mathematics

Article

Simultaneous Contour Method of a Trigonometric Integral by Prudnikov et al.

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Abstract: In this present work we derive, evaluate and produce a table of definite integrals involving logarithmic and exponential functions. Some of the closed form solutions derived are expressed in terms of elementary or transcendental functions. A substantial part of this work is new.

Keywords: entries of Gradshteyn and Ryzhik; Lerch function; Mellin transform

Citation: Reynolds, R.; Stauffer, A. Simultaneous Contour Method of a Trigonometric Integral by Prudnikov et al. *Mathematics* **2021**, *9*, 1453. https://doi.org/10.3390/math9121453

Academic Editor: Simeon Reich

Received: 31 March 2021
Accepted: 18 June 2021
Published: 21 June 2021

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1. Significance Statement

The Fourier cosine and Mellin transforms are used in a wide range of fields in science and engineering. Some of these applications are in software engineering [1], digital processing for the purposes of pattern recognition and Wiener filtering [2], distribution of sums of independent random variables [3], application to range radar profiles of naval vessels [4], gravity effect of a buried sphere and two-dimensional horizontal circular cylinder [5], and stress distribution in an infinite wedge [6].

The usage of these transforms is vast and covers all aspects of everyday life. In this present work, we aim to expand on the current compendium of tables of these transforms available to researchers in order to provide additional formula to AI in new research where these new formula are applicable. We also provide formal derivations of existing formula, which is a useful exercise for determining if these formula are correct. We derive new Fourier cosine and Mellin transforms and summarize these derivations in a table for easy perusal by interested readers and researchers.

2. Introduction

We apply the simultaneous contour integral method [7] to an integral in Prudnikov et al. and use it to derive closed forms for a Fourier cosine transform, Mellin transform, provide formal derivations for integrals in Gradshteyn and Ryzhik originally derived by Erdéyli et al. and Bierens de Haan, and present a table of definite integrals of results new and known.

In this paper, we derive the definite integral given by

$$\int_0^\infty \frac{e^{ax}\left(e^{imx}(\log(c)+ix)^k+e^{-imx}(\log(c)-ix)^k\right)}{(e^{ax}+1)^2}dx \tag{1}$$

and used it to achieve several objectives. We derive formula for the Mellin transform, Fourier cosine, and sine transform. We present formal derivations for some definite integrals in [8]. We also summarize our derivations with a table of transforms not listed in current literature. The Mellin transform is used in almost all areas of science and engineering, such as statistics and analysis of different kinds of geophysical data to name a few. Regarding our definite integral, the constants $a, k, c,$ and m in Equation (1) are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [7]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant [7] has the same value at the end points of the contour. This method involves using a form of Equation (2) then multiplying both sides by a function and then takes a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then, we multiply both sides of Equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2.1. Definite Integral of the Contour Integral

Using the method in [7] involving Cauchy's integral Equation (2), we replace y by $\log(c) + ix$ and multiply both sides by e^{mxi}. Then, we derive a second equation by replacing x by $-x$ and adding, followed by multiplying both sides by $\frac{e^{-ax}}{2(e^{-ax}+1)^2}$ to get

$$\begin{aligned}
&\int_0^\infty \frac{e^{-ax}\left(e^{imx}(\log(c)+ix)^k + e^{-imx}(\log(c)-ix)^k\right)}{4k!(e^{-ax}+1)^2} dx \\
&= \frac{1}{2\pi i} \int_0^\infty \int_C \frac{e^{-ax}c^w w^{-k-1}\cos(x(m+w))}{(e^{-ax}+1)^2} dwdx \\
&= \frac{1}{2\pi i} \int_C \int_0^\infty \frac{e^{-ax}c^w w^{-k-1}\cos(x(m+w))}{(e^{-ax}+1)^2} dxdw \\
&= \frac{1}{2\pi i} \int_C \frac{\pi c^w w^{-k-1}(m+w)\operatorname{csch}\left(\frac{\pi(m+w)}{a}\right)}{2a^2} dw
\end{aligned} \tag{3}$$

from Equation (2.5.34.7) in [9] where $0 < Im(\frac{w+m}{a})$. The logarithmic function is given, for example, in Section 4.1 in [10]. We are able to switch the order of integration over $w+m$ and x using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

2.2. The Lerch Function

We use Equation (1.11.3) in [11] where $\Phi(z,s,v)$ is the Lerch function, which is a generalization of the Hurwitz zeta and Polylogarithm functions. The Lerch function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, -2, -3, \ldots$, and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z} dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

2.3. Infinite Sum of the Contour Integral

2.3.1. Derivation of the First Contour Integral

In this section we will derive the contour integral given by

$$\frac{1}{2\pi i} \int_C \frac{\pi m c^w w^{-k-1}\operatorname{csch}\left(\frac{\pi(m+w)}{a}\right)}{2a^2} dw \tag{6}$$

Again, we use the method in [7]. Using Equation (2) replace y by $\frac{\pi(2y+1)}{a} + \log(c)$ and multiply both sides by $-\frac{\pi m}{a^2} e^{\frac{\pi m(2y+1)}{a}}$. Next, we take the infinite sum over $y \in [0, \infty)$ and simplify in terms of the Lerch function to get

$$
\begin{aligned}
&-\frac{2^k\pi^{k+1}m\left(\frac{1}{a}\right)^{k+2}e^{\frac{\pi m}{a}}\Phi\left(e^{\frac{2m\pi}{a}},-k,\frac{a\log(c)+\pi}{2\pi}\right)}{k!}\\
&=-\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{\pi m c^w w^{-k-1}e^{\frac{\pi(2y+1)(m+w)}{a}}}{a^2}dw\\
&=-\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{\pi m c^w w^{-k-1}e^{\frac{\pi(2y+1)(m+w)}{a}}}{a^2}dw\\
&=\frac{1}{2\pi i}\int_C\frac{\pi m c^w w^{-k-1}\operatorname{csch}\left(\frac{\pi(m+w)}{a}\right)}{2a^2}dw
\end{aligned} \tag{7}
$$

from (1.232.3) in [8] and $Im(\frac{m+w}{a})>0$ for convergence of the sum.

2.3.2. Derivation of the Second Contour Integral

In this section we will derive the contour integral given by

$$
\frac{1}{2\pi i}\int_C\frac{\pi c^w w^{-k}\operatorname{csch}\left(\frac{\pi(m+w)}{a}\right)}{2a^2}dw \tag{8}
$$

Again, we use the method in [7]. Using Equation (2), replace y by $\frac{\pi(2y+1)}{a}+\log(c)$ and multiply both sides by $-\frac{\pi}{a^2}e^{\frac{\pi m(2y+1)}{a}}$ and simplify in terms of the Lerch function to get

$$
\begin{aligned}
&-\frac{2^{k-1}\pi^k\left(\frac{1}{a}\right)^{k+1}e^{\frac{\pi m}{a}}\Phi\left(e^{\frac{2m\pi}{a}},1-k,\frac{a\log(c)+\pi}{2\pi}\right)}{(k-1)!}\\
&=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C\frac{\pi c^w w^{-k-1}e^{\frac{\pi(2y+1)(m+w)}{a}}}{2a^2}dw\\
&=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}\frac{\pi c^w w^{-k-1}e^{\frac{\pi(2y+1)(m+w)}{a}}}{2a^2}dw\\
&=\frac{1}{2\pi i}\int_C\frac{\pi c^w w^{-k}\operatorname{csch}\left(\frac{\pi(m+w)}{a}\right)}{2a^2}dw
\end{aligned} \tag{9}
$$

from (1.232.3) in [8] and $Im(\frac{m+w}{a})>0$ for convergence of the sum.

3. Main Results

Definite Integral in Terms of the Lerch Function

Theorem 1. *$k,c\in\mathbb{C}, Re(m/a)\le 0$,*

$$
\begin{aligned}
&\int_0^{\infty}\frac{e^{-ax}\left(e^{imx}(\log(c)+ix)^k+e^{-imx}(\log(c)-ix)^k\right)}{(e^{-ax}+1)^2}dx\\
&=k(2\pi)^k\left(\frac{1}{a}\right)^{k+1}\left(-e^{\frac{\pi m}{a}}\right)\Phi\left(e^{\frac{2m\pi}{a}},1-k,\frac{a\log(c)+\pi}{2\pi}\right)\\
&\quad-(2\pi)^{k+1}m\left(\frac{1}{a}\right)^{k+2}e^{\frac{\pi m}{a}}\Phi\left(e^{\frac{2m\pi}{a}},-k,\frac{a\log(c)+\pi}{2\pi}\right)
\end{aligned} \tag{10}
$$

Proof. Since the right-hand side of Equation (3) is equal to the sum of the right-hand sides of Equations (7) and (9), we can equate the left-hand sides and simplify the factorials. □

4. Derivation of a Fourier Cosine Transform

Theorem 2. *For all $k,c\in\mathbb{C}, Re(m/a)=0$,*

$$
\begin{aligned}
&\int_0^{\infty}\operatorname{sech}^2\left(\frac{ax}{2}\right)\cos(mx)\left((\log(c)-ix)^k+(\log(c)+ix)^k\right)dx\\
&=-2^{k+1}\pi^k\left(\frac{1}{a}\right)^{k+2}e^{-\frac{\pi m}{a}}\left(ak\Phi\left(e^{-\frac{2m\pi}{a}},1-k,\frac{a\log(c)+\pi}{2\pi}\right)\right.\\
&\qquad-2\pi m\Phi\left(e^{-\frac{2m\pi}{a}},-k,\frac{a\log(c)+\pi}{2\pi}\right)\\
&\quad\left.+e^{\frac{2\pi m}{a}}\left(ak\Phi\left(e^{\frac{2m\pi}{a}},1-k,\frac{a\log(c)+\pi}{2\pi}\right)+2\pi m\Phi\left(e^{\frac{2m\pi}{a}},-k,\frac{a\log(c)+\pi}{2\pi}\right)\right)\right)
\end{aligned} \tag{11}
$$

Proof. Use Equation (10) and form a second equation by replacing m by $-m$ and add to Equation (10) and simplify. Note we can derive the Fourier sine transform by taking the first partial derivative with respect to m of Equation (11). □

4.1. Derivation of a Mellin Transform

Theorem 3. *For all $k \in \mathbb{C}, Re(m/a) = 0$,*

$$\begin{aligned}\int_0^\infty x^{k+1}\operatorname{sech}^2(ax)\sin(mx)dx = \tfrac{1}{2}\pi^{k+1}\left(\tfrac{1}{a}\right)^{k+3} e^{-\frac{\pi m}{2a}} \sec\left(\tfrac{\pi k}{2}\right)\Big(\pi m\Phi\Big(e^{-\frac{m\pi}{a}},\\ -k-1,\tfrac{1}{2}\Big) - a(k+1)\Phi\Big(e^{-\frac{m\pi}{a}},-k,\tfrac{1}{2}\Big)\\ +e^{\frac{\pi m}{a}}\Big(\pi m\Phi\Big(e^{\frac{m\pi}{a}},-k-1,\tfrac{1}{2}\Big)\\ +a(k+1)\Phi\Big(e^{\frac{m\pi}{a}},-k,\tfrac{1}{2}\Big)\Big)\Big)\end{aligned} \tag{12}$$

Proof. Use Equation (11) replace a by $2a$ and set $c=1$ and simplify the left-hand side. □

4.2. Derivation of a Mellin Transform

Theorem 4. *For all $k \in \mathbb{C}, Re(m/a) = 0$,*

$$\begin{aligned}&\int_0^\infty x^k\operatorname{sech}^2\left(\tfrac{ax}{2}\right)\cos(mx)dx\\ &= (2\pi)^k\left(-\left(\tfrac{1}{a}\right)^{k+2}\right)e^{-\frac{\pi m}{a}}\sec\left(\tfrac{\pi k}{2}\right)\Big(ak\Phi\Big(e^{-\frac{2m\pi}{a}},1-k,\tfrac{1}{2}\Big)\\ &\qquad -2\pi m\Phi\Big(e^{-\frac{2m\pi}{a}},-k,\tfrac{1}{2}\Big)\\ &\qquad +e^{\frac{2\pi m}{a}}\Big(ak\Phi\Big(e^{\frac{2m\pi}{a}},1-k,\tfrac{1}{2}\Big)+2\pi m\Phi\Big(e^{\frac{2m\pi}{a}},-k,\tfrac{1}{2}\Big)\Big)\Big)\end{aligned} \tag{13}$$

Proof. Use Equation (12) and take the first partial derivative with respect to m and simplify. □

Derivation of Entry 4.118 in [8]

Theorem 5. *For allm $\in \mathbb{C}$,*

$$\int_0^\infty x\operatorname{sech}^2(x)\sin(mx)dx = \frac{1}{4}\pi\left(\pi m\coth\left(\frac{\pi m}{2}\right)-2\right)\operatorname{csch}\left(\frac{\pi m}{2}\right) \tag{14}$$

Proof. Use Equation (12) and set $k=0, a=1$ and use entries (2) and (3) in Table below (64:12:7) in [12] to simplify. □

Derivation of Entry 3.982.1 in [8]

Theorem 6. *For all $m \in \mathbb{C}$,*

$$\int_0^\infty \operatorname{sech}^2\left(\frac{ax}{2}\right)\cos(mx)dx = \frac{2\pi m\operatorname{csch}\left(\frac{\pi m}{a}\right)}{a^2} \tag{15}$$

Proof. Use Equation (13) and set $k=0$ and use entries (1) and (2) in Table below (64:12:7) in [12] to simplify. □

4.3. Derivation of an Integral with Combinations of Trigonometric and Rational Functions

Theorem 7. *For all $c \in \mathbb{C}, Re(m/a) = 0$,*

$$\begin{aligned}\int_0^\infty \frac{\operatorname{sech}^2\left(\frac{ax}{2}\right)\cos(mx)}{c^2+x^2}dx = \frac{e^{-\frac{\pi m}{a}}}{2\pi ac}\Big(&2\pi m\Phi\left(e^{-\frac{2m\pi}{a}},1,\frac{ac+\pi}{2\pi}\right)\\ &+a\Phi\left(e^{-\frac{2m\pi}{a}},2,\frac{ac+\pi}{2\pi}\right)\\ &+e^{\frac{2\pi m}{a}}\left(a\Phi\left(e^{\frac{2m\pi}{a}},2,\frac{ac+\pi}{2\pi}\right)\right.\\ &\left.-2\pi m\Phi\left(e^{\frac{2m\pi}{a}},1,\frac{ac+\pi}{2\pi}\right)\right)\Big)\end{aligned} \tag{16}$$

Proof. Use Equation (11) set $k=-1$ and replace c by e^c and simplify. □

4.4. Derivation of Logarithmic Functions and Powers

Theorem 8. *For all $c \in \mathbb{C}, Re(m/a) = 0$,*

$$\begin{aligned}\int_0^\infty \operatorname{sech}^2\left(\frac{ax}{2}\right)\log\left(c^2+x^2\right)\cos(mx)dx = \frac{e^{-\frac{\pi m}{a}}}{a^2}\Big(&-2e^{\frac{2\pi m}{a}}\left(a\Phi\left(e^{\frac{2m\pi}{a}},1,\frac{ac+\pi}{2\pi}\right)\right.\\ &\left.-2\pi m\Phi'\left(e^{\frac{2\pi m}{a}},0,\frac{ac+\pi}{2\pi}\right)\right)\\ &+4\pi m\left(\log\left(\frac{2\pi}{a}\right)\left(\coth\left(\frac{\pi m}{a}\right)+1\right)\right.\\ &\left.-\Phi'\left(e^{-\frac{2\pi m}{a}},0,\frac{ac+\pi}{2\pi}\right)\right)\\ &-2a\Phi\left(e^{-\frac{2m\pi}{a}},1,\frac{ac+\pi}{2\pi}\right)\Big)\end{aligned} \tag{17}$$

Proof. Use Equation (11) and take the first partial derivative with respect to k then set $k=0$ and use entries (2) and (3) in Table below (64:12:7) in [12] to simplify. □

5. A General Case

In this section, we will derive a formula for the general case for deriving equations in Table 4.376 in [8].

Theorem 9. *For all $k,\gamma,\beta \in \mathbb{C}, Re(m/a) = 0$,*

$$\begin{aligned}&\int_0^\infty x^k\log(x)\operatorname{sech}^2(ax)((2\gamma+1)\cosh(mx)-\beta x\sinh(mx))dx\\ &=\frac{1}{4}\pi^k\left(\frac{1}{a}\right)^{k+3}e^{-\frac{i\pi m}{2a}}\sec\left(\frac{\pi k}{2}\right)\left(-i\pi\beta\left(-2a(k+1)\Phi'\left(e^{-\frac{i\pi m}{a}},-k,\frac{1}{2}\right)\right.\right.\\ &-i\pi m\left(\pi\tan\left(\frac{\pi k}{2}\right)\Phi\left(e^{-\frac{im\pi}{a}},-k-1,\frac{1}{2}\right)-2\Phi'\left(e^{-\frac{i\pi m}{a}},-k-1,\frac{1}{2}\right)\right)\\ &+e^{\frac{i\pi m}{a}}\left(2a(k+1)\Phi'\left(e^{\frac{i\pi m}{a}},-k,\frac{1}{2}\right)-i\pi m\left(\Phi\left(e^{\frac{im\pi}{a}},-k-1,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)\right.\right.\right.\\ &\left.\left.\left.\left.+\pi\tan\left(\frac{\pi k}{2}\right)\right)-2\Phi'\left(e^{\frac{i\pi m}{a}},-k-1,\frac{1}{2}\right)\right)\right)\right)-2i\pi m\log\left(\frac{\pi}{a}\right)\Phi\left(e^{-\frac{im\pi}{a}},-k-1,\frac{1}{2}\right)\\ &+a\Phi\left(e^{-\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2(k+1)\log\left(\frac{\pi}{a}\right)+\pi(k+1)\tan\left(\frac{\pi k}{2}\right)+2\right)\\ &-ae^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2(k+1)\log\left(\frac{\pi}{a}\right)+\pi(k+1)\tan\left(\frac{\pi k}{2}\right)+2\right)\Big)\\ &-a(2\gamma+1)\left(-2ak\Phi'\left(e^{-\frac{i\pi m}{a}},1-k,\frac{1}{2}\right)+2i\pi m\Phi'\left(e^{-\frac{i\pi m}{a}},-k,\frac{1}{2}\right)\right.\\ &-2ake^{\frac{i\pi m}{a}}\Phi'\left(e^{\frac{i\pi m}{a}},1-k,\frac{1}{2}\right)-2i\pi me^{\frac{i\pi m}{a}}\Phi'\left(e^{\frac{i\pi m}{a}},-k,\frac{1}{2}\right)\\ &-i\pi m\Phi\left(e^{-\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)+\pi\tan\left(\frac{\pi k}{2}\right)\right)\\ &+i\pi me^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)+\pi\tan\left(\frac{\pi k}{2}\right)\right)\\ &+a\Phi\left(e^{-\frac{im\pi}{a}},1-k,\frac{1}{2}\right)\left(2k\log\left(\frac{\pi}{a}\right)+\pi k\tan\left(\frac{\pi k}{2}\right)+2\right)\\ &\left.+ae^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},1-k,\frac{1}{2}\right)\left(2k\log\left(\frac{\pi}{a}\right)+\pi k\tan\left(\frac{\pi k}{2}\right)+2\right)\right)\Big)\end{aligned} \tag{18}$$

Proof. Use Equation (12) and first replace m by mi, multiply both sides by $\frac{\beta}{i}$, and take the first partial derivative with respect to k to get

$$\begin{aligned}
&\int_0^\infty \beta x^{k+1}\log(x)\operatorname{sech}^2(ax)\sinh(mx)dx \\
&= \frac{1}{4}i\beta\pi^{k+1}\left(\frac{1}{a}\right)^{k+3} e^{-\frac{i\pi m}{2a}}\sec\left(\frac{\pi k}{2}\right)\left(-2a(k+1)\Phi'\left(e^{-\frac{i\pi m}{a}},-k,\frac{1}{2}\right)\right. \\
&-i\pi m\left(\pi\tan\left(\frac{\pi k}{2}\right)\Phi\left(e^{-\frac{im\pi}{a}},-k-1,\frac{1}{2}\right)-2\Phi'\left(e^{-\frac{i\pi m}{a}},-k-1,\frac{1}{2}\right)\right) \\
&+e^{\frac{i\pi m}{a}}\left(2a(k+1)\Phi'\left(e^{\frac{i\pi m}{a}},-k,\frac{1}{2}\right)\right. \\
&-i\pi m\left(\Phi\left(e^{\frac{im\pi}{a}},-k-1,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)+\pi\tan\left(\frac{\pi k}{2}\right)\right)-2\Phi'\left(e^{\frac{i\pi m}{a}},-k-1,\frac{1}{2}\right)\right)\Bigg) \\
&-2i\pi m\log\left(\frac{\pi}{a}\right)\Phi\left(e^{-\frac{im\pi}{a}},-k-1,\frac{1}{2}\right) \\
&+a\Phi\left(e^{-\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2(k+1)\log\left(\frac{\pi}{a}\right)+\pi(k+1)\tan\left(\frac{\pi k}{2}\right)+2\right) \\
&\left.-ae^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2(k+1)\log\left(\frac{\pi}{a}\right)+\pi(k+1)\tan\left(\frac{\pi k}{2}\right)+2\right)\right)
\end{aligned} \tag{19}$$

Next, using Equation (13), we multiply both sides by $2\gamma+1$, replace m by mi then take the first partial derivative with respect to m to get

$$\begin{aligned}
&\int_0^\infty (2\gamma+1)x^k\log(x)\operatorname{sech}^2(ax)\cosh(mx)dx \\
&= -\frac{1}{4}(2\gamma+1)\pi^k\left(\frac{1}{a}\right)^{k+2} e^{-\frac{i\pi m}{2a}}\sec\left(\frac{\pi k}{2}\right)\left(-2ak\Phi'\left(e^{-\frac{i\pi m}{a}},1-k,\frac{1}{2}\right)\right. \\
&+2i\pi m\Phi'\left(e^{-\frac{i\pi m}{a}},-k,\frac{1}{2}\right)-2ake^{\frac{i\pi m}{a}}\Phi'\left(e^{\frac{i\pi m}{a}},1-k,\frac{1}{2}\right) \\
&-2i\pi me^{\frac{i\pi m}{a}}\Phi'\left(e^{\frac{i\pi m}{a}},-k,\frac{1}{2}\right) \\
&-i\pi m\Phi\left(e^{-\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)+\pi\tan\left(\frac{\pi k}{2}\right)\right) \\
&+i\pi me^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},-k,\frac{1}{2}\right)\left(2\log\left(\frac{\pi}{a}\right)+\pi\tan\left(\frac{\pi k}{2}\right)\right) \\
&+a\Phi\left(e^{-\frac{im\pi}{a}},1-k,\frac{1}{2}\right)\left(2k\log\left(\frac{\pi}{a}\right)+\pi k\tan\left(\frac{\pi k}{2}\right)+2\right) \\
&\left.+ae^{\frac{i\pi m}{a}}\Phi\left(e^{\frac{im\pi}{a}},1-k,\frac{1}{2}\right)\left(2k\log\left(\frac{\pi}{a}\right)+\pi k\tan\left(\frac{\pi k}{2}\right)+2\right)\right)
\end{aligned} \tag{20}$$

Take the difference between Equations (19) and (20) and simplify. □

Derivation of Entry 4.376.2 in [8]

Using Equation (18), set $a=m=\beta=1$, replace k by μ and γ by $\mu/2$, and simplify in terms of the Hurwitz zeta function (12.7) in [13] using entry (4) in Table below (64:12:7) in [12] to get

$$\int_0^\infty x^\mu\log(x)\operatorname{sech}(x)(\mu-x\tanh(x)+1)dx = -2^\mu\pi^{\mu+1}\left(\zeta\left(-\mu,\frac{1}{4}\right)-\zeta\left(-\mu,\frac{3}{4}\right)\right)\sec\left(\frac{\pi\mu}{2}\right) \tag{21}$$

Derivation of Entry 4.376.3 in [8]

Using Equation (21) and simplifying in terms of the Bernoulli polynomials $B_\mu(x)$ using Equation (12.11.17) in [13], we get

$$\int_0^\infty x^\mu\log(x)\operatorname{sech}(x)(\mu-x\tanh(x)+1)dx = -\frac{e^{\frac{i\pi\mu}{2}}(2\pi)^{\mu+1}B_{\mu+1}\left(\frac{3}{4}\right)}{\mu+1} \tag{22}$$

Derivation of Entry 4.376.7 in [8]

Using Equation (18), replace β by a, k by $2n$, and γ by n and simplify using the Hurwitz zeta function (12.7) in [13] and entry (4) in Table below (64:12:7) in [12] to get

$$\begin{aligned}&\int_0^\infty x^{2n}\log(x)\operatorname{sech}(ax)(-ax\tanh(ax)+2n+1)dx\\&=-4^n\pi^{2n+1}\left(\frac{1}{a}\right)^{2n+1}\left(\zeta\left(-2n,\frac{1}{4}\right)-\zeta\left(-2n,\frac{3}{4}\right)\right)\sec(\pi n)\end{aligned} \tag{23}$$

Derivation of Entry 4.376.8 in [8]

For this entry, we derive the solution which is continued analytically using the Hurwitz zeta function from Equation (9.521.1) in [8].

Theorem 10. *For all $a,k\in\mathbb{C}$,*

$$\begin{aligned}&\int_0^\infty x^k\log(x)\operatorname{sech}^2(ax)(-2ax\tanh(ax)+2n+1)dx\\&=-4^{-k}\left(\frac{1}{a}\right)^{k+1}\left(\zeta(k)\Gamma(k)\left(2\left(2^k-2\right)k\log\left(\frac{\pi}{a}\right)(k-2n)-4\left(2^k-2\right)n\right.\right.\\&\left.+4k\left(2^k+k\log(2)-n\log(4)-2\right)+\pi\left(2^k-2\right)k(k-2n)\tan\left(\frac{\pi k}{2}\right)\right)\\&\left.-\left(2^k-2\right)k(2\pi)^k(k-2n)\sec\left(\frac{\pi k}{2}\right)\zeta'(1-k)\right)\end{aligned} \tag{24}$$

Proof. Use Equation (13) and multiply both sides by $-(2n+1)$ and set $m=0$, then take the first partial derivative with respect to k, simplify using entry (2) in Table below (64:7) and entry (4) in Table below (64:12:7) in [12] to get

$$\begin{aligned}&\int_0^\infty(-2n-1)x^k\log(x)\operatorname{sech}^2(ax)dx\\&=4^{-k}(2n+1)\left(\frac{1}{a}\right)^{k+1}\left(\zeta(k)\Gamma(k)\left(-2\left(\left(2^k-2\right)k\log\left(\frac{\pi}{a}\right)+2^k+k\log(4)-2\right)\right.\right.\\&\left.\left.-\pi\left(2^k-2\right)k\tan\left(\frac{\pi k}{2}\right)\right)+\left(2^k-2\right)k(2\pi)^k\sec\left(\frac{\pi k}{2}\right)\zeta'(1-k)\right)\end{aligned} \tag{25}$$

Next, use Equation (13), take the first partial derivative with respect to a and set $m=0$ and simplify using entry (2) in Table below (64:7) and entry (4) in Table below (64:12:7) in [12] to get

$$\begin{aligned}&\int_0^\infty 2ax^{k+1}\log(x)\tanh(ax)\operatorname{sech}^2(ax)dx\\&=\left(\frac{1}{a}\right)^{k+1}\left(4^{-k}\zeta(k)\Gamma(k)\left(2\left(\left(2^k-2\right)k(k+1)\log\left(\frac{\pi}{a}\right)+2^k+2k\left(2^k+k\log(2)-2+\log(2)\right)-2\right)\right.\right.\\&\left.\left.+\pi\left(2^k-2\right)k(k+1)\tan\left(\frac{\pi k}{2}\right)\right)-\left(2^k-2\right)k(k+1)\left(\frac{\pi}{2}\right)^k\sec\left(\frac{\pi k}{2}\right)\zeta'(1-k)\right)\end{aligned} \tag{26}$$

Next, add Equations (25) and (26) and simplify. □

Derivation of Entry 4.376.8 in [8]

Using Equation (24) and replace k by $2n$, and simplifying we get

$$\begin{aligned}&\int_0^\infty x^{2n}\log(x)\operatorname{sech}^2(ax)(-2ax\tanh(ax)+2n+1)dx\\&=\begin{cases}-4^{1-2n}(4^n-2)n\left(\frac{1}{a}\right)^{2n+1}\zeta(2n)\Gamma(2n), & \text{for } n\neq 0\\ -\frac{1}{a}, & \text{for } n=0\end{cases}\end{aligned} \tag{27}$$

Note the the formula quoted in [8] is in error.

5.1. Derivation of More Examples

Example 2

Using Equation (17) and setting $m=\pi i, c=1, a=\pi$, simplify to get

$$\int_0^\infty\log\left(x^2+1\right)(\pi x\tanh(\pi x)+1)\operatorname{sech}(\pi x)dx=-3+\pi+\log\left(\frac{81\Gamma\left(-\frac{3}{4}\right)^4}{64\Gamma\left(-\frac{1}{4}\right)^4}\right) \tag{28}$$

5.2. Integral in Terms of the Polylogarithmic Function

In this section we will derive formula using the Polylogarithms $Li_k(z)$ given in Section (25:12) in [12]. Polylogarithms are also known as Jonquière's functions (Ernest Jean Philippe Fauque de Jonquières, 1820–1901, French naval officer and mathematician), they appear in the Feynman diagrams of particle physics.

Theorem 11. *For all $c, k, m \in \mathbb{C}$,*

$$\begin{aligned} &\int_0^\infty \left((c-ix)^k+(c+ix)^k\right)\operatorname{sech}^2\left(\frac{\pi x}{2c}\right)\cos(mx)dx \\ &= -\frac{2^{k+1}c^{k+1}e^{-cm}}{\pi}\left(kLi_{1-k}\left(e^{2cm}\right)+e^{2cm}\left(kLi_{1-k}\left(e^{-2cm}\right)-2cmLi_{-k}\left(e^{-2cm}\right)\right)\right. \\ &\left.+2cmLi_{-k}\left(e^{2cm}\right)\right) \end{aligned} \tag{29}$$

Proof. Use Equation (11) and replace a by $\frac{\pi}{c}$ and simplify using Equation (64:12:2) in [12]. □

5.3. Definite Integral in Terms of the Hurwitz Zeta Function

Theorem 12. *For all $a, k, c \in \mathbb{C}$,*

$$\begin{aligned} &\int_0^\infty \operatorname{sech}\left(\frac{ax}{2}\right)\left((c-ix)^k+(c+ix)^k\right)dx \\ &= 2^{k+2}\pi^{k+1}\left(\frac{1}{a}\right)^{k+1}\left(2^k\zeta\left(-k,\frac{ac+\pi}{4\pi}\right)-2^k\zeta\left(-k,\frac{1}{2}\left(\frac{ac+\pi}{2\pi}+1\right)\right)\right) \end{aligned} \tag{30}$$

Proof. Use Equation (11) and replace m by $\frac{ai}{2}$ and simplify using entry (4) in Table below (64:12:7). □

Derivation of Entry 4.373.1 in [8]

Using Equation (30) and taking the first partial derivative with respect to k and setting $k=0$ simplifying using Equation (64:10:2) in [12], we get

$$\int_0^\infty \log\left(a^2+x^2\right)\operatorname{sech}(bx)dx = \frac{\pi}{b}\log\left(\frac{2\pi\Gamma\left(\frac{ab}{2\pi}+\frac{3}{4}\right)^2}{b\Gamma\left(\frac{2ab+\pi}{4\pi}\right)^2}\right) \tag{31}$$

Derivation of Entry 4.373.2 in [8]

Using Equation (31) and setting $a=1, b=\pi/2$ we get

$$\int_0^\infty \log\left(x^2+1\right)\operatorname{sech}\left(\frac{\pi x}{2}\right)dx = 2\log\left(\frac{4}{\pi}\right) \tag{32}$$

6. Definite Integral Involving the Arctangent Function in Terms of the Stieltjes Constant γ_1

Theorem 13. *For all $a, c \in \mathbb{C}$,*

$$\begin{aligned} &\int_0^\infty \frac{\operatorname{sech}\left(\frac{ax}{2}\right)\left(c\log\left(c^2+x^2\right)+2x\tan^{-1}\left(\frac{x}{c}\right)\right)}{c^2+x^2}dx \\ &= -\gamma_1\left(\frac{1}{4}\left(\frac{ac}{\pi}+3\right)\right)+\gamma_1\left(\frac{ac+\pi}{4\pi}\right) \\ &+\log\left(\frac{4\pi}{a}\right)\left(\psi^{(0)}\left(\frac{1}{4}\left(\frac{ac}{\pi}+3\right)\right)-\psi^{(0)}\left(\frac{ac+\pi}{4\pi}\right)\right) \end{aligned} \tag{33}$$

Proof. Use Equation (30) and take the first partial derivative with respect to k followed by applying L'Hopital's rule to the right-hand side as $k \to -1$ and simplify the left-hand side for the addition and subtraction of logarithmic functions and using Equation (64:4:1) in [12] and Equation (3.2) in [14]. Note the digamma function $\psi^0(z)$ is defined Section 6.3 in [10]. □

Example in Terms of the Riemann Zeta Function $\zeta(k)$

Theorem 14. *For all $k \in \mathbb{C}$,*

$$\int_0^\infty \left((1-ix)^k + (1+ix)^k\right) \operatorname{sech}\left(\frac{\pi x}{2}\right) dx = -2^{k+2}\left(2^{k+1}-1\right)\zeta(-k) \tag{34}$$

Proof. Use Equation (30) and replace c by $\frac{\pi}{a}$ and simplify using entry (2) in Table below (64:7) in [12]. □

Derivation of Entry 4.373.1 in [8]

Theorem 15. *For all $a, c \in \mathbb{C}$,*

$$\int_0^\infty \operatorname{sech}^2\left(\frac{ax}{2}\right) \log\left(c^2+x^2\right) dx = \frac{4\left(\psi^{(0)}\left(\frac{ac+\pi}{2\pi}\right) + \log\left(\frac{2\pi}{a}\right)\right)}{a} \tag{35}$$

Proof. Use Equation (11) set $m = 0$, replace c by e^c and take the first partial derivative with respect to k followed by applying L'Hopital's rule to the right-hand side as $k \to 0$ and simplify using Equation (64:4:1) and entry (4) in Table below (64:12:7) in [12]. □

Derivation of Entry 4.372.2 in [8]

Using Equation (35) and setting $c = 0, a = 2$ and simplifying we get

$$\int_0^\infty \log(x) \operatorname{sech}^2(x) dx = -\gamma - \log\left(\frac{4}{\pi}\right) \tag{36}$$

where γ is Euler's constant given in Section 8.367 in [8].

Please See Table 1 for the table of integrals.

Table 1. Table of Integrals.

$f(x)$	$\int_0^\infty f(x)dx$
$x\operatorname{sech}^2(x)\sin(mx)$	$\frac{1}{4}\pi\left(\pi m \coth\left(\frac{\pi m}{2}\right) - 2\right)\operatorname{csch}\left(\frac{\pi m}{2}\right)$
$\operatorname{sech}^2\left(\frac{ax}{2}\right)\cos(mx)$	$\frac{2\pi m}{a^2}\operatorname{csch}\left(\frac{\pi m}{a}\right)$
$x^\mu \log(x)\operatorname{sech}(x)(\mu - x\tanh(x)+1)$	$-\frac{1}{\mu+1}e^{\frac{i\pi\mu}{2}}(2\pi)^{\mu+1}B_{\mu+1}\left(\frac{3}{4}\right)$
$x^\mu \log(x)\operatorname{sech}(x)(\mu - x\tanh(x)+1)$	$-2^\mu \pi^{\mu+1}\left(\zeta\left(-\mu,\frac{1}{4}\right) - \zeta\left(-\mu,\frac{3}{4}\right)\right)\sec\left(\frac{\pi\mu}{2}\right)$
$x^{2k}\log(x)\operatorname{sech}(ax)(-ax\tanh(ax)+2k+1)$	$-4^k\pi^{2k+1}\left(\frac{1}{a}\right)^{2k+1}\left(\zeta\left(-2k,\frac{1}{4}\right) - \zeta\left(-2k,\frac{3}{4}\right)\right)\sec(\pi k)$
$x^{2n}\log(x)\operatorname{sech}^2(ax)(-2ax\tanh(ax)+2n+1)$	$-4^{1-2n}(4^n-2)n\left(\frac{1}{a}\right)^{2n+1}\zeta(2n)\Gamma(2n)$
$\log(x^2+1)(\pi x\tanh(\pi x)+1)\operatorname{sech}(\pi x)$	$-3+\pi+\log\left(\frac{81\Gamma\left(-\frac{3}{4}\right)^4}{64\Gamma\left(-\frac{1}{4}\right)^4}\right)$
$\operatorname{sech}(ax)(\log(c-ix)+\log(c+ix))$	$\frac{\pi}{a}\log\left(\frac{2\pi\Gamma\left(\frac{ac}{2\pi}+\frac{3}{4}\right)^2}{a\Gamma\left(\frac{2ac+\pi}{4\pi}\right)^2}\right)$
$\log(a^2+x^2)\operatorname{sech}(bx)$	$\frac{\pi}{b}\log\left(\frac{2\pi\Gamma\left(\frac{ab}{2\pi}+\frac{3}{4}\right)^2}{b\Gamma\left(\frac{2ab+\pi}{4\pi}\right)^2}\right)$
$\log(x^2+1)\operatorname{sech}\left(\frac{\pi x}{2}\right)$	$2\log\left(\frac{4}{\pi}\right)$
$\left((1-ix)^k+(1+ix)^k\right)\operatorname{sech}\left(\frac{\pi x}{2}\right)$	$-2^{k+2}\left(2^{k+1}-1\right)\zeta(-k)$
$\operatorname{sech}^2\left(\frac{ax}{2}\right)\log(c^2+x^2)$	$\frac{4}{a}\left(\psi^{(0)}\left(\frac{ac+\pi}{2\pi}\right)+\log\left(\frac{2\pi}{a}\right)\right)$
$\log(x)\operatorname{sech}^2(x)$	$-\gamma-\log\left(\frac{4}{\pi}\right)$

7. Mathematica Code

In this section, we give the Mathematica code [15] for the Table in Section 20. In some cases Mathematica is not able to evaluate the integrals, in which cases our formulae are provided.

```
Integrate[Cos[m x] Sech[(a x)/2]^2, {x, 0, \[Infinity]}]

ConditionalExpression[(2 m \[Pi] Csch[(m \[Pi])/a])/a^2,
 Im[Global`m] <
   Re[Global`a] && (Re[(-1)^(1/Global`a)] >= 1 ||
    Re[(-1)^(1/Global`a)] <= 0 || (-1)^(1/Global`a) \[NotElement]
      Reals) && Im[Global`m] + Re[Global`a] > 0]

Integrate[x Sech[x]^2 Sin[m x], {x, 0, \[Infinity]}]

 ConditionalExpression[
 1/4 \[Pi] (-2 + m \[Pi] Coth[(m \[Pi])/2]) Csch[(m \[Pi])/2], -2 <
  Im[m] < 2]

Integrate[
 x^\[Mu] Log[x] Sech[x] (1 + \[Mu] - x Tanh[x]), {x,
  0, \[Infinity]}], -2^\[Mu] \[Pi]^(
 1 + \[Mu]) (HurwitzZeta[-\[Mu], 1/4] -
   HurwitzZeta[-\[Mu], 3/4]) Sec[(\[Pi] \[Mu])/2]

Integrate[
 x^\[Mu] Log[x] Sech[x] (1 + \[Mu] - x Tanh[x]), {x,
  0, \[Infinity]}], -((E^((I \[Pi] \[Mu])/2) (2 \[Pi])^(1 + \[Mu]))/(1 + \[Mu]))
   BernoulliB[1 + \[Mu], 3/4]

Integrate[
 x^(2 n) Log[x] Sech[a x] (1 + 2 n - a x Tanh[a x]), {x,
  0, \[Infinity]}], -4^n (1/a)^(1 + 2 n) \[Pi]^(
 1 + 2 n) (HurwitzZeta[-2 n, 1/4] - HurwitzZeta[-2 n, 3/4]) Sec[
  n \[Pi]]

Integrate[
 x^\[Mu] Log[x] Sech[x] (1 + \[Mu] - x Tanh[x]), {x,
  0, \[Infinity]}], -2^\[Mu] \[Pi]^(
 1 + \[Mu]) (HurwitzZeta[-\[Mu], 1/4] -
   HurwitzZeta[-\[Mu], 3/4]) Sec[(\[Pi] \[Mu])/2]

Integrate[
 x^(2 k) Log[x] Sech[a x] (1 + 2 k - a x Tanh[a x]), {x,
  0, \[Infinity]}], -4^k (1/a)^(1 + 2 k) \[Pi]^(
 1 + 2 k) (HurwitzZeta[-2 k, 1/4] - HurwitzZeta[-2 k, 3/4]) Sec[
  k \[Pi]]

Integrate[
 x^(2 n) Log[x] Sech[a x]^2 (1 + 2 n - 2 a x Tanh[a x]), {x,
  0, \[Infinity]}], -4^(1 - 2 n) (-2 + 4^n) (1/a)^(
 1 + 2 n) n Gamma[2 n] Zeta[2 n]

Integrate[
 Log[1 + x^2] Sech[\[Pi] x] (1 + \[Pi] x Tanh[\[Pi] x]), {x,
  0, \[Infinity]}], -3 + \[Pi] +
 Log[81/64 (Gamma[-(3/4)]/Gamma[-(1/4)])^4]

Integrate[(Log[c - I x] + Log[c + I x]) Sech[a x], {x,
  0, \[Infinity]}], (\[Pi] Log[(
```

```
  2 \[Pi] Gamma[3/4 + (a c)/(2 \[Pi])]^2)/(
  a Gamma[(2 a c + \[Pi])/(4 \[Pi])]^2)])/a

Integrate[
 Log[a^2 + x^2] Sech[b x], {x, 0, \[Infinity]}], (\[Pi] Log[(
  2 \[Pi] Gamma[3/4 + (a b)/(2 \[Pi])]^2)/(
  b Gamma[(2 a b + \[Pi])/(4 \[Pi])]^2)])/b

Integrate[
 Log[1 + x^2] Sech[(\[Pi] x)/2], {x, 0, \[Infinity]}], 2 Log[4/\[Pi]]

Integrate[((1 - I x)^k + (1 + I x)^k) Sech[(\[Pi] x)/2], {x,
  0, \[Infinity]}], -2^(2 + k) (-1 + 2^(1 + k)) Zeta[-k]

Integrate[
 Log[x^2 + c^2] Sech[(a x)/2]^2, {x,
  0, \[Infinity]}], (4 (Log[(2 \[Pi])/a] +
   PolyGamma[0, (\[Pi] + a c)/(2 \[Pi])]))/a

Integrate[
 Log[x] Sech[x]^2, {x, 0, \[Infinity]}], -(EulerGamma + Log[4/\[Pi]])
```

8. Discussion

In this work, using our contour integration method, we achieve a few objectives from the definite integral in terms of the Lerch function formula. Firstly, we derived Fourier cosine and Fourier sine transforms along with two Mellin transforms. Furthermore we produce formal derivations for a few definite integrals in the book of Gradshteyn and Ryzhik [8]. We also derive a few new interesting definite integrals in terms of fundamental constants. We also provide extended evaluation for integrals through analytic continuation where possible. Finally, we summarized our work in the form a table of integrals for easy perusal by interested readers. We will be using our method to derive more formulae to produce more tables of integrals. We used Wolfram Mathematica to assist with the numeric evaluation of these formulae for complex values of the parameters.

Author Contributions: Conceptualization, R.R.; Funding acquisition, A.S.; Supervision, A.S. Both authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by Natural Sciences and Engineering Research Council of Canada (NSERC), Grant No. 504070.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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