_n^{GL}(x) \cdot f^{(2n)}\left(\frac{a+b}{2}\right) \\ & \leq \int_a^b f(t)\,dt - \sum_{k=1}^{2n} A_k^{GL}(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ & - \sum_{k=1, k\,odd}^{2n} B_k^{GL}(x) f^{(k-1)}\left(\frac{a+b}{2}\right) \leq U_n^{GL}(x) \cdot \left[\frac{1}{2} f^{(2n)}(a) + \frac{1}{2} f^{(2n)}(b)\right], \end{aligned} \tag{20}$$

where

$$\begin{aligned} U_n^{GL}(x) &= \frac{b^{2n+1} - a^{2n+1}}{(2n+1)!} \\ &- \sum_{k=1}^{2n} A_k^{GL}(x) \frac{x^{2n-k+1} + (-1)^{k-1}(a+b-x)^{2n-k+1}}{(2n-k+1)!} \\ &- \sum_{k=1, k\,odd}^{2n} B_k^{GL}(x) \frac{(a+b)^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}. \end{aligned} \tag{21}$$

If f is a $(2n+2)$-concave function, then inequalities (20) hold with reversed inequality signs.

Proof. A special case of Theorem 5 for $w(t) = 1$, $t \in [a,b]$, and a nonnegative function W_{2n}^{GL} defined by (19). □

If we assume that the polynomials $L_{j,x}(t)$ are such that

$$\begin{aligned} L_{0,x}(t) &= 0, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \\ L_{1,x}(x) &= x - a - \frac{(b-a)^3}{6(a+b-2x)^2}, \\ L_{j,x}(x) &= \frac{(x-a)^j}{j!}, \quad j = 2,3,4,5,6, \\ L_{j,x}(t) &= \sum_{k=1}^{6 \wedge j} L_{k,x}(x) \frac{(t-x)^{j-k}}{(j-k)!}, \quad \text{for } t \in \left[x, \frac{a+b}{2}\right], \quad j = 1, \dots, n, \end{aligned} \tag{22}$$

we get $A_1^{GL}(x) = \frac{(b-a)^3}{6(a+b-2x)^2}$, $A_k^{GL}(x) = 0$, for $k = 2,3,4,5,6$, $B_1^{GL}(x) = b - a - 2A_1^{GL}(x)$ and $B_3^{GL}(x) = 0$. Thus, we obtain the following non-weighted three-point quadrature formulae:

$$\begin{aligned} \int_a^b f(t)\,dt &= \frac{(b-a)^3}{6(a+b-2x)^2}[f(x) + f(a+b-x)] \\ &+ \left(b - a - \frac{(b-a)^3}{3(a+b-2x)^2}\right) f\left(\frac{a+b}{2}\right) \\ &+ T_n^{GL}(x) + (-1)^n \int_a^b W_n^{GL}(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{23}$$

where

$$\begin{aligned} T_n^{GL}(x) &= \sum_{k=7}^{n} A_k^{GL}(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(a+b-x)\right) \\ &+ \sum_{k=5, odd\, k}^{n} B_k^{GL}(x) f^{(k-1)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{24}$$

In particular, according to Remark 3, for $[a,b] = [-1,1]$ and $x = \frac{-\sqrt{15}}{5}$, we get $B_5^{GL}(x) = 0$, and there follows a generalization of the Gauss–Legendre three-point formula. Now, we derive Hermite–Hadamard–Fejér-type estimates for this generalization of the Gauss–Legendre three-point formula.

If the assumptions of Corollary 1 hold for $w(t) = 1$, $t \in [-1,1]$, and if $f : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, we derive:

$$\begin{aligned} & U_n^{GL}\left(\frac{-\sqrt{15}}{5}\right) \cdot f^{(2n)}(0) \\ & \le \int_{-1}^{1} f(t)\,dt - \frac{1}{9}\left[5f\left(\frac{-\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right] - T_{2n}^{GL}\left(\frac{-\sqrt{15}}{5}\right) \\ & \le U_n^{GL}\left(\frac{-\sqrt{15}}{5}\right) \cdot \left[\frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1)\right], \end{aligned} \tag{25}$$

where

$$\begin{aligned} U_n^{GL}\left(\frac{-\sqrt{15}}{5}\right) &= \frac{2 \cdot 5^{n-1} - 2(2n+1) \cdot 3^{n-2}}{5^{n-1}(2n+1)!} \\ &- \sum_{k=7}^{2n} A_k^{GL}\left(\frac{-\sqrt{15}}{5}\right) \frac{(-\sqrt{15})^{2n-k+1} + (-1)^{k-1}(\sqrt{15})^{2n-k+1}}{5^{2n-k+1}(2n-k+1)!}. \end{aligned}$$

In a special case, for $n = 3$, we get

$$\begin{aligned} & \frac{1}{15,750} \cdot f^{(6)}(0) \\ & \le \int_{-1}^{1} f(t)\,dt - \frac{1}{9}\left[5f\left(\frac{-\sqrt{15}}{5}\right) + 8f(0) + 5f\left(\frac{\sqrt{15}}{5}\right)\right] \\ & \le \frac{1}{15,750} \cdot \left[\frac{1}{2} f^{(6)}(-1) + \frac{1}{2} f^{(6)}(1)\right]. \end{aligned} \tag{26}$$

3.2. *Gauss–Chebyshev Three-Point Quadrature Formula of the First Kind*

Let us assume that $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$ and $x \in [-1,0)$.

From Theorem 4, there follow:

$$W_{n,w}^{GC1}(t,x) = \begin{cases} w_{1n}(t) = \frac{1}{(n-1)!} \int\limits_{-1}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\, ds, & t \in [-1,x], \\ w_{2n}(t) = \frac{1}{(n-1)!} \int\limits_{x}^{t} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\, ds + L_{n,x}(t), & t \in (x,0], \\ w_{3n}(t) = -\frac{1}{(n-1)!} \int\limits_{t}^{-x} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\, ds + (-1)^n L_{n,x}(-t), & t \in (0,-x], \\ w_{4n}(t) = -\frac{1}{(n-1)!} \int\limits_{t}^{1} \frac{(t-s)^{n-1}}{\sqrt{1-s^2}}\, ds, & t \in (-x,1], \end{cases} \tag{27}$$

$$A_k^{GC1}(x) = (-1)^{k-1} \left[\frac{(x+1)^{k-1/2}\sqrt{\pi}}{\sqrt{2}\Gamma\left(\frac{1}{2}+k\right)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+k, \frac{x+1}{2}\right) - L_{k,x}(x) \right], k \geq 1,$$

and

$$B_k^{GC1}(x) = 2\left[\frac{(-1)^{k-1}}{(k-1)!} \int\limits_{x}^{0} \frac{s^{k-1}}{\sqrt{1-s^2}}\, ds + L_{k,x}(0) \right], \quad \text{for odd } k \geq 1,$$

and

$$B_k^{GC1}(x) = 0, \quad \text{for even } k > 1.$$

Corollary 3. *Let $w_{2,2n}(t) \geq 0$, for all $t \in (x,0]$ and for $n \in \mathbb{N}$. If $f : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function and $f^{(2n)}$ is piecewise continuous on $[-1,1]$, then*

$$\begin{aligned} & U_n^{GC1}(x) \cdot f^{(2n)}(0) \\ & \leq \int\limits_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt - \sum_{k=1}^{2n} A_k^{GC1}(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x)\right) \\ & - \sum_{k=1, k\, odd}^{2n} B_k^{GC1}(x) f^{(k-1)}(0) \leq U_n^{GC1}(x) \cdot \left[\frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1)\right], \end{aligned} \tag{28}$$

where

$$\begin{aligned} U_n^{GC1}(x) &= \frac{1}{(2n)!} B\left(\frac{1}{2}, \frac{1}{2}+n\right) \\ &- \sum_{k=1}^{2n} A_k^{GC1}(x) \frac{x^{2n-k+1} + (-1)^{k-1}(-x)^{2n-k+1}}{(2n-k+1)!}. \end{aligned} \tag{29}$$

If f is a $(2n+2)$-concave function, then inequalities (28) hold with reversed inequality signs.

Proof. A special case of Theorem 5 for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$, and a nonnegative function $W_{2n,w}^{GC1}$ defined by (27). □

If we assume that the polynomials $L_{j,x}(t)$ are such that

$$
\begin{aligned}
L_{0,x}(t) &= 0, \quad \text{for } t \in [x,0], \\
L_{1,x}(x) &= \arcsin x + \frac{\pi}{2} - \frac{\pi}{4x^2}, \\
L_{j,x}(x) &= \frac{(x+1)^{j-1/2}\sqrt{\pi}}{\sqrt{2}\Gamma\left(\frac{1}{2}+j\right)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+j, \frac{x+1}{2}\right), \; j = 2,3,4,5,6, \\
L_{j,x}(t) &= \sum_{k=1}^{6\wedge j} L_{k,x}(x) \frac{(t-x)^{j-k}}{(j-k)!}, \; \text{for } t \in [x,0], \; j = 1,\dots,n,
\end{aligned}
$$

we calculate $A_1^{GC1}(x) = \frac{\pi}{4x^2}$, $A_k^{GC1}(x) = 0$, for $k = 2,3,4,5,6$, $B_1^{GC1}(x) = \pi - \frac{\pi}{2x^2}$ and $B_3^{GC1}(x) = 0$.

Now, we derive

$$
\begin{aligned}
\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt &= \frac{\pi}{4x^2} f(x) + \left(\pi - \frac{\pi}{2x^2}\right) f(0) + \frac{\pi}{4x^2} f(-x) \\
&+ T_{n,w}^{GC1}(x) + (-1)^n \int_{-1}^{1} W_{n,w}^{GC1}(t,x) f^{(n)}(t)\, dt,
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
T_{n,w}^{GC1}(x) &= \sum_{k=7}^{n} A_k^{GC1}(x) \left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x) \right) \\
&+ \sum_{k=5, odd\, k}^{n} B_k^{GC1}(x) f^{(k-1)}(0).
\end{aligned} \tag{31}
$$

In particular, there follows a generalization of the Gauss–Chebyshev three-point quadrature formula of the first kind for $x = -\frac{\sqrt{3}}{2}$. Now, we derive Hermite–Hadamard-type estimates for the Gauss–Chebyshev three-point quadrature formula of the first kind.

If the assumptions of Corollary 1 hold for $w(t) = \frac{1}{\sqrt{1-t^2}}$, $t \in (-1,1)$, and if $f : [-1,1] \to \mathbb{R}$ is a $(2n+2)$-convex function, we get

$$
\begin{aligned}
& U_n^{GC1}\left(\frac{-\sqrt{3}}{2}\right) \cdot f^{(2n)}(0) \\
& \le \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt - \frac{\pi}{3}\left[f\left(\frac{-\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right] - T_{2n,w}^{GC1}\left(\frac{-\sqrt{3}}{2}\right) \\
& \le U_n^{GC1}\left(\frac{-\sqrt{3}}{2}\right) \cdot \left[\frac{1}{2} f^{(2n)}(-1) + \frac{1}{2} f^{(2n)}(1)\right],
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
U_n^{GC1}\left(\frac{-\sqrt{3}}{2}\right) &= \frac{1}{(2n)!} B\left(\frac{1}{2}, \frac{1}{2}+n\right) - \frac{\pi \cdot 3^{n-1}}{2^{2n-1}(2n)!} \\
&- \sum_{k=7}^{2n} A_k^{GC1}\left(\frac{-\sqrt{3}}{2}\right) \frac{(-\sqrt{3})^{2n-k+1} + (-1)^{k-1}(\sqrt{3})^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}.
\end{aligned}
$$

In a special case, for $n = 3$, we obtain

$$\begin{aligned}&\frac{\pi}{23,040}\cdot f^{(6)}(0)\\&\leq\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,dt-\frac{\pi}{3}\left[f\left(\frac{-\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]\\&\leq\frac{\pi}{23,040}\cdot\left[\frac{1}{2}f^{(6)}(-1)+\frac{1}{2}f^{(6)}(1)\right].\end{aligned} \tag{33}$$

3.3. Gauss–Chebyshev Three-Point Quadrature Formula of the Second Kind

Assuming $w(t)=\sqrt{1-t^2}$, $t\in[-1,1]$ and $x\in[-1,0)$ and using Theorem 4, we obtain

$$W_{n,w}^{GC2}(t,x)=\begin{cases}w_{1n}(t)=\frac{1}{(n-1)!}\int\limits_{-1}^{t}(t-s)^{n-1}\sqrt{1-s^2}\,ds, & t\in[-1,x],\\ w_{2n}(t)=\frac{1}{(n-1)!}\int\limits_{x}^{t}(t-s)^{n-1}\sqrt{1-s^2}\,ds+L_{n,x}(t), & t\in(x,0],\\ w_{3n}(t)=-\frac{1}{(n-1)!}\int\limits_{t}^{-x}(t-s)^{n-1}\sqrt{1-s^2}\,ds+(-1)^nL_{n,x}(-t), & t\in(0,-x],\\ w_{4n}(t)=-\frac{1}{(n-1)!}\int\limits_{t}^{1}(t-s)^{n-1}\sqrt{1-s^2}\,ds, & t\in(-x,1],\end{cases} \tag{34}$$

$$A_k^{GC2}(x)=(-1)^{k-1}\left[\frac{(x+1)^{k+1/2}\sqrt{2\pi}}{\Gamma\left(\frac{3}{2}+k\right)}F\left(-\frac{1}{2},\frac{3}{2},\frac{3}{2}+k,\frac{x+1}{2}\right)-L_{k,x}(x)\right],\ k\geq 1,$$

$$B_k^{GC2}(x)=2\left[\frac{(-1)^{k-1}}{(k-1)!}\int\limits_{x}^{0}s^{k-1}\sqrt{1-s^2}\,ds+L_{k,x}(0),\right],\quad \text{for odd } k\geq 1,$$

and

$$B_k^{GC2}(x)=0,\quad \text{for even } k>1.$$

Corollary 4. *Let $w_{2,2n}(t)\geq 0$, for all $t\in(x,0]$ and for $n\in\mathbb{N}$. If $f:[-1,1]\to\mathbb{R}$ is a $(2n+2)$-convex function and $f^{(2n)}$ is piecewise continuous on $[-1,1]$, then*

$$\begin{aligned}&U_n^{GC2}(x)\cdot f^{(2n)}(0)\\&\leq\int_{-1}^{1}f(t)\sqrt{1-t^2}\,dt-\sum_{k=1}^{2n}A_k^{GC2}(x)\left(f^{(k-1)}(x)+(-1)^{k-1}f^{(k-1)}(-x)\right)\\&-\sum_{k=1,k\,odd}^{2n}B_k^{GC2}(x)f^{(k-1)}(0)\leq U_n^{GC2}(x)\cdot\left[\frac{1}{2}f^{(2n)}(-1)+\frac{1}{2}f^{(2n)}(1)\right],\end{aligned} \tag{35}$$

where

$$\begin{aligned}U_n^{GC2}(x)&=\frac{1}{(2n)!}B\left(\frac{3}{2},\frac{1}{2}+n\right)\\&-\sum_{k=1}^{2n}A_k^{GC2}(x)\frac{x^{2n-k+1}+(-1)^{k-1}(-x)^{2n-k+1}}{(2n-k+1)!}.\end{aligned} \tag{36}$$

If f is a $(2n+2)$-concave function, then inequalities (35) hold with reversed inequality signs.

Proof. A special case of Theorem 5 for $w(t)=\sqrt{1-t^2}$, $t\in[-1,1]$, and a nonnegative function $W_{2n,w}^{GC2}$ defined by (34). □

If the polynomials $L_{j,x}(t)$ are such that

$$\begin{aligned}
L_{0,x}(t) &= 0, \ \text{for } t \in [x,0], \\
L_{1,x}(x) &= \frac{1}{2}\left(\arcsin x + \frac{\pi}{2} - \frac{\pi}{8x^2} + \frac{x\sqrt{1-x^2}}{2}\right), \\
L_{j,x}(x) &= \frac{(x+1)^{j+1/2}\sqrt{2\pi}}{\Gamma\left(\frac{3}{2}+j\right)} F\left(-\frac{1}{2}, \frac{3}{2}, \frac{3}{2}+j, \frac{x+1}{2}\right), \ j = 2,3,4,5,6, \\
L_{j,x}(t) &= \sum_{k=1}^{6\wedge j} L_{k,x}(x)\frac{(t-x)^{j-k}}{(j-k)!}, \ \text{for } t \in [x,0], \ j = 1,\dots,n,
\end{aligned}$$

we have $A_1^{GC2}(x) = \frac{x\sqrt{1-x^2}}{4} - \frac{\pi}{16x^2}$, $A_k^{GC2}(x) = 0$, for $k = 2,3,4,5,6$, $B_1^{GC2}(x) = \frac{\pi}{2} - \frac{x\sqrt{1-x^2}}{2} + \frac{\pi}{8x^2}$ and $B_3^{GC2}(x) = 0$, so we obtain

$$\begin{aligned}
\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt &= A_1^{GC2}(x)[f(x)+f(-x)] + B_1^{GC2}(x)f(0) \\
&+ T_{n,w}^{GC2}(x) + (-1)^n \int_{-1}^{1} W_{n,w}^{GC2}(t,x) f^{(n)}(t)\,dt,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
T_{n,w}^{GC2}(x) &= \sum_{k=7}^{n} A_k^{GC2}(x)\left(f^{(k-1)}(x) + (-1)^{k-1} f^{(k-1)}(-x)\right) \\
&+ \sum_{k=5, odd\, k}^{n} B_k^{GC2}(x) f^{(k-1)}(0).
\end{aligned} \tag{38}$$

In particular, a generalization of the Gauss–Chebyshev three-point quadrature formula of the second kind follows for $x = -\frac{\sqrt{2}}{2}$. Now, we derive Hermite–Hadamard-type estimates for the Gauss–Chebyshev three-point quadrature formula of the second kind.

Applying Corollary 1 to $w(t) = \sqrt{1-t^2}$, $t \in [-1,1]$, $x = -\frac{\sqrt{2}}{2}$, and a $(2n+2)$-convex function f, we obtain

$$\begin{aligned}
& U_n^{GC2}\left(\frac{-\sqrt{2}}{2}\right) \cdot f^{(2n)}(0) \\
& \le \int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt - \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right] - T_{2n,w}^{GC2}\left(\frac{-\sqrt{2}}{2}\right) \\
& \le U_n^{GC2}\left(\frac{-\sqrt{2}}{2}\right) \cdot \left[\frac{1}{2}f^{(2n)}(-1) + \frac{1}{2}f^{(2n)}(1)\right],
\end{aligned}$$

where

$$\begin{aligned}
U_n^{GC2}\left(\frac{-\sqrt{2}}{2}\right) &= \frac{1}{(2n)!}B\left(\frac{3}{2}, \frac{1}{2}+n\right) - \frac{\pi}{2^{n+2}(2n)!} \\
&- \sum_{k=7}^{2n} A_k^{GC2}\left(\frac{-\sqrt{2}}{2}\right) \frac{(-\sqrt{2})^{2n-k+1} + (-1)^{k-1}(\sqrt{2})^{2n-k+1}}{2^{2n-k+1}(2n-k+1)!}.
\end{aligned}$$

As a special case, for $n = 3$, we obtain

$$\frac{\pi}{92,160} \cdot f^{(6)}(0)$$
$$\le \int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt - \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right]$$
$$\le \frac{\pi}{92,160} \cdot \left[\frac{1}{2}f^{(6)}(-1) + \frac{1}{2}f^{(6)}(1)\right].$$

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