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Some Families of Apéry-Like Fibonacci and Lucas Series

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Abstract: In this paper, the authors investigate two special families of series involving the reciprocal central binomial coefficients and Lucas numbers. Connections with several familiar sums representing the integer-valued Riemann zeta function are also pointed out.

Keywords: Riemann Zeta function; central binomial coefficient; Catalan numbers; Fibonacci numbers; Lucas numbers; harmonic numbers; Lerch's transcendent (or the Hurwitz–Lerch zeta function)

MSC: primary 11B37; 11B39; secondary 11B68; 11M06; 11M35

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1. Introduction and Motivation

One of the important and widely and extensively investigated functions in Analytic Number Theory is the Riemann Zeta function $\zeta(s)$, which is defined $\big($for $\Re(s) > 1\big)$ by

$$\zeta(s) := \begin{cases} \sum\limits_{k=1}^{\infty} \dfrac{1}{k^s} = \dfrac{1}{1-2^{-s}} \sum\limits_{k=1}^{\infty} \dfrac{1}{(2k-1)^s} & \big(\Re(s) > 1\big) \\ \dfrac{1}{1-2^{1-s}} \sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases} \tag{1}$$

and $\big($for $\Re(s) \leqq 1;\ s \neq 1\big)$ by its *meromorphic* continuation (see, for details, [1]).

Elegant *single-series* representations are known for such zeta values as $\zeta(2)$, $\zeta(3)$ and $\zeta(4)$, but no such single-series expressions are known for $\zeta(n)$ when $n \geqq 5$. In particular, the following series for $\zeta(3)$:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \tag{2}$$

played a key role in the celebrated proof of the irrationality of $\zeta(3)$ by Apéry [2]. In fact, as pointed out by Chen and Srivastava [3] [pp. 180–181] (see also [4] [p. 333]), the representation (2) was derived *independently* by (among others) Hjortnaes [5], Gosper [6], and Apéry [2]. With a view to honoring and felicitating Roger Apéry (1916–1994), $\zeta(3)$ is popularly known as the Apéry constant and sums of the type in (2) is referred to as Apéry-like series.

The computation of infinite series containing reciprocal central binomial coefficients is a challenging issue and it is currently an active field in number theory and experimental mathematics. The interest in these series comes from the existence of connections to certain generating functions of the zeta function, special zeta values and other important constants such as the golden ratio. On the other hand, central binomial coefficients are directly linked to the famous Catalan numbers, which are interesting on their own (see [7] for an introduction). In addition to (2), some examples of series with reciprocal central binomial coefficient are recalled here (see [8–10]):

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{2k}{k}} = \frac{\pi}{3\sqrt{3}}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} = \frac{\zeta(2)}{3}, \tag{3}$$

as well as the alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\binom{2k}{k}} = \frac{2\ln\alpha}{\sqrt{5}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2\binom{2k}{k}} = 2(\ln\alpha)^2 \qquad \left(\alpha := \frac{1+\sqrt{5}}{2}\right), \tag{4}$$

with α being the golden ratio.

The following two series evaluations appeared very recently in [11]:

$$\sum_{k=1}^{\infty} \frac{F_{2k}}{k\binom{2k}{k}} = 2\pi\sqrt{\frac{\alpha}{25\sqrt{5}}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{L_{2k}}{k\binom{2k}{k}} = 2\pi\alpha^2\sqrt{\frac{\alpha}{25\sqrt{5}}}, \tag{5}$$

where F_k and L_k are Fibonacci and Lucas numbers, respectively, and the golden ratio α is given in (4). For further information on series involving reciprocal central binomial coefficients, one can see also the papers by Boyadzhiev [12], Glasser [13], Rivoal [14] and Uhl [15].

The goal of this paper is to study two families of Apéry-like series with coefficients involving Fibonacci (Lucas) numbers. One such family of series is evaluated here in closed form. The other family of Apéry-like series, in addition, involves harmonic numbers. Here, we derive an equivalent expression for the series and relate a special case to the Lerch transcendent (or the Hurwitz–Lerch zeta function) $\Phi(z,s,a)$ defined by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{6}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right),$$

where $\mathbb{Z}_0^-$ denotes the set of *non-positive* integers.

2. The First Set of Main Results

Before stating our first main result, we recall some basic facts about Fibonacci and Lucas numbers (see [7] for more information). Both of these number sequences satisfy the second-order recurrence relation:

$$G_{n+1} = G_n + G_{n-1},$$

but they have different initial terms. Fibonacci numbers F_n start with $F_0 = 0$ and $F_1 = 1$; Lucas numbers L_n have the initial values $L_0 = 2$ and $L_1 = 1$. Their Binet forms are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n \qquad (n \geqq 0),$$

where α and β are roots of the quadratic equation $x^2 - x - 1 = 0$, that is,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Some basic arithmetical properties of the Fibonacci numbers are listed below:

$$\gcd(F_n, F_{n+1}) = 1 \quad \text{and} \quad \gcd(F_n, F_{n+2}) = 1,$$

$$F_{mn} \equiv 0 \pmod{F_m} \quad (m, n \geqq 1) \qquad \text{and} \qquad F_m | F_n \iff m | n.$$

The next lemma will be used in proving our next theorem.

Lemma 1. *For each $j \geqq 0$, each of the following congruences holds true*:

$$\begin{aligned} F_j &\equiv 0 \pmod{4} \iff j \equiv 0 \pmod{6} \\ F_j &\equiv 1 \pmod{4} \iff j \equiv 1, 2, 5 \pmod{6} \\ F_j &\equiv 2 \pmod{4} \iff j \equiv 3 \pmod{6} \\ F_j &\equiv 3 \pmod{4} \iff j \equiv 4 \pmod{6}. \end{aligned} \tag{7}$$

Proof. This lemma can be proved by applying the principle of mathematical induction. □

Theorem 1. *For each $j \geqq 1$, it is asserted that*

$$\begin{aligned} &\frac{1}{\sqrt{5}10F_{2j}}\left(\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2+\alpha^{2j}}{5k^2-\alpha^{2j}}\prod_{m=1}^{k-1}\left(1-\frac{\alpha^{2j}}{5m^2}\right) - \frac{15k^2+\alpha^{-2j}}{5k^2-\alpha^{-2j}}\prod_{m=1}^{k-1}\left(1-\frac{\alpha^{-2j}}{5m^2}\right)\right]\right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{25n^4-5n^2L_{2j}+1} \\ &= \begin{cases} \frac{1}{2} - \frac{\pi}{2\sqrt{5}L_j \sin\left(\frac{\pi}{2\sqrt{5}}L_j\right)}, & \textit{if } j \equiv 0, 3 \pmod{6}; \\ \frac{1}{2} - \frac{\pi}{10F_j \cos\left(\frac{\pi}{2\sqrt{5}}L_j\right)}, & \textit{if } j \equiv 1, 4, 5 \pmod{6}; \\ \frac{1}{2} + \frac{\pi}{10F_j \cos\left(\frac{\pi}{2\sqrt{5}}L_j\right)}, & \textit{if } j \equiv 2 \pmod{6}. \end{cases} \end{aligned} \tag{8}$$

Proof. Our proof of Theorem 1 starts with the following formula [9] (Equatiion (31)), which holds true for all complex values of x other than an integer:

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} \frac{3k^2+x^2}{k^2-x^2} \prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-x^2} \\ &= \frac{\pi}{2x\sin(\pi x)} - \frac{1}{2x^2}. \end{aligned} \tag{9}$$

Setting $x = \frac{\alpha^j}{\sqrt{5}}$ $(j \geqq 1)$ gives

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} \frac{15k^2+\alpha^{2j}}{5k^2-\alpha^{2j}} \prod_{m=1}^{k-1}\left(1-\frac{\alpha^{2j}}{5m^2}\right) &= 10\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n^2-\alpha^{2j}} \\ &= \frac{\sqrt{5}\pi}{\alpha^j \sin\left(\frac{\pi\,\alpha^j}{\sqrt{5}}\right)} - \frac{5}{\alpha^{2j}}. \end{aligned} \tag{10}$$

Similarly, with $x = \frac{\beta^j}{\sqrt{5}}$ $(j \geqq 1)$, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} \frac{15k^2+\beta^{2j}}{5k^2-\beta^{2j}} \prod_{m=1}^{k-1}\left(1-\frac{\beta^{2j}}{5m^2}\right) &= 10\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{5n^2-\beta^{2j}} \\ &= \frac{\sqrt{5}\pi}{\beta^j \sin(\pi\beta^j/\sqrt{5})} - \frac{5}{\beta^{2j}}. \end{aligned} \tag{11}$$

Now, using the Binet form, we find that

$$\begin{aligned}
&\frac{1}{\sqrt{5}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2+\alpha^{2j}}{5k^2-\alpha^{2j}}\prod_{m=1}^{k-1}\left(1-\frac{\alpha^{2j}}{5m^2}\right)-\frac{15k^2+\alpha^{-2j}}{5k^2-\alpha^{-2j}}\prod_{m=1}^{k-1}\left(1-\frac{\alpha^{-2j}}{5m^2}\right)\right]\right)\\
&\quad=10F_{2j}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{25n^4-5n^2L_{2j}+1}\\
&\quad=\pi(-1)^{j+1}\frac{\alpha^j\sin\left(\frac{\pi\alpha^j}{\sqrt{5}}\right)-\beta^j\sin\left(\frac{\pi\beta^j}{\sqrt{5}}\right)}{\sin\left(\frac{\pi\alpha^j}{\sqrt{5}}\right)\sin\left(\frac{\pi\beta^j}{\sqrt{5}}\right)}+5F_{2j}.
\end{aligned}$$

Next, we make use of the following relations:

$$\frac{\alpha^j}{\sqrt{5}}=\frac{1}{2}F_j+\frac{1}{2\sqrt{5}}L_j \qquad \text{and} \qquad \frac{\beta^j}{\sqrt{5}}=-\frac{1}{2}F_j+\frac{1}{2\sqrt{5}}L_j,$$

so that

$$\sin\left(\frac{\pi\alpha^j}{\sqrt{5}}\right)=\sin\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\cos\left(\frac{\pi}{2}\,F_j\right)+\cos\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\sin\left(\frac{\pi}{2}\,F_j\right)$$

and

$$\sin\left(\frac{\pi\beta^j}{\sqrt{5}}\right)=\sin\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\cos\left(\frac{\pi}{2}\,F_j\right)-\cos\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\sin\left(\frac{\pi}{2}\,F_j\right).$$

As

$$\sin\left(\frac{\pi}{2}\,n\right)=\begin{cases}0, & \text{if } n\equiv 0,2 \pmod 4;\\ 1, & \text{if } n\equiv 1 \pmod 4;\\ -1, & \text{if } n\equiv 3 \pmod 4)\end{cases}$$

and

$$\cos(\frac{\pi}{2}n)=\begin{cases}0, & \text{if } n\equiv 1,3 \pmod 4;\\ 1, & \text{if } n\equiv 0 \pmod 4;\\ -1, & \text{if } n\equiv 2 \pmod 4,\end{cases}$$

we find from Lemma 1 that

$$\sin\left(\frac{\pi}{2}\,F_j\right)=\begin{cases}0, & \text{if } j\equiv 0,3 \pmod 6;\\ 1, & \text{if } j\equiv 1,2,5 \pmod 6;\\ -1, & \text{if } j\equiv 4 \pmod 6\end{cases}$$

and

$$\cos\left(\frac{\pi}{2}\,F_j\right)=\begin{cases}0, & \text{if } j\equiv 1,2,4,5 \pmod 6;\\ 1, & \text{if } j\equiv 0 \pmod 6;\\ -1, & \text{if } j\equiv 3 \pmod 6;\end{cases}$$

Now, the cases can be treated separately. As a showcase, we consider the case when

$$j\equiv 0 \pmod 6 \iff F_j\equiv 0 \pmod 4.$$

Here, we have

$$\sin\left(\frac{\pi}{2}\,F_j\right)=0 \qquad \text{and} \qquad \cos\left(\frac{\pi}{2}\,F_j\right)=1,$$

and

$$\sin\left(\frac{\pi\alpha^j}{\sqrt{5}}\right)=\sin\left(\frac{\pi\beta^j}{\sqrt{5}}\right)=\sin\left(\frac{\pi}{2\sqrt{5}}\,L_j\right),$$

so that

$$10F_{2j} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{25n^4 - 5n^2 L_{2j} + 1} = \pi(-1)^{j+1} \frac{\sqrt{5}F_j}{\sin\left(\frac{\pi}{2\sqrt{5}}\, L_j\right)} + 5F_{2j}.$$

Dividing by $10F_{2j}$ and using the relation $F_n L_n = F_{2n}$ establishes the result. The other cases are treated similarly. □

Theorem 2. *For each $j \geqq 0$, the following identity holds true*:

$$\begin{aligned}
&\frac{1}{10}\left(\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2+\alpha^{2j}}{5k^2-\alpha^{2j}} \prod_{m=1}^{k-1}\left(1-\frac{\alpha^{2j}}{5m^2}\right) + \frac{15k^2+\alpha^{-2j}}{5k^2-\alpha^{-2j}} \prod_{m=1}^{k-1}\left(1-\frac{\alpha^{-2j}}{5m^2}\right)\right]\right) \\
&\quad = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{10n^2 - L_{2j}}{25n^4 - 5n^2 L_{2j} + 1} \\
&\quad = \begin{cases} \dfrac{\frac{\pi}{2\sqrt{5}}\, L_j}{\sin\left(\frac{\pi}{2\sqrt{5}}\, L_j\right)} - \frac{1}{2} L_{2j}, & \textit{if} \;\; j \equiv 0, 3 \pmod 6; \\[2ex] \dfrac{\frac{\pi}{2} F_j}{\cos\left(\frac{\pi}{2\sqrt{5}}\, L_j\right)} - \frac{1}{2} L_{2j}, & \textit{if} \;\; j \equiv 1, 4, 5 \pmod 6; \\[2ex] -\dfrac{\frac{\pi}{2}\, F_j}{\cos\left(\frac{\pi}{2\sqrt{5}}\, L_j\right)} - \frac{1}{2} L_{2j}, & \textit{if} \;\; j \equiv 2 \pmod 6. \end{cases}
\end{aligned} \tag{12}$$

Proof. Theorem 2 can be proved by using the same arguments as in the proof of Theorem 1. □

Theorem 3. *For each $j \geqq 1$, it is asserted that*

$$\begin{aligned}
&\frac{1}{10\sqrt{5}\, F_{2j}}\left(\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2-\alpha^{2j}}{5k^2+\alpha^{2j}} \prod_{m=1}^{k-1}\left(1+\frac{\alpha^{2j}}{5m^2}\right) - \frac{15k^2-\alpha^{-2j}}{5k^2+\alpha^{-2j}} \prod_{m=1}^{k-1}\left(1+\frac{\alpha^{-2j}}{5m^2}\right)\right]\right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{25n^4 + 5n^2 L_{2j} + 1} \\
&= -\frac{1}{2} + (-1)^j \frac{\pi}{10F_{2j}} \frac{L_j \sinh\left(\frac{\pi}{2}\, F_j\right) \cosh\left(\frac{\pi}{2\sqrt{5}}\, L_j\right) + \sqrt{5}F_j \sinh\left(\frac{\pi}{2\sqrt{5}}\, L_j\right) \cosh\left(\frac{\pi}{2}\, F_j\right)}{\sinh^2\left(\frac{\pi}{2\sqrt{5}}\, L_j\right) - \sinh^2\left(\frac{\pi}{2}\, F_j\right)}.
\end{aligned} \tag{13}$$

Proof. Setting $x = \mathrm{i}y \;\; (\mathrm{i} = \sqrt{-1})$ in (9), we get

$$\begin{aligned}
\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} \frac{3k^2-y^2}{k^2+y^2} \prod_{m=1}^{k-1}\left(1+\frac{y^2}{m^2}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+y^2} \\
&= -\frac{\pi}{2y\sinh(\pi y)} + \frac{1}{2y^2}.
\end{aligned} \tag{14}$$

Now, as in our proof of Theorem 1, we put

$$y = \frac{\alpha^j}{\sqrt{5}} \quad \text{and} \quad y = \frac{\beta^j}{\sqrt{5}} \qquad (j \geqq 1).$$

In order to simplify, we use the the following elementary identities:

$$\sinh(A+B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

and

$$\sinh(A) + \sinh(B) = 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right).$$

The result is

$$\begin{aligned}
&\frac{1}{10\sqrt{5}\,F_{2j}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2-\alpha^{2j}}{5k^2+\alpha^{2j}}\prod_{m=1}^{k-1}\left(1+\frac{\alpha^{2j}}{5m^2}\right)-\frac{15k^2-\alpha^{-2j}}{5k^2+\alpha^{-2j}}\prod_{m=1}^{k-1}\left(1+\frac{\alpha^{-2j}}{5m^2}\right)\right]\right)\\
&\quad=\sum_{n=1}^{\infty}\frac{(-1)^n}{25n^4+5n^2L_{2j}+1} \qquad (15)\\
&\quad=-\frac{1}{2}+(-1)^j\,\frac{\pi}{10F_{2j}}\,\frac{L_j\sinh\left(\frac{\pi}{2}\,F_j\right)\cosh\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)+\sqrt{5}F_j\sinh\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\cosh\left(\frac{\pi}{2}\,F_j\right)}{\left[\sinh\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\cosh\left(\frac{\pi}{2}\,F_j\right)\right]^2-\left[\sinh\left(\frac{\pi}{2}\,F_j\right)\cosh\left(\frac{\pi}{2\sqrt{5}}\,L_j\right)\right]^2}.
\end{aligned}$$

The denominator in the last expression can be simplified further using

$$\cosh^2(A)-\sinh^2(A)=1.$$

□

For $j=1$, we have

$$\begin{aligned}
&\frac{1}{10\sqrt{5}}\left(\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\left[\frac{15k^2-\alpha^{2}}{5k^2+\alpha^{2}}\prod_{m=1}^{k-1}\left(1+\frac{\alpha^{2}}{5m^2}\right)-\frac{15k^2-\alpha^{-2}}{5k^2+\alpha^{-2}}\prod_{m=1}^{k-1}\left(1+\frac{\alpha^{-2}}{5m^2}\right)\right]\right)\\
&\quad=\sum_{n=1}^{\infty}\frac{(-1)^n}{25n^4+15n^2+1} \qquad (16)\\
&\quad=-\frac{1}{2}-\frac{\pi}{10}\,\frac{\sinh\left(\frac{\pi}{2}\right)\cosh\left(\frac{\pi}{2\sqrt{5}}\right)+\sqrt{5}\sinh\left(\frac{\pi}{2\sqrt{5}}\right)\cosh\left(\frac{\pi}{2}\right)}{\sinh^2\left(\frac{\pi}{2\sqrt{5}}\right)-\sinh^2\left(\frac{\pi}{2}\right)}.
\end{aligned}$$

Finally, it is worth mentioning that, upon differentiating (9) and (14) with respect to the variables x and y, respectively, if we set $y=x$ in the resulting equation, we are led to the following identities:

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)\left(\frac{8k^2x}{(k^2-x^2)^2}-\frac{3k^2+x^2}{k^2-x^2}\sum_{m=1}^{k-1}\frac{2x}{m^2-x^2}\right)\\
&\quad=\sum_{n=1}^{\infty}\frac{4(-1)^{n-1}\,x}{(n^2-x^2)^2} \qquad (17)\\
&\quad=\frac{2}{x^3}-\pi\,\frac{\sin(\pi x)+\pi x\cos(\pi x)}{[x\sin(\pi x)]^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\prod_{m=1}^{k-1}\left(1+\frac{x^2}{m^2}\right)\left(\frac{3k^2-x^2}{k^2+x^2}\sum_{m=1}^{k-1}\frac{2x}{m^2+x^2}-\frac{8k^2x}{(k^2+x^2)^2}\right) \qquad (18)\\
&\quad=\sum_{n=1}^{\infty}\frac{4(-1)^{n}\,x}{(n^2+x^2)^2}\\
&\quad=\pi\,\frac{\sinh(\pi x)+\pi x\cosh(\pi x)}{[x\sinh(\pi x)]^2}-\frac{2}{x^3}.
\end{aligned}$$

Setting $x=\frac{1}{2}$ in (17) and noting that

$$\sum_{m=1}^{k-1}\frac{1}{4m^2-1}=\frac{1}{2}\sum_{m=1}^{k-1}\left(\frac{1}{2m-1}-\frac{1}{2m+1}\right)=\frac{1}{2}\left(\frac{2k-2}{2k-1}\right),$$

we can obtain

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\frac{-24k^4+12k^3+26k^2+k+1}{(4k^2-1)^2}\prod_{m=1}^{k-1}\left(1-\frac{1}{4m^2}\right)$$
$$=8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(4n^2-1)^2}=4-\pi.$$

The identities (17) and (18) also allow us to express special cases of the generalized alternating Mathieu series from [16,17] in an Apéry-like fashion.

3. A Further Main Result

Our Theorem 4 below is concerned with a family of reciprocal series involving Fibonacci (Lucas) numbers and harmonic numbers.

Theorem 4. *For each $j \geqq 1$, each of the following identities holds true:*

$$\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{2k-j(k+1)}}{(2k+1)\binom{2k}{k}}h_{k+1}F_{j(k+1)}$$
$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\frac{(-1)^{j(k+1)}\sum_{m=0}^{k+1}\binom{k+1}{m}(-1)^{jm}\,2^{jm}\,F_{jm}}{\left[2^{2j}+2^j\,L_j+(-1)^j\right]^{k+1}} \tag{19}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{2k-j(k+1)}}{(2k+1)\binom{2k}{k}}h_{k+1}L_{j(k+1)}$$
$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\frac{(-1)^{j(k+1)}\sum_{m=0}^{k+1}\binom{k+1}{m}(-1)^{jm}\,2^{jm}\,L_{jm}}{\left[2^{2j}+2^jL_j+(-1)^j\right]^{k+1}}, \tag{20}$$

where

$$h_n=\sum_{k=0}^{n}\frac{1}{2k-1}=H_{2n}-\frac{1}{2}H_n,$$

and H_n given by

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

denotes the nth harmonic number. In particular, we have the following relations:

$$\sum_{k=0}^{\infty}\frac{(-1)^k\,2^k}{(2k+1)\binom{2k}{k}}h_{k+1}F_{k+1}=\frac{4}{5}\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2}\,\frac{1}{5^k}=\frac{1}{20}\Phi\left(\frac{1}{5},2,\frac{1}{4}\right) \tag{21}$$

as well as

$$\sum_{k=0}^{\infty}\frac{(-1)^k\,2^k}{(2k+1)\binom{2k}{k}}h_{k+1}L_{k+1}=\frac{4}{5}\sum_{k=0}^{\infty}\frac{1}{(4k+3)^2}\,\frac{1}{5^k}=\frac{1}{20}\Phi\left(\frac{1}{5},2,\frac{3}{4}\right), \tag{22}$$

where $\Phi(z,s,a)$ is the Lerch transcendent $($or the Hurwitz–Lerch function$)$ defined by (6)*.*

Proof. The following identity, which is valid for $-\frac{1}{2}<x<1$, is due to Ramanujan (see [18] [p. 293, Entry 34]):

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left(\frac{x}{1+x}\right)^{k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{2k}\,(k!)^2\,h_{k+1}}{(2k+1)!}\;x^{k+1}. \tag{23}$$

For $j \geqq 1$, if we first set

$$x = \left(\frac{\alpha}{2}\right)^j \qquad \text{and} \qquad x = \left(\frac{\beta}{2}\right)^j,$$

and then combine the two results according to the Binet form, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{1}{\sqrt{5}} \left[\left(\frac{\alpha^j}{2^j + \alpha^j} \right)^{k+1} - \left(\frac{\beta^j}{2^j + \beta^j} \right)^{k+1} \right] \\
&\qquad = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{1}{\sqrt{5}} \left(\frac{\left[(2\alpha)^j + (-1)^j\right]^{k+1} - \left[(2\beta)^j + (-1)^j\right]^{k+1}}{(2^{2j} + 2^j L_j + (-1)^j)^{k+1}} \right).
\end{aligned}$$

By applying the binomial theorem, we straightforwardly find that

$$\begin{aligned}
&\frac{1}{\sqrt{5}} \left(\left[(2\alpha)^j + (-1)^j\right]^{k+1} - \left[(2\beta)^j + (-1)^j\right]^{k+1} \right) \\
&\qquad = (-1)^{j(k+1)} \sum_{m=0}^{k+1} \binom{k+1}{m} (-1)^{jm} \, 2^{jm} \, F_{jm}.
\end{aligned}$$

As the right-hand side of the last equation is obvious, the proof of the first identity is completed. If $j = 1$, then we have

$$\begin{aligned}
(-1)^{k+1} \sum_{m=0}^{k+1} \binom{k+1}{m} (-1)^m \, 2^m \, F_m &= \frac{(-1)^{k+1}}{\sqrt{5}} \left[(-\sqrt{5})^{k+1} - (\sqrt{5})^{k+1} \right] \\
&= (\sqrt{5})^k (1 + (-1)^k).
\end{aligned}$$

Furthermore, the left-hand side of (23) simplifies to the following form:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{(\sqrt{5})^k \left[1 + (-1)^k\right]}{5^{k+1}} &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \, 5^{-\frac{k+2}{2}} \left[1 + (-1)^k\right] \\
&= \sum_{k=0}^{\infty} \frac{2}{(4k+1)^2} 5^{-(k+1)}.
\end{aligned}$$

The Lucas series is similar. This completes our demonstration of Theorem 4. □

4. Concluding Remarks and Observations

From the following known series representation of the Lerch transcendent (or the Hurwitz–Lerch zeta function) $\Phi(z, s, a)$ defined by (6):

$$\Phi(z, s, a) = \frac{1}{1-z} \sum_{n=0}^{\infty} \left(-\frac{z}{1-z} \right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+a)^s}$$

$$\left(s \in \mathbb{C}; \ \Re(z) < \frac{1}{2} \right),$$

we immediately get

$$\sum_{k=0}^{\infty} \frac{(-1)^k \, 2^k}{(2k+1)\binom{2k}{k}} \, h_{k+1} \, F_{k+1} = \sum_{k=0}^{\infty} (-1)^k \, 2^{-2k} \sum_{m=0}^{k} (-1)^m \, \frac{\binom{k}{m}}{(4m+1)^2} \tag{24}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k \, 2^k}{(2k+1)\binom{2k}{k}} \, h_{k+1} \, L_{k+1} = \sum_{k=0}^{\infty} (-1)^k \, 2^{-2k} \sum_{m=0}^{k} (-1)^m \, \frac{\binom{k}{m}}{(4m+3)^2}. \tag{25}$$

Moreover, the relation $L_{k+1} - F_{k+1} = 2F_k$ yields

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\, 2^k}{(2k+1)\binom{2k}{k}}\, h_{k+1}\, F_k = \frac{16}{5} \sum_{k=0}^{\infty} \frac{2k+1}{(4k+1)^2(4k+3)^2}\, \frac{1}{5^k}$$
$$= \frac{1}{40}\left(\Phi\left(\frac{1}{5}, 2, \frac{1}{4}\right) - \Phi\left(\frac{1}{5}, 2, \frac{3}{4}\right)\right). \tag{26}$$

The case $j = 2$ finally gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\binom{2k}{k}} h_{k+1} F_{2k+2} = \frac{4}{29} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{\sum_{m=0}^{k+1} \binom{k+1}{m} 4^m\, F_{2m}}{29^k} \tag{27}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\binom{2k}{k}} h_{k+1} L_{2k+2} = \frac{4}{29} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{\sum_{m=0}^{k+1} \binom{k+1}{m} 4^m\, L_{2m}}{29^k}. \tag{28}$$

It is remarkable that the inner sums on the right-hand sides of (27) and (28), involving the Fibonacci numbers and the Lucas numbers, respectively, do not seem to possess simple closed forms. It is possible, however, to express the right-hand sides in terms of the Lerch transcendent (or the Hurwitz–Lerch zeta function) as follows:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\binom{2k}{k}} h_{k+1} F_{2k+2}$$
$$= \frac{4}{29} \frac{1+4\alpha^2}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{1+4\alpha^2}{29}\right)^k$$
$$- \frac{4}{29} \frac{1+4\beta^2}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{1+4\beta^2}{29}\right)^k$$
$$= \frac{7+2\sqrt{5}}{29\sqrt{5}} \Phi\left(\frac{7+2\sqrt{5}}{29}, 2, \frac{1}{2}\right) - \frac{7-2\sqrt{5}}{29\sqrt{5}} \Phi\left(\frac{7-2\sqrt{5}}{29}, 2, \frac{1}{2}\right) \tag{29}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\binom{2k}{k}} h_{k+1} L_{2k+2}$$
$$= \frac{4(1+4\alpha^2)}{29} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{1+4\alpha^2}{29}\right)^k$$
$$+ \frac{4(1+4\beta^2)}{29} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\frac{1+4\beta^2}{29}\right)^k$$
$$= \frac{7+2\sqrt{5}}{29} \Phi\left(\frac{7+2\sqrt{5}}{29}, 2, \frac{1}{2}\right) + \frac{7-2\sqrt{5}}{29} \Phi\left(\frac{7-2\sqrt{5}}{29}, 2, \frac{1}{2}\right). \tag{30}$$

We conclude this article by remarking further that the Fibonacci and Lucas numbers, as well as other widely studied numbers and polynomials, are potentially useful in both pure and applied mathematical sciences (see, for example, [19–21]). Indeed, at least partially, the motivation for the results presented in this article can be derived from these as well as other developments in the literature.

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