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Inverse Problem for the Sobolev Type Equation of Higher Order

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Abstract: The article investigates the inverse problem for a complete, inhomogeneous, higher-order Sobolev type equation, together with the Cauchy and overdetermination conditions. This problem was reduced to two equivalent problems in the aggregate: regular and singular. For these problems, the theory of polynomially bounded operator pencils is used. The unknown coefficient of the original equation is restored using the method of successive approximations. The main result of this work is a theorem on the unique solvability of the original problem. This study continues and generalizes the authors' previous research in this area. All the obtained results can be applied to the mathematical modeling of various processes and phenomena that fit the problem under study.

Keywords: Sobolev type equation; inverse problem; high-order equation; method of successive approximations; polynomial boundedness of operator pencils

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1. Introduction

Let $\mathcal{U}, \mathcal{F}, \mathcal{Y}$ be Banach spaces, operators $A, B_0, B_1, ..., B_{n-1} \in \mathcal{L}(\mathcal{U};\mathcal{F})$, i.e., linear and continuous operators defined on $\mathcal{U}$ and acting to $\mathcal{F}$, $\ker A \neq \{0\}$, $C \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, given functions $\chi : [0,T] \to \mathcal{L}(\mathcal{Y};\mathcal{F})$, $f : [0,T] \to \mathcal{F}$, $\Psi : [0,T] \to \mathcal{Y}$. Consider the following problem with $t \in [0,T]$

$$Av^{(n)}(t) = B_{n-1}v^{(n-1)}(t) + ... + B_1 v'(t) + B_0 v(t) + q(t)\chi(t) + f(t), \tag{1}$$

$$v(0) = v_0,\ v'(0) = v_1,\ ...,\ v^{(n-1)}(0) = v_{n-1}, \tag{2}$$

$$Cv(t) = \Psi(t). \tag{3}$$

The problem of finding a pair of functions $v(t) \in C^n([0,T];\mathcal{U})$ (n times continuously diferentiable) and $q(t) \in C^1([0,T];\mathcal{Y})$ (continuously diferentiable) from relations (1)–(3) is called an inverse problem. At present, the authors have obtained the result when studying the inverse problem, but only in the case of the second-order, Sobolev type equation [1].

The degeneracy of the operator A allows us to classify Equation (1) as a Sobolev type equation. Additionally, one can see that this equation is complete, since all the components $v(t)$, $v'(t)$, ..., $v^{(n)}(t)$ are present. In addition, the Cauchy condition (2) is posed. The overdetermination condition (3) arises due to the need to restore the parameter $q(t)$ of the equation.

The study of Sobolev type equations was carried out repeatedly [1–12]. There are articles devoted to both the first [2–4], the second [1,5,6], the third [7], and the higher [8–10] order. In [2], sufficient conditions for the existence of positive solutions to the Showalter–Sidorov and the Cauchy problem for an abstract linear equation of this type were presented. The linear representatives of Sobolev type equations, such as the Barenblatt–Zheltov–Kochina equation and the Hoff equation are studied in [3]. The paper [7] contains a condition for the existence of a weak, local, timely solution to the Cauchy problem for a model Sobolev type equation. In the study of the direct problem for a higher-order, Sobolev type equation, the phase space method was used [10]. Papers [11,12] are among the first investigations of Sobolev type equations, and the recent works devoted to applications of Sobolev type equations to real-life models are as follows: [13,14].

The works [1,5,15–24] were devoted to the consideration of inverse problems. In [15], the process of unsteady flow of a viscous incompressible fluid in a pipe with a permeable wall was considered. The dependence on the choice of the boundary of the rectangular region and the unique solvability of the inverse problem were investigated in [16]. The uniqueness criterion for the Lavrent'ev–Bitsadze equation is established in [17]. The correctness in Sobolev spaces of the problem of determining the function of sources in the heat and mass transfer Navier–Stokes system was proved [18]. The problem was finding the area where the vector of boundary displacements and forces is given in parametric form [19]. In [20], the inverse boundary value problem for the heat equation was studied, and the error of the obtained approximate solution was estimated.

The article consists of four sections. The second section combines the necessary, previously obtained, results of the theory of polynomially A-bounded of operator pencils formulated in the form of definitions, theorems and lemmas. Section «Results» has three subsections. The first one presents the result of applying the splitting theorem; thus, the original problem is divided into two equivalent problems in the aggregate: regular and singular. In the second subsection, we study the unique solvability of the regular problem by reducing it to an equivalent problem of the first order and achieving the necessary smoothness for the required function q using the method of successive approximations. The third subsection generalizes the result of studying the singular problem obtained earlier in the work [9], thus obtaining the theorem on the existence and uniqueness of the solution to the problem (1)–(3). In the last section, the significance of the obtained results is given in both the development of the studied theory and their practical application.

2. Preliminary Information

To find a pair of functions $v(t)$ and $q(t)$, we use the results obtained in the research into higher-order, Sobolev type equations [8]. Thus, we will apply the theory of polynomially A-bounded operator pencils. Denote by $\vec{B}$ the pencil of operators $B_0, B_1, ..., B_{n-1}$.

Definition 1. *The sets*

$$\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1} B_{n-1} - ... - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathcal{F}; \mathcal{U})\}$$

and $\sigma^A(\vec{B}) = \bar{\mathbb{C}} \backslash \rho^A(\vec{B})$ will be called the A-resolvent set and the A-spectrum of the pencil $\vec{B}$, respectively.

Definition 2. *The operator-function complex variable*

$$R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1} B_{n-1} - ... - \mu B_1 - B_0)^{-1}$$

with domain $\rho^A(\vec{B})$ will be called the A-resolvent of the pencil $\vec{B}$.

Definition 3. *Let the pencil $\vec{B}$ be polynomially A-bounded if*

$$\exists a \in \mathbb{R}_+ \ \ \forall \mu \in \mathbb{C} \ \ (|\mu| > a) \Rightarrow \Big(R^A_\mu(\vec{B}) \in \mathcal{L}(\mathcal{F}; \mathcal{U}) \Big).$$

Let the pencil $\vec{B}$ be polynomially A-bounded. Introduce an important condition

$$\int_\gamma \mu^k R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O}, \ \ k = 0, 1, ..., n-2, \tag{4}$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Lemma 1. *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled. Then, the operators*

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_\mu(\vec{B})\mu^{n-1}Ad\mu \ \in \ \mathcal{L}(\mathcal{U}),$$

$$Q = \frac{1}{2\pi i}\int\limits_{\gamma} \mu^{n-1}AR^A_\mu(\vec{B})d\mu \ \in \ \mathcal{L}(\mathcal{F})$$

are projectors.

Put $\mathcal{U}^0 = \ker P$, $\mathcal{F}^0 = \ker Q$, $\mathcal{U}^1 = \mathrm{im}\, P$, $\mathcal{F}^1 = \mathrm{im}\, Q$. From the previous Lemma it follows that $\mathcal{U} = \mathcal{U}^0 \oplus \mathcal{U}^1$, $\mathcal{F} = \mathcal{F}^0 \oplus \mathcal{F}^1$. Let $A^k(B^k_l)$ denote the restriction of the operator $A(B_l)$ onto $\mathcal{U}^k$, $k = 0,1; l = 0,1,...,n-1$.

Theorem 1. *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled. Then, the actions of the operators split:*

1. $A^k \in \mathcal{L}(\mathcal{U}^k;\mathcal{F}^k),\ k = 0,1;$
2. $B^k_l \in \mathcal{L}(\mathcal{U}^k;\mathcal{F}^k),\ k = 0,1;\ l = 0,1,...,n-1;$
3. *There exists an operator* $(A^1)^{-1} \in \mathcal{L}(\mathcal{F}^1;\mathcal{U}^1);$
4. *There exists an operator* $(B^0_0)^{-1} \in \mathcal{L}(\mathcal{F}^0;\mathcal{U}^0).$

Definition 4. *Define the family of operators $\{K^1_q, K^2_q, ..., K^n_q\}$ as follows*

$$K^1_1 = H_0,\ K^2_1 = -H_1,\ ...,\ K^n_1 = -H_{n-1},$$

$$K^1_{q+1} = K^n_q H_0,\ K^2_{q+1} = K^1_q - K^n_q H_1,\ ...,\ K^n_{q+1} = K^{n-1}_q - K^n_q H_{n-1};\ q = 1,2,...,$$

where $H_0 = (B^0_0)^{-1}A^0,\ H_1 = (B^0_0)^{-1}B^0_1,\ ...,\ H_{n-1} = (B^0_0)^{-1}B^0_{n-1}.$

Definition 5. *The point ∞ is called*

1. *Removable singular point of the A-resolvent of pencil $\vec{B}$, if $K^1_1 \equiv \mathbb{O},\ K^2_1 \equiv \mathbb{O},\ ...,\ K^n_1 \equiv \mathbb{O}$;*
2. *A pole of order $p \in \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, if $\exists\ p$ such, that $K^1_p \not\equiv \mathbb{O}$, $K^2_p \not\equiv \mathbb{O},\ ...,\ K^n_p \not\equiv \mathbb{O}$, but $K^1_{p+1} \equiv \mathbb{O},\ K^2_{p+1} \equiv \mathbb{O},\ ...,\ K^n_{p+1} \equiv \mathbb{O}$;*
3. *An essentially singular point of the A-resolvent of the pencil $\vec{B}$, if $K^n_p \not\equiv \mathbb{O}$ for any $p \in \mathbb{N}$.*

3. Results

3.1. Reduction of the Initial Inverse Problem

Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled, then $v(t)$ can be represented as $v(t) = Pv(t) + (I-P)v(t) = u(t) + \omega(t)$. Suppose that $\mathcal{U}^0 \subset \ker C$. Then, by virtue of Theorem 1 and Lemma 1 problem (1)–(3) is equivalent to the problem of finding the functions $u \in C^n([0,T];\mathcal{U}^1)$, $\omega \in C^n([0,T];\mathcal{U}^0)$, $q \in C^1([0,T];\mathcal{Y})$ from the relations

$$u^{(n)}(t) = S_{n-1}u^{(n-1)}(t) + ... + S_1u'(t) + S_0u(t) + q(t)(A^1)^{-1}Q\chi(t) + (A^1)^{-1}Qf(t), \quad (5)$$

$$u(0) = u_0,\ \ u'(0) = u_1,\ \ ...,\ \ u^{(n-1)}(0) = u_{n-1}, \quad (6)$$

$$Cu(t) = \Psi(t) \equiv Cv(t), \quad (7)$$

$$\begin{gathered} H_0\omega^{(n)}(t) = H_{n-1}\omega^{(n-1)}(t) + ... + H_2\omega''(t) + H_1\omega'(t) + \omega(t) + \\ + q(t)(B^0_0)^{-1}(I-Q)\chi(t) + (B^0_0)^{-1}(I-Q)f(t), \end{gathered} \quad (8)$$

$$\omega(0) = \omega_0,\ \ \omega'(0) = \omega_1,\ \ ...,\ \ \omega^{(n-1)}(0) = \omega_{n-1}, \quad (9)$$

where

$$S_0 = (A^1)^{-1}B_0^1,\ S_1 = (A^1)^{-1}B_1^1,\ \ldots,\ S_{n-1} = (A^1)^{-1}B_{n-1}^1,$$

$$u_0 = Pv_0,\ u_1 = Pv_1,\ \ldots,\ u_{n-1} = Pv_{n-1},$$

$$\omega_0 = (I-P)v_0,\ \omega_1 = (I-P)v_1,\ \ldots,\ \omega_{n-1} = (I-P)v_{n-1},\ t \in [0,T].$$

The inverse problem (5)–(7) is called regular, and problem (8), (9) is called singular.

3.2. Solution of the Regular Inverse Problem

Rewrite problem (5)–(7) in the notation [25]. Let $\mathcal{X} = \mathcal{U}^1$, operators $S_0, S_1, \ldots, S_{n-1} \in \mathcal{C}l(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, operator-function $\Phi : [0,T] \to \mathcal{L}(\mathcal{Y};\mathcal{X})$, functions $h : [0,T] \to \mathcal{X}$, $\Psi : [0,T] \to \mathcal{Y}$

$$u^{(n)}(t) = S_{n-1}u^{(n-1)}(t) + \ldots + S_1u'(t) + S_0u(t) + q(t)\Phi(t) + h(t),\ t \in [0,T], \tag{10}$$

$$u(0) = u_0,\ u'(0) = u_1,\ \ldots,\ u^{(n-1)}(0) = u_{n-1}, \tag{11}$$

$$Cu(t) = \Psi(t). \tag{12}$$

Theorem 2. *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled; moreover, $C \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\Phi \in C^1([0,T];\mathcal{L}(\mathcal{Y};\mathcal{X}))$, $h \in C^1([0,T];\mathcal{X})$, $\Psi \in C^{n+1}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ the operator $C\Phi(t)$ be invertible and $(C\Phi)^{-1} \in C^1([0,T];\mathcal{L}(\mathcal{Y}))$. If the compatibility condition $Cu_{n-1} = \Psi^{n-1}(0)$ is satisfied, then the solution to the inverse problem (10)–(12) exists and is unique in the class of functions $q \in C^1([0,T];\mathcal{Y})$, $u \in C^n([0,T];\mathcal{X})$.*

Proof of Theorem 2. Reduce problem (10)–(12) to the problem for the first-order equation

$$z'(t) = Az(t) + q(t)Q(t) + H(t),\ t \in [0,T], \tag{13}$$

$$z(0) = z_0, \tag{14}$$

$$Bz(t) = \bar{\Psi}(t), \tag{15}$$

where $z(t) = \begin{pmatrix} u(t) \\ \vdots \\ u^{(n-2)}(t) \\ u^{(n-1)}(t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \\ S_0 & S_1 & \ldots & S_{n-1} \end{pmatrix}$, $Q(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi(t) \end{pmatrix}$,

$H(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h(t) \end{pmatrix}$, $z(0) = \begin{pmatrix} u(0) \\ \vdots \\ u^{(n-2)}(0) \\ u^{(n-1)}(0) \end{pmatrix}$, $z_0 = \begin{pmatrix} u_0 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}$, $B = \begin{pmatrix} 0 & \ldots & 0 & C \end{pmatrix}$,

$\bar{\Psi}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Psi^{(n-1)}(t) \end{pmatrix}$.

Put $R(t) = -(C\Phi(t))^{-1}$. Therefore, all the conditions of Theorem 6.2.3 from [25], are fulfilled, and the function $q(t)$ satisfies the integral equation

$$\begin{gathered} q(t) = q_0(t) + R(t)\Bigg(CS_0\int_0^t V_{1,n}(t-s)q(s)\Phi(s)ds + \\ +CS_1\int_0^t V_{2,n}(t-s)q(s)\Phi(s)ds + \ldots + CS_{n-1}\int_0^t V_{n,n}(t-s)q(s)\Phi(s)ds\Bigg), \end{gathered} \tag{16}$$

where

$$q_0(t) = -R(t)\Bigg(\Psi^{(n)}(t) - CS_0V_{1,1}(t)u_0 - CS_1V_{2,1}(t)u_0 - \ldots - CS_{n-1}V_{n,1}(t)u_0 -$$

$$-CS_0V_{1,2}(t)u_1 - CS_1V_{2,2}(t)u_1 - \ldots - CS_{n-1}V_{n,2}(t)u_1 - \ldots -$$

$$-CS_0V_{1,n}(t)u_{n-1} - CS_1V_{2,n}(t)u_{n-1} - \ldots - CS_{n-1}V_{n,n}(t)u_{n-1} -$$

$$-CS_0\int\limits_0^t V_{1,n}(t-s)h(s)ds - CS_1\int\limits_0^t V_{2,n}(t-s)h(s)ds -$$

$$-\ldots - CS_{n-1}\int\limits_0^t V_{n,n}(t-s)h(s)ds - Ch(t)\Bigg).$$

Thus, there exists a unique solution $q \in C^1([0,T];\mathcal{Y})$, $z \in C^1([0,T];\mathcal{X}^n)$ to the inverse problem (13)–(15). And we obtain that the solution to the regular inverse problem (10)–(12) exists and is unique, with $q \in C^1([0,T];\mathcal{Y})$, $u \in C^n([0,T];\mathcal{X})$. □

In order to obtain a solution to a singular problem, we need a greater smoothness of the function q from the solution of a regular problem than class $C^1([0,T];\mathcal{Y})$. Next, we need the following Lemma from [1].

Lemma 2. *Let $l \in \mathbb{N}$, $V \in C^{l-1}([0,T];\mathcal{L}(\mathcal{X}))$, $g \in C^l([0,T];\mathcal{X})$. Then*

$$\left(\int\limits_0^t V(t-s)g(s)ds\right)^{(l)} = \sum_{k=0}^{l-1} V^{(l-k-1)}(t)g^{(k)}(0) + \int\limits_0^t V(t-s)g^{(l)}(s)ds.$$

The following theorem provides sufficient conditions for the existence of a more smooth (as $p \in \mathbb{N}$) solution $q \in C^{p+n}([0,T],\mathcal{Y})$ of a regular problem.

Theorem 3. *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled, $p \in \mathbb{N}_0$; moreover, $C \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\Phi \in C^{p+n}([0,T];\mathcal{L}(\mathcal{Y};\mathcal{X}))$, $h \in C^{p+n}([0,T];\mathcal{X})$, $\Psi \in C^{p+2n}([0,T];\mathcal{Y})$, for any $t \in [0,T]$ operator $C\Phi(t)$ be invertible, with $(C\Phi)^{-1} \in C^{p+n}([0,T];\mathcal{L}(\mathcal{Y}))$ and the compatibility condition $Cu_{n-1} = \Psi^{(n-1)}(0)$ be satisfied for some $u_{n-1} \in \mathcal{U}^1$. Then there exists and a unique solution of (10)–(12) and $q \in C^{p+n}([0,T];\mathcal{Y})$.*

Proof of Theorem 3. Write the propagators of the homogeneous Equation (10) in a matrix, denoting the resolving group of homogeneous Equation (13)

$$V(t) = \begin{pmatrix} V_{1,1}(t) & V_{1,2}(t) & \ldots & V_{1,n-1}(t) & V_{1,n}(t) \\ V_{2,1}(t) & V_{2,2}(t) & \ldots & V_{2,n-1}(t) & V_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{n-1,1}(t) & V_{n-1,2}(t) & \ldots & V_{n-1,n-1}(t) & V_{n-1,n}(t) \\ V_{n,1}(t) & V_{n,2}(t) & \ldots & V_{n,n-1}(t) & V_{n,n}(t) \end{pmatrix} = \frac{1}{2\pi i}\int\limits_\gamma R^A_\mu(\vec{B})\times$$

$$\times\left(\begin{matrix} \mu^{n-1}A - \mu^{n-2}B_{n-1} - \ldots - B_1 & \mu^{n-2}A - \mu^{n-3}B_{n-1} - \ldots - B_2 & \ldots \\ B_0 & \mu^{n-1}A - \mu^{n-2}B_{n-1} - \ldots - \mu B_2 & \ldots \\ \vdots & \vdots & \ddots \\ \mu^{n-3}B_0 & \mu^{n-3}B_1 + \mu^{n-4}B_0 & \ldots \\ \mu^{n-2}B_0 & \mu^{n-2}B_1 + \mu^{n-3}B_0 & \ldots \end{matrix}\right.$$

$$\begin{matrix} \dots & \mu A - B_{n-1} & \mathbb{I} \\ \dots & \mu^2 A - \mu B_{n-1} & \mu\mathbb{I} \\ \ddots & \vdots & \vdots \\ \dots & \mu^{n-1}A - \mu^{n-2}B_{n-1} & \mu^{n-2}\mathbb{I} \\ \dots & \mu^{n-2}B_{n-2} + \mu^{n-3}B_{n-3} + \dots + B_0 & \mu^{n-1}\mathbb{I} \end{matrix} \Bigg) e^{\mu t} d\mu,$$

where $\mathbb{I}$ is the identity operator. Earlier, in the proof of Theorem 2, it was established that the function $q(t)$ satisfies the integral Equation (16). Take the natural number $l \leq p+n$. Assuming that $q \in C^l([0,T];\mathcal{Y})$ by Lemma 2, we obtain the equality

$$q^{(l)}(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t)(q\Phi)^{(m)}(0) +$$

$$+ \sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{1,n}(t-s) q^{(l-k-m)}(s)\Phi^{(m)}(s)ds +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(t)(q\Phi)^{(m)}(0) +$$

$$+ \sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{2,n}(t-s) q^{(l-k-m)}(s)\Phi^{(m)}(s)ds + \dots +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(t)(q\Phi)^{(m)}(0) +$$

$$+ \sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_{n-1} \int_0^t V_{n,n}(t-s) q^{(l-k-m)}(s)\Phi^{(m)}(s)ds,$$

where $C_l^k = \frac{l!}{k!(l-k)!}$, $C_l^{k,m} = \frac{l!}{k!m!(l-k-m)!}$ and

$$q_0^{(l)}(t) = -\sum_{k=0}^{l} C_l^k R^{(k)}(t) \Bigg(\Psi^{(l-k+n)}(t) -$$

$$- C S_0 V_{1,1}^{(l-k)}(t) u_0 - C S_1 V_{2,1}^{(l-k)}(t) u_0 - \dots - C S_{n-1} V_{n,1}^{(l-k)}(t) u_0 -$$

$$- C S_0 V_{1,2}^{(l-k)}(t) u_1 - C S_1 V_{2,2}^{(l-k)}(t) u_1 - \dots - C S_{n-1} V_{n,2}^{(l-k)}(t) u_1 - \dots -$$

$$- C S_0 V_{1,n}^{(l-k)}(t) u_{n-1} - C S_1 V_{2,n}^{(l-k)}(t) u_{n-1} - \dots - C S_{n-1} V_{n,n}^{(l-k)}(t) u_{n-1} -$$

$$- C S_0 \int_0^t V_{1,n}(t-s) h^{(l-k)}(s) ds - C S_1 \int_0^t V_{2,n}(t-s) h^{(l-k)}(s) ds - \dots -$$

$$- C S_{n-1} \int_0^t V_{n,n}(t-s) h^{(l-k)}(s) ds - C h^{(l-k)}(t) \Bigg) +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t) h^{(m)}(0) +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(t) h^{(m)}(0) + \dots +$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(t) h^{(m)}(0)$$

exists from the conditions of this theorem for $l = 0, 1, ..., p+n$.

Show that $q \in C^{p+n}([0,T], \mathcal{Y})$; for this purpose, denote $r_0 = q_0(0)$, and for $l = 1, 2, ..., p+n$, determine the following values

$$r_l = q_0^{(l)}(0) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(0) C S_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(0) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) +$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(0) C S_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(0) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) + ... +$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(0) C S_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(0) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0).$$

Consider the system of integral equations

$$\tilde{q}_0(t) = q_0(t) + R(t)\Bigg(C S_0 \int_0^t V_{1,n}(t-s) \tilde{q}_0(s) \Phi(s) ds +$$

$$+ C S_1 \int_0^t V_{2,n}(t-s) \tilde{q}_0(s) \Phi(s) ds + ... + C S_{n-1} \int_0^t V_{n,n}(t-s) \tilde{q}_0(s) \Phi(s) ds \Bigg),$$

$$\tilde{q}_l(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) +$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) + ... +$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) +$$

$$+\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_0 \int_0^t V_{1,n}(t-s) \tilde{q}_{l-k-m}(s) \Phi^{(m)}(s) ds +$$

$$+\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_1 \int_0^t V_{2,n}(t-s) \tilde{q}_{l-k-m}(s) \Phi^{(m)}(s) ds + ... +$$

$$+\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) C S_{n-1} \int_0^t V_{n,n}(t-s) \tilde{q}_{l-k-m}(s) \Phi^{(m)}(s) ds,$$

$$l = 1, 2, ..., p+n. \tag{17}$$

Reduce (17) to the Volterra equation of the second kind

$$g(t) = g_0(t) + \int_0^t K(t,s) g(s) ds$$

on the space $(C([0,T];\mathcal{Y}))^{p+n+1}$ with a matrix operator function $K(t,s)$, given on the triangle $\Delta = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le s \le t\}$. By virtue of the continuity of all data of system (17), this has a unique solution

$$(\tilde{q}_0, \tilde{q}_1, ..., \tilde{q}_{p+n}) \in (C([0,T];\mathcal{Y}))^{p+n+1}.$$

This solution will be the limit of the sequence of approximations

$$\tilde{q}_{0,i}(t) = q_0(t) + R(t)\Bigg(CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{0,i-1}(s)\Phi(s)ds+$$

$$+CS_1\int_0^t V_{2,n}(t-s)\tilde{q}_{0,i-1}(s)\Phi(s)ds + ... + CS_{n-1}\int_0^t V_{n,n}(t-s)\tilde{q}_{0,i-1}(s)\Phi(s)ds\Bigg),$$

$$\tilde{q}_{l,i}(t) = q_0^{(l)}(t) + \sum_{k=0}^{l-1} C_l^k R^{(k)}(t)CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j}\Phi^{(j)}(0)+$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(t)CS_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j}\Phi^{(j)}(0) + ...+$$

$$+\sum_{k=0}^{l-1} C_l^k R^{(k)}(t)CS_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j}\Phi^{(j)}(0)+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m,i-1}(s)\Phi^{(m)}(s)ds+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t)CS_1\int_0^t V_{2,n}(t-s)\tilde{q}_{l-k-m,i-1}(s)\Phi^{(m)}(s)ds + ...+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t)CS_{n-1}\int_0^t V_{n,n}(t-s)\tilde{q}_{l-k-m,i-1}(s)\Phi^{(m)}(s)ds,$$

$$l = 1,2,...,p+n;\ i \in \mathbb{N}, \tag{18}$$

which for $i \to \infty$ on the interval $[0,T]$ converge uniformly to the functions $\tilde{q}_l$, $l = 0,1,...,p+n$. Set the initial approximation $\tilde{q}_{l,0} \equiv 0$; $l = 0,1,...,p+n$, then $\tilde{q}_{l+1,0} = \tilde{q}'_{l,0}$; $l = 0,1,...,p+n-1$. In addition, from (18), it follows that

$$\tilde{q}_{l,i}(0) = r_l;\ l = 0,1,...,p+n;\ i \in \mathbb{N}. \tag{19}$$

Assume that for all $\tau = 1,2,...,i$ the equalities $\tilde{q}_{l+1,\tau}(t) = \tilde{q}'_{l,\tau}(t)$, $l = 0,1,...,p+n-1$ are true. Then, using Lemma 2 and equalities (18), we obtain

$$\frac{d}{dt}\Bigg(\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m,i}(s)\Phi^{(m)}(s)ds\Bigg) =$$

$$= \sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k+1)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m,i}(s)\Phi^{(m)}(s)ds+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t)CS_0 V_{1,n}(t)\tilde{q}_{l-k-m,i}(0)\Phi^{(m)}(0)+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k}C_l^{k,m}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m,i}(s)\Phi^{(m+1)}(s)ds+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k}C_l^{k,m}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m+1,i}(s)\Phi^{(m)}(s)ds=$$

$$=\sum_{k=1}^{l+1}\sum_{m=0}^{l-k+1}C_l^{k-1,m}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m+1,i}(s)\Phi^{(m)}(s)ds+$$

$$+\sum_{k=0}^{l}C_l^kR^{(k)}(t)CS_0V_{1,n}(t)\sum_{m=0}^{l-k}C_{l-k}^m r_{l-k-m}\Phi^{(m)}(0)+$$

$$+\sum_{k=0}^{l}\sum_{m=1}^{l-k+1}C_l^{k,m-1}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m+1,i}(s)\Phi^{(m)}(s)ds+$$

$$+\sum_{k=0}^{l}\sum_{m=0}^{l-k}C_l^{k,m}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m+1,i}(s)\Phi^{(m)}(s)ds. \quad (20)$$

Denote by

$$a_{k,m}=R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m+1,i}(s)\Phi^{(m)}(s)ds,\ \ l=2,3,\ldots,p+n.$$

Taking into account the equalities

$$C_l^k+C_l^{k-1}=C_{l+1}^k,\ \ C_l^{k,m}+C_l^{k-1,m}+C_l^{k,m-1}=C_{l+1}^{k,m}$$

we obtain

$$\sum_{k=0}^{l}\sum_{m=0}^{l-k}C_l^{k,m}a_{k,m}+\sum_{k=1}^{l+1}\sum_{m=0}^{l-k+1}C_l^{k-1,m}a_{k,m}+\sum_{k=0}^{l}\sum_{m=1}^{l-k+1}C_l^{k,m-1}a_{k,m}=$$

$$=\left(\sum_{k=1}^{l}\sum_{m=1}^{l-k}C_l^{k,m}a_{k,m}+\sum_{k=1}^{l}C_l^{k,0}a_{k,0}+\sum_{m=0}^{l}C_l^{0,m}a_{0,m}\right)+$$

$$+\left(\sum_{k=1}^{l}\sum_{m=1}^{l-k}C_l^{k-1,m}a_{k,m}+\sum_{k=1}^{l}C_l^{k-1,0}a_{k,0}+\sum_{k=1}^{l}C_l^{k-1,l-k+1}a_{k,l-k+1}+C_l^{l,0}a_{l+1,0}\right)+$$

$$+\left(\sum_{k=1}^{l}\sum_{m=1}^{l-k}C_l^{k,m-1}a_{k,m}+\sum_{m=1}^{l+1}C_l^{0,m-1}a_{0,m}+\sum_{k=1}^{l}C_l^{k,l-k}a_{k,l-k+1}\right)=$$

$$=\sum_{k=1}^{l}\sum_{m=1}^{l-k}C_{l+1}^{k,m}a_{k,m}+\sum_{k=1}^{l}C_{l+1}^{k,0}a_{k,0}+\sum_{m=1}^{l}C_{l+1}^{0,m}a_{0,m}+\sum_{k=1}^{l}C_{l+1}^{k,0}a_{k,l-k+1}+$$

$$+C_l^{0,0}a_{0,0}+C_l^{0,l}a_{0,l+1}+C_l^{l,0}a_{l+1,0}=\sum_{k=0}^{l+1}\sum_{m=0}^{l-k+1}C_{l+1}^{k,m}a_{k,m}. \quad (21)$$

For $l=0,1$ fullment of (21) can be checked directly.
From (20) and (21), it follows that

$$\frac{d}{dt}\left(\sum_{k=0}^{l}\sum_{m=0}^{l-k}C_l^{k,m}R^{(k)}(t)CS_0\int_0^t V_{1,n}(t-s)\tilde{q}_{l-k-m,i}(s)\Phi^{(m)}(s)ds\right)=$$

$$= \sum_{k=0}^{l+1} \sum_{m=0}^{l-k+1} C_{l+1}^{k,m} R^{(k)}(t) CS_0 \int_0^t V_{1,n}(t-s) \tilde{q}_{l-k-m+1,i}(s) \Phi^{(m)}(s) ds +$$

$$+ \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS_0 V_{1,n}(t) \sum_{m=0}^{l-k} C_{l-k}^m r_{l-k-m} \Phi^{(m)}(0). \tag{22}$$

Similarly, we obtain the result for the subsequent integral element from (18)

$$\frac{d}{dt}\left(\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS_1 \int_0^t V_{2,n}(t-s) \tilde{q}_{l-k-m,i}(s) \Phi^{(m)}(s) ds \right) =$$

$$= \sum_{k=0}^{l+1} \sum_{m=0}^{l-k+1} C_{l+1}^{k,m} R^{(k)}(t) CS_1 \int_0^t V_{2,n}(t-s) \tilde{q}_{l-k-m+1,i}(s) \Phi^{(m)}(s) ds +$$

$$+ \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS_1 V_{2,n}(t) \sum_{m=0}^{l-k} C_{l-k}^m r_{l-k-m} \Phi^{(m)}(0). \tag{23}$$

Continuing the procedure for all subsequent integral elements (18), we present the result for the last

$$\frac{d}{dt}\left(\sum_{k=0}^{l} \sum_{m=0}^{l-k} C_l^{k,m} R^{(k)}(t) CS_{n-1} \int_0^t V_{n,n}(t-s) \tilde{q}_{l-k-m,i}(s) \Phi^{(m)}(s) ds \right) =$$

$$= \sum_{k=0}^{l+1} \sum_{m=0}^{l-k+1} C_{l+1}^{k,m} R^{(k)}(t) CS_{n-1} \int_0^t V_{n,n}(t-s) \tilde{q}_{l-k-m+1,i}(s) \Phi^{(m)}(s) ds +$$

$$+ \sum_{k=0}^{l} C_l^k R^{(k)}(t) CS_{n-1} V_{n,n}(t) \sum_{m=0}^{l-k} C_{l-k}^m r_{l-k-m} \Phi^{(m)}(0). \tag{24}$$

Changing the summation indices and re-grading the sums, we obtain

$$\frac{d}{dt}\left(\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) \right) =$$

$$= \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) +$$

$$+ \sum_{k=0}^{l-1} C_l^k R^{(k+1)}(t) CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) =$$

$$= \sum_{k=0}^{l-1} C_l^k R^{(k)}(t) CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) +$$

$$+ \sum_{k=1}^{l} C_l^{k-1} R^{(k)}(t) CS_0 \sum_{m=0}^{l-k} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) =$$

$$= \left(\sum_{k=1}^{l-1} C_l^k R^{(k)}(t) CS_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) + \right.$$

$$\left. + C_l^0 R(t) CS_0 \sum_{m=0}^{l-1} V_{1,n}^{(l-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) \right) +$$

$$+\left(\sum_{k=1}^{l-1} C_l^{k-1} R^{(k)}(t) C S_0 \sum_{m=0}^{l-k-1} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) + \right.$$

$$+\sum_{k=1}^{l-1} C_l^{k-1} R^{(k)}(t) C S_0 V_{1,n}(t) \sum_{j=0}^{l-k} C_{l-k}^{l} r_{l-k-j} \Phi^{(j)}(0) +$$

$$\left. + C_l^{l-1} R^{(l)}(t) C S_0 V_{1,n}(t) C_0^0 r_0 \Phi(0) \right) =$$

$$= \sum_{k=0}^{l} C_{l+1}^{k} R^{(k)}(t) C S_0 \sum_{m=0}^{l-k} V_{1,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) -$$

$$- \sum_{k=0}^{l} C_l^k R^{(k)}(t) C S_0 V_{1,n}(t) \sum_{m=0}^{l-k} C_{l-k}^{m} r_{l-k-m} \Phi^{(m)}(0). \tag{25}$$

Similarly, we obtain the result for the next non-integral element from (18)

$$\frac{d}{dt}\left(\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_1 \sum_{m=0}^{l-k-1} V_{2,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0)\right) =$$

$$= \sum_{k=0}^{l} C_{l+1}^{k} R^{(k)}(t) C S_1 \sum_{m=0}^{l-k} V_{2,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) -$$

$$- \sum_{k=0}^{l} C_l^k R^{(k)}(t) C S_1 V_{2,n}(t) \sum_{m=0}^{l-k} C_{l-k}^{m} r_{l-k-m} \Phi^{(m)}(0). \tag{26}$$

Continuing the procedure for all subsequent non-integral elements (18), we present the result for the last

$$\frac{d}{dt}\left(\sum_{k=0}^{l-1} C_l^k R^{(k)}(t) C S_{n-1} \sum_{m=0}^{l-k-1} V_{n,n}^{(l-k-m-1)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0)\right) =$$

$$= \sum_{k=0}^{l} C_{l+1}^{k} R^{(k)}(t) C S_{n-1} \sum_{m=0}^{l-k} V_{n,n}^{(l-k-m)}(t) \sum_{j=0}^{m} C_m^j r_{m-j} \Phi^{(j)}(0) -$$

$$- \sum_{k=0}^{l} C_l^k R^{(k)}(t) C S_{n-1} V_{n,n}(t) \sum_{m=0}^{l-k} C_{l-k}^{m} r_{l-k-m} \Phi^{(m)}(0). \tag{27}$$

Differentiating (18), and also using (22)–(27), we obtain the equalities $\tilde{q}'_{l,i+1} = \tilde{q}_{l+1,i+1}$; $l = 0, 1, ..., p+n-1$. Thus, the sequence $\tilde{q}_{0,i}$ converges as $i \to \infty$ to the function $\tilde{q}_0$ uniformly on the interval $[0, T]$, and the sequence $\tilde{q}'_{0,i} = \tilde{q}_{1,i}$ converges as $i \to \infty$ to the function $\tilde{q}_1$ uniformly on the segment $[0, T]$. Therefore, the function $\tilde{q}_0$ is continuously differentiable and $\tilde{q}'_0 = \tilde{q}_1$. The equalities of $\tilde{q}'_l = \tilde{q}_{l+1}$; $l = 1, 2, ..., p+n-1$, are proved in the same way, which implies that $\tilde{q}_0 \equiv q \in C^{p+n}([0, T]; \mathcal{Y})$ and, therefore, $q^{(l)} = \tilde{q}_l$; $l = 1, 2, ..., p+n$. $\square$

3.3. Solvability of the Original Inverse Problem

Theorem 4. *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (4) be fulfilled; moreover, the ∞ be a pole of order $p \in \mathbb{N}_0$ of the A-resolvent of the pencil $\vec{B}$, operator $C \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $\mathcal{U}^0 \subset \ker C$, $\chi \in C^{p+n}([0, T]; \mathcal{L}(\mathcal{Y}; \mathcal{F}))$, $f \in C^{p+n}([0, T]; \mathcal{F})$, $\Psi \in C^{p+2n}([0, T]; \mathcal{Y})$, for any $t \in [0, T]$ operator $C(A^1)^{-1}Q\chi$ be invertible, with $(C(A^1)^{-1}Q\chi)^{-1} \in C^{p+n}([0, T]; \mathcal{L}(\mathcal{Y}))$, the* *condition* $Cu_{n-1} = \Psi^{(n-1)}(0)$ *be satisfied at some initial value* $u_{n-1} \in \mathcal{U}^1$*, and the initial values* $w_k = (I - P)v_k \in \mathcal{U}^0$ *satisfy*

$$w_k = -\sum_{j=0}^{p} K_j^n (B_0^0)^{-1} \frac{d^{j+k}}{dt^{j+k}} \Big[(I - Q)(q(0)\chi(0) + f(0)) \Big], \quad k = 0, 1, ..., n-1.$$

Then, there exists a unique solution (v, q) *of inverse problem (1)–(3), where* $q \in C^{p+n}([0, T]; \mathcal{Y})$*,* $v = u + w$*, whence* $u \in C^n([0, T]; \mathcal{U}^1)$ *is the solution of (5)–(7), and the function* $w \in C^n([0, T]; \mathcal{U}^0)$ *is a solution of (8) and (9) given by*

$$w(t) = -\sum_{j=0}^{p} K_j^n (B_0^0)^{-1} \frac{d^j}{dt^j} \Big[(I - Q)(q(t)\chi(t) + f(t)) \Big]. \tag{28}$$

Proof of Theorem 4. The conditions of Theorems 2 and 3 are satisfied, and, therefore, there exists a unique solution (q, u) to problem (5)–(7), where $q \in C^{p+n}([0, T]; \mathcal{Y})$, $u \in C^n([0, T]; \mathcal{U}^1)$.

Using the result of [9] and the required smoothness of the function q, we obtain that there exists a unique solution $w \in C^n([0, T]; \mathcal{U}^0)$ to (8), (9), given by (28). □

4. Discussion

The results obtained in the article can be applied to various mathematical models, such as a model of oscillation of a rotating viscous fluid using the viscosity coefficient, a model of gravitational-gyroscopic and internal waves, and a model of sound waves in smectics, since these mathematical models can be reduced to the Sobolev type equations of higher order. One of the most typical examples of the application of the Sobolev type equations theory is the Boussinesq–Love model [5]:

$$(\lambda - \Delta)v_{tt} = \alpha(\Delta - \lambda')v_t + \beta(\Delta - \lambda'')v + qf, \tag{29}$$

with initial conditions

$$v(x, 0) = v_0(x), \; v_t(x, 0) = v_1(x),$$

boundary condition

$$v(x, t)|_{\partial\Omega} = 0$$

and overdetermination condition

$$\int_{\Omega} v(x, t)K(x)dx = \Phi(t), \tag{30}$$

where $v_0(x), v_1(x), K(x), \Phi(t)$ are given functions, $v(x, t)$ is a searched function and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a boundary $\partial\Omega$ of class C^∞. Equation (29) describes longitudinal vibrations in a thin elastic rod, taking into account the inertia and external load. The coefficients $\lambda, \alpha, \lambda', \beta, \lambda''$ characterize the properties of the rod material and relate such quantities as Young's modulus, Poisson's ratio, material density and radius of gyration relative to the center of gravity, in addition, the function f sets a known part of the external load (if known). The integral overdetermination condition (30) arises at the moment when, in addition to finding the function v, it is necessary to restore the component of the external load q. In addition, it is planned to use the obtained results for the development of numerical methods, to find approximate solutions to some of the previously presented models.

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