mathematics

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Mortality/Longevity Risk-Minimization with or without Securitization

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Abstract: This paper addresses the risk-minimization problem, with and without mortality securitization, *à la* Föllmer–Sondermann for a large class of equity-linked mortality contracts when no model for the death time is specified. This framework includes situations in which the correlation between the market model and the time of death is arbitrary general, and hence leads to the case of a market model where there are two levels of information—the public information, which is generated by the financial assets, and a larger flow of information that contains additional knowledge about the death time of an insured. By enlarging the filtration, the death uncertainty and its entailed risk are fully considered without any mathematical restriction. Our key tool lies in our optional martingale representation, which states that any martingale in the large filtration stopped at the death time can be decomposed into precise orthogonal local martingales. This allows us to derive the dynamics of the value processes of the mortality/longevity securities used for the securitization, and to decompose any mortality/longevity liability into the sum of orthogonal risks by means of a risk basis. The first main contribution of this paper resides in quantifying, as explicitly as possible, the effect of mortality on the risk-minimizing strategy by determining the optimal strategy in the enlarged filtration in terms of strategies in the smaller filtration. Our second main contribution consists of finding risk-minimizing strategies with insurance securitization by investing in stocks and one (or more) mortality/longevity derivatives such as longevity bonds. This generalizes the existing literature on risk-minimization using mortality securitization in many directions.

Keywords: time of death/random horizon/default; progressively enlarged filtration; optional martingale representation; risk decomposition; unit-linked mortality contracts; risk-minimization; mortality/longevity risk; insurance securitization

Citation: Choulli, T.; Daveloose, C.; Vanmaele, M. Mortality/Longevity Risk-Minimization with or without Securitization. *Mathematics* **2021**, *9*, 1629. https://doi.org/10.3390/math9141629

Academic Editor: Emanuela Rosazza Gianin

Received: 7 June 2021
Accepted: 6 July 2021
Published: 10 July 2021

1. Introduction

In this paper, we manage the risk of a life insurance portfolio that faces two main types of risk, financial risk and mortality or longevity risk, by designing quadratic hedging strategies *à la* Föllmer–Sondermann, introduced in [1], with and without mortality securization. Hereto, we identify the hedgeable part of a product's payoff and identify the remaining types of risk. This is of importance in a fair valuation study of hybrid financial and actuarial products in life insurance as in, for example, [2–6]. We consider a financial setting consisting of an initial market, characterized by its flow of information $\mathbb{F}$ and its underlying traded assets S, and a random time—the death time τ—that might not be observed through $\mathbb{F}$ when it occurs. The financial risk originates from the investment in the risky assets, while the mortality risk follows from the uncertainty of the death time and can be split into a systematic and an unsystematic part, see, for example, [7,8] and the references therein. Longevity risk refers to the risk that the reference population might, on average, live longer than anticipated. The unsystematic mortality risk, that is, the risk corresponding to individual mortality rates, can be diversified by increasing the size of the portfolio while systematic mortality risk and longevity risk cannot be diversified away by pooling. The market for mortality-linked instruments, the so-called life market, to transfer such illiquid risks into financial markets as an alternative to the classical actuarial form of risk mitigation, is in full development. In [9] (see also [10,11] and the references therein) the authors were the first to advocate the use of mortality-linked securities for hedging purposes. The first longevity bonds were sold in the late 1990s. The longevity derivatives market has since expanded to include forward contracts, options and swaps. A detailed account of this evolution can be found in [12–14]. The development of the life market entails questions about the engineering of mortality-linked securities or derivatives as well as their pricing and finding their dynamics. As the authors of [15] state, 'The pricing of any mortality linked derivative security begins with the choice of a mortality model'; these prices obviously depend heavily on the chosen mortality model and the method used to price those securities. Since the Lee–Carter model introduced in [16], many mortality models have been suggested. They can be classified into two main groups, depending on whether the obtained model was inspired by credit risk modelling or interest rate modelling. The first approach is based on the strong similarity between mortality and default and hence uses the arguments of credit risk theory, while the second approach follows the interest rate term structure approach, such as in [17]. However, model misspecification can have a significant impact on the performance of hedging strategies for mortality or longevity risk. Recently, in [18] (see also [19,20] for related discussion), the authors use the CAPM and the CCAPM to price longevity bonds and concluded that this pricing does not accurately reflect reality; they suggest that there might be a kind of 'mortality premium puzzle' *à la* Mehra and Prescott [21]. While this mortality premium puzzle might exist, the 'poor and/or bad' specification of the model for the mortality plays an important role in getting those wrong prices for longevity bonds. In [22], they propose a robust mean-variance hedging approach to deal with parameter uncertainty and model misspecification.

Our aim is to position ourselves in a context *without mortality specification* and to derive the dynamics of the security's price process and design the risk-minimizing hedging strategies.

To further elaborate our main aim in this paper and its relation to the literature, we introduce some notations that are valid throughout the whole paper. The tuplet $(\Omega, \mathcal{F}, \mathbb{F}, S, P)$ represents mathematically the initial financial market model. Herein, the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfies the usual condition (i.e., filtration is complete and right continuous) with $\mathcal{F}_t \subset \mathcal{F}$, and S is an $\mathbb{F}$-semimartingale representing the discounted price process of d risky assets. The mortality is modelled with the death time of the insured, τ, which is mathematically an arbitrary random time (i.e., a $[0, +\infty]$-valued random variable). The flow of information generated by the public flow $\mathbb{F}$ enlarged by the random time will be denoted by $\mathbb{G}$, where the relationship between the three components, $\mathbb{F}$, τ and $\mathbb{G}$, will be specified in the next section.

To our knowledge, apart from the recent paper [23], all the existing literature about mortality and/or longevity assumes a specific model for mortality and derives the dynamics for longevity bond prices accordingly. For an up-to-date extensive list of relevant papers, see [22]. We follow the approach of [23]. Even though $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ with τ, as in credit risk theory, the death time is kept arbitrary general with no assumption at all. This translates into the fact that the survival probabilities over time constitute a general non-negative supermartingale. To capture which risk to hedge under $\mathbb{G}$ we use the classification of [23], where a $\mathbb{G}$-risk up to τ is expressed as a functional of pure financial risk (PF), pure mortality risk (PM) and correlation risk (CR) intrinsic to the correlation between the financial market and the mortality. Mathematically, these risks can be expressed as (local) martingales due to arbitrage theory. In [23], we elaborated an optional martingale representation for martingales in the large filtration $\mathbb{G}$, stopped at the death time under no assumption of any kind. This representation states that any martingale in the large filtration $\mathbb{G}$ stopped at the death time can be decomposed into precise orthogonal local martingales. By means of this optional martingale representation we derive the dynamics of the value processes of the mortality/longevity securities and decompose any mortality/longevity liability into the sum of orthogonal risks by means of a risk basis. Then, *our main objective* lies, when one considers the quadratic hedging *à la* Föllmer–Sondermann, in determining the optimal hedging strategy $\xi^{\mathbb{G}}$ for the whole risk encountered under $\mathbb{G}$ on $[\![0,\tau]\!]$ as a functional Ξ such that

$$\xi^{\mathbb{G}} = \Xi\Big(\xi^{\mathrm{pf}}, \xi_1^{\mathrm{pm}}, \ldots, \xi_k^{\mathrm{pm}}, \xi_1^{\mathrm{cr}}, \ldots, \xi_l^{\mathrm{cr}}\Big), \tag{1}$$

where ξ^{pf} is the optimal hedging strategy for the pure financial risk, ξ_i^{pm}, $i = 1, \ldots, k$, are the optimal hedging strategies for the pure mortality risks, and ξ_j^{cr}, $j = 1, \ldots, l$, are the optimal hedging strategies for the correlation risks. Even though our results can be extended to more general quadratic hedging approaches, we opted to focus on the Föllmer–Sondermann method to well illustrate our main ideas. The literature addressing this objective has become quite rich in the last decade, while the existing literature makes assumptions on the triplet $(\mathbb{F}, S, \tau)$ that can be translated, in one way or another, to a sort of independence and/or no correlation between the financial market—represented by the pair $(\mathbb{F}, S)$—and the mortality represented by the death time τ. This independence feature, with its various degrees, has been criticized in the literature by both empirical and theoretical studies. In fact, a recent stream of financial literature highlights several links between demography and financial variables when dealing with longevity risk, see [24,25] and references therein.

We have two main contributions that are intimately related to each other and that realize the aforementioned main objective by providing a rigorous and precise formulation for (1). Our first main contribution lies in quantifying, as explicitly as possible, the effect of mortality on the risk-minimizing strategy by determining the optimal strategy in the enlarged filtration $\mathbb{G}$ in terms of strategies in the smaller filtration $\mathbb{F}$. Our second main contribution resides in finding risk-minimizing strategies with securization by investing in stocks and one (or more) insurance contracts, such as longevity bonds.

Concerning the literature about risk-minimization with or without mortality securitization, we cite [7,8,17,26–33] and the references therein to cite a few. In [8,32,33], the authors assume independence between the financial market and the insurance model, a fact that was criticized in [34]. The works of [27,29,35] assume 'the H-hypothesis', which guarantees that the mortality has no effect on the martingale structure at all (i.e., every $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale). This assumption can be viewed as a strong no correlation condition between the financial market and the mortality. In [17], the author weakens this assumption by considering the following two assumptions:

$$\text{Either } \tau \text{ avoids } \mathbb{F}\text{-stopping times or all } \mathbb{F}\text{-martingales are continuous,} \tag{2}$$

and

$$M^{\mathbb{G}} \text{ is given by } M_t^{\mathbb{G}} := \mathbb{E}[h_\tau \mid \mathcal{G}_t] \text{ where } h \text{ is } \mathbb{F}\text{-predictable with suitable integrability.} \tag{3}$$

However, these assumptions are also very restrictive. It is clear that for the popular and simple discrete time market models the assumption (2) fails. Furthermore, for most models in insurance (if not all), a Poisson process is an important component in the modelling, and hence for these models the second part of assumption (2) fails, while its first part can be viewed as a kind of 'independence' assumption between the mortality (i.e., the random time τ) and the financial market (i.e., the pair $(\mathbb{F}, S)$). In [36] and the references therein, the authors treat many death-related claims and liabilities in (life) insurance whose payoff process h fails (3). In [8,17,27,29,32], the authors assumed that the mortality has a hazard rate process, mimicking the intensity-based approach of credit risk, while in [26], the author uses the interest rate modelling of Heath–Jarrow–Morton. To our knowledge, all the literature considers the Brownian setting for the financial market except for [17]. In [30,31], the authors accounted for the mutual dependence between the financial and the insurance markets (and also for partial information on the mortality intensity of the policy holder) but assumed a particular hazard rate process for the mortality and worked in a Brownian setting. The application to the semimartingale case via the weaker concept of local risk minimization in [30] is by the continuity of the processes essentially reduced to finding the Galtchouk–Kunita–Watanabe decomposition under the minimal martingale measure.

This paper contains five sections, including the current section, and an appendix. The aim of the next section (Section 2) is to introduce the mathematical model, the optional martingale representation, and the Föllmer–Sondermann optimization criterion. The third and the fourth sections are the principal innovative sections of the paper and deal with quadratic hedging for mortality/longevity risks, in the spirit of Föllmer–Sondermann, in the cases where mortality/longevity securitization is incorporated or not. The last section summarizes our analysis and our contributions. For the sake of easy exposition, the proof of some intermediate technical lemmas are relegated to the appendix.

2. Mathematical Model and Preliminaries

This section presents our mathematical model, which is constituted by the initial market model and a death time, and recalls our optional martingale representation result that we use throughout the paper. We conclude this section by briefly reviewing the quadratic hedging criterion of Föllmer–Sondermann.

2.1. Time of Death, Enlargement of Filtration and a Martingale Representation Theorem

In addition to the initial market model $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, S, P)$, we consider from now on an $\mathcal{F}$-measurable random time τ that represents the *time of death* of an insured, which might not be an $\mathbb{F}$-stopping time. The knowledge about this time of death is limited. The right-continuous and non-decreasing process indicating whether death has occurred or not is denoted by D, while the enlarged filtration of $\mathbb{F}$ associated with the couple $(\mathbb{F}, \tau)$ is denoted by $\mathbb{G}$, and they are defined by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} := (\mathcal{G}_t)_{t\geq 0}, \quad \mathcal{G}_t = \cap_{s>0}(\mathcal{F}_{s+t} \vee \sigma(D_u,\ u \leq s+t)). \tag{4}$$

Thus, starting from the filtration $\mathbb{F}$, which represents the flow of public information, $\mathbb{G}$ is the *progressively enlarged* filtration by incorporating the information included in the process D. $\mathbb{G}$ is the smallest filtration, which contains $\mathbb{F}$ and makes τ a $\mathbb{G}$-stopping time.

We recall some notations that we will use throughout the paper. For any filtration $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$, we denote by $\mathcal{A}(\mathbb{H})$ (respectively $\mathcal{M}(\mathbb{H})$) the set of $\mathbb{H}$-adapted processes with $\mathbb{H}$-integrable variation (respectively that are $\mathbb{H}$-uniformly integrable martingales). For any process X, ${}^{o,\mathbb{H}}X$ (respectively ${}^{p,\mathbb{H}}X$) is the $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) projection of X. For an increasing process V, the process $V^{o,\mathbb{H}}$ (respectively $V^{p,\mathbb{H}}$) represents its dual $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) projection. For a filtration $\mathbb{H}$, $\mathcal{O}(\mathbb{H})$, $\mathcal{P}(\mathbb{H})$ and $\mathrm{Prog}(\mathbb{H})$ denote the $\mathbb{H}$-optional, the $\mathbb{H}$-predictable and the $\mathbb{H}$-progressive σ-fields respectively on $\Omega \times [0, +\infty]$. For an $\mathbb{H}$-semimartingale X, we denote by $L(X, \mathbb{H})$ the set of all X-integrable processes in Ito's sense, and for $H \in L(X, \mathbb{H})$, the resulting integral is a one dimensional $\mathbb{H}$-semimartingale denoted by $H \bullet X := \int_0^{\cdot} H_u dX_u$. If $\mathcal{C}(\mathbb{H})$ is a set of processes that is adapted to $\mathbb{H}$, then $\mathcal{C}_{\mathrm{loc}}(\mathbb{H})$—except when it is stated otherwise—is the set of processes, X, for which there exists a sequence of $\mathbb{H}$-stopping times, $(T_n)_{n\geq 1}$, which increases to infinity and X^{T_n} belongs to $\mathcal{C}(\mathbb{H})$, for each $n \geq 1$. The quadratic covariation of two $\mathbb{H}$-local martingales M and N is denoted by $[M, N]$ and the predictable quadratic covariation, also called the angle bracket process, of two $\mathbb{H}$-local martingales M and N is denoted by $\langle M, N\rangle^{\mathbb{H}}$. We recall the definition of orthogonality between local martingales.

Definition 1. *Let M and N be two $\mathbb{H}$-local martingales. Then M is said to be orthogonal to N whenever MN is also an $\mathbb{H}$-local martingale, or equivalently, $[M, N]$ is an $\mathbb{H}$-local martingale.*

The public, who have access to the filtration $\mathbb{F}$, can only obtain information about τ through the survival probabilities denoted by G_t and $\widetilde{G}_t$, and are given by

$$G_t := {}^{o,\mathbb{F}}(I_{[\![0,\tau[\![})_t = P(\tau > t|\mathcal{F}_t), \quad \widetilde{G}_t := {}^{o,\mathbb{F}}(I_{[\![0,\tau]\!]})_t = P(\tau \geq t|\mathcal{F}_t), \quad \text{and} \quad m := G + D^{o,\mathbb{F}}. \tag{5}$$

The processes G and $\widetilde{G}$ are known as Azéma supermartingales (G is right-continuous with left limits, while in general $\widetilde{G}$ has right and left limits only), while m is a BMO $\mathbb{F}$-martingale (An $\mathbb{F}$-martingale L is called a BMO $\mathbb{F}$-martingale if there exists a positive constant such that for any bounded $\mathbb{F}$-stopping times θ and σ such that $\theta \geq \sigma$ P-a.s., we have $E[|L_T - L_{\sigma-}| \mid \mathcal{F}_\sigma] \leq C$ P-a.s..). For more details about these, we refer the reader to ([37] paragraph 74, Chapter XX). Below we recall a class of $\mathbb{G}$-local martingales, resulting from $\mathbb{F}$-local martingales via a transformation, which plays a vital role in decomposing risk under $\mathbb{G}$.

Theorem 1 ([38] **Theorem 3).** *For any $\mathbb{F}$-local martingale M, the following,*

$$\mathcal{T}(M) := M^\tau - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [M, m] + I_{]\!]0,\tau]\!]} \bullet \left(\Delta M_{\widetilde{R}} I_{[\![\widetilde{R},+\infty[\![}\right)^{p,\mathbb{F}}, \tag{6}$$

is a $\mathbb{G}$-local martingale. Here,

$$R := \inf\{t \geq 0 : G_t = 0\}, \quad \textit{and} \quad \widetilde{R} := R I_{\{\widetilde{G}_R = 0 < G_{R-}\}} + \infty I_{\{\widetilde{G}_R = 0 < G_{R-}\}^c}. \tag{7}$$

To derive the risk-minimizing strategies for a mortality claim we will make use of the optional martingale representation for a $\mathbb{G}$-martingale, introduced in [23], which states that the risk can be decomposed into three types of risks.

Theorem 2 ([23] **Theorem 2.17).** *Let $h \in L^2(\mathcal{O}(\mathbb{F}), P \otimes dD)$, and M^h be given by*

$$M^h_t := {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dD^{o,\mathbb{F}}_u\right)_t = \mathbb{E}\Big[\int_0^\infty h_u dD^{o,\mathbb{F}}_u \mid \mathcal{F}_t\Big]. \tag{8}$$

Then the $\mathbb{G}$-martingale $H_t := {}^{o,\mathbb{G}}(h_\tau)_t = \mathbb{E}[h_\tau|\mathcal{G}_t]$ admits the following representation.

$$H - H_0 = \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(M^h) - \frac{M^h_- - (h \bullet D^{o,\mathbb{F}})_-}{G^2_-} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(m) + \frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G} I_{]\!]0,R[\![} \bullet N^{\mathbb{G}}, \tag{9}$$

where $R := \inf\{t \geq 0 : G_t = 0\}$ and both processes

$$\frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G_-} I_{]\!]0,R[\![} \bullet N^{\mathbb{G}} \;\textit{and}\; \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(M^h) - \frac{M^h_- - (h \bullet D^{o,\mathbb{F}})_-}{G^2_-} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(m)$$

are square integrable and orthogonal $\mathbb{G}$-martingales.

The first term in the RHS of (9) represents the 'pure' financial risk, while the second term represents the risk resulting from the correlation between the market model and the death time τ. Both terms are expressed in terms of $\mathbb{G}$-local martingales that belong to the class of Theorem 1. The third term in the RHS of (9) models the *pure mortality risk of type one* (see ([23] Theorem 2.11) for details), where the process $N^{\mathbb{G}}$ is given by

$$N^{\mathbb{G}} := D - \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet D^{o,\mathbb{F}}, \tag{10}$$

which is a $\mathbb{G}$-martingale with integrable variation. This pure mortality risk is called a pure default martingale in ([23] Definition 2.2), and it quantifies the uncertainty in τ (or equivalently in D defined in (4)) that cannot be seen through $\mathbb{F}$. For other types of pure mortality risks (local martingale), and for further details about pure mortality (or default) local martingales, we refer the reader to [23]. Our decomposition (9) extends [39] to an arbitrary general pair $(\mathbb{F}, \tau)$ and to the case where h is $\mathbb{F}$-optional, as is the case for some examples in [36].

2.2. The Quadratic Risk-Minimizing Method

In this subsection, we quickly review the main ideas of risk-minimizing strategies, a concept that was introduced in [1] for financial contingent claims and extended in [33] for insurance payment processes. Note that [1] assumed that the discounted risky asset is a square-integrable martingale under the original measure P. In [40], the results are proved under the weaker assumption that X is only a local P-martingale that does not need to be locally square integrable. Throughout this subsection, we consider an $\mathbb{H}$-adapted process X with values in $\mathbb{R}^d$ representing the discounted assets' price process. $\mathbb{H}$ is any filtration satisfying the usual condition (usually $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$). Throughout the paper, we denote by $x^{tr}y$ the inner product of x and y, for any $x, y \in \mathbb{R}^d$.

Definition 2. *Suppose that $X \in \mathcal{M}_{loc}(\mathbb{H})$.*
(a) *A 0-admissible trading strategy is any pair $\rho := (\xi, \eta)$ where $\xi \in L^2(X^T)$ with $L^2(X^T)$ the space of all $\mathbb{R}^d$-valued predictable processes ξ such that*

$$\|\xi\|_{L^2(X)} := \left(\mathbb{E}\left[\int_0^T \xi_u^{tr} d[X]_u \xi_u\right]\right)^{1/2} < \infty,$$

and η is a real-valued adapted process such that the discounted value process

$$V(\rho) = \xi^{tr} X^T + \eta \ \text{ is right-continuous and square-integrable, and } V_T(\rho) = 0, \quad P\text{-a.s.}. \tag{11}$$

(b) *ρ is called risk-minimizing for the square integrable $\mathbb{H}$-adapted payment process $A = (A_t)_{t \geq 0}$, if it is a 0-admissible strategy and for any 0-admissible strategy $\tilde{\rho}$, we have*

$$\mathcal{R}_{t\wedge T}(\rho) \leq \mathcal{R}_{t\wedge T}(\tilde{\rho}) \quad P\text{-a.s. for every } t \geq 0, \tag{12}$$

where

$$\mathcal{R}_t(\rho) := \mathbb{E}[(C_T(\rho) - C_{t\wedge T}(\rho))^2 \mid \mathcal{H}_t] \ \text{ and } \ C(\rho) := V(\rho) - \xi \bullet X^T + A^T.$$

It is known in the literature that the Galtchouk–Kunita–Watanabe decomposition (called hereafter the GKW decomposition) plays a central role in determining the risk-minimizing strategy.

Theorem 3. *Let $M, N \in \mathcal{M}^2_{loc}(\mathbb{H})$. Then there exist $\theta \in L^2_{loc}(N)$ and $L \in \mathcal{M}^2_{0,loc}(\mathbb{H})$ such that*

$$M = M_0 + \theta \bullet N + L, \quad \text{and} \quad \langle N, L\rangle^{\mathbb{H}} \equiv 0. \tag{13}$$

Furthermore, $M \in \mathcal{M}^2(\mathbb{H})$ if and only if $M_0 \in L^2(\mathcal{H}_0, P)$, $\theta \bullet N \in \mathcal{M}^2_0(\mathbb{H})$ and $L \in \mathcal{M}^2_0(\mathbb{H})$.

For more about the GKW decomposition, we refer the reader to [41,42], and the references therein. The following theorem was proved for a single payoff in [40], and was extended to payment processes in [33].

Theorem 4. *Suppose that $X \in \mathcal{M}_{loc}(\mathbb{H})$, and let $A = (A_t)_{t\geq 0}$ be the payment process that is square integrable. Then the following holds.*
(a) *There exists a unique risk-minimizing strategy $\rho^* = (\xi^*, \eta^*)$ for A given by*

$$\xi^* := \xi^A \quad \text{and} \quad \eta_t^* := \mathbb{E}[A_T - A_{t\wedge T} \mid \mathcal{H}_t] - \xi_t^{*tr} X_{t\wedge T}, \tag{14}$$

where $(\xi^A, L^A) = (\xi^A I_{[\![0,T]\!]}, (L^A)^T)$ is the pair resulting from the GKW decomposition of $\mathbb{E}[A_T \mid \mathcal{F}_t]$ with respect to X with $\xi^A \in L^2(X^T)$ and $L^A \in \mathcal{M}^2_0(\mathbb{H})$ satisfying $\langle L^A, \theta \bullet X\rangle \equiv 0$, for all $\theta \in L^2(X)$.
(b) *The remaining (undiversified) risk is L^A, while the optimal cost, risk and value processes are*

$$C_t(\rho^*) = \mathbb{E}[A_T \mid \mathcal{H}_0] + L_t^A,\ \mathcal{R}_t(\rho^*) = \mathbb{E}[(L_T^A - L_t^A)^2 \mid \mathcal{H}_t],\ \textit{and}\ V_t(\rho^*) = \mathbb{E}[A_T - A_{t\wedge T} \mid \mathcal{H}_t]. \tag{15}$$

The next two sections contain the main two contributions of this paper and deal with hedging mortality liabilities *à la* Föllmer–Sondermann.

3. Hedging Mortality Risk without Securitization

In this section, we hedge the mortality liabilities without mortality securitization. In this context, our aim lies in quantifying—as explicitly as possible—the effect of mortality uncertainty on the risk-minimizing strategy. This will be achieved by determining the $\mathbb{G}$-optimal strategy in terms of $\mathbb{F}$-strategies for a large class of mortality contracts. This section contains four subsections. The first subsection discusses the main factors that generate the various risks induced by τ and its correlation with the initial model $(S, \mathbb{F})$. The second subsection deals with the general setting, where it quantifies the various risks induced by the mortality and emphasizes the impact on the optimal strategy. The third subsection focuses more on the interplay between randomness in τ and in benefit policies, and illustrates this with two popular mortality-linked contracts. The last subsection proves some of the main results of the second and third subsections. **Throughout the rest of the paper, we consider a given finite time horizon** $T > 0$.

3.1. Classification of Risks for the Triplet $(\tau, \mathbb{F}, S)$*: Preliminary Discussion*

Our modelling for the death time is very general, as it falls into the case of general reduced form with the general parametrization by the pair $(\widetilde{G}, G)$, or equivalently the pair $(m, D^{o,\mathbb{F}})$. For more details about this discussion and the decomposition of $\mathbb{G}$-risks, we refer the reader to [23]. In fact, in ([23] Theorems 2.17 and 2.20), we proved that the randomness of τ bears three types of orthogonal risks, namely the correlation risks and the pure mortality risks of type one and two, and

$$\mathbb{G}\text{-risk} = \underbrace{\text{'pure-financial-risk'}}_{\mathcal{T}(M)} + \underbrace{\text{'correlation-risk'}}_{\text{generated by } m} + \underbrace{\text{'pure-mortality1'}}_{H\bullet N^{\mathbb{G}}} + \underbrace{\text{'pure-mortality2'}}_{k\bullet D},$$

where M spans the set of $\mathbb{F}$-local martingales, H spans the set of all bounded $\mathbb{F}$-optional processes, and k spans the space $L^1_{loc}(\widetilde{\Omega}, \text{Prog}(\mathbb{F}), P\otimes D)$ and satisfies $E[k_\tau I_{\{\tau<+\infty\}} | \mathcal{F}_\tau] = 0$ P-a.s..

The correlation risks are more complex to study, to single out, and even to classify as the interplay between the randomness of τ and that from the whole financial model—represented by $(S, \mathbb{F})$—can be very involved and depends on the nature of the problems and the objectives addressed. It is clear from the decomposition above that the correlation risks, which naturally appeared in the literature up to now, are all generated by the process m only. This fact was stressed in studies on the effect of τ on arbitrage and viability in [43,44], where the effect of m, besides the positivity of G, are vital.

Herein, especially in the third subsection (Section 3.3), another type of correlation risk naturally arises and is generated by $D^{o,\mathbb{F}}$ instead. This is one of the innovative parts of this subsection and the paper as a whole. This novel class of correlation risks, which appeared naturally in the interplay between the randomness of τ and that of the benefit policies, is generated by the random measure $D^{o,\mathbb{F}}$. Precisely, this class of correlation risk takes the form of $H \bullet D^{o,\mathbb{F}}$, where H spans the set of $\mathbb{F}$-optional processes that are $D^{o,\mathbb{F}}$-integrable. This shows that, in our current setting, both processes m and $D^{o,\mathbb{F}}$ play crucial roles in generating correlation risks. However, the correlation risks generated by m remain the complex and the most influential, and appear in the form of $[m, X]$ when X spans the set of $\mathbb{F}$-local martingales. In virtue of the studies about portfolios in [45], we understand that these correlation risks can be classified into subclasses depending on whether the "correlation" of τ is with S or the part in $\mathbb{F}$ that are strongly orthogonal with S. Herein, we go deeper and classify the correlation risks between τ and S into subclasses.

In order to better understand these finer classifications, we appeal to the powerful statistical notion of *predictable characteristics* associated with the pair $(S,\mathbb{F})$. This notion starts by representing S with the pair (S^c,μ), where S^c is the continuous-local martingale part of S, and μ is the random measure of the jumps of S given by $\mu(dt,dx) = \sum_{s>0} I_{\{\Delta S_t\neq 0\}}\delta_{(t,\Delta S_t)}(ds,dx)$. Here, δ_a is the Dirac mass at the point a. On the one hand, in virtue of ([46] Theorem 3.75) and to ([47] Lemma 4.24), it is well known that any $\mathbb{F}$-local martingale L can be uniquely represented by a quadruplet $(\beta, f, g, L^{\perp})$ such that β is a predictable process, $f(t,\omega,x)$ is $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$-measurable functional, $g(t,\omega,x)$ is $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$-measurable ($\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$), and $L^{\perp}\in\mathcal{M}_{loc}(\mathbb{F})$ such that

$$\beta^2\bullet\langle S^c,S^c\rangle\in\mathcal{A}^+_{loc},\quad (g^2\star\mu)^{1/2}\in\mathcal{A}^+_{loc},\quad (f^2\star\mu)^{1/2}\in\mathcal{A}^+_{loc},\quad [S,L^{\perp}]\equiv 0, \tag{16}$$

$$L = L_0 + \underbrace{\beta\bullet S^c + U_L\star(\mu-\nu)}_{\text{generated by } S} + \underbrace{g\star\mu}_{\substack{\text{generated by } \mu\\ \text{and orthogonal to } S}} + L^{\perp}, \tag{17}$$

where

$$U_L(t,x) := f(t,x) + \frac{\int f(t,y)\nu(\{t\},dy))}{1-\nu(\{t\},\mathbb{R})}, \tag{18}$$

ν is the $\mathbb{F}$-predictable random measure compensator of μ such that $\mu-\nu$ is the martingale measure, while for the two integrals $U_L\star(\mu-\nu)$ and $g\star\mu$ we refer to ([46] Definitions 3.63 and 3.73, Chapter 3, Section 3).

The decomposition (17) is called hereafter Jacod's decomposition, and it claims that L is the sum of an $\mathbb{F}$-martingale generated by S, an $\mathbb{F}$-local martingale generated by μ and is orthogonal to S, and an $\mathbb{F}$-local martingale strongly orthogonal to S, that is, $[L^{\perp},S]=0$. This leads to the following definition, which extends ([45] Definition 5.3).

Definition 3. *Let X be an $\mathbb{F}$-semimartingale.*

(a) *τ is said to be weakly non-correlated to X if $[X,m]$ is an $\mathbb{F}$-local martingale.*

(b) *τ is said to be strongly non-correlated to X if $[X,m]\equiv 0$.*

Furthermore, in virtue of (17), this class of correlation risks that is generated by m can be split into several kinds of correlation risks using Jacod's decomposition of m as follows.

Proposition 1. *The following assertions hold.*

(a) *There exists a unique tuplet $(\beta_m, f_m, g_m, m^{\perp})$ satisfying the conditions (16), and*

$$m = m_0 + \underbrace{\beta_m\bullet S^c + U_m\star(\mu-\nu)}_{(\tau,S)\text{-correlation risk}} + \underbrace{g_m\star\mu}_{\substack{(\tau,S)\text{-weak}\\ \text{non-correlation risk}}} + \underbrace{m^{\perp}}_{\substack{(\tau,S)\text{-strong}\\ \text{non-correlation risk}}} . \tag{19}$$

(b) *τ is weakly non-correlated to S if and only if the pair (β_m, f_m) satisfies*

$$c\beta_m + \int x f_m(x)F(dx) = 0,\quad P\otimes A - a.e. \tag{20}$$

(c) *τ is strongly non-correlated to S if and only if $(\beta_m, f_m, g_m)\equiv(0,0,0)$.*

The proof of this proposition is straightforward and will be omitted here. Throughout the paper, due to the nature of the minimization criterion we will adopt, we will assume that τ is weakly-non-correlated to S. The following lemma parametrizes the remaining correlation between S and τ as follows.

Lemma 1. *Suppose that*
$$S \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{F}) \quad \text{and} \quad \langle S, m\rangle^{\mathbb{F}} \equiv 0. \tag{21}$$
Then $S^\tau \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{G})$ and the following assertions hold.

(a) *For any $L \in \mathcal{M}_{\mathrm{loc}}(\mathbb{F})$ that is orthogonal to S, the $\mathbb{G}$-martingale $\mathcal{T}(L)$ is orthogonal to S^τ.*

(b) *It holds that*
$$U := I_{\{G_->0\}} \bullet [S, m] \in \mathcal{M}^2_{loc}(\mathbb{F}), \tag{22}$$
and there exist $\varphi^{(m)} \in L^2_{\mathrm{loc}}(S, \mathbb{F})$ and $L^{(m)} \in \mathcal{M}^2_{0,\mathrm{loc}}(\mathbb{F})$ orthogonal to S such that
$$U = \varphi^{(m)} \bullet S + L^{(m)}, \quad \text{and} \quad]\!]0,\tau]\!] \subseteq \{G_- > 0\} \subseteq \{G_- + \varphi^{(m)} > 0\}, \quad P\text{-a.s.}. \tag{23}$$

(c) *We have $\mathcal{T}(U) = G_-\widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet U$ and*
$$(G_- + \varphi^{(m)}) \bullet \mathcal{T}(S) = G_- \bullet S^\tau - \mathcal{T}(L^{(m)}). \tag{24}$$

The proof of this lemma is relegated to Appendix A for the sake of simple exposition.

3.2. Impact's Quantification of Mortality Risks: A General Formula

This subsection considers a portfolio consisting of life insurance liabilities depending on the random time of death τ of a single insured. For the sake of simplicity, we assume that the policyholder of a contract is the insured itself. In the financial market, there is a risk-free asset and a multidimensional risky asset at hand. The price of the risk-free asset follows a strictly positive, continuous process of finite variation, and the risky asset follows a real-valued RCLL $\mathbb{F}$-adapted stochastic process. The discounted value of the risky asset is denoted by S. In order to reach our goal of expressing the $\mathbb{G}$-optimal strategy in terms of $\mathbb{F}$-strategies via the Föllmer–Sondermann method, we need to assume for the pair (S, τ) the following conditions:
$$S \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{F}), \quad \langle S, m\rangle^{\mathbb{F}} \equiv 0, \quad \text{and} \quad \{\Delta S \neq 0\} \cap \{\widetilde{G} = 0 < G_-\} = \emptyset. \tag{25}$$

The assumption $\{\Delta S \neq 0\} \cap \{\widetilde{G} = 0 < G_-\} = \emptyset$ guarantees the structure conditions for $(S^\tau, \mathbb{G})$, and hence the quadratic risk-minimization problem can have a solution for this model. This assumption holds when the hazard rate (i.e., $G > 0$) exists, for instance. We recall that the conditions $S \in \mathcal{M}^2_{\mathrm{loc}}(\mathbb{F})$ and $\langle S, m\rangle^{\mathbb{F}} \equiv 0$ are dictated by the method used for risk minimization. In particular, the assumption $\langle S, m\rangle^{\mathbb{F}} \equiv 0$ (or equivalently τ is weakly-non-correlated to S) implies that the risk generated by m cannot be hedged in the model $(S, \mathbb{F})$. The risk-minimizing method is the quadratic hedging approach à la Föllmer and Sondermann, which requires that the discounted price processes for the underlying assets are locally square integrable martingales. In fact, under these two latter conditions, both models $(S, P, \mathbb{F})$ and $(S^\tau, P, \mathbb{G})$ are local martingales, and hence the Föllmer–Sondermann method will be applied simultaneously for both models. These two assumptions in (25) can be relaxed at the expense of considering the quadratic hedging method considered in [48,49], and the references therein. For the risk-minimization framework of these papers, the assumption $\sup_{0\leq t\leq \cdot} |S_t|^2 \in \mathcal{A}^+_{loc}(\mathbb{F})$ will suffice together with some "no-arbitrage or viability" assumption on (S, τ), developed in [44].

The following is our main result of this section.

Theorem 5. *Suppose that (25) holds, and let $h \in L^2(\mathcal{O}(\mathbb{F}), P \otimes dD)$. Then the following assertions hold.*

(a) *The risk-minimizing strategy for the claim h_τ, at term T under $(S^\tau, \mathbb{G})$, is denoted by $\xi^{(h,\mathbb{G})}$ and is given by*
$$\xi^{(h,\mathbb{G})} := \xi^{(h,\mathbb{F})} \left(G_- + \varphi^{(m)}\right)^{-1} I_{]\!]0,\tau]\!]}. \tag{26}$$
Here $\xi^{(h,\mathbb{F})}$ is the risk-minimizing strategy under $(S, \mathbb{F})$ for the claim $\mathbb{E}\Big[\int_0^\infty h_u dD^{o,\mathbb{F}}_u \mid \mathcal{F}_T\Big]$.

(b) *The remaining (undiversified) risk for the mortality claim h_τ, at term T under the model $(S^\tau, \mathbb{G})$, is denoted by $L^{(h,\mathbb{G})}$ and is given by*

$$
\begin{aligned}
L^{(h,\mathbb{G})} :=& \underbrace{\frac{-\xi^{(h,\mathbb{F})} G_-^{-1}}{G_- + \varphi^{(m)}} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(L^{(m)})}_{(S,\tau)\text{-correlation risk}} + \underbrace{\frac{-M^h_- + (h \bullet D^{o,\mathbb{F}})_-}{G_-^2} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m)}_{(\mathbb{F},\tau)\text{-correlation risk}} \\
&+ \underbrace{\frac{I_{[\![0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(h,\mathbb{F})})}_{\substack{\text{pure financial and} \\ (h,\tau)\text{-correlation risks}}} + \underbrace{\left(Gh - M^h + h \bullet D^{o,\mathbb{F}}\right) G^{-1} I_{]\!]0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T}_{\text{pure mortality risk type one}}.
\end{aligned}
\tag{27}
$$

Here, $L^{(h,\mathbb{F})}$ is the remaining (undiversified) risk under $(S, \mathbb{F})$ for the claim $\mathbb{E}\left[\int_0^\infty h_u dD^{o,\mathbb{F}}_u \mid \mathcal{F}_T\right] = M^h_T$, while M^h and $(\varphi^{(m)}, L^{(m)})$ are defined in (8) and (23), respectively.

(c) *The value of the risk-minimizing portfolio $V(\rho^{*,\mathbb{G}})$ for the claim h_τ, under $(S^\tau, \mathbb{G})$, is given by*

$$
V(\rho^{*,\mathbb{G}}) = h_\tau I_{[\![\tau,+\infty[\![} + G^{-1}\, {}^{o,\mathbb{F}}\!\left(h_\tau I_{]\!]0,\tau[\![}\right) I_{]\!]0,\tau[\![} - h_\tau I_{[\![T]\!]}. \tag{28}
$$

The proof of this theorem is relegated to Section 3.4, while herein we discuss its importance and meanings.

It is important to mention that the mortality claim h_τ has three types of risks. These are the mortality risks generated by the uncertainty of τ, the pure financial risk that is generated by the randomness of the benefit process that is "purely" generated by the flow $\mathbb{F}$, and the correlation risks resulting from the interplay between the randomness of τ and the flow $\mathbb{F}$ in general. Theorem 5 clearly singles out the various types of correlation risks via the quantification of the pair $(\xi^{\mathbb{G}}, L^{\mathbb{G}})$ of risk-minimization strategy and remaining/non-diversified risk. In virtue of the assumption $\langle m, S\rangle^{\mathbb{F}} \equiv 0$, the correlation risk between S and τ is generated by $[S, m]$, which is parametrized by the pair $(\varphi^{(m)}, L^{(m)})$ given in (23). The claim $\mathbb{E}\left[\int_0^\infty h_u dD^{o,\mathbb{F}}_u \mid \mathcal{F}_T\right]$ clearly bears the pure financial risk intrinsic to the randomness of the $\mathbb{F}$-based policy h and the correlation risk resulting from the interplay between this randomness and τ. Thus, the pair $(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})})$ represents, roughly speaking, the market price of risk and the remaining risk associated with both the pure financial and the correlation risks generated by the pair (h, τ). The abstract form of $(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})})$ in Theorem 5 is mainly due to the full generality of h, while a deeper disintegration of these risks requires more specifications on the benefit policy. This will be the aim of the next subsection. To see clearly this economic interpretation for the pair $(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})})$, we consider the case when the benefit policy is constantly equal to h_0 units and, for simplicity, we assume that $\tau < +\infty$ P-a.s.. Therefore, in this case, we easily calculate $\lim_{t\longrightarrow+\infty} G_t =: G_{\infty-} = P(\tau = +\infty|\mathcal{F}_{\infty-}) = 0$ and

$$
\int_0^\infty h_s dD^{o,\mathbb{F}}_s = h_0(D^{o,\mathbb{F}}_{\infty-} - D^{o,\mathbb{F}}_0) = h_0(m_{\infty-} - D^{o,\mathbb{F}}_0), \quad \text{and} \quad M^h = h_0(m - m_0) + h_0 G_0.
$$

This proves that, in this case, we have $(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}) = (0, h_0(m - m_0))$. This is very consistent with the fact that, in this case, due to the fact that the benefit is constant in time and not random, the pure financial risk is null and the correlation risk for the pair (h, τ) is also null. This leads to the corresponding market price of risk $\xi^{(h,\mathbb{F})}$ being null and the remaining risk is a linear transformation of the correlation risk m only, that is, it coincides with $h_0(m - m_0)$.

The life insurance liabilities where the claim h_τ is determined by an optional process h appear, typically, in the form of unit-linked insurance products. In these types of term insurance contracts, the insurer pays an amount K_τ at the time of death τ, if the policyholder dies before or at the term of the contract T, or equivalently the discounted payoff is $I_{\{\tau\le T\}}K_\tau$, where $K \in L^2(\mathcal{O}(\mathbb{F}), P\otimes dD)$. As a result, the payoff process for this case is

$$h_t := I_{\{t\le T\}}K_t, \quad \text{where} \quad K \in \mathcal{O}(\mathbb{F}) \quad \text{and} \quad \mathbb{E}\Big[|K_\tau|^2 I_{\{\tau<+\infty\}}\Big] < +\infty. \tag{29}$$

For this case, the pair $\left(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}\right)$ in Theorem 5 consists of the minimizing strategy and the remaining risk for the payoff $\int_0^T K_t dD_t^{o,\mathbb{F}}$ under the model $(S,\mathbb{F})$, while the value process $V(\rho^{*\mathbb{G}})$ under the model $(S^\tau,\mathbb{G})$ becomes

$$V(\rho^{*\mathbb{G}}) = G^{-1}\,{}^{o,\mathbb{F}}\Big(h_\tau I_{[\![0,\tau[\![}\Big) I_{[\![0,\tau[\![}. \tag{30}$$

Hereto, by considering the payment process $A = K_\tau I_{[\![\tau,+\infty[\![}$, we derive $A_T = h_\tau$ and for $t\in[0,T]$

$$A_T - A_t = I_{\{\tau\le T\}}K_\tau - I_{\{\tau\le t\}}K_\tau = I_{\{t<\tau\}}I_{\{\tau\le T\}}K_\tau = I_{\{t<\tau\}}h_\tau.$$

Thus, $V(\rho^{*\mathbb{G}}) = {}^{o,\mathbb{G}}(h_\tau I_{[\![0,\tau[\![})$, which is exactly the second term on the RHS of (28). This extends the results of [29], where the authors assume that K does not jump at τ (i.e., so that $I_{\{\tau\le t\}}K_\tau = I_{\{\tau\le t\}}K_{\tau-}$), and hence they can treat it as a predictable case. More precisely, they consider a life insurance payment process A with $A_t = I_{\{\tau\le t\}}\bar{A}_t$ with $\bar{A}$ a predictable process given by $\bar{A}_t = K_{t-}$ for $0<t\le T$.

Below, we illustrate the results of Theorem 5 in this setting where the payoff process h is $\mathbb{F}$-predictable.

Corollary 1. *Suppose that (25) holds, and consider $h \in L^2(\mathcal{P}(\mathbb{F}), P\otimes dD)$. Let m^h be given by*

$$m^h := {}^{o,\mathbb{F}}\Big(\int_0^\infty h_u dF_u\Big), \quad \textit{where} \quad F := 1-G. \tag{31}$$

Then the risk-minimizing strategy and the remaining risk for the mortality claim h_τ, at term T under $(S^\tau,\mathbb{G})$, are denoted by $\xi^{(h,\mathbb{G})}$ and $L^{(h,\mathbb{G})}$ and are given by

$$\xi^{(h,\mathbb{G})} := \xi^{(h,\mathbb{F})}\big(G_- + \varphi^{(m)}\big)^{-1} I_{]\!]0,\tau]\!]}, \tag{32}$$

$$\begin{aligned} L^{(h,\mathbb{G})} :=\ & \frac{-G_-^{-1}\xi^{(h,\mathbb{F})}}{G_- + \varphi^{(m)}} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(L^{(m)}) + \frac{hG_- - m_-^h + (h\bullet F)_-}{G_-^2} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m) \\ & + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(h,\mathbb{F})}) + \frac{hG - m^h + h\bullet F}{G} I_{]\!]0,R[\![} \bullet \Big(N^{\mathbb{G}}\Big)^T. \end{aligned} \tag{33}$$

Here, $\left(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}\right)$ stands for the risk-minimizing strategy and the remaining risk for the claim $\mathbb{E}\big[\int_0^\infty h_u dF_u | \mathcal{F}_T\big]$ under $(S,\mathbb{F})$, and the pair $(\varphi^{(m)}, L^{(m)})$ is given in (23).

The proof of this corollary mimics the proof of Theorem 5, and will be omitted.

Corollary 2. *Suppose $S \in \mathcal{M}^2_{loc}(\mathbb{F})$. Let $h \in L^2(\mathcal{O}(\mathbb{F}), P\otimes D)$, M^h be defined in (8) and $\left(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}\right)$ be the pair of $(S,\mathbb{F})$-risk-minimizing strategy and remaining risk for the claim M_T^h. Then, the following holds.*

(a) *If τ is strongly non-correlated to S, that is, $[S, m] \equiv 0$, then the risk-minimizing strategy and the remaining risk for h_τ at term T under $(S^\tau, \mathbb{G})$, denoted by $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$, are given by $\xi^{(h,\mathbb{G})} := \xi^{(h,\mathbb{F})} G_-^{-1} I_{]\!]0,\tau]\!]}$ and*

$$L^{(h,\mathbb{G})} := \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet L^{(h,\mathbb{F})} + \frac{hG_- - m^h_- + (h \bullet F)_-}{G_-^2} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m) + \frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T.$$

(b) *If τ is a pseudo-stopping time, that is, $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ for any $\mathbb{F}$-martingale M, then the pair $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$ of risk-minimizing strategy and remaining risk for the claim h_τ, at term T under $(S^\tau, \mathbb{G})$, is given by*

$$\xi^{(h,\mathbb{G})} := \frac{\xi^{(h,\mathbb{F})}}{G_-} I_{]\!]0,\tau]\!]} \quad \textit{and} \quad L^{(h,\mathbb{G})} := \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet L^{(h,\mathbb{F})} + \frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T.$$

(c) *If τ is independent of $\mathcal{F}_\infty := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ such that $P(\tau > T) > 0$ and h is a deterministic function, then*

$$\xi^{(h,\mathbb{G})} = 0 \quad L^{(h,\mathbb{G})} = \frac{hG - (h \bullet D^{o,\mathbb{F}})_\infty + h \bullet D^{o,\mathbb{F}}}{G} \bullet \left(N^{\mathbb{G}}\right)^T.$$

Thanks to [50] , it is clear that τ is a pseudo-stopping time if and only if $m \equiv m_0$, and this clearly implies that τ is strongly non-correlated to S. However, the reverse implication is wrong in general, except when S has the $\mathbb{F}$-predictable representation property. In fact, when this latter property is violated and S is continuous, there exists an $\mathbb{F}$-local martingale L that is not null and is orthogonal to S. Thus, the random time τ associated with the non-negative supermartingale $G = \min(1, \mathcal{E}(L)^+)$ is not a pseudo-stopping time and is strongly orthogonal to S. Furthermore, if τ is a pseudo-stopping time, then $\xi^{(h,\mathbb{G})}$ remains the same as in assertion (a), while the term $\mathcal{T}(m)$, in the formula of $L^{(h,\mathbb{G})}$, vanishes. Thus, the class of τ pseudo-stopping times is much smaller than the class of random times strongly non-correlated to S.

Proof of Corollary 2. The proof of assertion (a) follows immediately from combining Theorem 5 and the fact that the pair $(\varphi^{(m)}, L^{(m)})$ defined in (23) is null as soon as $[S, m] \equiv 0$. Thanks to [50], we deduce that $m \equiv m_0$ (constant process) if and only if τ is a pseudo-stopping time. This implies that U, defined in (22), is a null process. Therefore, we conclude that

$$\varphi^{(m)} \equiv 0, \quad L^{(m)} \equiv 0, \quad I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(m) \equiv 0 \quad \text{and} \quad \mathcal{T}(M) = M^\tau \quad \text{for any} \quad M \in \mathcal{M}(\mathbb{F}).$$

Then, by inserting these into (26) and (27), the proof of the corollary follows immediately. □

It is worth mentioning that the pseudo-stopping time model for τ covers the case when τ is independent of $\mathcal{F}_\infty$ (no correlation between the financial market and the death time), the case when τ is an $\mathbb{F}$-stopping time (i.e., the case of full correlation between the financial market and the death time), and the case when there is arbitrary moderate correlation such as in the immersion case of $\tau := \inf\{t \geq 0 \mid S_t \geq E\}$ with E as a random variable that is independent of $\mathcal{F}_\infty$. For more details about pseudo-stopping times, we refer the reader to [50].

Theorem 5 and Corollary 1 give the general relation between the $\mathbb{G}$-risk-minimizing strategy for the claim h_τ at term T under the model $(S^\tau, \mathbb{G})$ and the $\mathbb{F}$-risk-minimizing strategy for the claim $\mathbb{E}[\int_0^\infty h_u dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T]$ (or $\mathbb{E}[\int_0^\infty h_u dF_u | \mathcal{F}_T]$ when h is $\mathbb{F}$-predictable) at term T $(S, \mathbb{F})$. In the next subsections, we further establish the arising $\mathbb{F}$-risk-minimizing strategies for certain specific mortality contracts to highlight the impact of the interplay/correlation between the benefit policy and the death time.

Corollary 2 extends the results of ([17] Chapter 5) and [27–29] to broader models of $(S, \mathbb{F}, \tau)$ and to a larger class of mortality liabilities. Indeed, in these papers, the authors study the risk-minimization of life insurance liabilities consisting of two main building blocks: pure endowment and term insurance only, while an annuity contract can be a combination of both in general. This literature derives the results by applying the hazard rate approach of credit derivatives (see, e.g., [51]) under the following several assumptions:

(a) The random time τ is assumed to avoid $\mathbb{F}$-stopping times. This allows τ to be a totally inaccessible $\mathbb{G}$-stopping time, and $\Delta X_\tau = 0$ for any $\mathbb{F}$-adapted RCLL process X.
(b) The process G is strictly positive, that is, the stopping time $R = +\infty$ P-a.s..
(c) The payment processes and payoff processes are predictable processes.
(d) The H-hypothesis holds (i.e., M^τ is a $\mathbb{G}$-local martingale for any $\mathbb{F}$-local martingale M).

This H-hypothesis is relaxed in [17], while the author maintains the assumptions (a), (b) and (c), as his approach relies essentially on the martingale decomposition of [39].

3.3. Interplay between Mortality and Random Benefit Policies

This subsection highlights the risks resulting from the interplay between the randomness of the benefit policy and that of τ (i.e., what we previously called the second class of correlation risks). As we mentioned before, this is possible by analyzing (mainly two) examples of the benefit policies. To this end, throughout the rest of the paper, we consider the following survival probabilities:

$$F_t(s) := P(\tau \leq s|\mathcal{F}_t), \quad \text{and} \quad G_t(s) := P(\tau > s|\mathcal{F}_t) = 1 - F_t(s), \quad \forall\ s, t \in [0, T], \tag{34}$$

and below we define mathematically the benefit policies that we deal with herein.

Definition 4. *Consider $T \in (0, +\infty)$, $g \in L^1(\mathcal{F}_T)$ and $K \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$.*

(a) *A zero-coupon longevity bond is an insurance contract that pays the conditional survival probability at term T (i.e., an insurance contract with payoff $G_T = P(\tau > T|\mathcal{F}_T)$).*
(b) *A pure endowment insurance, with benefit g, is an insurance contract that pays g at term T if the insured survives (i.e., an insurance contract with payoff $gI_{\{\tau>T\}}$).*
(c) *An endowment insurance contract with benefit pair (g, K) is an insurance contract that pays g at term T if the insured survives and pays K_τ at the time of death if the insured dies before or at the maturity (i.e., its payoff is $gI_{\{\tau>T\}} + K_\tau I_{\{\tau\leq T\}}$).*

Our first result, on the interplay between the benefit policy and τ, analyzes the pure endowment with benefit g, while its proof is relegated to Section 3.4.

Theorem 6. *Suppose that (25) holds, and consider h given by*

$$h_t := gI_{]\!]T,+\infty[\![}(t), \qquad g \in L^2(\mathcal{F}_T, P). \tag{35}$$

Then, the following assertions hold.
(a) *The risk-minimizing strategy for the claim h_τ, under $(S^\tau, \mathbb{G})$, takes the form of*

$$\xi^{(h,\mathbb{G})} := \left(G_-(T)\xi^{(g,\mathbb{F})} + U^g_-\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})}\right)\left(G_- + \varphi^{(m)}\right)^{-1} I_{]\!]0,\tau]\!]}, \tag{36}$$

and the corresponding remaining risk is given by

$$\begin{aligned} L^{(h,\mathbb{G})} := &-\frac{G_-(T)\xi^{(g,\mathbb{F})} + U^g_-\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})}}{G_-\left(G_- + \varphi^{(m)}\right)} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(L^{(m)}) + I_{]\!]0,\tau]\!]}\frac{G_-(T)}{G_-} \bullet \mathcal{T}(L^{(g,\mathbb{F})}) \\ &+ \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(U^g_- \bullet L^{(G_T,\mathbb{F})} + L^{(\mathrm{Cor}_T,\mathbb{F})}) - \frac{M^{(g)}_-}{G^2_-} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m) - \frac{M^{(g)}}{G} I_{]\!]0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T. \end{aligned} \tag{37}$$

Here, the correlation process $\mathrm{Cor} = (\mathrm{Cor}_t)_{t\geq 0}$ *is given by*

$$\mathrm{Cor}_t := [G(T), U^g]_t + \mathrm{Cov}\Big(I_{\{\tau>T\}}, g \mid \mathcal{F}_t\Big), \tag{38}$$

and U^g *and* $M^{(g)}$ *are given by* $U^g_t := \mathbb{E}[g \mid \mathcal{F}_t]$ *and* $M^{(g)}_t := \mathbb{E}[gG_T \mid \mathcal{F}_t]$ *respectively. The pairs* $\Big(\xi^{(g,\mathbb{F})}, L^{(g,\mathbb{F})}\Big)$, $\Big(\xi^{(G_T,\mathbb{F})}, L^{(G_T,\mathbb{F})}\Big)$, *and* $\Big(\xi^{(\mathrm{Cor}_T,\mathbb{F})}, L^{(\mathrm{Cor}_T,\mathbb{F})}\Big)$ *are the risk-minimizing strategies and the remaining risks, under* $(S, \mathbb{F})$*, for the claims* g*,* G_T*, and* Cor_T*, respectively, while* $(\varphi^{(m)}, L^{(m)})$ *is defined in (23).*

(b) *The value process,* $V(\rho^{*,\mathbb{G}})$*, of the risk-minimizing portfolio under the model* $(S^\tau, \mathbb{G})$*, is given by*

$$V(\rho^{*,\mathbb{G}}) = G^{-1}\ {}^{o,\mathbb{F}}\Big(h_\tau I_{[\![0,\tau[\![}\Big) I_{[\![0,\tau\wedge T[\![}. \tag{39}$$

The amount g of a pure endowment is purely financial. The $\mathbb{F}$-strategy and the remaining risk for the claim $\mathbb{E}\Big[\int_0^\infty h_u dF_u \;\Big|\; \mathcal{F}_T\Big] = g(1-F_T) = gG_T$ are expressed as functions of the corresponding strategy and the risk for this pure financial claim g, for the mortality claim G_T and the correlation Cor_T between the pure financial market and the mortality model, including the time of death.

When g is deterministic then $M^{(g)} = gG(T)$, the martingale $U^{(g)}$ is equal to g, and the correlation process $(\mathrm{Cor}_t)_{0\leq t\leq T}$ is a null process. Thus, we get $(\xi^{(g,\mathbb{F})}, L^{(g,\mathbb{F})}) = (\xi^{(\mathrm{Cor}_T,\mathbb{F})}, L^{(\mathrm{Cor}_T,\mathbb{F})}) \equiv (0,0)$, and conclude that, in this case, the pair $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$ takes the following form:

$$\begin{aligned}
\xi^{(h,\mathbb{G})} &:= g\xi^{(G_T,\mathbb{F})}\big(G_- + \varphi^{(m)}\big)^{-1} I_{]\!]0,\tau]\!]},\\
L^{(h,\mathbb{G})} &:= -\frac{g\xi^{(G_T,\mathbb{F})} I_{]\!]0,\tau]\!]}}{G_-\big(G_- + \varphi^{(m)}\big)} \bullet \mathcal{T}(L^{(m)}) + \frac{gI_{]\!]0,\tau\wedge T]\!]}}{G_-} \bullet \mathcal{T}(L^{(G_T,\mathbb{F})} - \frac{G_-(T)}{G_-} \bullet m)\\
&\quad - \frac{gG(T)}{G} I_{]\!]0,R[\![} \bullet \Big(N^{\mathbb{G}}\Big)^T.
\end{aligned}$$

This remaining risk $L^{(h,\mathbb{G})}$ contains integrals with respect to $N^{\mathbb{G}}$ and $\mathcal{T}(m)$ that represent the unsystematic component of the mortality risk and a combination of systematic and unsystematic mortality risk respectively. For this particular case of a pure endowment contract, we further compare the pair $\Big(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}\Big)$, obtained in [17,27–29], with our pair given by (36)–(37). In [17], the author assumes that the financial market is independent of the mortality model in the sense that the process Cor, defined in (38), is null. Further, he assumes that $G(T)$ is strongly orthogonal to S, meaning that the systematic risk mortality component cannot be hedged by investing in S, or equivalently $\Big(\xi^{(G_T,\mathbb{F})}, L^{(G_T,\mathbb{F})}\Big) = (0, G(T))$. In [27–29], it is also assumed that $G(T)$ is driven by an $\mathbb{F}$-local martingale Y which is strongly orthogonal to S, but they follow a slightly different approach. They derive a predictable decomposition of $M^{(g)}$ with respect to S and Y instead of our natural decomposition below:

$$M^{(g)} = G_0(T)U^g_0 + G_-(T) \bullet U^g + U^g_- \bullet G(T) + \mathrm{Cor}.$$

Hence, the authors do not distinguish between the three components (pure financial, pure mortality and correlation) as we do. In ([17] Chapter 5), the author studies risk-minimization for a pure endowment contract under strict independence between the financial and insurance markets and other further assumptions on $(S, \mathbb{F}, \tau)$. It is clear that by imposing additional specifications such as τ as an $\mathbb{F}$-pseudo stopping time, or τ avoids all $\mathbb{F}$-stopping times, or all $\mathbb{F}$-martingales are continuous, our result boils down to that of ([17] Proposition 5.1). Therefore, we conclude that Theorem 6 generalizes [17] in several directions.

Corollary 3. *Consider the mortality claim h_τ, where h is given by (35), and the square integrable $\mathbb{F}$-martingale $U^g_t := \mathbb{E}[g \mid \mathcal{F}_t]$. Then the following assertions hold.*
(a) *Suppose that τ is a pseudo-stopping time. Then the pair $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$, of (36)–(37), becomes*

$$\xi^{(h,\mathbb{G})} := \Big(\frac{G_-(T)}{G_-}\xi^{(g,\mathbb{F})} + \frac{U^g_-}{G_-}\xi^{(G_T,\mathbb{F})} + \frac{1}{G_-}\xi^{(\mathrm{Cor}_T,\mathbb{F})}\Big) I_{]\!]0,\tau]\!]}, \tag{40}$$

$$L^{(h,\mathbb{G})} := \frac{G_-(T)I_{]\!]0,\tau]\!]}}{G_-}\bullet L^{(g,\mathbb{F})} + I_{]\!]0,\tau]\!]}\frac{U^g_-}{G_-}\bullet L^{(G_T,\mathbb{F})} + \frac{I_{]\!]0,\tau]\!]}}{G_-}\bullet L^{(\mathrm{Cor}_T,\mathbb{F})} - \frac{M^{(g)}}{G} I_{]\!]0,R[\![}\bullet \left(N^{\mathbb{G}}\right)^T. \tag{41}$$

(b) *Suppose τ is independent of $\mathcal{F}_\infty$ and $P(\tau > T) > 0$. Then $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$ takes the following form*

$$\xi^{(h,\mathbb{G})}_t := \frac{P(\tau > T)}{P(\tau \geq t)}\xi^{(g,\mathbb{F})}_t I_{\{t\leq\tau\}},\ L^{(h,\mathbb{G})}_t := \int_0^{t\wedge\tau}\frac{P(\tau > T)}{P(\tau \geq s)}dL^{(g,\mathbb{F})}_s - \int_0^{t\wedge T}\frac{P(\tau > T)U^g_s}{P(\tau > s)}dN^{\mathbb{G}}_s. \tag{42}$$

(c) *Suppose τ is independent of $\mathcal{F}_\infty$ such that $P(\tau > T) > 0$, and g is deterministic. Then we have*

$$\xi^{(h,\mathbb{G})}_t := 0, \quad L^{(h,\mathbb{G})}_t := -\int_0^{t\wedge T}\frac{P(\tau > T)g}{P(\tau > s)}dN^{\mathbb{G}}_s. \tag{43}$$

Proof. It is clear that, when τ is independent of $\mathcal{F}_\infty$, we have

$$G_t(T) = P(\tau > T) = G_{t-}(T),\quad G_t = P(\tau > t),\quad G_{t-} = P(\tau \geq t) = \widetilde{G}_t,\quad m \equiv m_0,\quad \mathrm{Cor} \equiv 0.$$

As a consequence, τ is a pseudo-stopping time and

$$\xi^{(G_T,\mathbb{F})} \equiv 0, \quad L^{(G_T,\mathbb{F})} \equiv 0, \quad \xi^{(\mathrm{Cor}_T,\mathbb{F})} \equiv 0, \quad L^{(\mathrm{Cor}_T,\mathbb{F})} \equiv 0.$$

Thus, by plugging these in (40) and (41) and using the facts that $R > T$ P-a.s. (due to the assumption $G_T = P(\tau > T) > 0$) and $M^{(g)}_t := \mathbb{E}[gG_T|\mathcal{F}_t] = P(\tau > T)U^{(g)}_t$, assertion (b) follows immediately from assertion (a). Hence, the rest of the proof focuses on proving assertion (a). To this end, recall that when τ is a pseudo-stopping time, we have $m \equiv m_0$, and as a consequence we get

$$\varphi^{(m)} \equiv 0, \quad L^{(m)} \equiv 0, \quad \text{and} \quad \mathcal{T}(M) = M^\tau \quad \text{for any} \quad M \in \mathcal{M}_{loc}(\mathbb{F}).$$

Hence, by inserting these into (36) and (37), assertion (a) follows immediately, and the proof of the corollary is completed. □

Our second result, on the interplay between benefit policy and τ, addresses endowment insurance. This contract is an annuity paid until the time of death of the policyholder, or until the end of the contract T, whatever occurs first. Thus, we consider an $\mathbb{F}$-optional and square integrable (with respect to $P \otimes D$) process $C := (C_t)_{t\geq 0}$, where C_t represents the discounted accumulated amount up to time t paid by the insurer, with $C_0 = 0$. Then, the discounted payoff of the endowment insurance contract is $I_{\{\tau>T\}}C_T + I_{\{\tau\leq T\}}C_\tau$.

Theorem 7. *Suppose that (25) holds, and h is given by*

$$h_t := I_{\{t>T\}}C_T + I_{\{t\leq T\}}C_t. \tag{44}$$

Let U^K be the $\mathbb{F}$-martingale $U^K_t := \mathbb{E}[K \mid \mathcal{F}_t]$ for any $K \in L^2(\mathcal{F}_T, P)$. Then the following assertions hold.

(a) *The risk-minimizing strategy and the remaining risk for the claim h_τ, under $(S^\tau, \mathbb{G})$, are given by*

$$\xi^{(h,\mathbb{G})} := \frac{G_-(T)\xi^{(C_T,\mathbb{F})} + U_-^{C_T}\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})} + \xi^{(\widetilde{C}_T,\mathbb{F})}}{G_- + \varphi^{(m)}} I_{]\!]0,\tau]\!]}. \tag{45}$$

and

$$\begin{aligned} L^{(h,\mathbb{G})} := & -\frac{\xi^{(\widetilde{C}_T,\mathbb{F})} + G_-(T)\xi^{(C_T,\mathbb{F})} + U_-^{C_T}\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})}}{G_-\left(G_- + \varphi^{(m)}\right)} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(L^{(m)}) \\ & + I_{]\!]0,\tau]\!]}\frac{G_-(T)}{G_-} \bullet \mathcal{T}(L^{(C_T,\mathbb{F})}) + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(\widetilde{C}_T,\mathbb{F})}) + I_{]\!]0,\tau]\!]}\frac{U_-^{C_T}}{G_-} \bullet \mathcal{T}(L^{(G_T,\mathbb{F})}) \\ & + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(\mathrm{Cor}_T,\mathbb{F})}) - \frac{M_-^{(C_T)}}{G_-^2} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m) - \frac{M^{(C_T)}}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T. \end{aligned} \tag{46}$$

Herein, $M_t^{(C_T)} := \mathbb{E}[C_T G_T \mid \mathcal{F}_t]$,

$$\mathrm{Cor}_t := [G(T), U^{C_T}]_t + \mathrm{Cov}\left(I_{\{\tau>T\}}, C_T \middle| \mathcal{F}_t\right), \quad \textit{and} \quad \widetilde{C}_t := \int_0^t C_u dD_u^{o,\mathbb{F}}. \tag{47}$$

The pairs of processes $\left(\xi^{(C_T,\mathbb{F})}, L^{(C_T,\mathbb{F})}\right)$, $\left(\xi^{(G_T,\mathbb{F})}, L^{(G_T,\mathbb{F})}\right)$, $\left(\xi^{(\mathrm{Cor}_T,\mathbb{F})},\ L^{(\mathrm{Cor}_T,\mathbb{F})}\right)$, *and* $\left(\xi^{(\widetilde{C}_T,\mathbb{F})}, L^{(\widetilde{C}_T,\mathbb{F})}\right)$ *are the risk-minimizing strategies and the remaining (undiversified) risk, under the model $(S, \mathbb{F})$, for the claims C_T, G_T, Cor_T, and $\widetilde{C}_T$ respectively, and the pair $(\varphi^{(m)}, L^{(m)})$ is given by Lemma 1.*

(b) *The value of the risk-minimizing portfolio $V(\rho^{*,\mathbb{G}})$ under the model $(S^\tau, \mathbb{G})$, is given by*

$$V(\rho^{*,\mathbb{G}}) = G^{-1\ o,\mathbb{F}}\!\left(h_\tau I_{[\![0,\tau[\![}\right) I_{[\![0,\tau[\![} - I_{[T]} G^{-1\ o,\mathbb{F}}\!\left(I_{\{\tau\leq T\}} C_\tau I_{[\![0,\tau[\![}\right) I_{[\![0,\tau[\![}. \tag{48}$$

Proof. Thanks to Theorem 5, the above theorem will follow immediately as long as we prove that

$$\begin{aligned} \xi^{(h,\mathbb{F})} &= G_-(T)\xi^{(C_T,\mathbb{F})} + U_-^{C_T}\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T(C_T),\mathbb{F})} + \xi^{(U^{\widetilde{C}_T},\mathbb{F})}, \\ L^{(h,\mathbb{F})} &= G_-(T) \bullet L^{(C_T,\mathbb{F})} + L^{(\widetilde{C}_T,\mathbb{F})} + U_-^{C_T} \bullet L^{(G_T,\mathbb{F})} + L^{(\mathrm{Cor}_T,\mathbb{F})}, \end{aligned} \tag{49}$$

where $\left(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})}\right)$ is the risk-minimizing strategy and the remaining (undiversified) risk of the payoff $\mathbb{E}\left[\int_0^\infty h_u dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T\right]$ under the model $(S, \mathbb{F})$. To prove (49), we first remark that $h = h^{(1)} + h^{(2)}$, where $h^{(1)}$, which has the same form as the payoff process of (35), and $h^{(2)}$ are given by

$$h_t^{(1)} := C_T I_{\{t>T\}} \quad \text{and} \quad h_t^{(2)} = I_{\{t\leq T\}} C_t. \tag{50}$$

Thus, we derive

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty h_u dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T\right] &= \mathbb{E}\left[\int_0^\infty h_u^{(1)} dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T\right] + \int_0^T C_u dD_u^{o,\mathbb{F}} \\ &=: \mathbb{E}\left[\int_0^\infty h_u^{(1)} dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T\right] + \widetilde{C}_T, \end{aligned}$$

and deduce that

$$\xi^{(h,\mathbb{F})} = \xi^{(h^{(1)},\mathbb{F})} + \xi^{(\widetilde{C}_T,\mathbb{F})}, \quad \text{and} \quad L^{(h,\mathbb{F})} = L^{(h^{(1)},\mathbb{F})} + L^{(\widetilde{C}_T,\mathbb{F})}.$$

Therefore, by combining this with Theorem 6 with $g = C_T$, the proof of (49) follows immediately.

The value process $V(\rho^{*,\mathbb{G}})$ of the risk-minimizing strategy under the model $(S^\tau, \mathbb{G})$ also consists of two parts given by (39) and (30) for $h^{(1)}$ and $h^{(2)}$, respectively. This ends the proof of the theorem. $\square$

Similarly to the case of a pure endowment, for the annuity contract, in [29] the authors derive a predictable decomposition for the martingale $M^{(C_T)}$ in terms of S and of the $\mathbb{F}$-local martingale Y, which drives $G(T)$ and which is strongly orthogonal to S. Hence, their assumptions correspond to Cor $= 0$, $\xi^{(G_T,\mathbb{F})} = 0$ and there is a term in Y instead of our $L^{(G_T,\mathbb{F})}$ in (49). In [17], the author proceeded very differently and did not write the payoff of the annuity contract as a sum of a pure endowment and a term insurance, while he worked with the integral expression. This made the results very involved and hard to interpret. Again, for the annuity contract, [17] falls into the case where Cor $= 0$ and $G(T)$ is orthogonal to S (i.e., $\langle G(T), S\rangle^{\mathbb{F}} \equiv 0$).

Corollary 4. *Consider the mortality claim h_τ, where h is given by (44), and put $U^K_t := \mathbb{E}[K \mid \mathcal{F}_t]$ and $M^{(K)}_t := \mathbb{E}[G_T K|\mathcal{F}_t]$ for any $K \in L^2(\mathcal{F}_T, P)$. Then the following assertions hold.*

(a) *Suppose τ is a pseudo-stopping time. Then the pair $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$, of (45)–(46), becomes*

$$\xi^{(h,\mathbb{G})} := \left(\frac{G_-(T)}{G_-}\xi^{(C_T,\mathbb{F})} + \frac{U^{C_T}_-}{G_-}\xi^{(G_T,\mathbb{F})} + \frac{1}{G_-}\xi^{(\mathrm{Cor}_T,\mathbb{F})} + \frac{1}{G_-}\xi^{(\widetilde{C}_T,\mathbb{F})}\right) I_{]\!]0,\tau]\!]}, \tag{51}$$

$$\begin{aligned} L^{(h,\mathbb{G})} :=\, & I_{]\!]0,\tau]\!]}\frac{G_-(T)}{G_-} \bullet L^{(C_T,\mathbb{F})} + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet L^{(\widetilde{C}_T,\mathbb{F})} + I_{]\!]0,\tau]\!]}\frac{U^{C_T}_-}{G_-} \bullet L^{(G_T,\mathbb{F})} \\ & + G_-^{-1} I_{]\!]0,\tau]\!]} \bullet L^{(\mathrm{Cor}_T,\mathbb{F})} - \frac{M^{(C_T)}}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T. \end{aligned} \tag{52}$$

(b) *Suppose that τ is independent of $\mathcal{F}_\infty$, and $P(\tau > T) > 0$. Then we get*

$$\xi^{(h,\mathbb{G})}_t := \frac{P(\tau > T)\xi^{(C_T,\mathbb{F})}_t + \xi^{(\widetilde{C}_T,\mathbb{F})}_t}{P(\tau \geq t)} I_{\{t\leq\tau\}}, \tag{53}$$

$$L^{(h,\mathbb{G})}_t := \int_0^{t\wedge\tau} \frac{P(\tau > T)}{P(\tau \geq s)} dL^{(C_T,\mathbb{F})}_s + \int_0^t \frac{1}{P(\tau \geq s)} dL^{(\widetilde{C}_T,\mathbb{F})}_s - \int_0^{t\wedge T} \frac{P(\tau > T)}{P(\tau > s)} dN^{\mathbb{G}}_s. \tag{54}$$

(c) *Suppose that τ is independent of $\mathcal{F}_\infty$ such that $P(\tau > T) > 0$, and the benefit policy $C = (C_t)_{0\leq t\leq T}$ is a deterministic function of time. Then we get*

$$\xi^{(h,\mathbb{G})}_t = 0, \quad L^{(h,\mathbb{G})}_t = -\int_0^{t\wedge T} \frac{P(\tau > T)}{P(\tau > s)} dN^{\mathbb{G}}_s. \tag{55}$$

Proof. The proof of this corollary mimics the proof of Corollary 3 using Theorem 7 instead. $\square$

3.4. Proofs of Theorems 5 and 6

This subsection proves the two theorems. To this end, we start by singling out the main ideas of these proofs. Indeed, this idea lies in applying the risk-minimization under $(S, \mathbb{F})$ for the risk, resulting from the interplay between the benefit policy and τ, which is given by the process $M^h = {}^{o,\mathbb{F}}\left(\int_0^\infty h_u dD^{o,\mathbb{F}}_u\right)$, and using Lemma 1 to get the explicit form of the $\mathbb{G}$-strategy. Notice that the risk m cannot be hedged under the model $(S, \mathbb{F})$ due to the second assumption in (25). Once the strategy is described, we will prove that this strategy indeed belongs to $L^2(S^\tau, \mathbb{G})$ (i.e., it is 'admissible') afterwards, which follows from proving M^h is a square integrable $\mathbb{F}$-martingale.

Proof of Theorem 5. This proof is achieved in three steps. The first step proves that M^h is a square integrable $\mathbb{F}$-martingale, while the second step describes the $\mathbb{G}$-strategy explicitly and locally on a sequence of subsets that increases to $\Omega \times [0, +\infty)$. The third (last) step proves the admissibility of the $\mathbb{G}$-strategy and ends the proof of the theorem. By applying Theorem 2 to H, where $H_t = \mathbb{E}[h_\tau \mid \mathcal{G}_t]$ is a $\mathbb{G}$-square integrable martingale, we get the decomposition (9).

Step 1. Let $K \in L^\infty(\mathcal{F}_\infty, P)$, and consider the $\mathbb{F}$-martingale $K_t := \mathbb{E}[K \mid \mathcal{F}_t]$. Then, we derive

$$\begin{aligned}\mathbb{E}\left(K\int_0^\infty h_u dD_u^{o,\mathbb{F}}\right) &= \mathbb{E}\left(\int_0^\infty K_u h_u dD_u^{o,\mathbb{F}}\right) \leq \mathbb{E}\left(\int_0^\infty \sup_{0\leq t\leq u}|K_t||h_u|dD_u^{o,\mathbb{F}}\right)\\ &= \mathbb{E}\left(\int_0^\infty \sup_{0\leq t\leq u}|K_t||h_u|dD_u\right) = \mathbb{E}\left(\sup_{0\leq t\leq \tau}|K_t||h_\tau|I_{\{\tau<+\infty\}}\right)\\ &\leq \sqrt{\mathbb{E}(h_\tau^2 I_{\{\tau<+\infty\}})}\sqrt{\mathbb{E}\left(\sup_{t\geq 0}|K_t|^2\right)} \leq 2\sqrt{\mathbb{E}(h_\tau^2 I_{\{\tau<+\infty\}})}\sqrt{\mathbb{E}(|K|^2)},\end{aligned}$$

where the last inequality follows from Doob's inequality. Thus, this proves that $\int_0^\infty h_u dD_u^{o,\mathbb{F}}$ is a square integrable random variable for any $h \in L^2(\mathcal{O}(\mathbb{F}), P \otimes D)$. As a result, $M^h \in \mathcal{M}^2(\mathbb{F})$.

Step 2. By applying Theorem 3 to the pair (M^h, S) of elements of $\mathcal{M}^2_{\text{loc}}(\mathbb{F})$, we deduce the existence of the pair $(\xi^{(h,\mathbb{F})}, L^{(h,\mathbb{F})})$ such that

$$M^h = M_0^h + \xi^{(h,\mathbb{F})} \bullet S + L^{(h,\mathbb{F})}. \tag{56}$$

Hence $\xi^{(h,\mathbb{F})}$ is the risk-minimizing strategy and $L^{(h,\mathbb{F})}$ is the remaining risk, under $(S, \mathbb{F})$ for the claim $\mathbb{E}[\int_0^\infty h_u dD_u^{o,\mathbb{F}} \mid \mathcal{F}_T]$ at term T. As a result, we get $\mathcal{T}(M^h) = \xi^{(h,\mathbb{F})} \bullet \mathcal{T}(S) + \mathcal{T}(L^{(h,\mathbb{F})})$, and by inserting this into (9) we obtain

$$\begin{aligned}H = H_0 &+ \frac{\xi^{(h,\mathbb{F})}}{G_-} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(S) + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(h,\mathbb{F})}) - \frac{M_-^h - (h \bullet D^{o,\mathbb{F}})_-}{G_-^2} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(m)\\ &+ \frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T.\end{aligned} \tag{57}$$

Put

$$\Sigma_n := \left(\{|\xi^{(h,\mathbb{F})}| \leq n \ \&\ G_- + \varphi^{(m)} \geq 1/n\} \cap]\!]0, \tau]\!]\right) \bigcup\ ]\!]\tau, +\infty[\![, \tag{58}$$

and utilize (24) to derive

$$\begin{aligned}I_{\Sigma_n} \bullet H &= \frac{\xi^{(h,\mathbb{F})}}{G_- + \varphi^{(m)}} I_{]\!]0,\tau]\!]\cap\Sigma_n} \bullet S^\tau - \frac{\xi^{(h,\mathbb{F})} G_-^{-1}}{G_- + \varphi^{(m)}} I_{]\!]0,\tau]\!]\cap\Sigma_n} \bullet \mathcal{T}(L^{(m)}) + \frac{I_{]\!]0,\tau]\!]\cap\Sigma_n}}{G_-} \bullet \mathcal{T}(L^{(h,\mathbb{F})})\\ &\quad - \frac{M_-^h - (h \bullet D^{o,\mathbb{F}})_-}{G_-^2} I_{]\!]0,\tau]\!]\cap\Sigma_n} \bullet \mathcal{T}(m) + \frac{hG - M^h + h \bullet D^{o,\mathbb{F}}}{G} I_{[\![0,R[\![} I_{\Sigma_n} \bullet \left(N^{\mathbb{G}}\right)^T\\ &=: \xi^{(n,\mathbb{G})} \bullet S^\tau + L^{(n,\mathbb{G})},\end{aligned}$$

where

$$\xi^{(n,\mathbb{G})} := \xi^{(h,\mathbb{F})}\big(G_- + \varphi^{(m)}\big)^{-1} I_{]\!]0,\tau]\!]\cap\Sigma_n} \text{ and}$$

$$\begin{aligned} L^{(n,\mathbb{G})} := &\frac{-\xi^{(h,\mathbb{F})} I_{]\!]0,\tau]\!]\cap\Sigma_n}}{G_-\big(G_- + \varphi^{(m)}\big)} \bullet \mathcal{T}(L^{(m)}) - \frac{M^h_- - (h\bullet D^{o,\mathbb{F}})_-}{G^2_-} I_{]\!]0,\tau]\!]\cap\Sigma_n} \bullet \mathcal{T}(m) \\ &+ \frac{I_{]\!]0,\tau]\!]\cap\Sigma_n}}{G_-} \bullet \mathcal{T}(L^{(h,\mathbb{F})}) + \frac{hG - M^h + h\bullet D^{o,\mathbb{F}}}{G} I_{[\![0,R[\![} I_{\Sigma_n} \bullet \Big(N^{\mathbb{G}}\Big)^T. \end{aligned}$$

Step 3. Here, we prove that $\xi^{(h,\mathbb{G})} := \lim_{n\longrightarrow+\infty} \xi^{(n,\mathbb{G})}$ belongs in fact to $L^2(S^\tau, \mathbb{G})$. To this end, we remark that $[\xi^{(n,\mathbb{G})}\bullet S^\tau, L^{(n,\mathbb{G})}] = \xi^{(n,\mathbb{G})}\bullet[S^\tau, L^{(n,\mathbb{G})}]$ is a $\mathbb{G}$-local martingale, and we consider a sequence of $\mathbb{G}$-stopping times $(\sigma(n,k))_{k\geq 1}$ that goes to infinity with k such that $[\xi^{(n,\mathbb{G})}\bullet S^\tau, L^{(n,\mathbb{G})}]^{\sigma(n,k)}$ is a uniformly integrable martingale. Then, we get

$$\mathbb{E}\big[[I_{\Sigma_n}\bullet H]_{\sigma(n,k)}\big] = \mathbb{E}\big[[\xi^{(n,\mathbb{G})}\bullet S^\tau]_{\sigma(n,k)}\big] + \mathbb{E}\big[[L^{(n,\mathbb{G})}]_{\sigma(n,k)}\big] \leq \mathbb{E}\big[[H,H]_\infty\big] < +\infty.$$

Thus, by combining this with Fatou's lemma (we let k go to infinity and then let n go to infinity afterwards) and the fact that $\xi^{(n,\mathbb{G})}$ converges pointwise to $\xi^{(h,\mathbb{G})}$, we conclude that $\xi^{(h,\mathbb{G})} \in L^2(S^\tau, \mathbb{G})$, and $\xi^{(n,\mathbb{G})}\bullet S^\tau$ converges to $\xi^{(h,\mathbb{G})}\bullet S^\tau$ in $\mathcal{M}^2(\mathbb{G})$. Since $I_{\Sigma_n}\bullet H$ converges to $H - H_0$ in the space of $\mathcal{M}^2(\mathbb{G})$, we conclude that $L^{(n,\mathbb{G})}$ converges in the space $\mathcal{M}^2(\mathbb{G})$, and its limit $L^{(h,\mathbb{G})}$ is orthogonal to S^τ. As a result, we deduce $\xi^{(h,\mathbb{F})}\Big(G_- + \varphi^{(m)}\Big)^{-1} I_{]\!]0,\tau]\!]}$ is $\mathcal{T}(L^{(m)})$-integrable and the resulting integral is a $\mathbb{G}$-local martingale. This proves assertions (a) and (b), while assertion (c) is immediate from the fact that the $\mathbb{G}$-payment process corresponding to the claim h_τ at term T is $A_t = I_{\{t=T\}}h_\tau$ and the value process of the portfolio is given by

$$V_t(\rho^{*,\mathbb{G}}) = \mathbb{E}[A_T|\mathcal{G}_t] - A_t = \mathbb{E}[h_\tau|\mathcal{G}_t] - I_{\{t=T\}}h_\tau = H_t - I_{\{t=T\}}h_\tau,$$

where the $\mathbb{G}$-martingale H is decomposed as

$$H_t = h_\tau I_{\{\tau\leq t\}} + \frac{I_{\{t<\tau\}}}{G_t}\mathbb{E}\Big[h_\tau I_{\{t<\tau\}} \mid \mathcal{F}_t\Big]. \tag{59}$$

This ends the proof of this theorem. □

Proof of Theorem 6. Notice that for the payoff process h given in (35), we have $h\equiv 0$ on $[0,T]$, $\mathbb{E}[\int_0^\infty h_u dD^{o,\mathbb{F}}_u|\mathcal{F}_T] = gG_T$ and

$$M^h_t = M^{(g)}_t := {}^{o,\mathbb{F}}(G_T g)_t = \mathbb{E}[I_{\{\tau>T\}}|\mathcal{F}_t]\mathbb{E}[g|\mathcal{F}_t] + \mathrm{Cov}\Big(I_{\{\tau>T\}}, g|\mathcal{F}_t\Big) := G_t(T)U^g_t + \mathrm{Cov}^g_t.$$

Therefore, in virtue of Corollary 1 (see also Theorem 5), the proof of Theorem 6 follows immediately as soon as we prove that

$$\begin{aligned} &\xi^{(h,\mathbb{F})} = G_-(T)\xi^{(g,\mathbb{F})} + U^g_-\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})} \quad \text{and} \\ &L^{(h,\mathbb{F})} = G_-(T)\bullet L^{(g,\mathbb{F})} + U^g_-\bullet L^{(G_T,\mathbb{F})} + L^{(\mathrm{Cor}_T,\mathbb{F})}. \end{aligned} \tag{60}$$

A direct application of the integration by parts formula to $G_t(T)U^g_t$ leads to

$$M^{(g)} = G_0(T)U^g_0 + G_-(T)\bullet U^g + U^g_-\bullet G(T) + \mathrm{Cor}, \tag{61}$$

where Cor is the process defined in (38). In order to apply the GKW decomposition for each of the $\mathbb{F}$-local martingales in the RHS term of (61), we need to prove that these local martingales are actually (locally) square integrable martingales. To this end, we remark that $0 \leq G_-(T) \leq 1$ and U^g is a square integrable $\mathbb{F}$-martingale. Furthermore, we derive $\sup_{0\leq t\leq T} |M_t^{(g)}| \leq \sup_{0\leq t\leq T} \mathbb{E}[\,|g| \mid \mathcal{F}_t] \in L^2(\Omega, \mathcal{F}, P)$ and

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T (U^g_{s-})^2 d[G(T)]_s\Big] \\
&\leq \mathbb{E}\left[\int_0^T \sup_{0\leq s<t} (U^g_s)^2 d[G(T)]_t\right] \\
&= \mathbb{E}\left[\int_0^T ([G(T)]_T - [G(T)]_t)\, d \sup_{0\leq s\leq t} (U^g_s)^2\right] + \mathbb{E}\Big[([G(T)]_T - [G(T)]_0)(U^g_0)^2\Big] \\
&\leq \mathbb{E}\left[\int_0^T \mathbb{E}([G(T)]_T - [G(T)]_t \mid \mathcal{F}_t)\, d \sup_{0\leq s\leq t} (U^g_s)^2\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq T} (U^g_s)^2\right] < +\infty.
\end{aligned}$$

In the last two inequalities, we used $\mathbb{E}[[G(T)]_T - [G(T)]_t \mid \mathcal{F}_t] = \mathbb{E}\big[(G_T)^2 - (G_t(T))^2 \mid \mathcal{F}_t\big] \leq 1$ and/or $\mathbb{E}[(U_0^g)^2] \leq \mathbb{E}[g^2]$. As a result, the three local martingales, $M^{(g)}$, $G(T)_- \bullet U^g$ and $U^g_- \bullet G(T)$, are square integrable martingales, and subsequently $G(T)_- \bullet U^g$, $U^g_- \bullet G(T)$ and Cor are square integrable martingales.

Therefore, by applying the GKW decomposition to U^g, $G(T)$ and Cor, we obtain

$$\begin{aligned}
M^{(g)} = M_0^{(g)} &+ \Big(G_-(T)\xi^{(g,\mathbb{F})} + U^g_-\xi^{(G_T,\mathbb{F})} + \xi^{(\mathrm{Cor}_T,\mathbb{F})}\Big) \bullet S \\
&+ G_-(T) \bullet L^{(g,\mathbb{F})} + U^g_- \bullet L^{(G_T,\mathbb{F})} + L^{(\mathrm{Cor}_T,\mathbb{F})},
\end{aligned}$$

and the proof of (60) follows immediately. This ends the proof of assertion (a).

Concerning the value process of the corresponding portfolio, we note that the payment process A is given by $A_t = I_{\{t=T\}} I_{\{\tau>T\}} g = I_{\{t=T\}} h_\tau$ such that $A_T - A_t = (1 - I_{\{t=T\}}) h_\tau$ and

$$V_t(\rho^{*,\mathbb{G}}) = (1 - I_{\{t=T\}}) H_t,$$

with H given by (59) where the first term is zero since we do not hedge beyond the term of the contract, thus $I_{\{\tau>T\}} I_{\{\tau\leq t\}} = 0$. This ends the proof of the theorem. □

4. Hedging Mortality Risk with Insurance Securitization

In this section, we address the hedging problem for mortality liabilities, using the risk-minimization criterion of Section 2.2, by investing in both the stock and one (or more) of the mortality/longevity securities defined in Definition 4 . This process is known, in the insurance literature, as insurance securitization. Thus, we need to specify the dynamics of these derivatives that will be used in the securitization process.

The following theorem, see ([23] Theorem 3.1), elaborates the prices' dynamics of these insurance contracts while allowing the death time τ to have an arbitrary model and under the assumption that *P is a risk-neutral probability for the model* $(\Omega, \mathbb{G})$. **This assumption means that all discounted price processes of traded securities in the market** $(\Omega, \mathbb{G})$ **are local martingales under** P. In our view, this assumption of the probability P is not a restriction. In fact, on the one hand, it is dictated by the Föllmer–Sondermann optimization method of Section 2.2, considered herein. On the other hand, one can calculate every process used in the theorem (starting with the processes G, $\widetilde{G}$, G_-) under a chosen risk-neutral measure, Q, for the informational model $(\Omega, \mathbb{G})$. It is important to mention that the following theorem is a direct application of our martingale representation theorems ([23] Theorems 2.17 and 2.20).

Theorem 8 ([23] **Theorem 3.2**)**.** *Suppose P is a risk-neutral probability for $(\Omega, \mathbb{G})$. Then the following holds.*

(a) *The discounted price process of the pure endowment insurance contract with benefit* $g \in L^1(\mathcal{F}_T, P)$ *at term* T*, is denoted by* $P^{(g)}$*, and is given by*

$$P^{(g)} = P_0^{(g)} + \frac{I_{]\!]0,\tau\wedge T]\!]}}{G_-} \bullet \mathcal{T}(M^{(g)}) - \frac{M_-^{(g)}}{G_-^2} I_{]\!]0,\tau\wedge T]\!]} \bullet \mathcal{T}(m) - \frac{M^{(g)}}{G} I_{]\!]0,R[\![} \bullet (N^{\mathbb{G}})^T, \quad (62)$$

$$\text{with } M_t^{(g)} := \mathbb{E}\big[gG_T \mid \mathcal{F}_t\big].$$

(b) *The discounted price process of the longevity bond, with term* T*, is denoted by* B *and satisfies*

$$\begin{aligned} B^\tau &= B_0 + \frac{I_{]\!]0,\tau\wedge T]\!]}}{G_-} \bullet \mathcal{T}(M^{(B)}) - \frac{M_-^{(B)} - \overline{D}_-^{o,\mathbb{F}}}{G_-^2} I_{]\!]0,T\wedge\tau]\!]} \bullet \mathcal{T}(m) \\ &\quad + \frac{\xi^{(G)} G - M^{(B)} + \overline{D}^{o,\mathbb{F}}}{G} I_{]\!]0,R[\![} I_{]\!]0,T]\!]} \bullet N^{\mathbb{G}} + \Big(\mathbb{E}[G_T \mid \mathcal{G}_\tau] - \xi_\tau^{(G)}\Big) I_{[\![\tau,+\infty[\![}, \end{aligned} \quad (63)$$

where

$$M_t^{(B)} := \mathbb{E}\Big[\overline{D}_\infty^{o,\mathbb{F}} - \overline{D}_0^{o,\mathbb{F}} \mid \mathcal{F}_{t\wedge T}\Big], \quad \xi^{(G)} := \frac{d\overline{D}^{o,\mathbb{F}}}{dD^{o,\mathbb{F}}}, \quad \overline{D}^{o,\mathbb{F}} := \Big(G_T I_{[\![\tau,+\infty[\![}\Big)^{o,\mathbb{F}}. \quad (64)$$

Throughout the rest of this section, we consider the following notation. Thanks to the GKW-decomposition, with respect to S, of $G(T)$ and $M^{(B)}$ defined in (34) and (64), respectively, we get

$$G(T) = G_0(T) + \varphi^{(E)} \bullet S + L^{(E)}, \quad M^{(B)} = M_0^{(B)} + \varphi^{(B)} \bullet S + L^{(B)}. \quad (65)$$

The superscripts E and B in the strategies $\varphi^{(\cdot,\mathbb{H})}$ and the remaining risks $L^{(\cdot,\mathbb{H})}$ refer to the type of contract (i.e., the letter "E" refers to the pure endowment, while the letter "B" refers to the longevity bond). Then, throughout this section, we consider

$$\begin{aligned} \varphi^{(E,\mathbb{G})} := \varphi^{(E)}\Big(G_- + \varphi^{(m)}\Big)^{-1} I_{]\!]0,\tau]\!]}, \quad L^{(E,\mathbb{G})} &:= G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(L^{(E)}) - \frac{\varphi^{(E,\mathbb{G})}}{G_-} \bullet \mathcal{T}(L^{(m)}) \\ &\quad - G(T) G^{-1} I_{[\![0,R[\![} \bullet \Big(N^{\mathbb{G}}\Big)^T, \end{aligned} \quad (66)$$

$$\varphi^{(B,\mathbb{G})} := \varphi^{(B)}\Big(G_- + \varphi^{(m)}\Big)^{-1} I_{]\!]0,\tau]\!]}, \quad L^{(B,\mathbb{G})} := \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(B)}) - \frac{\varphi^{(B,\mathbb{G})}}{G_-} \bullet \mathcal{T}(L^{(m)}) + L^{(1)}, \quad (67)$$

$$\begin{aligned} L^{(1)} := \Big(-M_-^{(B)} + \overline{D}_-^{o,\mathbb{F}}\Big) G_-^{-2} I_{]\!]0,T\wedge\tau]\!]} \bullet \mathcal{T}(m) + \Big[\xi^{(G)} + \Big(-M^{(B)} + \overline{D}^{o,\mathbb{F}}\Big) G^{-1} I_{[\![0,R[\![}\Big] \bullet \Big(N^{\mathbb{G}}\Big)^T \\ + \mathbb{E}\Big[G_T - \xi_\tau^{(G)} \mid \mathcal{G}_\tau\Big] I_{[\![\tau,+\infty[\![}. \end{aligned} \quad (68)$$

Here, $\xi^{(G)}$ and $\overline{D}^{o,\mathbb{F}}$ are given by (64). Now, we are in the stage of announcing the main result of this section.

Theorem 9. *Suppose that (25) holds, and let* $h \in L^2(\mathcal{O}(\mathbb{F}), P \otimes D)$*. Consider* $(\varphi^{(B,\mathbb{G})}, L^{(B,\mathbb{G})})$ *and* $(\varphi^{(E,\mathbb{G})}, L^{(E,\mathbb{G})})$ *defined in (65)–(66) and (65)–(67) respectively, and* $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$ *given by (26)–(27). Then the following assertions hold.*
(a) *Consider the market model* $(S^\tau, B^\tau, \mathbb{G})$*. Then the risk-minimizing strategy and the remaining risk in this market model, for the insurance contract with payoff* h_τ*, are denoted by* $(\xi^{(h,1)}, \xi^{(h,2)})$ *and* $L^{(\mathbb{G})}$*, respectively, satisfy*

$$H := {}^{o,\mathbb{G}}(h_\tau) = H_0 + \xi^{(h,1)} \bullet S^\tau + \xi^{(h,2)} \bullet B^\tau + L^{(\mathbb{G})},$$

and are given by

$$\xi^{(h,2)} := \frac{d\langle L^{(h,\mathbb{G})}, L^{(B,\mathbb{G})}\rangle^{\mathbb{G}}}{d\langle L^{(B,\mathbb{G})}\rangle^{\mathbb{G}}}, \quad \xi^{(h,1)} := \xi^{(h,\mathbb{G})} - \varphi^{(B,\mathbb{G})}\xi^{(h,2)}, \quad L^{(\mathbb{G})} := L^{(h,\mathbb{G})} - \xi^{(h,2)} \bullet L^{(B,\mathbb{G})}. \tag{69}$$

(b) *Consider the market model* $\left(S^\tau, P^{(1)}, \mathbb{G}\right)$. *Then the risk-minimizing strategy and the remaining risk in this market model, for the insurance contract with payoff* h_τ, *are denoted by* $(\widetilde{\xi}^{(h,1)}, \widetilde{\xi}^{(h,2)})$ *and* $\widetilde{L}^{(\mathbb{G})}$ *respectively, satisfy*

$$H := {}^{o,\mathbb{G}}(h_\tau) = H_0 + \widetilde{\xi}^{(h,1)} \bullet S^\tau + \widetilde{\xi}^{(h,2)} \bullet P^{(1)} + \widetilde{L}^{(\mathbb{G})},$$

and are given by

$$\widetilde{\xi}^{(h,2)} := \frac{d\langle L^{(h,\mathbb{G})}, L^{(E,\mathbb{G})}\rangle^{\mathbb{G}}}{d\langle L^{(E,\mathbb{G})}\rangle^{\mathbb{G}}}, \quad \widetilde{\xi}^{(h,1)} := \xi^{(h,\mathbb{G})} - \varphi^{(E,\mathbb{G})}\widetilde{\xi}^{(h,2)}, \quad \widetilde{L}^{(\mathbb{G})} := L^{(h,\mathbb{G})} - \widetilde{\xi}^{(h,2)} \bullet L^{(E,\mathbb{G})}. \tag{70}$$

(c) *Consider the market model* $\left(S^\tau, P^{(1)}, B^\tau, \mathbb{G}\right)$. *Then the risk-minimizing strategy and the remaining risk in this market, for the insurance contract with payoff* h_τ, *are denoted by* $(\overline{\xi}^{(h,1)}, \overline{\xi}^{(h,2)}, \overline{\xi}^{(h,3)})$ *and* $\overline{L}^{(\mathbb{G})}$, *respectively, and satisfy*

$$H := H_0 + \overline{\xi}^{(h,1)} \bullet S^\tau + \overline{\xi}^{(h,2)} \bullet P^{(1)} + \overline{\xi}^{(h,3)} \bullet B^\tau + \overline{L}^{(\mathbb{G})},$$

where

$$\overline{\xi}^{(h,2)} := \frac{\widetilde{\xi}^{(h,2)} - \psi^{(E,B)}\xi^{(h,2)}}{1 - \psi^{(E,B)}\theta^{(E,B)}} I_{\{\psi^{(E,B)}\theta^{(E,B)} \neq 1\}}, \quad \overline{\xi}^{(h,3)} := \frac{\xi^{(h,2)} - \theta^{(E,B)}\widetilde{\xi}^{(h,2)}}{1 - \psi^{(E,B)}\theta^{(E,B)}} I_{\{\psi^{(E,B)}\theta^{(E,B)} \neq 1\}},$$
$$\overline{\xi}^{(h,1)} := \xi^{(h,\mathbb{G})} - \varphi^{(E,\mathbb{G})}\overline{\xi}^{(h,2)} - \varphi^{(B,\mathbb{G})}\overline{\xi}^{(h,3)} \quad \overline{L}^{(\mathbb{G})} := L^{(h,\mathbb{G})} - \overline{\xi}^{(h,2)} \bullet L^{(E,\mathbb{G})} - \overline{\xi}^{(h,3)} \bullet L^{(B,\mathbb{G})}.$$

Here, $\theta^{(E,B)}$ *and* $\psi^{(E,B)}$ *are given by*

$$\theta^{(E,B)} := \frac{d\langle L^{(E,\mathbb{G})}, L^{(B,\mathbb{G})}\rangle^{\mathbb{G}}}{d\langle L^{(B,\mathbb{G})}\rangle^{\mathbb{G}}}, \quad \psi^{(E,B)} := \frac{d\langle L^{(E,\mathbb{G})}, L^{(B,\mathbb{G})}\rangle^{\mathbb{G}}}{d\langle L^{(E,\mathbb{G})}\rangle^{\mathbb{G}}}.$$

Proof. This proof is achieved in three parts where we prove assertions (a), (b) and (c), respectively.

Part 1: By combining (65) and (24) in (63), we derive

$$B^\tau = B_0 + \varphi^{(B,\mathbb{G})} \bullet S^\tau - \frac{\varphi^{(B,\mathbb{G})}}{G_-} \bullet \mathcal{T}(L^{(m)}) + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(B)}) + L^{(1)} = B_0 + \varphi^{(B,\mathbb{G})} \bullet S^\tau + L^{(B,\mathbb{G})}, \tag{71}$$

where $\varphi^{(B,\mathbb{G})}$ and $L^{(1)}$ are given in (67) and (68), respectively. Then, by inserting this equality in $H = H_0 + \xi^{(h,1)} \bullet S^\tau + \xi^{(h,2)} \bullet B^\tau + L^{(\mathbb{G})}$, we obtain

$$H = H_0 + \left[\xi^{(h,1)} + \varphi^{(B,\mathbb{G})}\xi^{(h,2)}\right] \bullet S^\tau + \xi^{(h,2)} \bullet L^{(B,\mathbb{G})} + L^{(\mathbb{G})}.$$

Thus, by comparing this resulting equation with

$$H = H_0 + \xi^{(h,\mathbb{G})} \bullet S^\tau + L^{(h,\mathbb{G})}, \tag{72}$$

where $\xi^{(h,\mathbb{G})}$ and $L^{(h,\mathbb{G})}$ are given by (26)–(27), we conclude that

$$\xi^{(h,\mathbb{G})} = \xi^{(h,1)} + \varphi^{(B,\mathbb{G})}\xi^{(h,2)}, \quad L^{(h,\mathbb{G})} = \xi^{(h,2)} \bullet L^{(B,\mathbb{G})} + L^{(\mathbb{G})}.$$

This is due to the fact that $L^{(\mathbb{G})}$ is orthogonal to (S^τ, B^τ) if and only if it is also orthogonal to

$(S^\tau, L^{(B,\mathbb{G})})$. Therefore, the proof of assertion (a) follows immediately.

Part 2: To prove assertion (b), similarly, we derive the following decomposition for $P^{(1)}$ in (62)

$$\begin{aligned}P^{(1)} &= P^{(1)}_0 + \varphi^{(E,\mathbb{G})} \bullet S^\tau - \frac{\varphi^{(E,\mathbb{G})}}{G_-} \bullet \mathcal{T}(L^{(m)}) + \frac{I_{]\!]0,\tau]\!]}}{G_-} \bullet \mathcal{T}(L^{(E)}) - \frac{G_-(T)}{G_-^2} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(m) \\ &\quad - \frac{G(T)}{G} I_{[\![0,R[\![} \bullet \left(N^{\mathbb{G}}\right)^T \\ &= P^{(1)}_0 + \varphi^{(E,\mathbb{G})} \bullet S^\tau + L^{(E,\mathbb{G})}. \end{aligned} \tag{73}$$

Then, by combining this with (72), the proof of assertion (b) follows immediately.

Part 3: Herein, we prove assertion (c). By inserting (71) and (73) in $H = H_0 + \overline{\xi}^{(h,1)} \bullet S^\tau + \overline{\xi}^{(h,2)} \bullet P^{(1)} + \overline{\xi}^{(h,3)} \bullet B^\tau + L^{(\mathbb{G})}$, we obtain

$$H = H_0 + \left[\overline{\xi}^{(h,1)} + \varphi^{(E,\mathbb{G})}\overline{\xi}^{(h,2)} + \varphi^{(B,\mathbb{G})}\overline{\xi}^{(h,3)}\right] \bullet S^\tau + \overline{\xi}^{(h,2)} \bullet L^{(E,\mathbb{G})} + \overline{\xi}^{(h,3)} \bullet L^{(B,\mathbb{G})} + \overline{L}^{(\mathbb{G})}.$$

Therefore, the proof of assertion (c) follows immediately from combining this with (72) and the fact that the orthogonality of $\overline{L}^{(\mathbb{G})}$ to $(S^\tau, P^{(1)}, B^\tau)$ is equivalent to the orthogonality of $\overline{L}^{(\mathbb{G})}$ to $(S^\tau, L^{(E,\mathbb{G})}, L^{(B,\mathbb{G})})$. This ends proof of the theorem. □

To our knowledge, Theorem 9 generalizes all the existing literature on risk-minimizing strategies using mortality securitization in many directions. Our approach in this theorem, which is based essentially on our optional martingale decomposition of Theorem 2 and the resulting risk decomposition of Theorem 8, allows us to work on any model (S, τ) fulfilling (25). As mentioned before, this assumption covers all the cases treated in the literature and goes beyond that. The reader can easily see this fact by comparing our framework to those considered in [26–29] and the references therein, to cite a few. Indeed, in [29] the assumptions include H-hypothesis (i.e., all $\mathbb{F}$-local martingales are $\mathbb{G}$-local martingale), τ avoids the $\mathbb{F}$-stopping times and the hazard rate exists, and/or the mortality follows affine models. In [27,28], the authors assume the independence between the stock price process and the mortality rate process, and consider the Brownian filtration. Barbarin assumes, in [26], that the mortality follows the Heath–Jarrow–Morton model, and considers the Brownian filtration for $\mathbb{F}$.

Furthermore, our results in Theorem 9 are very explicit and, more importantly, they explain the impact of the securitization on the pair of risk-minimizing strategies and the remaining risk in the following sense. For any securitization model $\mathcal{S} := (S^\tau, Y^{(1)}, Y^{(2)}, \mathbb{G})$, where $Y^{(i)}$ denotes the price process of the i^{th} mortality security, we describe in Theorem 9 very precisely how the pair of the risk-minimizing strategies and the remaining risk associated with this securitization model $(\xi^{(\mathcal{S})}, L^{(\mathcal{S})})$ are obtained from the pair of cases without securitization $(\xi^{(h,\mathbb{G})}, L^{(h,\mathbb{G})})$, and/or from the pair that is associated with the securitization model $(S^\tau, Y^{(i)}, \mathbb{G})$, $i = 1, 2$.

When τ is independent of $\mathcal{F}_\infty$ such that $P(\tau > T) > 0$, then on the one hand, (62) becomes

$$P^{(g)} = P^{(g)}_0 - \frac{gP(\tau > T)}{P(\tau > \cdot)} \bullet \left(N^{\mathbb{G}}\right)^T = P^{(g)}_0 - \frac{gP(\tau > T)}{P(\tau > \cdot)} \bullet \left(\overline{N}^{\mathbb{G}}\right)^T, \tag{74}$$

where $\overline{N}^{\mathbb{G}} := D - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet D^{p,\mathbb{F}}$ is the $\mathbb{G}$-martingale in the Doob–Meyer decomposition of D.

On the other hand, the longevity bond has a constant price process equal to G_T, and hence it cannot be used for hedging any risk! Thus, under the independence condition between τ and $\mathbb{F}$, the pure endowment insurance with benefit one (the contract that pays one dollar to the beneficiary if s/he survives) is more adequate to hedge pure mortality/longevity risk in insurance liabilities, while the longevity bond has no effect at all.

5. Conclusions

We addressed the risk-minimization problem for mortality liabilities by designing quadratic hedging strategies *à la* Föllmer–Sondermann with and without insurance securization when no model for the death time is specified. The correlation between the market model and the time of death is also kept arbitrary general, implying two levels of information: the public information $\mathbb{F}$, which is generated by the financial assets, and a larger flow of information $\mathbb{G}$ that contains additional knowledge about a death time of an insured. By enlarging the filtration, the death uncertainty and its entailed risk were fully considered without any mathematical restriction. When hedging mortality risk without securization, as a first main contribution we quantified—as explicitly as possible—the effect of mortality on the risk-minimizing strategy for mortality claims modelled by an optional process by determining the $\mathbb{G}$-optimal strategy in terms of the $\mathbb{F}$-strategies, where we singled out the various types of correlation risks. We distinguished three types of risk for a mortality claim: pure financial risk, pure mortality risk and correlation risk resulting from the interplay between the randomness of the time of death and the flow $\mathbb{F}$. The obtained general results are further established for specific mortality contracts (pure endowment and endowment insurance) to highlight the impact of the interplay/correlation between the benefit policy and the death time. As a second main contribution we derived risk-minimizing hedging strategies with insurance securitization by investing in stocks and one (or more) mortality/longevity derivatives such as longevity bonds. Our obtained results are very explicit and, more importantly, they explain the impact of the securitization on the pair of risk-minimizing strategies and the remaining risk. We provided detailed discussions on how our results generalize in as many directions as those in the literature, such as for example [26–29] and the references therein, on risk-minimizing strategies with and without mortality securization. We presented how broader models of the financial model and the time of death and a larger class of mortality liabilities fit into our framework and showed how, by imposing additional specifications, our results will boil down to those in the literature.

Author Contributions: Conceptualization, T.C., C.D. and M.V.; methodology, T.C., C.D. and M.V.; formal analysis, T.C., C.D. and M.V.; investigation, T.C., C.D. and M.V.; writing—original draft preparation, T.C., C.D. and M.V.; supervision, T.C. and M.V. All authors have read and agreed to the published version of the manuscript.

Funding: The research by Tahir Choulli was funded by NSERC (through grant NSERC RGPIN04987). The research by Catherine Daveloose was funded by the Agency for Innovation by Science and Technology in Flanders (IWT, grant number 111262).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Tahir Choulli is thankful to Michèle Vanmaele and the Department of Applied Mathematics, Computer Science, and Statistics (Ghent University), for their invitation and hospitality, where this work started. The three authors are very grateful to the anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Lemma 1

The proof of this lemma requires two lemmas that we start with.

Lemma A1. *Let V be an $\mathbb{F}$-adapted process with $\mathbb{F}$-locally integrable variation. Then we have*

$$(V^{\tau})^{p,\mathbb{G}} = (G_{-})^{-1} I_{]\!]0,\tau]\!]} \bullet \left(\widetilde{G} \bullet V\right)^{p,\mathbb{F}}. \tag{A1}$$

For the proof of this lemma, we refer the reader to ([43] Lemma 3.1).

Lemma A2. *For an $\mathbb{H}$-optional process ϕ such that $0 \leq \phi \leq 1$ and $V \in \mathcal{A}^+_{\rm loc}(\mathbb{H})$, the following hold.*
(i) There exists an $\mathbb{H}$-predictable process, ψ, satisfying

$$0 \leq \psi \leq 1 \quad \textit{and} \quad (\phi \bullet V)^{p,\mathbb{H}} = \psi \bullet V^{p,\mathbb{H}}.$$

(ii) If $P \otimes dV(\{\phi = 0\}) = 0$, then ψ can be chosen strictly positive for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.

Proof. (i) Since $\phi \leq 1$, it is clear that $d(\phi \bullet V)^{p,\mathbb{H}} \ll dV^{p,\mathbb{H}}$, P-a.s.. Hence, there exists a non-negative and $\mathbb{H}$-predictable process $\psi^{(1)}$ such that

$$(\phi \bullet V)^{p,\mathbb{H}} = \psi^{(1)} \bullet V^{p,\mathbb{H}}. \tag{A2}$$

As a result, we derive $0 = I_{\{\psi^{(1)}>1\}} \bullet \left[(\phi \bullet V)^{p,\mathbb{H}} - \psi^{(1)} \bullet V^{p,\mathbb{H}}\right] = \left((\phi - \psi^{(1)})I_{\{\psi^{(1)}>1\}} \bullet V\right)^{p,\mathbb{H}}$, and deduce that $P \otimes V^{p,\mathbb{H}}(\{\psi^{(1)} > 1\}) = 0$. Thus, by putting $\psi = \psi^{(1)} \wedge 1$, assertion (a) follows.

(ii) It is clear from (A2) that $0 = I_{\{\psi^{(1)}=0\}} \bullet (\phi \bullet V)^{p,\mathbb{H}} = (\phi I_{\{\psi^{(1)}=0\}} \bullet V)^{p,\mathbb{H}}$. This implies that $\{\psi^{(1)} = 0\} \subset \{\phi = 0\}$ $P \otimes V$-a.e.. Therefore, assertion (b) follows from putting $\psi = \psi^{(1)} \wedge 1 + I_{\{\psi^{(1)}=0\}}$, and the proof of the lemma is completed. □

Proof of Lemma 1. This proof consists of three steps, where we prove $S^\tau \in \mathcal{M}^2_{loc}(\mathbb{G})$, assertion (a), and assertions (b)-(c) respectively.

Step 1: Thanks to [52], $S^\tau - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \langle S, m\rangle^{\mathbb{F}}$ is a $\mathbb{G}$-local martingale. Thus, by combining this with the second assumption in (25) (i.e., $\langle S, m\rangle^{\mathbb{F}} \equiv 0$), we deduce that S^τ is $\mathbb{G}$-local martingale. Hence, $S^\tau \in \mathcal{M}^2_{loc}(\mathbb{G})$ follows from this and the fact that $\sup_{0\leq \leq t \cdot} |S_t|^2 \in \mathcal{A}^+_{loc}(\mathbb{F}) \subset \mathcal{A}^+_{loc}(\mathbb{G})$, which is a direct consequence of the first condition in (21).

Step 2: Due to the third assumption in (25), it holds that $\Delta S I_{[\![\widetilde{R}]\!]} = \Delta S I_{\{\widetilde{G}=0<G_-\}} \equiv 0$. Thus, for any $L \in \mathcal{M}_{loc}(\mathbb{F})$ orthogonal to S, we have

$$[\mathcal{T}(L), S^\tau] = G_- \widetilde{G}^{-1} I_{]\!]0,\tau]\!]} \bullet [L, S] + {}^{p,\mathbb{F}}(\Delta L I_{[\![\widetilde{R}]\!]}) \bullet S^\tau.$$

Since ${}^{p,\mathbb{F}}(\Delta L I_{[\![\widetilde{R}]\!]}) \bullet S^\tau$ a $\mathbb{G}$-local martingale and $\Delta L \Delta S I_{[\![\widetilde{R}]\!]} \equiv 0$, Lemma A1 implies that

$$\langle \mathcal{T}(L), S^\tau\rangle^{\mathbb{G}} = I_{[\![0,\tau]\!]} \bullet \langle L, S\rangle^{\mathbb{F}} \equiv 0.$$

This proves assertion (a).

Step 3: Since m is bounded and orthogonal to $S \in \mathcal{M}^2_{loc}(\mathbb{F})$, it is clear that $U := I_{\{G_->0\}} \bullet [S, m] \in \mathcal{M}^2_{0,loc}(\mathbb{F})$. Then, an application of the Galtchouk-Kunita-Watanabe decomposition of U with respect to S, we get the first property in (23). To prove the second property in (23), we remark that $[U, S] = \Delta m I_{\{G_->0\}} \bullet [S, S]$, and put

$$W := G_- \bullet [S, S] + [U, S] = \widetilde{G} I_{\{G_->0\}} \bullet [S, S] \quad \text{and} \quad V := I_{\{G_->0\}} \bullet [S, S].$$

A direct application of Lemma A2 to the pair $(V, \widetilde{G} + I_{\{\widetilde{G}=0\}})$ (it is easy to see that the assumptions of this lemma are fulfilled as $P \otimes V(\{\phi = 0\}) = P \otimes I_{\{G_->0\}} \bullet [S, S](\{\widetilde{G} = 0\}) = 0$ which follows from $I_{\{\widetilde{G}=0<G_-\}} \Delta S = 0$), we deduce that the existence of $\mathbb{F}$-predictable ψ such that $0 < \psi \leq 1$ and

$$W^{p,\mathbb{F}} = \psi I_{\{G_->0\}} \bullet \langle S, S\rangle^{\mathbb{F}} = (G_- + \varphi^{(m)}) I_{\{G_->0\}} \bullet \langle S, S\rangle^{\mathbb{F}}.$$

This completes the proof of assertion (b). Thus, the rest of the proof focuses on proving assertion (c). To this end, we notice that due to $\Delta U I_{\{\widetilde{G}=0<G_-\}} = -G_-\Delta S I_{\{\widetilde{G}=0<G_-\}} = 0$, it is clear that

$$\mathcal{T}(U) = U^\tau - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \bullet [U,m] = I_{[\![0,\tau]\!]} \bullet [S,m] - \widetilde{G}^{-1} I_{[\![0,\tau]\!]} \Delta m \bullet [S,m] = I_{]\!]0,\tau]\!]} G_- \widetilde{G}^{-1} \bullet U.$$

As a result, on the one hand, we get

$$\mathcal{T}(S) = S^\tau - G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(U). \tag{A3}$$

On the other hand, due to (23), we derive

$$\mathcal{T}(U) = \varphi^{(m)} \bullet \mathcal{T}(S) + \mathcal{T}(L^{(m)}) = \varphi^{(m)} \bullet S^\tau - \varphi^{(m)} G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(U) + \mathcal{T}(L^{(m)}).$$

Solving for $\mathcal{T}(U)$, we get

$$(G_- + \varphi^{(m)}) G_-^{-1} I_{]\!]0,\tau]\!]} \bullet \mathcal{T}(U) = \varphi^{(m)} \bullet S^\tau + \mathcal{T}(L^{(m)}).$$

By inserting this equality in (A3), (24) follows immediately, and the proof of the lemma is complete. □

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